NATURAL AXIOMS OF SET THEORY AND THE CONTINUUM PROBLEM

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ABSTRACT. As is well-known, Cantor's continuum problem, namely, what is the cardinality of $\mathbb{R}$? is independent of the usual ZFC axioms of Set Theory. K. Gödel ([12], [13]) suggested that new natural axioms should be found that would settle the problem and hinted at large-cardinal axioms as such. However, shortly after the invention of forcing, it was shown by Levy and Solovay [20] that the problem remains independent even if one adds to ZFC the usual large-cardinal axioms, like the existence of measurable cardinals, or even supercompact cardinals, provided, of course, that these axioms are consistent. While numerous axioms have been proposed that settle the problem–although not always in the same way–from the Axiom of Constructibility to strong combinatorial axioms like the Proper Forcing Axiom or Martin's Maximum, none of them so far has been recognized as a natural axiom and been accepted as an appropriate solution to the continuum problem. In this paper we discuss some heuristic principles, which might be regarded as *Meta-Axioms of Set Theory*, that provide a criterion for assessing the naturalness of the set-theoretic axioms. Under this criterion we then evaluate several kinds of axioms, with a special emphasis on a class of recently introduced set-theoretic principles for which we can reasonably argue that they constitute very natural axioms of Set Theory and which settle Cantor's continuum problem.

1. INTRODUCTION

> *There must be a first step in recognizing axioms, [...] a step which will make the axioms seem worth considering as axioms rather than merely as conjectures or speculations.*
>
> W.N. Reinhardt ([27])

Cantor's continuum problem, namely, what is the cardinality of $\mathbb{R}$? has been the central problem in the development of Set Theory. Since Cantor's formulation in 1878 of the Continuum Hypothesis (CH), which states that every infinite subset of $\mathbb{R}$ is either countable or has the same cardinality as $\mathbb{R}$ ([8]), very dramatic and unexpected advances have been made by Set Theory towards the solution of the problem. As is well-known, neither CH nor its negation can be proved from the usual ZFC axioms of Set Theory, provided they are consistent. In Gödel's constructible universe CH holds, while Cohen's method of forcing allows to build models of ZFC in which the cardinality of $\mathbb{R}$ can be any cardinal, subject only to the necessary requirement that it have uncountable cofinality.

This situation, however, is far from satisfactory. Admittedly, some mathematicians, including Cohen himself (see [9]), have expressed the belief that no further, more satisfactory solution is attainable, and that one should be content with the independence results. But this is a rather uncommon position among mathematicians, and set theorists in particular, with respect to the continuum problem. Drawing on a realistic approach to Mathematics, the most common by far among mathematicians, one can argue that the only thing the results of Gödel and Cohen show is that the ZFC axioms, while sufficient for developing most of classical Mathematics, constitute too weak a formal system for settling Cantor's problem and they should, therefore, be supplemented with additional axioms. Indeed, Gödel himself formulated a program ([12],[13]) of finding new natural axioms which, added to the ZFC axioms, would settle the continuum problem, and he hinted that large cardinal axioms would do it. This has been known as *Gödel's program*. Unfortunately, however, it was soon noticed by Levy and Solovay [20] that the usual large cardinal axioms, like the existence of measurable cardinals, or even supercompact or huge cardinals, would not be enough. But this does not mean that Gödel's program is no longer defensible. Quite the contrary. It is still perfectly possible that new kinds of large-cardinal axioms, different from the ones that have been considered so far, could be relevant to the solution of the continuum problem. In fact, recent work by Woodin ([38]) shows that under large cardinals, any reasonable extension of the ZFC axioms that would settle all questions of the same complexity of CH, in a strong logic known as Ω-logic, would refute CH (see [15] for a discussion of the relevance of Woodin's work on Gödel's program). But our purpose here is not to address the import of large-cardinal axioms to the continuum problem, at least not directly, but to introduce and discuss some heuristic principles, which might be regarded as *Meta-Axioms of Set Theory*, that provide a criterion for assessing the naturalness of the set-theoretic axioms. Under this criterion we then evaluate several kinds of axioms, including large cardinals, with a special emphasis on a class of set-theoretic principles that have been recently introduced, known as Bounded Forcing Axioms, for which one can reasonably argue that they constitute very natural axioms of Set Theory, and which settle Cantor's continuum problem.

2. Natural axioms of Set Theory

The central principle is the reflection principle, which presumably will be understood better as our experience increases.
K. Gödel ([36])

What should be counted as a *natural* axiom of Set Theory? Certainly any intuitively obvious fact about sets. Here we shall take for granted that the ZF axioms are of this sort. There is very little disagreement about this point. As for the Axiom of Choice, the reluctance regarding its full acceptance by some mathematicians is due more to some of its counter-intuitive consequences, rather than to its otherwise very natural character (see however [16]). It is a fact that no other universally (or almost-universally) accepted as intuitively obvious principles about sets have been proposed, perhaps with the only exception of the existence of small large cardinals, like the inaccessible cardinals.

If we accept that *being an intuitively obvious fact about sets* is a necessary requirement for a set-theoretic principle to be counted as an axiom, then no axioms other than the ZF (or ZFC) axioms, plus, perhaps, some small large-cardinal existence axioms should be accepted. So, if we were to look for additional axioms we should first try to sharpen our intuitions about sets until we were forced to accept some new principle as intuitively obvious, or at least intuitively reasonable. While this is *a priori* possible, and it would certainly be a remarkable achievement to discover such a new principle, there are at least two practical difficulties with this approach. First, it is well known that intuition may be easily confused with familiarity. For do we not end up finding reasonable whatever principle we have been using for a long time? Are we not eager to welcome as a new axiom any principle in which we have invested a considerable amount of time and effort, and for which we have developed, no doubt, a strong intuition? Second, in principle, incompatible intuitively reasonable principles could be found. For what prevents set-theoretic intuition to be developed in several irreconcilable ways? It may be replied that if this were the case, then all the better, for we would have several different set theories, all founded on intuition, albeit each on a different one. If this will be the case, then so be it. But we will see that, beyond intuition, there are other criteria which can be successfully used to find new axioms.

In his paper *What is Cantor's Continuum Problem?* ([12], [13]), Gödel considers two criteria for the acceptance of new axioms of Set Theory. One is that of *necessity or non-arbitrariness.* He uses this criterion to justify the existence of inaccessible cardinals. If we want to extend the operations of set formation beyond what is provable in ZFC, then we are forced to postulate the existence of an inaccessible cardinal (see our discussion of this point in section 4 below). Thus the existence of an inaccessible cardinal is a necessary, non-arbitrary assumption, for further extending the *set of* operation. Notice that the postulation of the existence of an inaccessible cardinal is analogous to the situation in which, starting from ZF-Infinity (i.e., Zermelo-Fraenkel Set Theory minus the Axiom of Infinity), we postulate the existence of an infinite set. Indeed, no matter how we extend the ZF-Infinity axioms by asserting the existence of new sets, we are forced to assert the existence of an infinite set, and so, in this sense, ZF is a necessary, non-arbitrary extension of ZF-Infinity. Once the existence of an inaccessible cardinal is accepted, then one is naturally led to the iteration of this principle, thus leading to hyperinaccessible cardinals, and beyond. But can larger cardinals be justified under the necessity criterion? In what sense, if any, are measurable cardinals necessary? We shall come back to this.

