ON THE METHOD OF IDENTITIES

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1. Introduction

The method of identities is a way of using partition calculus for proving compactness results for logics with cardinality quantifiers. In 1957 Andrzei Mostowski introduced the extension $L(\exists^{\geq\kappa})$ of first order logic. Here $\exists^{\geq\kappa}$ is the generalized quantifier

$$\mathfrak{M} \models \exists^{\geq\kappa} x\phi(x,\vec{a}) \iff |\{b \in M : \mathfrak{M} \models \phi(b,\vec{a})\}| \geq \kappa.$$

Mostowski asked whether $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact (i.e. every countable set of sentences, every finite subset of which has a model, has itself a model) and observed that $L(\exists^{\geq\aleph_0})$ is not. In 1963 Gerhard Fuhrken [5] proved that $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact if $\aleph_0$ is small for κ (i.e. if $\lambda_n < \kappa$ for $n < \omega$, then $\prod_{n<\omega} \lambda_n < \kappa$). His proof was based on the observation that the usual Łoś Lemma

$$\prod_{n<\omega} \mathfrak{M}_n/F \models \phi \iff \{n < \omega : \mathfrak{M}_n \models \phi\} \in F$$

for ultrafilters F on ω and first order sentences ϕ can be proved for $\phi \in L(\exists^{\geq\kappa})$ if $\aleph_0$ is small for κ. The $\aleph_0$-compactness follows from the Łoś Lemma immediately.

Vaught [19] proved $\aleph_0$-compactness of $L(\exists^{\geq\aleph_1})$ by proving what is now known as Vaught's Two-Cardinal Theorem and Chang [3] extended this to $L(\exists^{\geq\kappa^+})$ by proving $(\omega_1,\omega) \to (\kappa^+,\kappa)$, when $\kappa^{<\kappa} = \kappa$. Jensen [6] extended this to all κ under the assumption GCH+$\Box_k$, which he showed to follow from $V = L$. Keisler [7] proved with a different method $\aleph_0$-compactness of $L(\exists^{\geq\kappa})$ when κ is a singular strong limit cardinal. This led to the important observation that if V=L holds and every regular cardinal is a successor cardinal (i.e. there are no weakly inaccessible cardinals), then $L(\exists^{\geq\kappa})$ is $\aleph_0$-compact for all $\kappa > \omega$. We still do not know if this is provable in ZFC:

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Open Problem: *Is it provable in ZFC that* $L(\exists^{\geq\kappa})$ *is* $\aleph_0$*-compact for all* $\kappa > \omega$*? In particular, is it provable in ZFC that* $L(\exists^{\geq\aleph_2})$ *is* $\aleph_0$*-compact?*

The best result today towards solving this problem is:

Theorem 1. *(*[14]*) It is consistent, relative to the consistency of ZF that* $L(\exists^{\geq\aleph_1}, \exists^{\geq\aleph_2})$ *is not* $\aleph_0$*-compact.*

In this paper we investigate the more general question involving an infinite sequence $(\kappa_n)_{n<\omega}$ of uncountable cardinals:

Question: *For which sequences* $(\kappa_n)_{n<\omega}$ *of uncountable cardinals is the logic* $L(\exists^{\geq\kappa_n})_{n<\omega}$ $\aleph_0$*-compact?*

As the preceding discussion indicates we cannot expect a general solution in ZFC. Extreme cases are

(1) $\kappa_n = \aleph_1$ for all $n < \omega$.
(2) $\aleph_0$ is small for each κ_n.
(3) Some κ_n is the supremum of a subset of the others.

in which case we have a trivial solution (in case 2 we have Łoś Lemma and therefore $\aleph_0$-compactness, and in case 3 we have an easy counter-example to $\aleph_0$-compactness).

Let us call a logic **recursively compact** if every recursive set of sentences, every finite subset of which has a model, itself has a model. Naturally this concept is meaningful only for logics which possess a canonical Gödel numbering of its sentences. Let us call a logic **recursively axiomatizable** if the set of (Gödel numbers of) valid sentences of the logic is recursively enumerable. By a result of Per Lindström [9] any recursively axiomatizable logic of the form $L(\exists^{\geq\kappa_n})_{n\leq m}$ is actually recursively compact. This raises the question:

Question: *For which sequences* $(\kappa_n)_{n<\omega}$ *of uncountable cardinals is the logic* $L(\exists^{\geq\kappa_n})_{n<\omega}$ *recursively axiomatizable?*

We give in this paper an axiomatization $\mathcal{A}$ of $L(\exists^{\geq\kappa_n})_{n<\omega}$. We do not know in general whether this $\mathcal{A}$ is recursive (or r.e.). In fact, we do not even know if $\mathcal{A}$ is complete in the following sense: A logic L^* endowed with a set A^* of axioms and rules is **complete** if a sentence of the logic is valid (true in all models) if and only it it follows from the axioms of A^* by means of the rules of A^*. We give a combinatorial characterization of sequences $(\kappa_n)_{n<\omega}$ for which the $\mathcal{A}$ is complete.

In the presence of an axiomatization $\mathcal{A}$ we can redefine our compactness properties. Rather than requiring that every finite subtheory has a model we can require that every finite subtheory is **$\mathcal{A}$-consistent** in the sense that no contradiction can be derived from it by means of the axioms and rules of $\mathcal{A}$. It turns out that this change is not significant in the sense that in our main result we could use either. However, this modified concept of compactness reveals an interesting connection between completeness and compactness: we can think of completeness (every consistent sentence has a model) as a compactness property of one-element theories. In this sense recursive compactness is a strengthening of completeness.

Since we are discussing properties of logics in relation to their axiomatizations, if it useful to introduce a new concept: A **logic frame** is a pair $\langle L^*, A^*\rangle$ where L^* is a logic in the usual sense of abstract model theory and A^* is a set of axioms and rules for L^*. The idea is that A^* is an axiomatization for L^* but in many cases we do not know (in ZFC) whether the axiomatization is a genuine axiomatization in the sense that it is complete.

Let us call a logic frame $\langle L^*, A^*\rangle$ **recursively compact** if every set of sentences which is recursive in A^*, every finite subset of which has a model, itself has a model. We are consciously liberal here as to what recursiveness in A^* means as it is not really relevant for our results. If A^* is a set of sentences and L^* has a canonical Gödel numbering of its sentences, then recursive in A^* means recursive in the set of Gödel numbers of A^*. Let us say that a logic frame $\langle L^*, A^*\rangle$ has

- **Finite character** if $\langle L^*, A^*\rangle$ is $\aleph_0$-compact in every inner model and forcing extension in which it is complete.
- **Recursive character** if $\langle L^*, A^*\rangle$ is $\aleph_0$-compact in every inner model and forcing extension in which it is recursively compact.

For example, if A^* is the Keisler axiomatization (from [8]) for $L(\exists^{\geq\aleph_1})$, then it is consistent that $\langle L(\exists^{\geq\aleph_2}), A^*\rangle$ is complete (follows from GCH), and it is also consistent that $\langle L(\exists^{\geq\aleph_2}), A^*\rangle$ is incomplete (follows from $(\aleph_1, \aleph_0) \not\twoheadrightarrow (\kappa^+, \kappa)$ which is consistent by [10]). However, we know it has provably finite character (Proposition 24).

The main result of this paper (proved in Section 5.2) is the following:

Theorem 2. *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. There is an axiomatization $\mathcal{A}$ of $L(\exists^{\geq\kappa_n})_{n<\omega}$ such that the logic frame $\langle L(\exists^{\geq\kappa_n})_{n<\omega}, \mathcal{A}\rangle$ has recursive character.*

It is noteworthy that the above theorem is a result in ZFC. The proof is based on formulating a partition theoretic equivalent condition for the $\aleph_0$-compactness (equivalently recursive compactness) of $L(\exists^{\geq\kappa_n})_{n<\omega}$.

