THERE MAY BE NO HAUSDORFF ULTRAFILTERS

TOMEK BARTOSZYNSKI AND SAHARON SHELAH

Abstract. An ultrafilter U is Hausdorff if for any two functions $f, g \in \omega^\omega$, $f(U) = g(U)$ iff $f{\restriction}X = g{\restriction}X$ for some $X \in U$. We will show that it is consistent that there are no Hausdorff ultrafilters.

1. Introduction

For $f \in \omega^\omega$ and an ultrafilter U on ω define $f(U) = \{X \subseteq \omega : f^{-1}(X) \in U\}$. Let FtO be the collection of all finite-to-one functions $f \in \omega^\omega$.

Definition 1. *Let U be an ultrafilter on ω. We say that*

(1) *U is Hausdorff if for any two functions $f, g \in \omega^\omega$, if $f(U) = g(U)$ then $f{\restriction}X = g{\restriction}X$ for some $X \in U$.*
(2) *U is weakly Hausdorff if for any two functions $f, g \in \mathsf{FtO}$, if $f(U) = g(U)$ then $f{\restriction}X = g{\restriction}X$ for some $X \in U$.*

It is worth mentioning that the following appears as an exercise in [10].

Lemma 2. *If $f(U) = U$ then there exists $X \in U$ such that $f(n) = n$ for $n \in X$.*

Therefore, if U is not Hausdorff, then this is witnessed by two functions, both not one-to-one mod U. It follows from it that Ramsey ultrafilters are Hausdorff.

The notion of a Hausdorff ultrafilters was reintroduced and studied by Mauro Di Nasso, Marco Forti and others in a sequence of papers ([9], [8], [2] and [7]) in context of topological extensions. They used the name Hausdorff because Hausdorff ultrafilters are precisely those ultrafilters whose ultrapowers equipped with the standard topology are Hausdorff topological spaces. They asked whether the existence of a Hausdorff ultrafilter can be

1991 *Mathematics Subject Classification.* 03.
Key words and phrases. Hausdorff, ultrafilters, consistency.
This material is based on work supported while the first author was serving at NSF, and by NSF grant DMS 0200671 and by KBN grant 5 P03A 037 20. The final version of this paper was written at CRM Bellaterra, Spain.
The second author was partially supported by Israel Science Foundation. Publication 826.

proved in ZFC. We will show that, at least for ultrafilters on ω, the answer is negative. However such ultrafilters (with various extra properties) may be constructed under from additional set theoretical assumptions (see [8]).

2. Construction of the model

In this section we will show how to build a model where there are no Hausdorff ultrafilters modulo the proofs of theorems 4 and 6 below.

Definition 3. *An ultrafilter U is strongly non-Hausdorff if for every $f \in \mathsf{FtO}$, $f(U)$ is not weakly Hausdorff.*

Theorem 4. *Assume* CH*. There exists a strongly non-Hausdorff p-point.*

Definition 5. [3], [4], [6]. *Let* NCF *stand for the following statement: for any ultrafilters U, V on ω there exists $h \in \mathsf{FtO}$ such that $h(U) = h(V)$.*

Theorem 6. *There exists a proper forcing notion $\mathcal{P}$ such that*

(1) *There is an strongly non-Hausdorff filter U_0 in $\mathbf{V}$ such that $\mathbf{V}^{\mathcal{P}} \models U_0$ generates a strongly non-Hausdorff ultrafilter.*
(2) $\mathbf{V}^{\mathcal{P}} \models \mathsf{NCF}$.

Corollary 7. *Suppose that $\mathbf{V} \models \mathsf{GCH}$. Then in $\mathbf{V}^{\mathcal{P}}$ there are no weakly Hausdorff ultrafilters. In particular, there are no Hausdorff ultrafilters in this model.*

Proof. Let U_0 be a strongly non-Hausdorff ultrafilter in $\mathbf{V}$ given by theorem 6. Thus $\mathbf{V}^{\mathcal{P}}$ satisfies NCF and there is there is a strongly non-Hausdorff ultrafilter in $\mathbf{V}^{\mathbb{M}_{\omega_2}}$. So suppose that U is an ultrafilter in $\mathbf{V}^{\mathbb{M}_{\omega_2}}$. By NCF there exists $h \in \mathsf{FtO}$ such that $h(U) = h(U_0)$. Since U_0 is strongly non-Hausdorff in $\mathbf{V}^{\mathcal{P}}$ it follows that $h(U_0)$ is not Hausdorff. On the other hand if U were Hausdorff then $h(U)$ would be Hausdorff as well, a contradiction. $\square$