A second criterion used by Gödel in [12] for the acceptance as axioms of set-theoretic principles is *success*, that is, the *fruitfulness in their consequences*. This criterion is put forward as an alternative to *necessity* or *non-arbitrariness*. After over half a century of continued work on large cardinals, and especially since the discovery of the connections between large cardinals and determinacy in the eighties, it can be argued that the existence of large cardinals, at least up to Woodin cardinals, should be accepted as axioms of Set Theory, according to this criterion. Indeed, Martin and Steel [25] showed that the Axiom of Projective Determinacy (PD), and in fact the axiom $AD^{L(\mathbb{R})}$, which asserts that all sets of reals definable fom ordinals and real numbers as parameters are determined, follows from axioms of large cardinals. Woodin showed that the existence of infinitely many Woodin Cardinals plus a measurable cardinal larger than all of them would suffice and, furthermore, that infinitely many Woodin cardinals are necessary to obtain PD (see [37] and [34]). As it became clear during the seventies through the spectacular advances made by Descriptive Set Theory under the assumption of PD, this principle appears to be the right one for developing the theory of projective sets of real numbers. Indeed, PD gives an essentially complete theory of the projective sets. Moreover, any known set-theoretic principle of at least the consistency strength of PD – for instance, the Proper Forcing Axiom – implies PD, which strongly suggests its necessity. The fruitfulness of large cardinal axioms is further exemplified by their numerous consequences in infinitary combinatorics (see [18]). It is now plainly clear that many desirable consequences, not only in Set Theory, but in all areas of Mathematics where set-theoretic methods are applied, follow from large-cardinal assumptions. Thus, strong large-cardinal principles have done very well under the fruitfulness criterion. But is this sufficient for accepting them as axioms of Set Theory? This may be so for the existence of infinitely many Woodin cardinals, since they have been shown to be both sufficient and necessary to obtain PD, thus yielding a rich and elegant theory for the projective sets of real numbers which extends the classical ZFC theorems of Descriptive Set Theory. For stronger large-cardinal principles, the situation is much less clear. The main problem in accepting large cardinal axioms is their consistency. After all, some large cardinal principles have been shown to be inconsistent and consequently rejected. Nevertheless, the so-called *inner model program*, which attempts to build canonical models for large cardinals, has developed very sophisticated methods for showing that, at least for large cardinals up to infinitely-many Woodin cardinals, one can construct canonical inner models with a well-developed fine structure, thereby building confidence in their consistency. So, in spite of some diverging opinions, we can fairly say that it is a widespread belief among set theorists that large-cardinal principles should be accepted as axioms of Set Theory provided there is a sufficiently well-developed inner model theory for them. This is already the case for infinitely many Woodin cardinals, but no such inner model theory has been yet developed for, e.g., supercompact cardinals.

But as has been pointed out before, large-cardinal axioms, in spite of their extraordinary success, are not sufficient for settling Cantor's Continuum Problem. So in the absence of any further intuitively obvious axioms, the question is whether there are any other kinds of axioms that are non-arbitrary and, if possible, that also satisfy the fruitfulness criterion.

Although the value of an axiom will ultimately be determined by its success, the criterion of *success* can hardly be sufficient for accepting a new axiom. It should only be used to assess, *a posteriori*, the value of the axioms, which must be found according to other criteria.

H. Wang, in [35], and later in [36] section 8.7, quotes Gödel on his 1972 answer to the question of what should be the principles by which new axioms of Set Theory should be introduced. According to Gödel there are five such principles: *Intuitive Range*, the *Closure Principle*, the *Reflection Principle*, *Extensionalization*, and *Uniformity.* The first, *Intuitive Range*, is the principle of intuitive set formation, which is embodied into the ZFC axioms. The *Closure Principle* can be subsumed into the principle of *Reflection*, which may be summarized as follows: The universe V of all sets cannot be uniquely characterized, i.e., distinguished from all its initial segments, by any property expressible in any reasonable logic involving the membership relation. A weak form of this principle is the ZFC-provable reflection theorem of Montague and Levy (see [18]):

> *Any sentence in the first-order language of Set Theory that holds in V holds also in some V_α.*

Gödel's *Reflection* principle consists precisely of the extension of this theorem to higher-order logics, infinitary logics, etc.

The principle of *Extensionalization* asserts that V satisfies an extensional form of the Axiom of Replacement and it is introduced in order to justify the existence of inaccessible cardinals. We will explain its role in the next section.

The principle of *Uniformity* asserts that the universe V is uniform, in the sense that its structure is similar everywhere. In Gödel's words ([36], 8.7.5): *The same or analogous states of affairs reappear again and again (perhaps in more complicated versions).* He also says that this principle may also be called *the principle of proportionality of the universe*, according to which, analogues of the properties of small cardinals lead to large cardinals. Gödel claims that this principle makes plausible the introduction of measurable or strongly compact cardinals, insofar as those large-cardinal notions are obtained by generalizing to uncountable cardinals some properties of ω.

Thus, following Gödel, in the search for new axioms beyond ZFC, we are to be guided by the criteria of *Necessity*, *Success*, *Reflection*, *Extensionalization*, and *Uniformity*, to which we should add that of *Consistency*, which Gödel certainly took for granted. The new axioms should be necessary in order to extend the operations of set formation beyond what is provable in ZFC, they should take the form of reflection principles, they should imply some kind of uniformity in the universe of all sets, and they should be both consistent and fruitful in their consequences.

In the next section we will discuss and attempt to further clarify these criteria so that they can be actually applied in the testing of – and the search for – new axioms. We will argue that all criteria reduce essentially to two: *Maximality* and *Fairness*. *Consistency* and *Success* play a complementary role, the first as a regulator and the second as a final test for value. All together, the criteria may be regarded as an attempt to define what *being a natural axiom of Set Theory* actually means. They may as well be viewed as a test for *necessity* or *non-arbitrariness*, since any set theoretic statement that satisfies the criteria will, in a precise sense, be forced upon us if we want to extend ZFC.

3. Meta-axioms of Set Theory

We are searching for additional axioms of Set Theory that extend ZFC, that is, for a sentence (or a recursive set of sentences) in the first-order language of Set Theory. What are the criteria such a sentence should satisfy in order to be considered an axiom?

The first criterion is, of course, *Consistency*. We want the new axiom to be consistent with ZFC. Clearly, by Gödel's second incompleteness theorem, we can only hope for a proof of relative consistency. Namely, we should be able to prove that *if* ZFC is consistent, *then* so is ZFC plus the new axiom. There are many incompatible examples, e.g., $CON(ZFC)$ and $\neg CON(ZFC)$, the Axiom of Constructibility or its negation, the Continuum Hypothesis or its negation, Suslin's Hypothesis or its negation, etc. Thus, consistency cannot be the only criterion. Moreover, we should also entertain the possibility of accepting axioms whose consistency (modulo ZFC) cannot be proved in ZFC, simply because they can be shown to be, consistencywise, stronger than ZFC, but which nevertheless satisfy the other criteria.