The question of character is, of course, relevant for any logic. Let us consider as an example the **stationary logic**. Let

$$\mathfrak{M} \models Q^{St}xy\phi(x,y,\vec{a}) \iff \begin{array}{l}\{(a,b) : \mathfrak{M} \models \phi(a,b,\vec{a})\} \text{ is an } \aleph_1\text{-like} \\ \text{linear order with a stationary set of} \\ \text{initial segments which have a supremum.}\end{array}$$

We say that $\langle F_\alpha : \alpha < \omega_1 \rangle$ is a **filtration** of a linear order D if every F_α is an initial segment of D, $\alpha < \beta$ implies $F_\alpha \subseteq F_\beta$, and $F_\nu = \bigcup_{\alpha<\nu} F_\alpha$ for limit ν. Let

$$\mathfrak{M} \models Q^{St}_S xy\phi(x,y,\vec{a}) \iff \begin{array}{l}\{(a,b) : \mathfrak{M} \models \phi(a,b,\vec{a})\} \text{ is an } \aleph_1\text{-like} \\ \text{linear order with a filtration } \langle F_\alpha : \alpha < \omega_1 \rangle \\ \text{such that } \{\alpha : F_\alpha \text{ has a supremum}\} = S.\end{array}$$

Let us say stationary sets $S_n \subseteq \omega_1$ are **independent** if any non-trivial finite boolean combination of them is stationary. For such S_n there is a natural recursive axiomatization $\mathcal{A}_{aa}$ of $L(\exists^{\geq\aleph_1}, Q^{St}, Q^{St}_{S_n})_{n<\omega}$ based on the Pressing Down Lemma (as in [2]). The axiomatization is independent of the choice of the sets $S_n \subseteq \omega_1$ as long as they are stationary and independent. Moreover, the axiomatization $\mathcal{A}_{aa}$ is complete. However, it is shown in [18] that there are independent stationary sets $S_n \subseteq \omega_1$ such that $L(\exists^{\geq\aleph_1}, Q^{St}, Q^{St}_{S_n})_{n<\omega}$ is recursively compact but not $\aleph_0$-compact.

2. (κ, λ)-models

There is a basic reduction of generalized quantifiers of the form $\exists^{\geq\kappa}$ to first order logic. This was established by Fuhrken [4]. A model $\langle M, \ldots, A, <, \ldots\rangle$ is called **λ-like** if $\langle A, < \rangle$ is a λ-like linear order (i.e. of cardinality λ with all initial segments of cardinality $< \lambda$). Fuhrken established a canonical translation $\phi \mapsto \phi^+$ of $L(\exists^{\geq\kappa})$ to first order logic so that

$$\phi \text{ has a model} \iff \phi^+ \text{ has a } \kappa\text{-like model.}$$

Thus the questions of axiomatization and $\aleph_0$-compactness of $L(\exists^{\geq\kappa})$ were reduced to questions of axiomatization and $\aleph_0$-compactness of first order logic restricted to κ-like models.

If $\kappa = \lambda^+$ the reduction is slightly simpler. Then we can use **(κ, λ)-models**, i.e. models $\langle M, \ldots, A, \ldots \rangle$, where $|M| = \kappa$ and $|A| = \lambda$. The study of model theory of (κ, λ)-models makes, of course, sense also if $\kappa \neq \lambda^+$ even if this more general case does not arise from a reduction of $L(\exists^{\geq \kappa})$.

Let $\kappa > \lambda$. What would be a natural way to construct a (κ, λ)-model for a first order theory T in a vocabulary L? It makes sense to assume $|L| \leq \lambda$. Let L^* be the Skolem-expansion of L and T^* the Skolem-closure of T. Let c_α, $\alpha < \kappa$, be new constant symbols. Let P be the predicate symbol the interpretation of which we want to have cardinality λ. Consider the axioms

(T1) T^*.
(T2) $c_\alpha \neq c_\beta$ for $\alpha < \beta < \kappa$.
(T3) $P(c_\alpha)$ for $\alpha < \lambda$.

If T has for all $n < \omega$ a model with P of cardinality $\geq n$, then (T1)-(T3) have a model $\mathfrak{M}$ and the Skolem-hull $\mathfrak{N}$ of $\mathfrak{M}$ has cardinality κ. In $\mathfrak{N}$ the predicate P has cardinality $\geq \lambda$. Nothing that we have said so far prevents it from having cardinality $> \lambda$. Following an idea of Morley [11], further developed in [15], we fix an equivalence relation E on $[\kappa]^{<\omega}$ such that

(E1) Equivalent sets have the same cardinality.
(E2) There are at most λ equivalence classes.

Such an equivalence relation on $[\kappa]^{<\omega}$ is called a **(κ, λ)-pattern**. Now we add the following axioms to (T1)-(T3):

(T4)

$$t(c_{\alpha_0}, \ldots, c_{\alpha_n}) = \\ t(c_{\beta_0}, \ldots, c_{\beta_n}) \vee (\neg P(t(c_{\alpha_0}, \ldots, c_{\alpha_n})) \wedge \neg P(t(c_{\beta_0}, \ldots, c_{\beta_n})))$$

for all Skolem-terms t and all $\alpha_0 < \ldots < \alpha_n < \kappa, \beta_0 < \ldots < \beta_n < \kappa$ such that $\{\alpha_0, \ldots, \alpha_n\} E \{\beta_0, \ldots, \beta_n\}$.

Suppose E is chosen so that (T1)-(T4) have a model $\mathfrak{M}$. Let $\mathfrak{N}$ be again the Skolem-hull of $\mathfrak{M}$. Every element in $P^{\mathfrak{N}}$ is of the form $t(c_{\alpha_0}, \ldots, c_{\alpha_n})^{\mathfrak{N}}$ for some $\alpha_0 < \ldots < \alpha_n < \kappa$. This value of the Skolem-term depends only on the equivalence class of $\{\alpha_0, \ldots, \alpha_n\}$. As there are only $\leq \lambda$ equivalence classes, $P^{\mathfrak{N}}$ has cardinality exactly λ. Summa summarum, if we find an equivalence relation E such that (T1)-(T4) have a model, we get a (κ, λ)-model for the theory T.

How to find the (κ, λ)-pattern E? This is a crucial question. We can first observe that although we have to know E before we can write down (T1)-(T4), we only need to show that every finite subset of (T1)-(T4) has a model. So let us assume our starting theory T has the property that every finite subset has a (κ, λ)-model. Let Σ be an arbitrary finite subset of (T1)-(T4). Let $\mathfrak{M}$ be a (κ, λ)-model of $\Sigma \cap T^*$. Let $\{c_{\gamma_0}, \ldots, c_{\gamma_m}\}$ be the set of constants c_α occurring in Σ and let $D = \{\gamma_0, \ldots, \gamma_m\}$. Let us expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ by adding interpretations to all the constants c_α, $\alpha < \kappa$, in such a way that all sentences of the form (T2) or (T3) are satisfied in $\mathfrak{M}'$. A priori, we cannot be sure that the sentences (T4) are satisfied by the new constants, but after all, we need an interpretation for the constants $\{c_{\gamma_0}, \ldots, c_{\gamma_m}\}$ only. So we try to find interpretations for these constants from among the κ elements $c_\alpha^{\mathfrak{M}'}$, $\alpha < \kappa$.

