THE SIGNATURE AND THE ELLIPTIC GENUS OF EVEN 4-MANIFOLDS WITH S^1 ACTIONS

HAYDEÉ HERRERA* AND RAFAEL HERRERA†‡

ABSTRACT. We prove the vanishing of the signature of oriented smooth 4-manifolds with even intersection form and admitting circle actions. When the manifold is Spin, such a vanishing follows from the Atiyah-Hirzebruch vanishing of $\widehat{A}$-genus regardless of the parity of the intersection form. We prove the vanishing of the signature in the *non-Spin* case by proving the vanishing of the $\widehat{A}$-genus via the rigidity of the elliptic genus under S^1 actions. As a corollary we see that the Enriques surface and its n-fold connected sums admit no smooth S^1 actions.

1. INTRODUCTION

In the study of the topology of oriented smooth 4-manifolds, the intersection form is one of the classical invariants and determining which quadratic forms can represent such an intersection form is still a subject of central interest. In this paper we address this question on 4-manifolds with even intersection form and admitting smooth circle actions. More precisely, let E_8 be the unique irreducible negative definite quadratic form of rank eight and let H be the hyperbolic quadratic form. By the classification of quadratic forms any indefinite even intersection form Q is of the form $aE_8 \oplus bH$. Throughout the paper, we shall assume that the manifolds are oriented, compact, connected and smooth. We prove the following theorem.

Theorem 1.1. *Let M be an even 4-manifold admitting smooth (isometric) circle actions, and let $Q = aE_8 \oplus bH$ denote its intersection form. Then, the signature of M vanishes,* $\mathrm{sign}(M) = 0$*, i.e. the intersection form is* $Q = bH$.

---

*Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102, USA. E-mail: haydeeh@camden.rutgers.edu.

†Mathematics Department, Princeton University, Fine Hall, Washington Road, Princeton NJ 08544 USA. Mathematics Department, City University of New York, College of Staten Island, 2800 Victory Blvd., Staten Island, NY 10314, USA. E-mail: rherrera@math.princeton.edu.

‡Partially supported by a Guggenheim Fellowship, NSF grant DMS-0204002 and PSC-CUNY grant #66501- 00-35.

If the 4-manifold is Spin, the vanishing of the signature follows from the Atiyah-Hirzebruch vanishing of the $\widehat{A}$-genus [3] regardless of the parity of the intersection form. Therefore, the main goal of this paper is to prove the theorem in the non-Spin case. Note that even non-Spin manifolds are not simply-connected [1]. As an example, consider the oriented 4-manifold $S^2 \times S^2/(x,y) \sim (-x,-y)$, which is oriented, non-Spin, has even intersection form H, and admits smooth S^1 actions [1].

Since in dimension 4, the two characteristic numbers satisfy

$$\mathrm{sign}(M) = -8\,\widehat{A}(M),$$

we prove the vanishing of the signature by proving the vanishing of the $\widehat{A}$-genus. This, however, does not follow from index theory for the Dirac operator, since such an operator is not defined on non-Spin manifolds. In a similar fashion to that of [7, 8], we prove the vanishing of the $\widehat{A}$-genus on even non-Spin 4-manifolds admitting circle actions by means of the rigidity of the elliptic genus.

The elliptic genus was introduced as a topological genus by Ochanine [12, 13] and reinterpreted by Witten [15, 16] in a Quantum Field Theoretical context. Witten conjectured the rigidity property under S^1 actions on *Spin* manifolds which was proved by Taubes [14], Bott and Taubes [5] and Liu [11]. The rigidity theorem has been extended to Spin^c manifolds by Dessai [6] and to π_2-finite non-Spin manifolds by the authors [7, 8]. The latter extension allowed the authors to prove the vanishing of the $\widehat{A}$-genus of π_2-finite non-Spin manifolds admitting circle actions [7, 8], therefore extending Atiyah and Hirzebruch's vanishing theorem [3].

The note is organized as follows. In Section 2 we recall the definition of the elliptic genus and prove the rigidity theorem. In Section 3 we briefly recall the argument that renders the vanishing of the $\widehat{A}$-genus and in Section 4 we give some of applications.