Therefore, the criterion of *Consistency* can only play a regulatory role in the search and justification of new axioms. It puts a bound on the joint action of the other criteria. The mere fact that a set-theoretic principle can be shown to be consistent with ZFC does not make it automatically an axiom. But consistency with ZFC is certainly a necessary requirement. Moreover, if the new axiom is shown to be consistent modulo some large-cardinal assumption, then the consistency of such a large cardinal must follow from ZFC plus the new axiom, thus proving its necessity for the new axiom's consistency proof.

The second criterion is that of *Maximality*. Namely, the more sets the axiom asserts to exist, the better. Gödel already stated that: *...Only a maximum property would seem to harmonize with the concept of set..*(see [13]). The idea of maximizing has been defended by many people and it has been extensively discussed by P. Maddy (see [21] and [22]) in the context of her naturalistic philosophy of Set Theory. The maximality criterion has normally been used to provide a justification for the rejection of the Axiom of Constructibility, but here we intend to apply it systematically as a guiding criterion in the search for new axioms.

All large-cardinal axioms and all forcing axioms satisfy the *Maximality* criterion, in the weak sense that they all imply the existence of new sets. Thus, in such a generality this is clearly too vague a criterion, and therefore definitely useless. For if ZFC is consistent, then we can easily find statements that are consistent, modulo ZFC, and assert the existence of some new sets, but which are incompatible. Take, for instance, $CON(ZFC)$, which asserts the existence of a model of ZFC, and $\neg CON(ZFC)$, which asserts the existence of a (non-standard) proof of a contradiction from ZFC.

To attain a more concrete and useful form of the *Maximality* criterion it will be convenient to think about maximality in terms of models. Namely, suppose V is the universe of all sets as given by ZFC, and think of V as being properly contained in an *ideal* larger universe W which also satisfies ZFC and contains, of course, some sets that do not belong to V – and it may even contain V itself as a set – and whose existence, therefore, cannot be proved in ZFC alone. Now the new axiom should imply that some of those sets existing in W already exist in V, i.e., that some existential statements that hold in W hold also in V. Since the sets in V are already given we may as well allow for the existential statements to have parameters in V. Thus, *Maximality* leads to *Reflection* principles, namely, the existential statements (with parameters) that hold in the ideal extension W *reflect* to V.

By repeated application of *Reflection*, something which the *Maximality* criterion forces us to do, the universe of all sets becomes more uniform. For instance, if some set A is the solution of an existential sentence $\varphi(x)$ that holds in some ideal extension W of V, then we may consider the sentence $(\varphi(x) \wedge \neg x = A)$, which contains A as a parameter, and by applying *Reflection* again obtain another solution of $\varphi(x)$ different from A. Or if α is the rank of A, then by considering the sentence $(\varphi(x) \wedge rank(x) > \alpha)$ we obtain another solution of $\varphi(x)$ of higher rank, etc. Thus, *Reflection* leads to the existence of *many* solutions of any given existential statement, e.g., solutions of arbitrarily high rank. Gödel listed *Uniformity* as a separate principle. He understood it as a justification for the extrapolation to larger cardinals of some of the properties of small cardinals, like ω. We do not consider this by itself as a sound criterion, since we do not see any need for arbitrary properties of, say, ω to hold for some larger cardinals. Some of its properties certainly do not hold for larger cardinals, like the property of being countable. So, some criterion should be given for choosing among all the distinct properties. In our remarks below regarding particular kinds of axioms we will see how a strong form of *Uniformity* does follow from the systematic application of the criterion of *Maximality*.

Notice that not all existential statements are maximizing principles in the same sense. Indeed, CH is an existential statement which asserts the existence of a function on ω_1 that enumerates all the real numbers, but at the same time asserts the existence of *few* real numbers. So, does CH assert the existence of *more* sets or of *fewer* sets? On the other hand, not-CH is also an existential statement which asserts the existence of more than $\aleph_1$ many reals, while implying that, for instance, there are no diamond sequences. So, again it is unclear, *a priori*, whether not-CH is a *maximizing* or a *minimizing* principle. Which one of CH or its negation should we then accept according to the *Maximality* criterion? The difficulty of the question is best exemplified by the fact that it is easy to construct by forcing three models of ZFC, $M_1 \subseteq M_2 \subseteq M_3$, such that CH holds in both M_1 and M_3 and fails in M_2. The problem is that both CH and its negation are Σ_2 statements, and Σ_2 sentences, while asserting the existence of some sets, may in fact be limitative. The same applies to more complex existential sentences. The only unquestionably maximizing existential sentences are the Σ_1.

Another direct consequence of the *Maximality* criterion is Gödel's principle of *Extensionalization.* This can be stated as follows: We should require that V satisfies all instances of the Replacement Axiom for functions with domain some set in V and range contained in V that are available in some ideal extension of V. To what extent is this a reasonable assumption? It is reasonable insofar as this is what we would like to have for V itself. With V the problem is that, besides the set-functions, there are no more such functions available other than those that are definable in V. But when more functions become available, even if they are ideal functions, there is no reason, a priori, why they should be excluded.

We may thus conclude that Gödel's principles of *Reflection*, *Extensionalization*, and *Uniformity* arise naturally from the systematic application of the criterion of *Maximality.*

We need a third criterion to help us sort out among all possible set existence statements that hold in some ideal extensions of V those that will be taken as new axioms. Such a criterion may be called *Fairness*. We could also call it the *Equal Opportunity* criterion. It can be stated as:

One should not discriminate against sentences of the same logical complexity.

The rationale for this criterion is that in the absence of a clear intuition for the selection, among all the set-existence statements that hold in some ideal extension of the set-theoretic universe, of those that are true about sets, we have *a priori* no reason for accepting one or another. So, once we accept one, we must also accept all those that have the same logical complexity.

The logical complexity of a formula of the language of Set Theory is given by the Levy hierarchy, namely, the Σ_n and Π_n classes of formulas (see [17]).

If we are to allow parameters in our formulas, then we should also require that:

One should not discriminate against sets of the same complexity.

Now the complexity of a set may be defined in different ways, but the most natural measures of the complexity of a set are its rank and its hereditary cardinality.

Thus, a *fair* class of existential sentences will be one of the classes Σ_n with parameters in some V_α, α an ordinal, or some H_κ, κ a cardinal. Classes of higher-order formulas, like the Σ^m_n, or formulas pertaining to some infinitary logic could also be considered. Moreover, the language could also be expanded by allowing new constants or predicates, etc.

Finally, there is the criterion of *Success*. As was remarked before, its main use is for evaluating the axioms that have been found by following the other criteria. A new axiom should not only be natural, but it should also be useful. Now, usefulness may be measured in different ways, but a useful new axiom must be able at least to decide some natural questions left undecided by ZFC. If, in addition, the new axiom provides a clearer picture of the set-theoretic universe, or sheds new light into obscure areas, or provides new simpler proofs of known results, then all the better.