The model $\mathfrak{M}'$ and Σ **induce** in a canonical way a (κ, λ)-pattern E' as follows:

$$\begin{aligned}
&\{\alpha_0, \ldots, \alpha_n\} E' \{\beta_0, \ldots, \beta_n\} \iff \\
&\mathfrak{M}' \models t(c_{\alpha_0}, \ldots, c_{\alpha_n}) = t(c_{\beta_0}, \ldots, c_{\beta_n}) \vee \\
&(\neg P(t(c_{\alpha_0}, \ldots, c_{\alpha_n})) \wedge \neg P(t(c_{\beta_0}, \ldots, c_{\beta_n}))) \\
&\text{for all Skolem-terms } t \text{ occurring in } \Sigma.
\end{aligned}$$

Since Σ is finite, E' is indeed a (κ, λ)-pattern. What we need now is a one-one mapping

$$\pi : \{\gamma_0, \ldots, \gamma_m\} \to \kappa$$

such that

$$\{\alpha_0, \ldots, \alpha_n\} E \{\beta_0, \ldots, \beta_n\} \to \{\pi\alpha_0, \ldots, \pi\alpha_n\} E' \{\pi\beta_0, \ldots, \pi\beta_n\} \tag{1}$$

for all elements $\alpha_0 < \ldots < \alpha_n$ and $\beta_0 < \ldots < \beta_n$ of $\{\gamma_0, \ldots, \gamma_m\}$. If such a mapping π is found and we define

$$c_{\gamma_i}^{\mathfrak{M}''} = c_{\pi\gamma_i}^{\mathfrak{M}'}$$

for $i \leq n$, we get an expansion $\mathfrak{M}''$ of $\mathfrak{M}$ which satisfies Σ.

Thus we have to choose the (κ, λ)-pattern E already in the beginning so that for every $D \in [\kappa]^{<\omega}$ and every (κ, λ)-pattern E' there is a one-one mapping $\pi : D \to \kappa$ such that (1) holds for all $\alpha_0 < \ldots < \alpha_n$ and $\beta_0 < \ldots < \beta_n$ in D. Such a (κ, λ)-pattern is called **fundamental**.

To prove the existence of fundamental (κ, λ)-patterns we introduce the important concept of an identity. An **identity** is an equivalence relation I on $\mathcal{P}(D_I)$ for some finite set D_I such that equivalent sets have the same cardinality. An example of an identity is the restriction $E \restriction D$ of a (κ, λ)-pattern to a finite $D \subseteq \kappa$.

An identity I is a **subidentity** of another identity I', in symbols $I \leq I'$ if there is a one-one mapping $\pi : D_I \to D_{I'}$ such that

$$\{\alpha_0, \ldots, \alpha_n\} I \{\beta_0, \ldots, \beta_n\} \to \{\pi\alpha_0, \ldots, \pi\alpha_n\} I' \{\pi\beta_0, \ldots, \pi\beta_n\}$$

holds for elements $\alpha_0 < \ldots < \alpha_n$ and $\beta_0 < \ldots < \beta_n$ of D_I. We write $I \equiv I'$ is both $I \leq I'$ and $I' \leq I$. Thus a (κ, λ)-pattern E is fundamental if and only if for every finite $D \subseteq \kappa$ and for every (κ, λ)-pattern E' there is a finite $D' \subseteq \kappa$ such that $E \restriction D \leq E' \restriction D'$. The set $I(\kappa, \lambda)$ is defined as the set of identities I such that for every (κ, λ)-pattern E there is a finite $D \subseteq \kappa$ such that $I \leq E \restriction D$.

Lemma 3. [15] *If $\lambda^\omega = \lambda$ and $\kappa \geq \lambda$, then there is a fundamental (κ, λ)-pattern.*

Proof. Let I_n, $n < \omega$, be the list of all (up to $\equiv$) identities I_n such that for some (κ, λ)-pattern E_n we have $I_n \not\leq E_n \restriction D$ for all $D \in [\kappa]^{<\omega}$. Since $\lambda^\omega = \lambda$, we can easily construct a (κ, λ)-pattern E such that for all D and all E_n there is a finite $f_n(D)$ such that $E \restriction D \leq E_n \restriction f_n(D)$. Now E is fundamental, for suppose there were a finite D such that for some (κ, λ)-pattern E' we had $E \restriction D \not\leq E_n \restriction D'$ for all D'. Then there would be n such that $I_n \leq E \restriction D$. As $E \restriction D \leq E_n \restriction f_n(D)$, we contradict the assumption $I_n \not\leq E_n \restriction D'$ for all D'. □

Corollary 4. [15] *If $\lambda^\omega = \lambda$ and $\kappa \geq \lambda$, then first order logic on (κ, λ)-models is λ-compact. In particular, then $L(\exists^{\geq \lambda^+})$ is λ-compact.*

Another case that we can deal with is when $\beth_\omega(\lambda) \leq \kappa$. In this case *every* (κ, λ)-pattern is fundamental! To see how this is possible, let E be an arbitrary (κ, λ)-pattern and let $D \subseteq \kappa$ be finite, say $|D| = n$. Suppose E' is another (κ, λ)-pattern. Choose a set $X_0 \subseteq \kappa$ of cardinality κ such that all singletons $\{a\}$, $a \in X_0$, are E'-equivalent. By the Erdös-Rado Theorem there is $X_1 \subseteq X_0$ such that $|X_1| \geq \beth_n(\lambda)$ and all pairs in $[X_1]^2$ are E'-equivalent. By repeating this process $n-1$ times we arrive at an infinite set $X_{n-1} \subseteq \kappa$ such that all sets in $[X_{n-1}]^m$ are E'-equivalent for each $m \leq n$ individually. Now any one-one $\pi : D \to X_{n-1}$ demonstrates $E \restriction D \leq E' \restriction \pi'' D$. We have proved that E is fundamental.

Corollary 5. [20] *If $\beth_\omega(\lambda) \leq \kappa$, then first order logic on (κ, λ)-models is λ-compact.*

Yet, another case where we can construct a fundamental (κ, λ)-pattern is $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, λ singular, κ singular strong limit. Here the set $I(\kappa, \lambda)$ is interestingly independent of κ and λ, and recursive, as proved in [16].

Corollary 6. [16] *If* $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, λ *singular,* κ *singular strong limit, then first order logic on* (κ, λ)*-models is recursively axiomatizable and* λ*-compact.*

3. κ-like models

We digressed into (κ, λ)-models as they appeared to be simpler than κ-like models. Let us see how identities help us prove compactness results for κ-like models.

A natural way to construct a κ-like model for a first order theory T in a vocabulary L of cardinality $< \kappa$ is the following. Let L^* be the Skolem-expansion of L and T^* the Skolem-closure of T. Let c_α, $\alpha < \kappa$, be new constant symbols. Let $<$ be the predicate symbol the interpretation of which we want to be κ-like. Consider the axioms

(T1)′ T^* (Skolem-closure of T).
(T2)′ $c_\alpha < c_\beta$ for $\alpha < \beta < \kappa$.
(T3)′ $t(c_{\alpha_0}, \ldots, c_{\alpha_n}) < c_{h(\{\alpha_0,\ldots,\alpha_n\})}$ for $\alpha < \lambda$, where $\alpha_0 < \ldots < \alpha_n < \kappa$, t is a Skolem-term and h is a fixed function $[\kappa]^{<\omega} \to \kappa$.
(T4)′ $t(c_{\alpha_0}, \ldots, c_{\alpha_n}) = t(c_{\beta_0}, \ldots, c_{\beta_n}) \vee (c_\alpha \leq t(c_{\alpha_0}, \ldots, c_{\alpha_n}) \wedge c_\alpha \leq t(c_{\beta_0}, \ldots, c_{\beta_n}))$ for all Skolem-terms t and all $\alpha_0 < \ldots < \alpha_n < \kappa, \beta_0 < \ldots < \beta_n < \kappa$ such that $\{\alpha_0, \ldots, \alpha_n\} E_\alpha \{\beta_0, \ldots, \beta_n\}$.