3. A construction of non-Hausdorff ultrafilter

Let $I \subset \omega$ be a finite set and let $\Delta = \{(n, n) : n \in \omega\}$. Denote by $[I]^2 = (I \times I) \setminus \Delta$. For a set $X \subseteq [I]^2$ define

$$\|X\|_I = \min\left\{k : \exists\{A_i, B_i : i \le k\}\ \forall i \le k\ A_i \cap B_i = \emptyset \text{ and } X \subseteq \bigcup_{i \le k} A_i \times B_i\right\}.$$

We will drop the subscript I if it is clear from the context what it is.

Lemma 8. (1) $\left\|[I]^2\right\|_I \longrightarrow \infty$ *as* $|I| \to \infty$.
(2) $\|X \cup Y\|_I \le \|X\|_I + \|Y\|_I$,
(3) *if $Z \subseteq I$ and $X \subseteq [I]^2$, $\|X\|_I > 2$, then either $\left\|[Z]^2 \cap X\right\|_I \ge \|X\|_I/2 - 1$ or $\left\|[I \setminus Z]^2 \cap X\right\|_I \ge \|X\|_I/2 - 1$.*

Proof. If (1) fails then there is $k \in \omega$ and sets $\{A^n_j, B^n_j : n, j \le k\}$ such that $A^n_j \cap B^n_j = \emptyset$ for $j \le k$ and $[n]^2 = \bigcup_{j \le k} A^n_j \times B^n_j$. By compactness we get sets $\{A^\omega_j, B^\omega_j : j \le k\}$ such that $A^\omega_j \cap B^\omega_j = \emptyset$ for $j \le k$ and $[\omega]^2 = \bigcup_{j \le k} A^\omega_j \times B^\omega_j$, which is not possible.

A more direct argument shows that $\left\| [I]^2 \right\|_I \ge |I| - 2$.

(2) is obvious.

(3) Note that

$$\|X\|_I \le \left\| \big([Z]^2 \cup [I \setminus Z]^2 \cup (Z \times (I \setminus Z)) \cup ((I \setminus Z) \times Z)\big) \cap X \right\|_I \le \left\| [Z]^2 \cap X \right\|_I + \left\| [I \setminus Z]^2 \cap X \right\|_I + 1 + 1.$$

Thus

$$\left\| [Z]^2 \cap X \right\|_I + \left\| [I \setminus Z]^2 \cap X \right\|_I \ge \|X\|_I - 2.$$

□

For $I \in [\omega]^{<\omega}$ let $\pi_1, \pi_2 : [I]^2 \longrightarrow I$ be projections onto first and second coordinate respectively.

Lemma 9. *Suppose that* $X \subseteq [I]^2$, *and* $\|X\|_I > 2$. *Then* $\pi_0(X) \cap \pi_1(X) \neq \emptyset$.

Proof. Suppose that $\pi_0(X) = u$ and $\pi_1(X) = v$. If $u \cap v = \emptyset$ then $X \subseteq (u \times v) \cup (v \times u)$. Thus $\|X\|_I \le 2$. □

Next we define functions $f^0, g^0 \in \mathsf{FtO}$ that will witness that ultrafilter V_0 that we are about to construct is not Hausdorff.

Let $\{I_k, J_k : k \in \omega\}$ be two sequences of disjoint consecutive intervals such that for $k \in \omega$,

(1) $\left\| [I_k]^2 \right\|_{I_k} \ge 2^{2^k}$,
(2) $|J_k| = |[I_k]^2|$.

Bijection implicit in (2) allows us to define projections $\pi^k_0, \pi^k_1 : J_k \longrightarrow I_k$. Let $f^0 = \bigcup_k \pi^k_0$ and $g^0 = \bigcup_k \pi^k_1$. Note that $f^0(x) \neq g^0(x)$ for any $x \in J_k = [I_k]^2$, $k \in \omega$.