In conclusion, once we agree on what kind of ideal extensions of V we should be considering, by applying the three criteria above simultaneously (*Consistency, Maximality,* and *Fairness*), the crucial question becomes:

Find a (largest possible) fair class Σ of existential sentences such that the principle that asserts that all sentences in Σ that hold in an ideal extension are true can be stated as a sentence (or a recursive set of sentences) in the first-order language of Set Theory and is consistent with ZFC.

Once such a principle is found, we can reasonably argue that it constitutes a natural axiom of Set Theory. Its survival as a new axiom, in terms of being accepted and used by the set theorists, will then be largely determined by its success.

We shall now put to test our criteria in the case of large-cardinal axioms.

4. The naturalness of large-cardinal axioms

> *Whatever theory we have about what exists, it should be compatible with our understanding of our theory that the totality of existing things should be a set.*
>
> W.N. Reinhardt ([27])

Large cardinal axioms may be divided into two classes: the strong axioms of infinity, and the large cardinal axioms arising from elementary embeddings of V into transitive proper classes, i.e., the measurable cardinals and above.

4.1. **Strong axioms of infinity.** The *strong axioms of infinity* originate when one considers ideal extensions of the universe V of all sets, as given by ZFC, in which the transfinite sequence of all ordinals, and therefore the power set operation, is continued yet even further. In this ideal extension, the class OR^V of all ordinals in V would be an ordinal κ, and V itself would be a set. We thus imagine V to be actually some initial rank V_κ of a larger universe so that $V_\kappa \models ZFC$.

We can introduce new axioms stating that sentences in a given fair class Σ reflect to V_κ. These kinds of axiom, even though they satisfy our criteria, they may not have any large-cardinal strength and their consequences may be rather poor. For instance, the axiom that asserts that V_κ satisfies ZFC and reflects all Σ_n sentences, for some fixed n, follows from the existence of a stationary class of ordinals α such that V_α satisfies ZFC, a principle which has no large-cardinal strength and is consistent with the Axiom of Constructibility.

A crucial step forward in strength is obtained by requiring that κ is a regular cardinal. Notice that if V_κ is a model of ZFC, then $\langle V_\kappa, \in, \kappa\rangle \models$ "κ is a regular cardinal". But κ need not even be a cardinal in V. Requiring that κ is a regular cardinal in V amounts to requiring that V_κ satisfies a bit of the second-order Replacement Axiom. Namely, Replacement for all functions with domain some ordinal less than κ and values in κ, which need not be definable in V_κ. It turns out that since $V_\kappa \models ZFC$, satisfying this bit of second-order Replacement implies that V_κ satisfies the full second-order Replacement Axiom. This form of *extensional Replacement* is exactly the content of Gödel's principle of *Extensionalization*, which we have already discussed in the previous section; we argued its naturalness under the *Maximality* criterion.

Now for κ a regular cardinal, the following are equivalent:

(1) $V_\kappa \models ZFC$
(2) $V_\kappa \prec_{\Sigma_1} V$
i.e., V_κ reflects all Σ_1 sentences with parameters, which means that for every $a_1, ..., a_k \in V_\kappa$ and every Σ_1-formula $\varphi(x_1, ..., x_k)$,

$$V_\kappa \models \varphi(a_1, ..., a_k) \quad \text{iff} \quad \varphi(a_1, ..., a_k).$$

A regular cardinal satisfying (1) or (2) above is *inaccessible*. Thus according to our criteria the existence of an inaccessible cardinal is a natural axiom of Set Theory. If we want to continue, *yet one more step*, the iterative construction of V, we are forced to accept the existence of an inaccessible cardinal. The existence of an inaccessible cardinal is the first of the large cardinal axioms.

The existence of an inaccessible cardinal cannot be proved in ZFC, for if κ is inaccessible, then V_κ is a model of ZFC. Hence, the consistency of ZFC cannot imply the consistency of ZFC plus the existence of an inaccessible cardinal. The sentence that asserts the existence of an inaccessible cardinal κ, as every other large cardinal axiom, has greater consistency-strength than

ZFC. Therefore, it cannot satisfy the criterion of *Consistency* in its basic form, but of course it trivially satisfies it modulo large cardinals. It does however satisfy the other two criteria of *Maximality* and *Fairness* for the class of Σ_1 formulas with parameters in $V_\kappa = H_\kappa$.

The next step is to consider the class of Σ_2 sentences, namely, suppose that κ is inaccessible and

$$V_\kappa \prec_2 V$$

i.e., it reflects all Σ_2 sentences with parameters. Then κ is an inaccessible cardinal, a limit of inaccessible cardinals, and much more.

More generally, for every n one may consider the existence of a regular cardinal κ such that

$$V_\kappa \prec_n V$$

Such a cardinal is called *n-reflecting*. The axioms that assert the existence of n-reflecting cardinals do satisfy the criteria of *Maximality* and *Fairness*. But if $n < m$, then ZFC plus the existence of an m-reflecting cardinal implies the consistency of ZFC plus there is a n-reflecting cardinal. Thus, those axioms are strictly increasing in consistency strength.

Notice that since for $n < m$, if κ is an m-reflecting cardinal then it is also n-reflecting, asserting the existence of an m-reflecting cardinal makes the universe larger than just asserting the existence of an n-reflecting cardinal.

For each n, the sentence: *There exists a n-reflecting cardinal*, can be written as a first-order sentence. However, by Tarski's theorem on the undefinability of truth, there cannot be a definable κ such that V_κ reflects all sentences. Moreover, the sentence: *There exists a cardinal κ that reflects all Σ_n sentences, all n*, cannot even be written in the first-order language of Set Theory.

We conclude that the set of all sentences of the form: *There exists a n-reflecting cardinal*, n an integer, forms a recursive set of natural axioms of Set Theory (modulo its consistency with ZFC). In fact, by the same arguments, and following the principle of *Maximality*, we are led to the acceptance as a natural recursive set of axioms the set of all sentences of the form: *There exists a proper class of n-reflecting cardinals*, n an integer (modulo its consistency with ZFC).

A strengthening of the notion of inaccessibility is that of a Mahlo cardinal: κ is a *Mahlo* cardinal if it is regular and the set of inaccessible cardinals below κ is stationary, i.e., every closed and unbounded subset of κ contains an inaccessible cardinal. Notice that since inaccessible cardinals are regular, we cannot hope to have a club of inaccessible cardinals below κ, but we may have the next best thing, namely, a stationary set of them. This is a natural assumption according to the principle of *Maximality*. The point is that, provably in ZFC, every sentence φ that holds in V reflects to a club class of V_α. So, there should be an inaccessible cardinal κ such that V_κ satisfies φ. Once the existence of inaccessible cardinals is accepted, we should also accept that there are as many of them as possible, and this means a stationary class of them.

A Mahlo cardinal cardinal κ is inaccessible, and in V_κ there is a stationary class of Σ_ω-reflecting cardinals, i.e., Σ_n-reflecting for every n. Notice that κ is Mahlo iff κ is regular, $V_\kappa \models ZFC$, and the set of regular $\lambda < \kappa$ such that $V_\lambda \models ZFC$ is stationary. Thus, once inaccessible cardinals and reflecting cardinals are accepted, Mahlo cardinals are the next natural step in the process of extending the reflection properties of the universe of all sets.