So now we have for each $\alpha < \kappa$ an equivalence relation E_α with $< \kappa$ equivalence classes, and in addition a function $h : [\kappa]^{<\omega} \to \kappa$. The point of the function h is that it makes sure the interpretations of the constants c_α in the Skolem hull form a cofinal sequence in the domain of $<$. The equivalence relations E_α, $\alpha < \kappa$, make sure the set of predecessors of every c_α has cardinality $< \kappa$ in the Skolem hull.

We have now a new notion of pattern. A pair $\mathcal{E} = \langle \langle E_\alpha : \alpha < \kappa \rangle, h \rangle$, where

(E1)′ E_α is an equivalence relation on $[\kappa]^{<\omega}$ such that equivalent sets have the same cardinality.
(E2)′ There are $< \kappa$ equivalence classes in E_α.
(E3)′ $h : [\kappa]^{<\omega} \to \kappa$.

is called a **κ-pattern**. As in the context of (κ, λ)-models, we have to make a careful choice of a "fundamental" κ-pattern. To see what the requirements for this careful choice are, let us try to construct a model.

We assume again that our starting theory T has the property that every finite subset has a κ-like model. Let Σ be an arbitrary finite subset of (T1)′-(T4)′. Let $\mathfrak{M}$ be a (κ, λ)-model of $\Sigma \cap T^*$. Let $D = \{\gamma_0, \ldots, \gamma_m\}$ be the set of $\gamma < \kappa$ such that c_γ occurs in Σ. Let us expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ by adding interpretations to all the constants c_α, $\alpha < \kappa$, in such a way that they increase with α and are cofinal in $<^{\mathfrak{M}}$. The model $\mathfrak{M}'$ and Σ **induce** in a canonical way a κ-pattern $\mathcal{E}'$ as follows: If $\alpha < \kappa$, then let

$$\begin{aligned} &\{\alpha_0, \ldots, \alpha_n\} E'_\alpha \{\beta_0, \ldots, \beta_n\} \iff \\ &\mathfrak{M}' \models t(c_{\alpha_0}, \ldots, c_{\alpha_n}) = t(c_{\beta_0}, \ldots, c_{\beta_n}) \vee \\ &(c_\alpha \leq t(c_{\alpha_0}, \ldots, c_{\alpha_n}) \wedge c_\alpha \leq t(c_{\beta_0}, \ldots, c_{\beta_n})) \\ &\text{for all Skolem-terms } t \text{ occurring in } \Sigma. \end{aligned}$$

and

$$\begin{aligned} h(\{\alpha_0, \ldots, \alpha_n\}) \quad = \quad & \min\{\beta < \kappa : t(c_{\alpha_0}, \ldots, c_{\alpha_n})^{\mathfrak{M}'} < c_\beta^{\mathfrak{M}'} \\ & \text{for all Skolem-terms } t \text{ occurring in } \Sigma\} \end{aligned}$$

A **cardinal identity** is a triple $J = \langle \langle E_d : d \in D_J \rangle, <_J, h \rangle$ where $\langle D_J, <_J \rangle$ is a finite linear order, each E_d is an equivalence relation on $\mathcal{P}(D_J)$ such that equivalent sets have the same cardinality, and $h : \mathcal{P}(D_J) \to D_J$ is a partial function. An example of a cardinal identity is the restriction

$$\mathcal{E} \restriction D = \langle \langle E_\alpha \restriction D : \alpha \in D \rangle, < \restriction D, h \restriction D \rangle$$

of a κ-pattern to a finite $D \subseteq \kappa$.

A cardinal identity $J = \langle \langle E_d : d \in D_J \rangle, <_J, h \rangle$ is a **subidentity** of another cardinal identity $J' = \langle \langle E'_d : d \in D'_{J'} \rangle, <'_J, h' \rangle$, in symbols $J \leq J'$, if there is an order-preserving mapping $\pi : D_J \to D_{J'}$ such that

$$\{d_0, \ldots, d_n\} E_c \{d'_0, \ldots, d'_n\} \to \{\pi d_0, \ldots, \pi d_n\} E_{\pi c} \{\pi d'_0, \ldots, \pi d'_n\}$$

and

$$\pi h(\{d_0, \ldots, d_n\}) \leq_{J'} h'(\{\pi d_0, \ldots, \pi d_n\})$$

holds for elements $d_0 <_J \ldots <_J d_n$ and $d'_0 <_J \ldots <_J d'_n$ of D_J. We call a κ-pattern $\mathcal{E}$ **fundamental** if for every finite $D \subseteq \kappa$ and for every κ-pattern $\mathcal{E}'$ there is a finite $D' \subseteq \kappa$ such that $\mathcal{E} \restriction D \leq \mathcal{E}' \restriction D'$. We write $J \equiv J'$ is both $J \leq J'$ and $J' \leq J$. The set $I(\kappa)$ is defined as the set of cardinal identities J such that for every κ-pattern $\mathcal{E}$ there is a finite $D \subseteq \kappa$ such that $J \leq \mathcal{E} \restriction D$.

Suppose now that there is a fundamental κ-pattern $\mathcal{E}$. Let us see how we can finish the constructions of a κ-like model for T. We built up a κ-pattern $\mathcal{E}'$ from the model $\mathfrak{M}'$. Since $\mathcal{E}$ is fundamental, there is a finite set D' such that $\mathcal{E} \restriction D \leq \mathcal{E}' \restriction D'$. Thus $\mathfrak{M}$ can be expanded to a model of Σ.

What about the existence of fundamental κ-patterns? If $\aleph_0$ is small for κ, the construction of Lemma 3 gives a fundamental κ-pattern. Thus we have:

Corollary 7. [15] *If $\aleph_0$ is small for κ, then first order logic on κ-like models is λ-compact for all $\lambda < \kappa$. In particular, then $L(\exists^{\geq\kappa})$ is λ-compact for all $\lambda < \kappa$.*

If κ is singular strong limit, then $I(\kappa)$ is recursive and independent of κ [16] (see [12] for details). Thus we have:

Corollary 8. [16] *If κ is singular strong limit, then $L(\exists^{\geq\kappa})$ is λ-compact and recursively axiomatizable for each $\lambda < \kappa$.*

If κ is ω-Mahlo[1], then any κ-pattern is fundamental. The proof of this is based on a succinct repeated use of the Erdös-Rado Theorem.

Corollary 9. [13] *If κ is ω-Mahlo, then $L(\exists^{\geq\kappa})$is λ-compact and recursively axiomatizable for each $\lambda < \kappa$.*

4. $(\kappa_n)_{n<\omega}$-like models

After the preliminary investigation of (κ, λ)-models and κ-like models we now attack the more general case of an infinite sequence $(\kappa_n)_{n<\omega}$ of uncountable cardinals and the associated logic $L(\exists^{\geq\kappa_n})_{n<\omega}$. There is an immediate translation of the logic $L(\exists^{\geq\kappa_n})_{n<\omega}$ to first order logic on models that have a unary predicate P_n and a κ_n-like linear order $<_n$ on P_n for each $n < \omega$. Let us call such models $(\kappa_n)_{n<\omega}$**-like** models. Mutatis mutandis, our approach applies also to logics of the form $L(\exists^{\geq\kappa_n})_{n<m}$.