As a warm-up let us use these definitions to show the following:

Lemma 10. *Assume* CH. *There exists a p-point that is not weakly Hausdorff.*

Proof. We will need the following easy observation:

Lemma 11. *If* $f, g \in \mathsf{FtO}$ *and* U *is an ultrafilter then the following conditions are equivalent:*

(1) $f(U) \neq g(U)$,
(2) $f[X] \cap g[X] = \emptyset$ *for some* $X \in U$. □

We will build an ultrafilter V_0 on the set $\bigcup_k [I_k]^2$ which we identified with ω. Let $\{Z_\alpha : \alpha < \omega_1\}$ be enumeration of $[\omega]^\omega$.

We will build by induction a sequence $\{X_\alpha : \alpha < \omega_1\}$ so that

(1) $\forall \beta < \alpha \; X_\alpha \subseteq^\star X_\beta$,
(2) $X_{\alpha+1} \cap Z_\alpha = \emptyset$ or $X_{\alpha+1} \subseteq Z_\alpha$ for all α.
(3) for every $\alpha < \omega_1$, $f^0[X_\alpha] \cap g^0[X_\alpha] \neq \emptyset$.
(4) for every $\alpha < \omega_1$, $\limsup_k \|X_\alpha \cap J_k\|_{I_k} = \infty$.

Let $V_0 = \{X : \exists \alpha \; X_\alpha \subseteq^\star X\}$. Note that the conditions (1) and (2) guarantee that V_0 is a p-point, and lemma 11 and (3) implies that $f^0(V_0) = g^0(V_0)$. Finally, (4) is the requirement that (by lemma 9) implies (3).

Successor step. Suppose that X_α is given. Find a strictly increasing sequence $\{l_k : k \in \omega\}$ such that the set $A = \{k : \|X_\alpha \cap J_k\|_{I_k} = l_k\}$ is infinite. Let $A_0 = \{k : \|X_\alpha \cap Z_\alpha \cap J_k\|_{I_k} \geq l_k/2 - 1\}$ and $A_1 = \{k : \|(X_\alpha \setminus Z_\alpha) \cap J_k\|_{I_k} \geq l_k/2 - 1\}$. By lemma 8(3), one of these sets, say A_0, is infinite. Let $X_{\alpha+1} = \bigcup_{k \in A_0} X_\alpha \cap Z_\alpha \cap J_k$. The other case is the same.

Limit step. Given $\{X_\beta : \beta < \alpha < \omega_1\}$ let $\{\beta_k : k \in \omega\}$ be an increasing sequence cofinal in α. By finite modifications we can assume that $X_{\beta_{k+1}} \subseteq X_{\beta_k}$ for all k. Build by recursion a strictly increasing sequence $\{u_k : k \in \omega\}$ such that

$$\forall k \; \forall j \leq k \; \exists i \in [u_k, u_{k+1}) \; \left\|X_{\beta_j} \cap J_i\right\|_{I_i} \geq k,$$

and let

$$X_\alpha = \bigcup_k \left(X_{\beta_k} \cap \bigcup_{i \in [u_k, u_{k+1})} J_i \right).$$

It is clear that X_α satisfies (1) and (4). □

Observe that CH was only needed in the limit step. If we do not require that that V_0 is a p-point then we have the following:

Theorem 12. *There exists an ultrafilter that is not weakly Hausdorff.*

Proof. As in lemma 10, we will build an ultrafilter on the set $\bigcup_k [I_k]^2$. Let

$$\mathcal{I} = \left\{ X \subseteq \bigcup_k [I_k]^2 : \limsup_k \|X \cap J_k\|_{I_k} < \infty \right\}.$$

Note that $\mathcal{I}$ is an ideal, and let U be any ultrafilter orthogonal to $\mathcal{I}$. Functions f^0, g^0 witness that U is not Hausdorff. □

4. A construction of a strongly non-Hausdorff ultrafilter under CH

Now we are ready to prove Theorem 4 and to construct a p-point ultrafilter U_0 whose all finite-to-one images are not weakly Hausdorff.

Let $\mathbf{C}$ be the Cohen forcing interpreted as adding a function $e \in \mathsf{FtO}$. Specifically, the conditions are finite sequences of consecutive intervals $\{E_k : k < n\}$ and $e(i) = k \iff i \in E_k$. Let $X \subseteq 2^\omega$ be a non-meager set. It is well known that for every countable model M there exists $e \in X$ which is Cohen over M.