By allowing higher-order formulas one obtains the so-called *indescribable cardinals*, which form a hierarchy, according to the complexity and the order of the formulas reflected: κ is Σ^m_n-*indescribable* (Π^m_n-*indescribable*) if for every $A \subseteq V_\kappa$ and every Σ^m_n-sentence (Π^m_n-sentence) φ, if $\langle V_\kappa, \in, A\rangle \models \varphi$, then there is $\lambda < \kappa$ such that $\langle V_\lambda, \in, A \cap V_\lambda\rangle \models \varphi$.

We have that κ is Σ^1_1-indescribable iff it is inaccessible. A minimal strengthening of this property yields the Π^1_1-indescribable cardinals. Π^1_1-indescribable cardinals are also known as *weakly-compact cardinals*. Every weakly-compact cardinal κ is Mahlo and the set of Mahlo cardinals below κ is stationary.

Above all those cardinals are the *totally indescribable* cardinals. i.e., κ is totally indescribable if for every $A \subseteq V_\kappa$ and every sentence, of any complexity and any order, that holds in $\langle V_\kappa, \in, A\rangle$ it already holds in some $\langle V_\lambda, \in, A \cap V_\lambda\rangle$, $\lambda < \kappa$.

Totally indescribable cardinals seem to be the end in the direction of extending the reflection properties of V obtained by considering ideal extensions of the sequence of ordinals. We may have a stationary class of totally indescribable cardinals, but no stronger forms of reflection seem possible.

It can be shown that if the large cardinal axioms considered so far are consistent with ZFC, then they are also consistent with ZFC plus $V = L$. This is not surprising since those axioms arise without making any assumptions on the structure of V beyond ZFC, and for all we know V might just be L.

4.2. **Large cardinal axioms.** One obtains much stronger axioms by considering another kind of ideal extension of V. Even though V contains *all* sets, we may think of V as included in a larger transitive universe M having the same ordinals as V so that M is fatter than V, in the sense that for every ordinal α, V_α is included in M_α, and for some α – hence also for all ordinals greater than α – the inclusion is proper. According to the *Fairness* criterion, we would like to say that every Σ_1 sentence, possibly with parameters in V, that holds in M, already holds in V. But this is not possible. No transitive proper class V different from M can be a Σ_1-elementary substructure of M. The reason is that if this were the case, then $M_\alpha = V_\alpha$, for all α, contradicting the assumption that M was fatter than V. The problem here is twofold. On one hand we assumed M contains some sets that do not belong to V, while having the same ordinals. On the other hand we allowed arbitrary parameters in our Σ_1 sentences. But there is a more fundamental problem: in considering ideal extensions of V which contain the same ordinals, we just do not know what are the ideal sets that exist in M but not in V. In the case of the strong axioms of infinity, when we considered ideal extensions where the ordinals extended beyond all the ordinals of V, we knew what the new sets could be like, namely, the constructible sets built at the ideal ordinal stages. But in the present situation, where the ordinals of V and M are the same and V is contained in M, we just do not have any clue as to what the ideal sets in M might be. In other words, for all we know V, and therefore M, might just be L.

One possible way out of this difficulty is to take M to be a subclass of V, so that there are really no new sets, but still view V as properly contained in M. This is possible if we think of V as *embedded* into M. By transitively collapsing M we may just assume that M is transitive. So, suppose that M is a transitive class and there exists an embedding $j : V \to M$ which is not the identity and is Σ_1-elementary, i.e., for every Σ_1 sentence $\varphi(x_1, ..., x_n)$, and every $a_1, ..., a_n$,

$$\varphi(a_1, ..., a_n) \quad \text{iff} \quad M \models \varphi(j(a_1), ..., j(a_n)).$$

Then there is a least cardinal such that $j(\kappa) \neq \kappa$, called the critical point of j. κ is the first ordinal where $j''V$ and M start to differ. Indeed, we have that $j \restriction V_\kappa$ is the identity. Such a cardinal is measurable, i.e., there exists a two-valued κ-complete measure $\mathcal{U}$ on κ, namely $\mathcal{U} = \{X \subseteq \kappa : \kappa \in j(X)\}$. In fact, the existence of a measurable cardinal is equivalent to the existence of a Σ_1-elementary embedding, different from the identity, of V into a transitive class M. The class M is the transitive collapse of the ultrapower $V^\kappa/\mathcal{U}$, and the embedding is given by $j(x) = \pi([c_x]_\mathcal{U})$, where $c_x : \kappa \to \{x\}$ is the constant function x and π is the Mostowski transitive collapsing function.

If κ is a measurable cardinal, then it is the κ-th inaccessible cardinal. However, it need not even be Σ_2-reflecting.

As it turns out, if $j : V \to M$ is Σ_1-elementary, then it is fully elementary, i.e., for every formula $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n$,

$$\varphi(a_1, ..., a_n) \quad \text{iff} \quad M \models \varphi(j(a_1), ..., j(a_n)).$$

Although the sentence *There exists an embedding from V into M* is not first-order expressible, we can assert the existence of an elementary embedding from V into some class M just by asserting the existence of a measurable cardinal κ, which is first-order expressible.

Thus, we conclude that the axiom that asserts the existence of a measurable cardinal satisfies the criteria of *Maximality* and *Fairness* and is, therefore, a natural axiom of Set Theory (modulo its consistency with ZFC).

M cannot be V itself, since by a famous result of Kunen (see [17]), one cannot have a non-trivial elementary embedding $j : V \to V$. M cannot be L either, since as it was observed by Scott (see [17]) otherwise we would have $V = L$ and, if κ is the least measurable and j the associated embedding, by elementarity, in L $j(\kappa)$ would be the least measurable cardinal, thus contradicting the fact that $\kappa < j(\kappa)$. Thus, unlike in the case of Σ_n-reflecting cardinals, the existence of a measurable cardinal implies that $V \neq L$.

The larger M, the closer it is to V, the stronger is the axiom obtained. This is not surprising, since the richer M is, the richer is any substructure elementarily embedded into it. The upper bound is when M is V itself, which leads to inconsistency, by Kunen's result. Some possible strengthenings are the following: first, we may require that M contain arbitrarily large initial segments of V, namely,

There is a cardinal κ such that for every ordinal α there is an elementary embedding $j : V \to M$, M transitive, with critical point κ and with $V_\alpha \subseteq M$.

Such a cardinal κ is known as a *strong cardinal.* If κ is strong, then it is the κ-th measurable cardinal. Unlike the case of measurable cardinals, the existence of a strong cardinal κ cannot be formulated in terms of the existence of a certain measure on κ. However, a formulation in the first-order language of Set Theory is still possible, although somewhat more involved (see [18]). If there exists a strong cardinal, then $V \neq L(A)$, for every set A. In particular, $V \neq L(V_\alpha)$, for every α. Thus the existence of a strong cardinal could never be obtained by just ideally extending the ordinal sequence. A further strengthening is given by the following:

There is a cardinal κ such that for every ordinal α there is an elementary embedding $j : V \to M$, M transitive, with critical point κ and with ${}^{\alpha}M \subseteq M$.