Definition 10. *A triple*

$$\mathcal{F} = \langle\langle E_a : a \in \bigcup_{n<\omega} A_n\rangle, \langle\langle A_n, <_n\rangle : n < \omega\rangle, \langle h_n : n < \omega\rangle\rangle,$$

where

- **(E1)″** *The sets A_n are disjoint and for each n the structure $\langle A_n, <_n\rangle$ is a well-order of order type κ_n.*
- **(E2)″** *Each E_a is an equivalence relation on $[\bigcup_{n<\omega} A_n]^{<\omega}$ such that equivalent sets have the same cardinality.*
- **(E3)″** *If $a \in A_n$, the number of equivalence classes of E_a is $< \kappa_n$.*
- **(E4)″** $h_n : [\bigcup_{n<\omega} A_n]^{<\omega} \to A_n$.

*is called a $(\kappa_n)_{n<\omega}$***-pattern**.

The reason for using well-orders $\langle A_n, <_n\rangle$ of type κ_n rather than the cardinals κ_n themselves is merely to keep the domains A_n disjoint and thereby have easier notation. We say that $\langle a_0, \ldots, a_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ is **increasing** if its restriction to any $\langle A_m, <_m\rangle$ is increasing in $\langle A_m, <_m\rangle$.

[1] κ is 0-Mahlo if it is regular, $(n+1)$-Mahlo, if there is a stationary set of n-Mahlo cardinals below κ, and ω-Mahlo if it is n-Mahlo fo all $n < \omega$.

Let us now try to use the new pattern to construct a $(\kappa_n)_{n<\omega}$-like model. Let us again assume that our starting theory T has the property that every finite subset has a $(\kappa_n)_{n<\omega}$-like model. We assume the vocabulary L of T has cardinality $< \min\{\kappa_n : n < \omega\}$. Let L^* be the Skolem-expansion of L and T^* the Skolem-closure of T. Let c_a, $a \in \bigcup_{n<\omega} A_n$, be new constant symbols. Let $<_n$ be the predicate symbol the interpretation of which we want to be κ_n-like. Consider the axioms

(T1)″ T^* (Skolem-closure of T).
(T2)″ $c_\alpha <_n c_\beta$ for $\alpha <_n \beta$ in A_n.
(T3)″ $P_n(c_a)$ for $a \in A_n$
(T4)″ $P_m(t(c_{a_0}, \ldots, c_{a_n})) \to t(c_{a_0}, \ldots, c_{a_n}) <_m c_{h_m(\{a_0,\ldots,a_n\})}$, where $\langle a_0, \ldots, a_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ is increasing and t is a Skolem-term.
(T5)″ $t(c_{a_0}, \ldots, c_{a_n}) = t(c_{b_0}, \ldots, c_{b_n}) \vee (\neg(t(c_{a_0}, \ldots, c_{a_n}) <_m c_a) \wedge \neg(t(c_{b_0}, \ldots, c_{b_n}) <_m c_a))$ for all Skolem-terms t and all increasing $\langle a_0, \ldots, a_n\rangle, \langle b_0, \ldots, b_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$ such that $\{a_0, \ldots, a_n\} E_a \{b_0, \ldots, b_n\}$, whenever $a \in A_m$.

Let Σ be an arbitrary finite subset of (T1)″-(T5)″. Let $\mathfrak{M}$ be a $(\kappa_n)_{n<\omega}$-like model of $\Sigma \cap T^*$. Let D_m be the set of $a \in A_m$ such that c_a occurs in Σ. Let us expand $\mathfrak{M}$ to a model $\mathfrak{M}'$ by adding interpretations to all the constants c_a, $a \in \bigcup_{n<\omega} A_n$, in such a way that they increase in $\langle P_m^{\mathfrak{M}}, <_m^{\mathfrak{M}}\rangle$ with $a \in P_m^{\mathfrak{M}}$ and are cofinal in $<_m^{\mathfrak{M}}$. The model $\mathfrak{M}'$ and Σ **induce** in a canonical way a $(\kappa_n)_{n<\omega}$-pattern

$$\mathcal{F}' = \langle\langle E'_a : a \in \bigcup_{n<\omega} A_n\rangle, \langle\langle A_n, <_n\rangle : n < \omega\rangle, \langle h'_n : n < \omega\rangle\rangle \tag{2}$$

as follows: If $a \in A_m$, then define for increasing $\langle a_0, \ldots, a_n\rangle, \langle b_0, \ldots, b_n\rangle \in [\bigcup_{n<\omega} A_n]^{<\omega}$

$$\begin{aligned}&\{a_0, \ldots, a_n\} E'_a \{b_0, \ldots, b_n\} \iff\\ &\mathfrak{M}' \models t(c_{a_0}, \ldots, c_{a_n}) = t(c_{b_0}, \ldots, c_{b_n}) \vee\\ &(\neg(t(c_{a_0}, \ldots, c_{a_n}) <_m c_a) \wedge \neg(t(c_{b_0}, \ldots, c_{b_n}) <_m c_a))\\ &\text{for all Skolem-terms } t \text{ occurring in } \Sigma.\end{aligned}$$

and

$$\begin{aligned}h'_m(\{a_0, \ldots, a_n\}) \quad = \quad &\min\{b \in A_m : t(c_{a_0}, \ldots, c_{a_n})^{\mathfrak{M}'} <_m c_b^{\mathfrak{M}'}\\ &\text{for all Skolem-terms } t \text{ occurring in } \Sigma\}\end{aligned}$$

An **ω-cardinal identity** is a triple

$$\mathfrak{I} = \langle \langle E_a : a \in \bigcup_{n<\omega} D_n \rangle, \langle \langle D_n, <_n \rangle : n < \omega \rangle, \langle h_n : n < \omega \rangle \rangle \qquad (3)$$

where

- **(I1)** The $\langle D_m, <_m \rangle$ are disjoint finite linear orders, $D_m = \emptyset$ for all but finitely many m. The cardinality of $\bigcup_{n<\omega} D_n$ is called the **size** of $\mathfrak{I}$.
- **(I2)** Each E_a, $a \in D_m$, is an equivalence relation on $\mathcal{P}(D_m)$ such that equivalent sets have the same cardinality.
- **(I3)** $h_m : [\bigcup_{n<\omega} D_n]^{<\omega} \to D_m$ is a partial function.

An example of an ω-cardinal identity is the restriction

$$\mathcal{F} \restriction D = \langle \langle E_a \restriction D : a \in D \cap \bigcup_{n<\omega} D_n \rangle, \langle \langle D_n, <_n \rangle \restriction D : n < \omega \rangle, \langle h_n \restriction D : n < \omega \rangle \rangle$$

of $(\kappa_n)_{n<\omega}$-pattern to a finite D. An ω-cardinal identity

$$\mathfrak{I} = \langle \langle E_a : a \in \bigcup_{n<\omega} D_n \rangle, \langle \langle D_n, <_n \rangle : n < \omega \rangle, \langle h_n : n < \omega \rangle \rangle$$

is a **subidentity** of another ω-cardinal identity

$$\mathfrak{I}' = \langle \langle E'_a : a \in \bigcup_{n<\omega} D'_n \rangle, \langle \langle D'_n, <_n \rangle : n < \omega \rangle, \langle h'_n : n < \omega \rangle \rangle,$$

in symbols $\mathfrak{I} \leq \mathfrak{I}'$, if there is an order-preserving mapping $\pi : \bigcup_{n<\omega} D_n \to \bigcup_{n<\omega} D'_n$ such that

- **(S1)** $\pi \restriction D_m : \langle D_m, <_m \rangle \to \langle D'_m, <'_m \rangle$ is order-preserving.
- **(S2)** $\{d_0, \ldots, d_n\} E_a \{d'_0, \ldots, d'_n\} \to \{\pi d_0, \ldots, \pi d_n\} E_{\pi a} \{\pi d'_0, \ldots, \pi d'_n\}$ holds for $\{d_0, \ldots, d_n\}, \{d'_0, \ldots, d'_n\} \in [\bigcup_{n<\omega} D_n]^n$.
- **(S3)** $\pi h_m(\{d_0, \ldots, d_n\}) \leq'_m h'_m(\{\pi d_0, \ldots, \pi d_n\})$ if $\{d_0, \ldots, d_n\} \in [\bigcup_{n<\omega} D_n]^n$.