Acknowledgements. The authors wish to thank Tian-Jun Li for suggesting the problem. The first named author wishes to thank the Mathematical Sciences Research Institute (Berkeley) and the Instituto de Matemáticas of UNAM (Mexico) for their hospitality and support. The second named author wishes to thank Marcel Nicolau for stimulating conversations, as well as the Centre de Recerca Matemàtica (Barcelona) and the Mathematical Sciences Research Institute (Berkeley) for their hospitality and support.

2. Rigidity of the Elliptic Genus

Let $D\colon \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over the manifold M. The index of D is the virtual vector space $\operatorname{ind}(D) = \ker(D) - \operatorname{coker}(D)$. If M admits a circle action preserving D, i.e. such that S^1 acts on E and F, and commutes with D, $\operatorname{ind}(D)$ admits a Fourier decomposition into complex 1-dimensional irreducible representations of S^1 $\operatorname{ind}(D) = \sum a_m L^m$, where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $e^{i\theta} \mapsto e^{im\theta}$. The elliptic operator D is called *rigid* if $a_m = 0$ for all $m \neq 0$, i.e. $\operatorname{ind}(D)$ consists only of the trivial representation with multiplicity a_0. The elliptic operator D is called *universally rigid* if it is rigid under any S^1 action on M by isometries.

Let $\bigwedge_c^{\pm}$ be the even and odd complex differential forms on the oriented, compact, smooth 4-manifold M under the Hodge $*$-operator, respectively. The signature operator

$$d_s^M = d - *d* \ : \ \textstyle\bigwedge_c^+ \longrightarrow \bigwedge_c^-$$

is elliptic and the virtual dimension of its index equals the signature of M, $\operatorname{sign}(M)$. If W is a complex vector bundle on M endowed with a connection, we can *twist* the signature operator to forms with values in W

$$d_s^M \otimes W \colon \textstyle\bigwedge_c^+(W) \longrightarrow \bigwedge_c^-(W).$$

This operator is also elliptic and the virtual dimension of its index is denoted by $\operatorname{sign}(M, W)$.

Definition 2.1. *Let $T = TM \otimes \mathbb{C}$ denote the complexified tangent bundle of M and let R_i be the sequence of bundles defined by the formal series*

$$R(q, T) = \sum_{i=0}^{\infty} q^i R_i = \bigotimes_{i=1}^{\infty} \textstyle\bigwedge_{q^i} T \otimes \displaystyle\bigotimes_{j=1}^{\infty} S_{q^j} T,$$

where $S_t T = \sum_{k=0}^{\infty} t^k S^k T$, $\bigwedge_t T = \sum_{k=0}^{\infty} t^k \bigwedge^k T$, and $S^k T$, $\bigwedge^k T$ denote the k-th symmetric and exterior tensor powers of T, respectively. The elliptic genus *of M is defined as*

$$\Phi(M) = \operatorname{ind}(d_s^M \otimes R(q, T)) = \sum_{i=0}^{\infty} q^i \cdot \operatorname{sign}(M, R_i). \qquad (1)$$

The first few terms of the sequence $R(q, T)$ are $R_0 = 1$, $R_1 = 2T$, $R_2 = 2(T^{\otimes 2} + T)$, etc. In particular, the constant term of $\Phi(M)$ is $\operatorname{sign}(M)$.

Theorem 2.1. *Let M be an oriented, compact, connected, even, smooth* 4-*manifold admitting smooth S^1 actions. Then, each of the operators $d_s \otimes R_i$ is universally rigid.*

Proof. Without loss of generality let us assume that the S^1 action is effective.

The proof is carried out along the lines of [5] from which we recall the main arguments and give appropriate modifications. By the Atiyah-Segal G-signature theorem [4]

$$\Phi(M) = \sum_{\{P\}} \mu(P)$$

where P runs over the connected components of the fixed point set of the S^1 action [5, p. 155]. Note that in this dimension the connected components P are oriented, totally geodesic submanifolds of even codimension, i.e. oriented surfaces or isolated fixed points. The contribution $\mu(P)$ of P to $\Phi(M)$ is given by the index of the signature operator on P twisted by an appropriate power series of vector bundles on P; namely,

$$\begin{aligned}\mu(P) &= \frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})}\prod_{l=1}^{\infty}\frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{-m_1})(1-q^k\lambda^{-m_1})}\\ &\times \frac{(1+\lambda^{m_2})}{(1-\lambda^{m_2})}\prod_{l=1}^{\infty}\frac{(1+q^k\lambda^{m_2})(1+q^k\lambda^{-m_2})}{(1-q^k\lambda^{-m_2})(1-q^k\lambda^{-m_2})},\end{aligned}$$