Such a κ is called a *supercompact cardinal.* If κ is supercompact, then it is strong. Consistency-wise, the existence of a supercompact cardinal is much stronger than the existence of a strong cardinal. Many other variations and further strengthenings are possible (see [18]), yielding ever stronger axioms. Specially important for their essential role in Descriptive Set Theory are the Woodin cardinals, which are consistency-wise between strong and supercompact cardinals.

We already remarked that the upper limit of the axioms of this sort is given by Kunen's proof of the impossibility of having a non-trivial elementary embedding $j : V \to V$. But by fusing together the two kinds of ideal extensions of V considered so far, namely, the extension of the ordinal sequence and the existence of elementary embeddings of V into some transitive classes, we could ask for the existence of some non-trivial elementary embedding $j : V_\alpha \to V_\alpha$, for some α. This turns out to be an extremely strong axiom, although so far no inconsistency has been derived from it. But this axiom does satisfy the two criteria of *Maximality* and *Fairness*, and so, modulo its consistency, is a natural axiom of Set Theory.

As with the axioms of strong infinity, in the case of axioms of large cardinals, once we are led to the acceptance of the existence of a certain large cardinal, by applying the principle of *Maximality* we are naturally led to the acceptance of a (stationary) proper class of them.

Let us stop here our discussion of the axioms of large cardinals, since the above examples are sufficient for our present purposes. We just wanted to illustrate the fact that the usual large cardinal axioms are nothing else but the natural axioms – *natural* meaning that they satisfy the criteria of *Maximality* and *Fairness* – one obtains by asserting the existence of those sets that would exist in ideal extensions of V obtained by either expanding the ordinal sequence or by viewing V as embedded in yet a larger universe having the same ordinals, but which is, in fact, a subclass of V. It has been repeatedly argued that the remarkable fact that large cardinal axioms, in spite of the initially different motivations for their introduction, have been shown to fall into a linearly ordered hierarchy, lends them naturalness and contributes to their justification as additional axioms of Set Theory. But this is a misleading perspective. There is nothing remarkable about the fact that the large cardinal axioms fall into a linear hierarchy, for this is an immediate consequence of their being equivalent to ever stronger reflection principles from ideal expansions of the universe into V. What are remarkable, in any case, are the results that characterize them as reflection principles, thus revealing their true nature.

Another possible solution to the difficulties of finding fair axioms arising from ideal extensions of V which contain the same ordinals is provided by the method of *forcing*. Forcing is actually the only general method we know of which, starting with a model of ZFC, allows to build a larger new model of ZFC.

5. Suslin's Hypothesis and Forcing Axioms

Forcing is a method to make true statements about something of which we know nothing.

K. Gödel ([36])

Arguably, the second most important problem for the development of Set Theory (the first being, of course, Cantor's continuum problem) has been Suslin's Hypothesis: *Every complete dense and without endpoints linear ordering with the countable chain condition is order-isomorphic to* $\mathbb{R}$. The proof of its failure in L by Jensen led to his discovery of the $\diamondsuit$ principle and all the subsequent combinatorial principles in L, the development of fine structure theory, etc. On the other hand, the proof of its consistency by Solovay and Tennenbaum [30] gave birth to the theory of iterated forcing with all its developments and applications. The special relevance of Suslin's Hypothesis to our discussion lies in the fact that, as we shall see, it is in the proof of its consistency that we find the origin of the class of set-theoretic principles that we want to discuss.

The proof of the consistency of Suslin's Hypothesis using iterated forcing led to the isolation by D. Martin [24] of a set-theoretic principle which has been known as *Martin's Axiom* (MA). In spite of its name, at first glance the principle can be hardly recognized as an *axiom*. It states the following:

For every partially-ordered set $\mathbb{P}$ *with the countable chain condition, and for every family* $\mathcal{D}$ *of cardinality less than the cardinality of the continuum of dense open (in the order topology) subsets of* $\mathbb{P}$*, there is a filter* $\mathcal{F} \subseteq \mathbb{P}$ *that intersects all sets in* $\mathcal{D}$*.*

This axiom can also be seen as a generalization of the Baire Category Theorem, for it is equivalent to the following:

In every compact Hausdorff ccc space, the intersection of fewer than the cardinality of the continuum dense open sets is dense.

Since its formulation in 1970, MA has been widely used not only within Set Theory, but it has also been successfully applied to the solution of many problems in Combinatorics, General Topology, Measure Theory, Real Analysis, etc. (see [10]). However, in spite of its success as a technical tool, the prevalent opinion has been that it is by no means an axiom, in the same sense that the other ZFC axioms are, namely, an intuitively obvious fact about sets (see, for instance, [19]).

In the late seventies, and as an outgrowth of his study of Jensen's forcing which was used to prove the consistency of Suslin's Hypothesis with the generalized Continuum Hypothesis, Shelah introduced the notion of *Proper Forcing* (see [28]). *Properness* is a property of partially-ordered sets weaker than the countable chain condition (ccc). It is a rather natural notion that arises when one wants to perform forcing iterations with partial orderings that are not ccc without collapsing ω_1.

Several weaker notions than the ccc had already been considered in the literature before Shelah's notion of properness, and the corresponding stronger forms of MA had been formulated and applied. Especially successful was Baumgartner's Axiom A, a property of partial orderings weaker than the ccc which encompassed many of the partial orderings used in forcing constructions involving the continuum. Since properness is an even weaker condition than the Axiom A property, Baumgartner naturally formulated the *Proper Forcing Axiom* (PFA), that is, MA for the class of proper posets with the necessary restriction that the family $\mathcal{D}$ of dense open subsets of the partial ordering $\mathbb{P}$ be of cardinality at most $\aleph_1$. Without this restriction the axiom would just be inconsistent with ZFC. Baumgartner also showed that PFA is consistent with ZFC, assuming the consistency of ZFC with the existence of a supercompact cardinal.

An even weaker notion than properness was introduced by Shelah in [28], namely, *semi-properness*, which is essentially the weakest property that a partial ordering must have in order to iterate it without collapsing ω_1. The corresponding axiom, the *Semi-Proper Forcing Axiom* (SPFA), was subsequently formulated by Shelah and proved to be consistent modulo a supercompact cardinal. In a rather surprising result, however, Shelah [29] showed that SPFA was actually equivalent to the maximal possible extension of MA, introduced by Foreman, Magidor and Shelah in [23] and known as *Martin's Maximum* (MM). This is MA for the class of partial orderings that do not collapse stationary subsets of ω_1 (and for $\mathcal{D}$ of cardinality at most $\aleph_1$, a necessary assumption as it was pointed out before). Many consequences of MM are proved in [23], the most remarkable for our purposes being that the size of the continuum is $\aleph_2$.

Thus, MM, the strongest consistent (modulo the existence of a supercompact cardinal) generalization of MA settles the continuum problem, and in a way that was already predicted by Gödel, namely that its size is $\aleph_2$. This result was later improved by Todorčević and Veličković by showing that PFA (actually MA for a class much smaller than the Axiom A partial orderings, a principle consistent modulo the existence of a weakly-compact cardinal, suffices) implies already that the continuum has size $\aleph_2$ (see [7]). The question therefore arises as to what extent these are natural axioms of Set Theory.