Let $\mathfrak{I}(\mathcal{F})$ be the set of all subidentities of $\mathcal{F} \restriction D$ for finite D. We write

$$(\kappa_n)_{n<\omega} \to (\mathfrak{I}),$$

if $\mathfrak{I}$ belongs to $\mathfrak{I}(\mathcal{F})$ for every $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Let $\mathfrak{I}((\kappa_n)_{n<\omega})$ be the set of all $\mathfrak{I}$ such that $(\kappa_n)_{n<\omega} \to (\mathfrak{I})$, i.e.

$$\mathfrak{I}((\kappa_n)_{n<\omega}) = \bigcap \{\mathfrak{I}(\mathcal{F}) : \mathcal{F} \text{ is a } (\kappa_n)_{n<\omega}\text{-pattern}\}.$$

Definition 11. *A $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$ is* fundamental *if $\mathfrak{I}(\mathcal{F}) = \mathfrak{I}((\kappa_n)_{n<\omega})$.*

Suppose now that there is a fundamental $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Let us see how we can finish the constructions of a κ-like model for T. We built up a $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}'$ from the model $\mathfrak{M}'$. Since $\mathcal{F}$ is fundamental, there is a finite set D' such that $\mathcal{F} \restriction D \leq \mathcal{F}' \restriction D'$. Thus $\mathfrak{M}$ can be expanded to a model of Σ.

To sum up, we have proved the following result:

Theorem 12. *If there is a fundamental $(\kappa_n)_{n<\omega}$-pattern then first order logic on $(\kappa_n)_{n<\omega}$-models is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$. In particular, then $L(\exists^{\geq \kappa_n})_{n<\omega}$ is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$.*

The question of existence of fundamental $(\kappa_n)_{n<\omega}$-patterns is, of course, quite difficult. If $\aleph_0$ is small for each κ_n, the construction of Lemma 3 gives a fundamental $(\kappa_n)_{n<\omega}$-pattern.

Corollary 13. [15] *If $\aleph_0$ is small for each κ_n, then first order logic on $(\kappa_n)_{n<\omega}$-like models is λ-compact for all $\lambda < \kappa$. In particular, then $L(\exists^{\geq \kappa_n})_{n<\omega}$ is λ-compact for all $\lambda < \min\{\kappa_n : n < \omega\}$.*

If each κ_n is singular strong limit and no κ_n is a supremum of some of the others, then there is a fundamental $(\kappa_n)_{n<\omega}$-pattern $\mathcal{E}$, and $\mathfrak{I}((\kappa_n)_{n<\omega})$ is recursive and independent of the cardinals κ_n [16] (see [12] for details). Thus we have:

Corollary 14. [16] *If each κ_n is singular strong limit and no κ_n is a supremum of some of the others, then $L(\exists^{\geq \kappa_n})_{n<\omega}$ is λ-compact and recursively axiomatizable for each $\lambda < \min\{\kappa_n : n < \omega\}$.*

Example 15. *$L(\exists^{\geq \beth_{\omega \cdot n}})_{0<n<\omega}$ is λ-compact and recursively axiomatizable for all $\lambda < \beth_\omega$.*

Example 16. *The logic $L(\exists^{\geq \beth_{\omega \cdot n}})_{0<n\leq\omega}$ fails to be $\aleph_0$-compact for trivial reasons. Still every fragment containing only finitely many generalized quantifiers is $\aleph_0$-compact.*

If each κ_n is ω-Mahlo, then any κ-pattern is fundamental.

Corollary 17. [13] *If each κ_n is ω-Mahlo, then $L(\exists^{\geq \kappa_n})_{n<\omega}$ is λ-compact and recursively axiomatizable for each $\lambda < \min\{\kappa_n : n < \omega\}$.*

The results of this Section could have been proved also for a finite sequence $(\kappa_n)_{n<m}$ of uncountable cardinals, with obvious modifications.

5. The character of $L(\exists^{\geq \kappa_n})_{n<\omega}$

Our goal in this Section is to give the axioms $\mathcal{A}$ of $L(\exists^{\geq \kappa_n})_{n<\omega}$ and prove that $\langle L(\exists^{\geq \kappa_n})_{n<\omega}\, \mathcal{A}\rangle$ has recursive character. Since $L(\exists^{\geq \kappa_n})_{n<\omega}$ is the union of its fragments $L(\exists^{\geq\kappa_n})_{n<m}$, where $n<\omega$, we first introduce an axiomatization of $L(\exists^{\geq\kappa_n})_{n<m}$ and discuss its completeness.

5.1. **Logic with finitely many quantifiers.** Keisler gave a simple and elegant complete axiomatization for $L(\exists^{\geq\aleph_1})$ based on a formalization of the principle that if an uncountable set is divided into non-empty parts, then either there are uncountably many parts or one part is uncountable. If $\kappa=\kappa^{<\kappa}$, this works also for $L(\exists^{\geq\kappa^+})$, but it certainly does not work for $L(\exists^{\geq\kappa})$ if κ is singular. Keisler gave a different axiomatization for $L(\exists^{\geq\kappa})$ when κ is a singular strong limit cardinal. We give a general axiomatization $\mathcal{A}_m$ for $L(\exists^{\geq\kappa_n})_{n<m}$, whatever $(\kappa_n)_{n<m}$ is, plus a criterion when this is complete. The question whether $\mathcal{A}_m$ is a *recursive* axiomatization remains open. In certain cases we can assert its recursiveness. We use this axiomatization to prove the finite character of the logic frame $\langle L(\exists^{\geq\kappa_n})_{n<m},\mathcal{A}_m\rangle$.

In fact, we do not give the axioms of $\mathcal{A}_m$ explicitly but only give a criterion for their choice. Because of the nature of this criterion the set of Gödel numbers of the axioms is recursively enumerable. The method of "straightening Henkin-formulas" introduced by Barwise [1], could be used to turn our criterion into an explicit, albeit probably very complicated, set of axioms.

We defined above what it means for a $(\kappa_n)_{n<m}$-like model to induce a $(\kappa_n)_{n<m}$-pattern. If we have a model that is not $(\kappa_n)_{n<m}$-like, it still induces ω-cardinal identities. This is defined as follows: The model $\mathfrak{M}'$ and Σ **induce** any ω-cardinal identity that is a subidentity of

$$\mathfrak{I}=\langle\langle E_a : a\in\bigcup_{n<\omega}D_n\rangle,\langle\langle D_n,<_n\rangle : n<\omega\rangle,\langle h_n : n<\omega\rangle\rangle$$

defined as follows: Let D_n be the set of $a\in A_n$ for which c_a occurs in Σ. If $a\in D_m$, then define for increasing $\langle a_0,\ldots,a_n\rangle,\langle b_0,\ldots,b_n\rangle\in[\bigcup_{n<\omega}D_n]^{<\omega}$

$$\begin{array}{l}\{a_0,\ldots,a_n\}E'_a\{b_0,\ldots,b_n\} \iff\\ \mathfrak{M}'\models t(c_{a_0},\ldots,c_{a_n})=t(c_{b_0},\ldots,c_{b_n})\vee\\ (\neg(t(c_{a_0},\ldots,c_{a_n})<_m c_a)\wedge\neg(t(c_{b_0},\ldots,c_{b_n})<_m c_a)\\ \text{for all Skolem-terms } t \text{ occurring in } \Sigma.\end{array}$$

and

$$\begin{array}{rcl}h'_m(\{a_0,\ldots,a_n\}) &=& \min\{b\in D_m : t(c_{a_0},\ldots,c_{a_n})^{\mathfrak{M}'}<_m c_b^{\mathfrak{M}'}\\ && \text{for all Skolem-terms } t \text{ occurring in } \Sigma\},\\ && \text{(or undefined).}\end{array}$$

This concept is the heart of our axiom system $\mathcal{A}_m$. Suppose ϕ is a sentence in $L(\exists^{\geq\kappa_n})_{n<m}$. Fuhrken introduced a reduction method by means of which there is a first order sentence ϕ^+ in a larger vocabulary such that ϕ has a model if and only if ϕ^+ has a $(\kappa_n)_{n<m}$-like model.