if P is an isolated fixed point and $TM|_P = E^{\#}_{m_1} \oplus E^{\#}_{m_2}$, where $E^{\#}_{m_i}$ denotes the canonical underlying real bundle of the complex bundle E_{m_i} on which S^1 acts by sending λ to λ^{m_i}; or

$$\mu(P) = \mathrm{ind}\left(d_s^P \otimes \frac{\bigwedge_{\lambda^{m_2}} E_{m_2}}{\bigwedge_{-\lambda^{m_2}} E_{m_2}}\right),$$

when P is 2-dimensional, where $TM|_P = TP \oplus E^{\#}_{m_2}$, i.e. E_{m_1} is now a trivial representation ($m_1 = 0$) and d_s^P is the signature operator on P.

The contributions $\mu(P)$ are meromorphic functions on the 2-dimensional torus $\mathbb{T}_{q^2} = \mathbb{C}^*/q^2$ considered as the quotient of the multiplicative group of non-zero complex numbers $\mathbb{C}^*$ by the subgroup generated by the element $q^2 \neq 0$. The rigidity of the elliptic genus is equivalent to the function $\Phi(M) = \sum_{\{P\}} \mu(P)$ having no poles at all on $\mathbb{T}_{q^2}$.

Define the translation $t_a\Phi(M)$ of $\Phi(M)$ with $a \in \mathbb{C}^*$, by the map at the character level $\lambda \mapsto a\lambda$. The rigidity theorem for $\Phi(M)$ follows from showing that none of the translations $t_a\Phi(M)$, by points $a \in \mathbb{T}_{q^2}$ of finite order, has a pole on the circle $|\lambda| = 1$. Let $k \in \mathbb{N}$ and a be any root of the form

$$a = \alpha^s, \qquad \alpha^k = q,$$

with k and s relatively prime. Define the formal power series

$$\varphi_a(F) = \frac{\bigwedge_a F}{\bigwedge_{-a} F} \otimes \bigotimes_{n=1}^{\infty}\left(\frac{\bigwedge_{aq^n} F}{\bigwedge_{-aq^n} F} \otimes \frac{\bigwedge_{a^{-1}q^n} F^*}{\bigwedge_{-a^{-1}q^n} F^*}\right)$$

where F denotes a complex vector bundle and F^* its complex dual bundle, and define

$$\varphi_{\pm q^{1/2}}(E) = \frac{\bigotimes_{n=1}^{\infty} \bigwedge_{\pm q^{n-1/2}} E}{\bigotimes_{n=1}^{\infty} \bigwedge_{\mp q^{n-1/2}} E}$$

for a real vector bundle E. The translations $t_{\alpha^s}\Phi(M)$ can be expressed as *twists* of Φ on the connected components M_k of the fixed point submanifold of the subgroup $\mathbb{Z}_k = \{e^{2\pi il/k} | l = 1, ..., k\} \subset S^1$ which *do contain fixed points of the S^1 action*, as follows

$$t_{\alpha^s}\Phi(M) = \mathrm{ind}\left(d_s^{M'_k} \otimes R(q, TM'_k \otimes \mathbb{C}) \otimes \varphi_{\alpha^{\omega_{r_1}}}(T_{r_1}) \otimes \varphi_{\alpha^{\omega_{r_2}}}(T_{r_2})\right), \quad (2)$$

where M'_k is the submanifold M_k with a specific orientation and the bundles T_r are defined below.

Given (2), [5, Proposition 6.1] says that the translations $t_{\alpha^s}\Phi(M)$ converge on some annulus containing the unit circle $|\lambda| = 1$ to the Laurent series of a meromorphic function on $\mathbb{T}_{q^2}$ which has no poles on the unit circle. The function $t_{\alpha^s}\tau_q(M)$ on the variable q is regular on an annulus containing the unit circle for all $k \in \mathbb{N}$, so that $t_{\alpha^s}\Phi(M)$ has no poles on the unit circle $|\lambda| = 1$. Hence, $\Phi(M)$ has no poles at all on $\mathbb{T}_{q^2}$, and must be constant, i.e. the rigidity theorem follows.