On the one hand, they are generalizations of ZFC-provable statements, for they generalize $MA_{\aleph_1}$ which is itself a generalization of the Baire Category Theorem. Further, they have been shown to be consistent modulo some large cardinal axioms. But generalizing some ZFC theorems should certainly not be taken as a sufficient condition for being considered as axioms, for the simple reason that ZFC theorems may be generalized in incompatible ways. To be counted as natural axioms we need to see that they satisfy the criteria of *Maximality* and *Fairness*.

5.1. **Forcing axioms as principles of generic absoluteness.** We have already remarked that forcing axioms were regarded, until recently, as ad hoc principles, very useful indeed as technical tools for proving the consistency of mathematical statements without having to use forcing directly, but by no means real axioms. However, some recent results show that, in fact, certain *bounded* forms of the forcing axioms are real axioms. The first indication of this is a result first proved by J. Stavi and J. Väänänen, which shows that Martin's Axiom is equivalent to the following statement:

Every Σ_1 sentence with parameters in $H_{2^{\aleph_0}}$ that can be forced to hold by a ccc forcing notion, is true.

Unfortunately, the result remained unpublished for many years, but it was later independently discovered and first published in [4]. The Stavi-Väänänen paper containing the result has now also been published ([31]).

This result shows that by considering ideal forcing extensions of the universe, MA can be seen to satisfy the criteria of *Maximality* and *Fairness*.

As for stronger forcing axioms, S. Fuchino [11] gave the following surprising characterization of PFA in terms of potential embeddings:

PFA is equivalent to the statement that for any two structures $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ of cardinality $\aleph_1$, if a proper forcing notion forces that there is an embedding of $\mathcal{A}$ into $\mathcal{B}$, then such an embedding exists.

The same characterization holds for the axioms SPFA and MM, replacing *proper* by *semi-proper* or by *preserving stationary subsets of* ω_1, respectively.

Given two structures $\mathcal{A}$ and $\mathcal{B}$, the sentence: *There exists an embedding of $\mathcal{A}$ into $\mathcal{B}$*, is Σ_1 in the parameters $\mathcal{A}$ and $\mathcal{B}$. Thus, PFA satisfies to some extent the criterion of *Maximality*, for it asserts the existence of certain sets, namely, embeddings between structures, that would exist in an ideal forcing extension of the universe by a proper poset. But it does not seem to satisfy the *Fairness* criterion, since the class of existential sentences that assert the existence of embeddings between structures appears to be too restrictive. Similar considerations apply to the axioms SPFA and MM.

5.2. **Bounded Forcing Axioms.** PFA can also be formulated as follows: *For every proper partial ordering $\mathbb{P}$ and every family $\mathcal{D}$ of size $\aleph_1$ of maximal antichains of $\mathbb{B} =_{df} r.o.(\mathbb{P})\setminus\{\mathbf{0}\}$, there is a filter $\mathcal{F} \subseteq \mathbb{B}$ that intersects every antichain in $\mathcal{D}$.*

M. Goldstern and S. Shelah [14] introduced the Bounded Proper Forcing Axiom (BPFA) which is like PFA, as formulated above, but with the additional requirement that the maximal antichains of $\mathcal{D}$ have size at most $\aleph_1$. Fuchino's argument shows that BPFA is actually equivalent to the statement that for any two structures $\mathcal{A}$ and $\mathcal{B}$ of size $\aleph_1$, if a proper forcing notion forces that there is an embedding of $\mathcal{A}$ into $\mathcal{B}$, then such an embedding exists. Notice that in this formulation we may assume that the structures $\mathcal{A}$ and $\mathcal{B}$ belong to H_{ω_2}.

Unlike the case of structures of arbitrarily large size, the set of Σ_1-sentences that assert the existence of an embedding between structures of size $\aleph_1$ as parameters is not restrictive, for if any such sentence that can be forced is true, then the same applies to any other Σ_1 sentence with parameters in H_{ω_2}. Thus we have the following characterization of BPFA ([5]):

BPFA is equivalent to the statement that every Σ_1 sentence with parameters in H_{ω_2} that is forced by a proper forcing notion is true.

More generally, given a class of forcing notions Γ, let the Bounded Forcing Axiom for the class Γ, written $BFA(\Gamma)$, be the following statement:

Every Σ_1 sentence with parameters in H_{ω_2} that is forced by a forcing notion in Γ is true.

That is, for every $\mathbb{P} \in \Gamma$, if φ is a Σ_1 sentence, possibly with parameters in H_{ω_2}, that has $r.o.(\mathbb{P})$-Boolean value $\mathbf{1}$, then φ holds.

Thus, MA for families of dense open sets of size $\aleph_1$ is just $BFA(\Gamma)$, where Γ is the class of ccc posets. Also, we can formulate the bounded forms of SPFA and MM. Namely: The Bounded Semi-proper Forcing Axiom (BSPFA) and the Bounded Martin's Maximum (BMM) are the axioms $BFA(\Gamma)$, where Γ is the class of semi-proper posets or the class of posets that preserve stationary subsets of ω_1, respectively.

Goldstern and Shelah ([14]) showed that BPFA is consistent relative to the consistency of the existence of a Σ_2-reflecting cardinal, and that this is its exact consistency strength. The same applies to BSPFA. Further, Woodin proved the consistency of BMM [38] relative to the existence of large cardinals much weaker than a supercompact ($\omega + 1$-many Woodin cardinals suffices). As for consistency strength, R. Schindler has shown that BMM implies that for every set X there is an inner model with a strong cardinal containing X. Thus, BMM is, consistency-wise, much stronger that SPFA and PFA. Schindler has also shown, modulo large cardinals, that BPFA does not imply BSPFA. Therefore, the axioms BPFA, BSPFA, and BMM form a strictly increasing chain in strength.

Of course, there are no real extensions of the universe of all sets, and therefore no real forcing extensions. But given a forcing notion $\mathbb{P}$, we can define the Boolean-valued model $V^{\mathbb{B}}$, where $\mathbb{B} = r.o.(\mathbb{P})$, and view V as contained in $V^{\mathbb{B}}$ via the canonical embedding given by $x \longmapsto \check{x}$. Thus, if we want to maximize all Σ_1 sentences that hold in $V^{\mathbb{B}}$ or, equivalently, that would hold in any ideal extension of V by $\mathbb{B}$, allowing both a fair class of parameters as large as possible and a class of forcing extensions as wide as possible, this is exactly what the Bounded Forcing Axioms do.