Definition 18. *A sentence ϕ of $L(\exists^{\geq\kappa_n})_{n<m}$ in the vocabulary L is said to be $\mathcal{A}_m$-*consistent*, if for all $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$ and all finite $\Sigma \subseteq \{\phi^+\}^*$ there is a model $\mathfrak{M}$ of Σ such that $\mathfrak{M}$ and Σ induce $\mathfrak{I}$. The set $\mathcal{A}_m$ of axioms of $L(\exists^{\geq\kappa_n})_{n<m}$ consists of all sentences ϕ of $L(\exists^{\geq\kappa_n})_{n<m}$ for which $\neg\phi$ is not $\mathcal{A}_m$-consistent.*

The definition of the axioms $\mathcal{A}_m$ may seem trivial as we take all "valid" sentences as axioms. However, whether all "valid" sentences are actually axioms depends on whether we can prove the completeness of our axioms. Also, while there is no obvious reason why the set of valid sentences should be recursively enumerable in $\mathfrak{I}((\kappa_n)_{n<\omega})$, the set $\mathcal{A}_m$ of axioms certainly is.

Lemma 19. *Suppose ϕ is a sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ and ϕ has a model. Then ϕ is $\mathcal{A}_m$-consistent.*

Proof. Suppose $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$ and $\Sigma \subseteq \{\phi^+\}^*$ is finite. Suppose $\mathfrak{M}$ is a $(\kappa_n)_{n<\omega}$-like model of Σ. Then $\mathfrak{M}$ and Σ induce a $(\kappa_n)_{n<\omega}$-pattern $\mathcal{F}$. Since $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<\omega})$, there is a finite D such that $\mathfrak{I} \leq \mathcal{F} \restriction D$. Thus $\mathfrak{M}$ and Σ induce $\mathfrak{I}$. □

Lemma 20. *If there is a fundamental $(\kappa_n)_{n<m}$-pattern, then every $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ has a model.*

Proof. Suppose ϕ is an $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<\omega}$. Let $\mathcal{F}$ be a fundamental $(\kappa_n)_{n<m}$-pattern. Let $T = \{\phi^+\}$. It suffices to show that the theory (T1)″-(T5)″ constructed from $\mathcal{F}$ and T is finitely consistent. Let Σ be a finite part of (T1)″-(T5)″ and let D be the set of $a \in \bigcup_{n<\omega} A_n$ for which c_a occurs in Σ. Note, that if we let $\mathfrak{I} = \mathcal{F} \restriction D$, then $\mathfrak{I} \in \mathfrak{I}((\kappa_n)_{n<m})$. By assumption, $\Sigma \cap T^*$ has a model $\mathfrak{M}$ such that $\mathfrak{M}$ and Σ induce $\mathcal{F} \restriction D$. Thus $\mathfrak{M}$ can be expanded to a model of Σ. □

Proposition 21. *If every $\mathcal{A}_m$-consistent sentence of $L(\exists^{\geq\kappa_n})_{n<m}$ has a model, then there is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Proof. Let $\mathfrak{I}$ be an arbitrary ω-cardinal identity, as in (3). Let the size of $\mathfrak{I}$ be k. Let $D_i = \{d_0, \ldots, d_k\}$. Below $\vec{s}$ ranges over sequences $\langle s_i : i \leq k\rangle$ of natural numbers $\leq k$. We say that $\{\alpha_0, \ldots, \alpha_n\} \in$ is of **type** $\vec{s}$ if the intersection of $\{\alpha_0, \ldots, \alpha_n\}$ with D_i has size $s(i)$ for each $i \leq k$. Consider the following sentences of $L(\exists^{\geq\kappa_n})_{n<m}$ in a vocabulary consisting of a unary

predicate P_i, a binary predicate $<_i$ and n-ary function symbols F_i^n and H_i^n for each $i < m$, and $n \leq k$. Let $\sigma_{\mathfrak{I}}$ be the conjunction of

1. $\langle P_n, <_n \rangle$ is a κ_n-like linear order for $n < m$,
2. $F_i^{\vec{s}}$ is a function mapping sets $\{a_0, \ldots, a_n\}$ of type $\vec{s}$ to P_i for $n < k$ and $i < m$.
3. The range of $F_i^{\vec{s}}$ is bounded in P_i.
4. $H_i^{\vec{s}}$ is a function mapping sets $\{a_0, \ldots, a_n\}$ of type $\vec{s}$ to P_i for $n < k$ and $i < m$.
5. There are no $x_0 \ldots x_k$ of type $\vec{s}$ which would satisfy
   (a) $F_i^{\vec{s}}(x_{r_0}, \ldots, x_{r_n}) = F_i^{\vec{s}}(x_{s_0}, \ldots, x_{s_n})$
   (b) $F_i^{\vec{s}}(x_{r_0}, \ldots, x_{r_n}) <_i d_a$
   whenever $\langle r_0, \ldots, r_n \rangle, \langle r_0, \ldots, r_n \rangle \in [\bigcup_{n<\omega} D_n]^{<\omega}$ are increasing and of type $\vec{s}$, $\{d_{r_0}, \ldots, d_{r_n}\} E_a \{d_{s_0}, \ldots, d_{s_n}\}$, $a \in D_i$, and
   (c) $x_{h(\{d_{r_0}, \ldots, d_{r_n}\})} \leq_i H_i^n(x_{r_0}, \ldots, x_{r_n})$
   whenever $h(\{d_{r_0}, \ldots, d_{r_n}\}) \in D_i$.

Any model $\mathfrak{M}$ of $\sigma_{\mathfrak{I}}$ and any choice of a cofinal suborder $\langle A'_n, <_n \rangle$ of $\langle P_n, <_n \rangle^{\mathfrak{M}}$ of type κ_n (for $n < \omega$) gives rise to a $(\kappa_n)_{n<m}$-pattern $\mathcal{F}'$ as in (2), where for $a \in A'_i$

$$\{a_0, \ldots, a_n\} E'_a \{b_0, \ldots, b_n\}$$

$$\iff$$

$$\text{If } (F_i^n)^{\mathfrak{M}}(a_0, \ldots, a_n) <_i a \text{ or } (F_i^n)^{\mathfrak{M}}(b_0, \ldots, b_n) <_i a \text{ then}$$

$$(F_i^n)^{\mathfrak{M}}(a_0, \ldots, a_n) = (F_i^n)^{\mathfrak{M}}(b_0, \ldots, b_n).$$

and

$$h'_i(\{a_0, \ldots, a_n\}) \quad = \quad (H_i^n)^{\mathfrak{M}}(a_0, \ldots, a_n).$$

We have written into the sentence $\sigma_{\mathfrak{I}}$ the condition that $\mathfrak{I}$ is not in $\mathfrak{I}(\mathcal{F}')$. On the other hand, if $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$, it is easy to construct a model of $\sigma_{\mathfrak{I}}$. Moreover, if $\mathfrak{I}_0, \ldots, \mathfrak{I}_n \notin \mathfrak{I}((\kappa_n)_{n<m})$, it is not hard to construct a model of $\sigma_{\mathfrak{I}_0} \wedge \ldots \wedge \sigma_{\mathfrak{I}_n}$.