In order to define the aforementioned twists, the submanifolds M_k must be orientable. The connected components of M_k that we are interested in are only those that contain S^1-fixed points since they are intermediate steps between M and the S^1-fixed point submanifolds P. The orientability of such components is confirmed as follows.

Since the transformations in $\mathbb{Z}_k \subset S^1$ are orientation-preserving, the codimension of the components of M_k is even, so that in our case, they are isolated points or surfaces. The surfaces containing S^1-fixed points are orientable since they either coincide with a S^1-fixed component P, or are S^1-invariant with positive Euler characteristic, i.e. they are 2-dimensional spheres. The subgroup $\mathbb{Z}_k$ acts on the normal bundle of M_k in M

$$\begin{cases} T|_{M_k} = T^{\#}_{r_1} \oplus T^{\#}_{r_2} & \text{if } M_k \text{ is an isolated fixed point, or} \\ T|_{M_k} = TM_k \oplus T^{\#}_{r_2} & \text{if } M_k \text{ is a surface,} \end{cases} \quad (3)$$

where $T^{\#}_r$ is an irreducible real representation of $\mathbb{Z}_k$ with $1 \leq r \leq [k/2]$, $[k/2]$ is the greatest integer smaller than or equal to $k/2$, and TM_k is a trivial representation. The S^1 action on M induces an S^1 action on M_k, whose differential induces an action on $T|_{M_k}$ preserving the decomposition, and making each $T^{\#}_r$ an S^1 bundle over M_k, for $1 \leq r \leq [(k-1)/2]$. Each $T^{\#}_r$, with $r \neq k/2$ if k is even, is endowed with a complex structure such that $\lambda \in S^1$ acts by λ^r, for $1 \leq r \leq [(k-1)/2]$. Hence, $T^{\#}_r$ comes from a complex vector bundle T_r. For k even, the action on $T^{\#}_{k/2} = T_{k/2}$ is multiplication by -1, and it does not necessarily come from a complex vector bundle, while the $T_r^\#$ inherit an orientation from the complex structure on T_r, for $r = 1, \dots, [(k-1)/2]$. Hence, if k is odd, TM_k has an induced orientation. If k is even, however, we only know that $TM_k \oplus T_{k/2}$ is orientable, if $T_{k/2}$ does appear in (3). On the other hand, we know that M_k is also orientable. Let us, therefore, choose an orientation. In this way, $T_{k/2}$ inherits an orientation from M and M_k.

We must choose the orientation of M_k' to be compatible with the orientations of the S^1-fixed point submanifolds P as follows. Recall the decomposition of TM along P

$$\begin{cases} TM|_P = E_{m_1} \oplus E_{m_2} & P \text{ is isolated fixed point,} \\ TM|_P = TP \oplus E_{m_2} & P \text{ is a surface.} \end{cases} \tag{4}$$

When k is odd, the decomposition (3) determines an orientation on TM_k denoted by $+1$. If $P \subset M_k$, choose the exponents along P so that each $m_j \not\equiv 0 \pmod{k}$ is congruent to some $r \in \{1, \dots, (k-1)/2\}$. Choose the orientation of TP and the sign of those $m_j \equiv 0 \pmod{k}$ so that the induced orientation on $TM|_P$ is the given one. The induced orientation on $TM_k|_P$ will be the $+1$ orientation. For m_1 and m_2 let $(l_1, \omega_1), (l_2, \omega_2) \in \mathbb{Z} \times \{1, \dots, k-1\}$ be such that

$$s \cdot m_1 = l_1 \cdot k + \omega_1, \qquad s \cdot m_2 = l_2 \cdot k + \omega_2 \tag{5}$$