It is worth noting that it is a theorem of ZFC that all Σ_1 sentences that hold in some Boolean-valued model $V^{\mathbb{B}}$, allowing only sets in H_{ω_1} as parameters, are true. So, the Bounded Forcing Axioms are just natural generalizations of this fact to H_{ω_2}. Moreover, this is the most we can hope for. We cannot have the same for Σ_2 formulas since, for instance, both CH and its negation are of this sort. Moreover, as we pointed out in the last section, V cannot be a Σ_1-elementary substructure of $V^{\mathbb{B}}$ for any non-trivial $\mathbb{B}$. In fact, for many $\mathbb{B}$ we cannot even allow as parameters of the Σ_1 formulas all sets in H_{ω_3} (see [6] for a thorough discussion of the limitations of Bounded Forcing Axioms). Furthermore, if we want Γ to be the class of *all* forcing notions, then we cannot even have ω_1 as a parameter, since we can easily collapse ω_1 to ω, and saying that ω_1 *is countable* is Σ_1 in the parameter ω_1. Even $BFA(\Gamma)$ for the class of forcing notions that preserve ω_1 is inconsistent with ZFC. For if S is a stationary and co-stationary subset of ω_1, then we can add a club $C \subseteq S$ by forcing and at the same time preserve ω_1. But saying that S *contains a club* is Σ_1 in the parameter S, and so the axiom would imply that such a club exists in the ground model, which is impossible.

So, a natural question is what is the maximal class Γ for which $BFA(\Gamma)$ is consistent with ZFC. This class has been singled out by D. Asperó [1]: Let Γ be the class of all posets $\mathbb{P}$ such that for every set X of cardinality $\aleph_1$ of stationary subsets of ω_1 there is a condition $p \in \mathbb{P}$ such that p forces that S is stationary for every $S \in X$. This class coincides with the class of forcing notions that preserve stationary subsets of ω_1 if and only if the ideal of the non-stationary subsets of ω_1 is ω_1-dense. The axiom $BFA(\Gamma)$ is maximal, i.e., if $\mathbb{P} \notin \Gamma$, then the Bounded Forcing Axiom for $\mathbb{P}$ fails. Asperó also shows that the axiom can be forced assuming the existence of a Σ_2-reflecting cardinal which is the limit of strongly compact cardinals.

We conclude that Bounded Forcing Axioms are *the* natural axioms of Set Theory arising from the application of the criteria of *Maximality* and *Fairness* to ideal forcing extensions of V. Bounded Forcing Axioms are axioms of generic absoluteness for H_{ω_2}. Generally speaking, an axiom of generic absoluteness asserts that whatever statement can be forced is true, subject only to the requirement that it be consistent. Axioms of generic absoluteness for H_{ω_1}, i.e., axioms that state that whatever statements with parameters in H_{ω_1} can be forced they are true, appear naturally in Descriptive Set Theory, and they are a consequence of large cardinals (see [6]). Thus, the Bounded Forcing Axioms constitute the next level, i.e., for H_{ω_2}, of this kind of axioms. Since the continuum problem is decided in H_{ω_2}, it is reasonable to expect that the Bounded Forcing Axioms will be the appropriate kind of axioms for solving the problem.

6. Bounded Forcing Axioms and the continuum problem

Many consequences, mostly combinatorial, of the axioms BPFA, BSPFA, and BMM are known (see [2] and [33]). But the relevance of Bounded Forcing Axioms to our present discussion is that, unlike the axioms of large cardinals, they do settle Cantor's continuum problem.

Woodin [38] showed that if there exists a measurable cardinal, then BMM implies that there is a well-ordering of the reals in length ω_2 which is definable in H_{ω_2} with an ω_1-sequence of stationary subsets of ω_1 as a parameter, and hence the cardinality of the continuum is $\aleph_2$. D. Asperó and P. Welch [3] obtained the same result from a weaker large-cardinal hypothesis. Finally, Todorcevic [32] proved that BMM implies that there is a well-ordering of the reals in length ω_2 which is definable in H_{ω_2} with a ω_1-sequence of real numbers as a parameter, and so the cardinality of the continuum is $\aleph_2$.

Showing that BMM implies that the size of the continuum is $\aleph_2$ requires some method for coding reals by ordinals less than ω_2. Two such methods were devised by Woodin – assuming the existence of a measurable cardinal – and Todorcevic, respectively. Very recently, Justin T. Moore [26] has discovered a new coding method which further improves on the aforementioned chain of results of Woodin, Asperó-Welch, and Todorcevic, namely: BPFA implies that there is a well-ordering of the reals in length ω_2 which is definable in H_{ω_2} with an ω_1-sequence of countable ordinals as a parameter, and hence the cardinality of the continuum is $\aleph_2$.

Since, as we have already argued, Bounded Forcing Axioms are natural axioms of Set Theory, the results that show that they imply that the cardinality of the continuum is $\aleph_2$ constitute a natural solution to Cantor's continuum problem.

There still remains the question of the consistency of the Bounded Forcing Axioms with ZFC. We already observed that BPFA and BSPFA are consistent relative to the existence of a Σ_2-reflecting cardinal, a very weak large-cardinal hypothesis in the large-cardinal hierarchy. The consistency strength of BMM is not known, this being one of the most interesting open questions in the area. BMM may even imply PD, i.e., that every projective set of real numbers is determined, and so its consistency strength would be roughly at the level of infinitely-many Woodin cardinals. It is also an open question whether Asperó's maximal bounded forcing axiom is actually equivalent to BMM. Further open questions are the following: It would be interesting to know whether there is any Bounded Forcing Axiom, for a natural class of forcing notions, that implies that the cardinality of the continuum is $\aleph_2$ and whose consistency strength is just ZFC. It would also be of great interest to find, under some form of Bounded Forcing Axiom, a coding of reals by ordinals less than ω_2 using a single real as parameter.

Bounded Forcing Axioms are at least as natural as the axioms of large cardinals. Both kinds of axioms satisfy the criteria of *Maximality* and *Fairness*. But Bounded Forcing Axioms are in a sense more natural than the axioms of large cardinals, for the ideal extensions on which they are based, namely, the ideal forcing extensions of the universe, are more intuitive than the ideal extensions obtained by viewing a transitive class M, which is already included in V, as an extension of V via the trick of embedding V into it.

All known large-cardinal axioms are compatible with Bounded Forcing Axioms. Thus it is reasonable to work with both kinds of axioms simultaneously. Woodin has isolated an axiom we may call *Woodin's Maximum* (WM), that brings together the power of large cardinals and the Bounded Forcing Axioms. WM has the astonishing property that it decides in Ω-logic the whole theory of H_{ω_2} (see [39]). WM asserts the following:

(1) There exists a proper class of Woodin cardinals, and
(2) A strong form of BMM holds in every inner model M of ZFC that contains H_{ω_2} and thinks that there is a proper class of Woodin cardinals.

The strong form of BMM of (2) says: Every Σ_1 sentence (with parameters) in the language of the structure $\langle H_{\omega_2}, \in, NS_{\omega_1}, X\rangle$ – where NS_{ω_1} is the non-stationary ideal and X is any set of reals in $L(\mathbb{R})$ – that holds in some (ideal) forcing extension of V via a forcing notion that preserves stationary subsets of ω_1 holds already in V.

Woodin [38] has shown that the consistency strength of WM is essentially that of the existence of infinitely-many Woodin cardinals. Moreover, assuming the existence of a proper class of Woodin cardinals and an inaccessible limit of Woodin cardinals, he proved that WM is Ω-consistent. So, if the Ω-conjecture is true, then WM holds in some (ideal) forcing extension of the universe V. This would certainly contribute to making WM, according to our criteria, a natural axiom of Set Theory.

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