Let $\mathfrak{I}_n$, $n < \omega$, be a list of all $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$. Without loss of generality, this list is recursive in $\mathcal{A}_m$. Suppose the set of valid $L(\exists^{\geq \kappa_n})_{n<m}$-sentences is r.e. in $\mathcal{A}_m$. Now we use an argument (due to Per Lindström [9]) from abstract model theory. Let A be a set of natural numbers which is co-r.e. in $\mathcal{A}_m$ but not r.e. in $\mathcal{A}_m$. Say,

$$n \in A \iff \forall k((n, k) \in B),$$

where B is recursive in $\mathcal{A}_m$. Let P be a new unary predicate symbol and θ_n the first order sentence which says that P has exactly n elements. Let T be the theory $\{\theta_n \to \sigma_{\mathfrak{I}_i} : \forall k \leq i((n,k) \in B)\}$, and $C = \{n : T \models \neg\theta_n\}$. We show that $C \subseteq A$. Suppose $T \models \neg\theta_n$. If $n \notin A$, then there is k such that $(n,k) \notin B$. Let $\mathfrak{M}$ be a model of $\{\sigma_{\mathfrak{I}_j} : i < k\} \cup \{\theta_m\}$. If $\theta_n \to \sigma_{\mathfrak{I}_i} \in T$, then $i < k$, whence $\mathfrak{M} \models \sigma_{\mathfrak{I}_i}$. So $\mathfrak{M} \models T$, a contradiction. Since C is r.e. in $\mathcal{A}$, there is $n \in A \setminus C$. Thus there is $\mathfrak{M} \models T$ such that $\mathfrak{M} \models \theta_n$. Since $\forall k((n,k) \in B)$, the sentence $\theta_n \to \sigma_{\mathfrak{I}_i}$ is in T, and thereby true in $\mathfrak{M}$ for every i. Since $\mathfrak{M} \models \theta_n$, $\mathfrak{M} \models \sigma_{\mathfrak{I}_i}$ for all i. Let $\mathcal{F}$ be the $(\kappa_n)_{n<m}$-pattern that $\mathfrak{M}$ gives rise to. $\mathcal{F}$ is necessarily a fundamental $(\kappa_n)_{n<m}$-pattern. □

Summing up:

Theorem 22. *Suppose $(\kappa_n)_{n<m}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\mathcal{A}_m$ is a complete axiomatization of $L(\exists^{\geq\kappa_n})_{n<m}$.*
2. *$\langle L(\exists^{\geq\kappa_n})_{n<m}, \mathcal{A}_m\rangle$ is recursively compact.*
3. *$L(\exists^{\geq\kappa_n})_{n<m}$ is λ-compact for all $\lambda < \min\{\kappa_0, \ldots, \kappa_{m-1}\}$.*
4. *There is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Corollary 23. *$\langle L(\exists^{\geq\kappa_n})_{n<m}, \mathcal{A}_m\rangle$ has finite character.*

We do not know if $\mathcal{A}_m$ is recursive, except in such special cases as in Corollaries 14 and 17.

Proposition 24.

1. *Suppose $I(\kappa^+, \kappa)$ is recursive, and either A^* is recursive or there is a universe $V' \supseteq V$ in which $\langle L(\exists^{\geq\kappa}), A^*\rangle$ is recursively compact, then $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ has finite character.*
2. *Suppose $\langle L(\exists^{\geq\aleph_1}), A^*\rangle$ is coherent (i.e. if a sentence has a model it is consistent with A^*). Then $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ has finite character.*

Proof. 1. Suppose $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ is complete. Let $\Phi \in L(\exists^{\geq\kappa})$ say in the language of set theory that $\sigma_{\mathfrak{I}}$ holds for all $\mathfrak{I} \notin \mathfrak{I}((\kappa_n)_{n<m})$. Since $I(\kappa^+, \kappa)$ is recursive, this can be written in $L(\exists^{\geq\kappa})$. We show that Φ is consistent with the axioms A^*: If A^* is recursive, $\langle L(\exists^{\geq\kappa}), A^*\rangle$ is recursively compact and there is a fundamental (κ^+, κ)-pattern, whence Φ is consistent with A^*. On the other hand, is there is a universe V' in which $\langle L(\exists^{\geq\kappa}), A^*\rangle$ is recursively compact, then in V' there is a fundamental (κ^+, κ)-pattern, and hence in V' the sentence Φ is consistent with A^*. Thus Φ is consistent with A^* also in V. By completeness Φ has a model. Thus there is a fundamental (κ^+, κ)-pattern and $\langle L(\exists^{\geq\kappa}), A^*\rangle$ is $\aleph_0$-compact.

2. Completeness implies $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$. We know that $I(\aleph_1, \aleph_0)$ is recursive ([17]). Since $(\aleph_1, \aleph_0) \to (\kappa^+, \kappa)$ implies $I(\aleph_1, \aleph_0) = I(\kappa^+, \kappa)$, also the latter is recursive. Now we use part 1. □

Corollary 25. *If A^* is the Keisler axiomatization for $L(\exists^{\geq\aleph_1})$, then $\langle L(\exists^{\geq\kappa^+}), A^*\rangle$ has finite character.*

5.2. **Logic with infinitely many quantifiers.** The axioms $\mathcal{A}$ are simply all the axioms $\mathcal{A}_m$ $m<\omega$, put together.

Proposition 26. *If $\langle L(\exists^{\geq\kappa_n})_{n<\omega},\mathcal{A}\rangle$ is recursively compact, then there is a fundamental $(\kappa_n)_{n<\omega}$-pattern.*

Proof. Let $\mathfrak{I}_n$, $n<\omega$, be a list of all $\mathfrak{I}\notin\mathfrak{I}((\kappa_n)_{n<\omega})$. Without loss of generality, this list is recursive in $\mathcal{A}$. Note that if $\mathfrak{I}\notin\mathfrak{I}((\kappa_n)_{n<\omega})$, then there is an m such that $\mathfrak{I}\notin\mathfrak{I}((\kappa_n)_{n<m})$, so we can use the sentences $\sigma_{\mathfrak{I}_n}$. Let T be the set of all $\sigma_{\mathfrak{I}_n}$, $n<\omega$. This theory is recursive in $\mathcal{A}$ and it is finitely consistent. Hence it has a model. The $(\kappa_n)_{n<\omega}$-pattern the model $\mathfrak{M}$ gives rise to is clearly fundamental. □

Theorem 27. *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\mathcal{A}$ is a complete axiomatization of $L(\exists^{\geq\kappa_n})_{n<\omega}$.*
2. *For every $m<\omega$ there is a fundamental $(\kappa_n)_{n<m}$-pattern.*

Theorem 28. *Suppose $(\kappa_n)_{n<\omega}$ is a sequence of uncountable cardinals. The following conditions are equivalent:*

1. *$\langle L(\exists^{\geq\kappa_n})_{n<\omega},\mathcal{A}\rangle$ is recursively compact.*
2. *$L(\exists^{\geq\kappa_n})_{n<\omega}$ is λ-compact for all $\lambda<\min\{\kappa_n : n<\omega\}$.*
3. *There is a fundamental $(\kappa_n)_{n<\omega}$-pattern.*

Corollary 29. *$\langle L(\exists^{\geq\kappa_n})_{n<\omega},\mathcal{A}\rangle$ has recursive character.*

Example 30. *If $\kappa_n=\beth_{\omega\cdot n}$ for $0<n\leq\omega$, then $\langle L(\exists^{\geq\kappa_n})_{n<\omega},\mathcal{A}\rangle$ is complete but not $\aleph_0$-compact and thereby does not have finite character.*

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