and define

$$\varepsilon(P) = l_1 + l_2. \tag{6}$$

The orientation for M_k' is now defined as $(+1) \cdot (-1)^{\varepsilon(P)}$, if $M_k \supseteq P$. Lemma 2.1 below ensures that this orientation is well defined. When k is even, $TM|_{M_k}$ decomposes according to (3), so that $TM_k \oplus T_{k/2}$ always inherits an orientation if the summand $T_{k/2}$ does appear in (3). Let us choose an orientation for TM_k and call it $+1$, which induces an orientation on $T_{k/2}$. If $P \subset M_k$, select the exponents at P as follows. If $m_j \not\equiv 0, k/2 \pmod{k}$, make the choice as before so that $(m_j)_{\mathrm{mod}\, k} \in \{1, \dots, k/2 - 1\}$. Choose the signs for those $m_j \equiv 0, k/2 \pmod{k}$ and the orientation of TP to make the induced orientation of $(TM_k \oplus T_{k/2})|_P$ correct. This ensures that the induced orientation of $TM|_P$ is correct. The induced orientation of $TM_k|_P$, however, *may not* be the correct one $(+1)$. Let $\varepsilon_0 = 0, 1$, with $\varepsilon_0 = 0$ if the induced orientation on $TM_k|_P$ is correct, and $\varepsilon_0 = +1$ if the induced orientation on $TM_k|_P$ is incorrect. For each m_j, define (l_j, ω_j) by (5) and set

$$\varepsilon(P) = \varepsilon_0 + l_1 + l_2. \tag{7}$$

The orientation of M_k' is given by $(+1) \cdot (-1)^{\varepsilon(P)}$, if $M_k \supset P$. The orientation of M_k' is well defined by the following lemma.

Lemma 2.1. *Let M be an oriented, compact, connected,* even *smooth 4-manifold admitting a smooth S^1 action. Let $k \in \mathbb{N}$ and M_k be a connected component of the fixed point set of $\mathbb{Z}_k$. Let $s \in \mathbb{Z}$ be relatively prime to k, and P, $P' \subset M_k$ be connected components of the S^1-fixed point set. Use the prescription* (6) *or* (7) *above to define the numbers $\varepsilon(P)$ and $\varepsilon(P')$. Then $(-1)^{\varepsilon(P)} = (-1)^{\varepsilon(P')}$.*

Proof. Lemma 2.1 is analogous to [5, Lemma 8.1], so we shall concentrate in the relevant changes to the proof. Let $k \in \mathbb{N}$, P be a connected fixed point submanifolds of S^1 contained in a connected component of M_k, which we shall also denote also by M_k. Just as before, given that we are working in dimension 4, there are three cases: (i) $\dim M_k = \dim P = 2$; (ii) $\dim M_k = 2$, $\dim P = 0$; (iii) $\dim M_k = \dim P = 0$. In (i) and (iii) the components P and M_k coincide so that we do not have to check any compatibility of the exponents $\varepsilon(P)$. In case (ii), we can have two isolated S^1-fixed points p and p'. Consider a path joining p and p' which avoids other S^1-fixed points. Let the path flow with the S^1 action to generate a sphere $S \cong S^2$ with "north" and "south" poles p and p' respectively (which, in fact, coincides with the component of M_k). Let the sets of integers $\{m_1, m_2\}$ and $\{m'_1, m'_2\}$ denote the exponents of the S^1 action on T_pM and $T_{p'}M$ respectively.

By [5, Lemmas 9.1, 9.2], the number

$$\varepsilon(P) - \varepsilon(P') \equiv c \cdot (m_1 + m_2 - m'_1 - m'_2) \pmod 2,$$

where c is a constant, while

$$(m_1 + m_2 - m'_1 - m'_2) = \int_S c_1(TM|_S),$$

so that the lemma is reduced to finding the parity of the integer represented by the last integral. Notice that

$$TM|_S = TS \oplus \nu,$$

where ν is the (real rank 2) normal bundle of S in M. The three bundles on S can be considered as complex vector bundles over S (cf. [5, p. 159]). Hence,

$$c_1(TM|_S) = c_1(TS) + c_1(\nu),$$

and

$$\int_S c_1(TM|_S) = \int_S c_1(TS) + \int_S c_1(\nu) = 2 + \int_S c_1(\nu),$$

so that the parity depends on the last integral only. Given that we are working in dimension 4

$$\int_S c_1(\nu) = S \cdot S \equiv 0 \pmod 2,$$

the self-intersection number of this sphere, which by the hypothesis on the intersection form is an even number. □

3. Vanishing of the signature

Proof of Theorem 1.1. Since in dimension 4

$$\mathrm{sign}(M) = -8\,\widehat{A}(M)$$

we shall prove Theorem 1.1 by proving the vanishing of $\widehat{A}(M)$.

Since we are also considering the case when M may be non-Spin, $\widehat{A}(M)$ may only be defined as a characteristic number and may not represent the index of an elliptic operator. Thus, $\widehat{A}(M)$ may, in principle, be a rational number.

Since S^1 acts on M, the equivariant genus $\Phi(M)_g$ is defined for any $g \in S^1$ as

$$\Phi(M)_g = \sum \mathrm{sign}(M, R_i)_g \cdot q^i,$$

where $\mathrm{sign}(M,R_i)_g = \mathbf{tr}|_g \ker(d_s \otimes R_i) - \mathbf{tr}|_g \mathrm{coker}(d_s \otimes R_i)$. The coefficients of its q-development are now equivariant twisted signatures. Thus, according to Theorem 2.1, the value of $\Phi(M)_g$ does not depend on g. Applying the Atiyah-Bott fixed point theorem [2], $\Phi(M)_g$ can be expressed in terms of the fixed point set M^g of g and the action of g on the normal bundle of $M^g \subset M$. In particular, let g be the orientation preserving involution in $\mathbb{Z}_2 \subset S^1$. We denote the transversal self-intersection of M_2 by $M_2 \circ M_2$. In [10, p. 315], Hirzebruch and Slodowy showed that

$$\Phi(M)_g = \Phi(M_2 \circ M_2).$$

On the other hand, applying Theorem 2.1, $\Phi(M) = \Phi(M)_g$, i.e.

$$\Phi(M) = \Phi(M_2 \circ M_2). \tag{8}$$

The codimension of M_2 is positive and even, so that the elliptic genus $\Phi(M)$ can now be computed from the elliptic genera of submanifolds of M of codimension at least 4, i.e. isolated points.

Let us now recall the expansion of $\Phi(M)$ at the other cusp [9]

$$\tilde{\Phi}(M) = \frac{1}{q^{\dim(M)/8}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j) \cdot q^j,$$

where R'_j is the sequence of virtual tensor bundles given by

$$R'(q,T) = \bigotimes_{k=2m+1} \bigwedge\nolimits_{-q^k} T \otimes \bigotimes_{k=2m+2} S_{q^k} T,$$

and the $\widehat{A}(M, R'_j) = \langle \widehat{A}(M) \cdot \mathrm{ch}(R'_j), [M] \rangle$ may only defined be as characteristic numbers. The first few terms of the sequence are $R'_0 = 1$, $R'_1 = -T$, $R'_2 = \bigwedge^2 T + T$, etc. This expansion is obtained by considering $q = e^{\pi i t}$ and changing the t coordinate in (1) by $t \to -1/t$, and then by $t \to 2t$ (cf. [9]). This expansion has, a priori, a pole of order 1/2. On the other hand, by (8) we also have

$$\tilde{\Phi}(M) = \tilde{\Phi}(M_2 \circ M_2), \tag{9}$$

whose right hand side has a pole of order at most $1/2 - 1/2 = 0$, since the dimension of any connected component of $M_2 \circ M_2$ is at most 0. Therefore (9) implies that the first coefficient on the left hand side vanishes,

$$\widehat{A}(M) = 0,$$

i.e.

$$\mathrm{sign}(M) = 0.$$

□

4. Applications

In dimension 4, given the modular nature of the elliptic genera, Φ and $\tilde{\Phi}$ satisfy

$$\begin{aligned} \Phi(M) &= \frac{\delta}{\varepsilon^{1/2}}\,\mathrm{sign}(M) = \frac{-8\,\delta}{\varepsilon^{1/2}}\,\widehat{A}(M), \\ \tilde{\Phi}(M) &= \frac{-8\,\delta}{(\delta^2 - \varepsilon)^{1/2}}\widehat{A}(M) = \frac{\delta}{(\delta^2 - \varepsilon)^{1/2}}\,\mathrm{sign}(M), \end{aligned}$$

where

$$\begin{aligned} \delta &= \frac{1}{4} + 6(q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + \ldots), \\ \varepsilon &= \frac{1}{16} - q + 7q^2 - 28q^3 + 71q^4 - 126q^5 + 196q^6 \pm \ldots, \end{aligned}$$

are the generators of degree 2 and 4 respectively of the space of modular forms for the subgroup

$$\Gamma_0(2) = \left\{ A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod 2 \right\} \subset SL_2(\mathbb{Z}).$$

Hence, we have also proved the following.

Corollary 4.1. *Let M be an oriented, compact, connected even 4-manifold admitting smooth S^1 actions. Then, the elliptic genus vanishes identically on M*

$$\Phi(M) = \mathrm{sign}(M) = 0 \quad \textit{and} \quad \tilde{\Phi}(M) = 0.$$

Furthermore, every genus vanishes on M and, in particular, the Witten genus [9]

$$\varphi_W(M) = \mathrm{ind}\left(d_s^M \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T\right) = 0$$

vanishes on M. □

Corollary 4.2. *The Enriques surface and the n-fold sum of Enriques surfaces admit no smooth S^1-actions.*

Proof. The Enriques surface has intersection form $E_8 \oplus H$, so it is incompatible with Theorem 1.1. Similarly for the n-fold connected sum of Enriques surfaces which have intersection form $nE_8 \oplus nH$. □

Finally, let us remark that the arguments in Sections 2 and 3, and particularly equation (2), prove the rigidity of the elliptic genus and the vanishing of the signature in a wider class of *non-Spin* 4-manifolds, including non-Spin π_2-finite manifolds [7, 8].

Theorem 4.1. *Let M be an oriented, compact, connected, smooth 4-manifold admitting a smooth S^1 action. Furthermore, assume that the normal bundle to every S^1-invariant 2-sphere contained in M is topologically trivial. Then,*

1. *the elliptic genus $\Phi(M)$ of M is rigid under the S^1 action;*
2. *the signature and $\widehat{A}$-genus of M vanish*

$$\mathrm{sign}(M) = 0, \quad \widehat{A}(M) = 0;$$

3. *the elliptic genus $\Phi(M)$ and all other genera vanish on M.*

□

References

[1] Acosta, D., Lawson, T.: Even non-Spin manifolds, spin^c structures, and duality. Enseign. Math. (2) **43** (1997), no. 1-2, 27–32.

[2] Atiyah, M.F., Bott, R.: The Lefschetz fixed point theorem for elliptic complexes. II. Applications. Ann. Math. **88**, 451–491 (1968)

[3] Atiyah, M.F., Hirzebruch, F.: Spin manifolds and group actions. Essays in Topology and Related Subjects, Springer-Verlag, Berlin, pp. 18–28 (1970)

[4] Atiyah, M.F., Segal, G.: The index of elliptic operators II. Ann. of Math. **86**, 531–545 (1968)

[5] Bott, R., Taubes, T.: On the rigidity theorems of Witten. J. AMS, **2**, No. 1, 137–186 (1989)

[6] Dessai, A.: Rigidity theorems for Spin^c-manifolds. Topology **39**, no. 2, 239–258 (2000)

[7] Herrera, H., Herrera, R.: *$\widehat{A}$-genus on non-spin manifolds with S^1 actions and the classification of positive quaternion-Kähler 12-manifolds.* J. Differential Geom. **61**, no. 3, 341–364 (2002)

[8] Herrera, H., Herrera, R.: *A result on the $\widehat{A}$ and elliptic genera on non-spin manifolds with circle actions.* C. R. Acad. Sci. Paris, Ser. I **335**, 371–374 (2002)

[9] Hirzebruch, F., Berger, T., Jung, R.: Manifolds and Modular Forms. Aspects of Mathematics, VIEWEG, 1992.

[10] Hirzebruch, F., Slodowy, P.: Elliptic genera, involutions, and homogeneous spin manifolds. Geometriae Dedicata **35**, 309–343 (1990)

[11] Liu, K.: On elliptic genera and theta-functions. Topology **35**, no. 3, 617–640 (1996)

[12] Ochanine, S.: Sur les genres multiplicatifs définis par des intégrales elliptiques. Topology **26**, 143–151 (1987)

[13] Ochanine, S.: Genres elliptiques équivariants. Elliptic curves and modular forms in ALgebraic Topology (P.S. Landweber ed.), Lect. Notes Math. 1326, pp.107–122. Berlin Heidelberg New York: Springer, 1988

[14] Taubes, C. H.: S^1 actions and elliptic genera. Comm. Math. Phys. **122**, no. 3, 455–526 (1989)

[15] Witten, E.: Elliptic genera and quantum field theory. Comm. Math. Phys. **109**, 525 (1987)

[16] Witten, E.: The index of the Dirac operator on loop space. Elliptic curves and modular forms in Algebraic Topology (P.S. Landweber ed.), Lect. Notes Math. 1326, pp.161–181. Berlin Heidelberg New York: Springer, 1988