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a tower of countable transitive elementary submodels of $\mathbf{H}(\chi)$ such that

(1) $M_\beta \subseteq M_\alpha$ for $\beta \leq \alpha < \omega_1$,
(2) $M_\alpha \models \bigcup_{\beta<\alpha} M_\beta$ is countable,
(3) $\mathbf{V} \cap \omega^\omega = \bigcup_{\alpha<\omega_1} M_\alpha \cap \omega^\omega$,
(4) $e_\alpha \in M_\alpha$ is a Cohen real over M_β for $\beta < \alpha$. (interpreted as generic object for $\mathbf{C}$ defined above).

We will build by induction sequence $\{X_\alpha : \alpha < \omega_1\}$ defining an ultrafilter $U_0 = \{X \in [\omega]^\omega : \exists \alpha\ X_\alpha \subseteq^\star X\}$. We will require that

(1) $\forall \beta < \alpha\ X_\alpha \subseteq^\star X_\beta$,
(2) $U^\alpha = \{X \in M_\alpha \cap [\omega]^\omega : X_{\alpha+1} \subseteq^\star X\}$ is an ultrafilter in M_α,
(3) for every $\beta < \alpha$ and for every $h \in M_\beta \cap \mathsf{FtO}$, $f^0 \circ e_\alpha, g^0 \circ e_\alpha$ witness that $h(U_0)$ is not Hausdorff.
(4) $\forall \beta < \alpha\ \forall h \in M_\beta \cap \mathsf{FtO}\ \forall \gamma < \omega_1\ f^0 \circ e_\alpha \circ h[X_\gamma] \cap g^0 \circ e_\alpha \circ h[X_\gamma] \neq \emptyset$.

As before, (1) and (2) guarantee that U_0 is a p-point, and (3) implies that U_0 is strongly non-Hausdorff, and (4) is a specific form of (3). Note that at the limit stages we only have to preserve the induction hypothesis.

Definition 13. *A finite set $Y \subseteq X_\alpha$ is a (n, h, β, α)-witness if there exists $k \in \omega$ such that $\|e_\beta \circ h[Y] \cap J_k\|_{I_k} \geq n$.*

To satisfy (3), we demand that for $\beta < \alpha < \omega_1$,

(5) for every $h \in M_\beta \cap \mathsf{FtO}$, $\limsup_k \|e_{\beta+1} \circ h[X_\alpha] \cap J_k\|_{I_k} = \infty$, or equivalently
(6) $\forall n\ \forall h \in M_\beta \cap \mathsf{FtO}\ \exists Y \in [X_\alpha]^{<\omega}\ Y$ is a (n, h, β, α)-witness.

Limit step.

Suppose that $\{X_\beta : \beta < \alpha\}$ are defined and α is a limit ordinal. Work in M_α and let $\{\beta_k : k \in \omega\}$ be an increasing sequence cofinal in α. By finite modifications we can assume that $X_{\beta_k} \subseteq X_{\beta_l}$ if $k \geq l$. Let

$$X_\alpha = \bigcup_k \left(X_{\beta_k} \cap e_\alpha^{-1}(J_k) \right).$$

It is clear that X_α satisfies (1) and (4).

Successor step.

Suppose that X_α satisfying (4) is already defined and we want to define $X_{\alpha+1}$ satisfying (2) and (5). Recall that by the induction hypothesis, for

$\beta < \alpha$,

$$\forall n \; \forall h \in M_\beta \; \exists Y \in [X_\alpha]^{<\omega} \; Y \text{ is a } (n, h, \beta, \alpha)\text{-witness.}$$

We will work in $M_{\alpha+1}$. Let $\{Z_n : n \in \omega\}$ be the enumeration of $M_\alpha \cap [\omega]^\omega$. We will build by induction sequence $\{X^n_\alpha : n \in \omega\}$. Let $X^0_\alpha = X_\alpha$ and suppose that $\{X^k_\alpha : k \leq n\}$ are already given and they satisfy the same inductive hypothesis as X_α.

CASE 1. If for some $\beta \leq \alpha$, and $h \in M_\beta \cap \mathsf{FtO}$,

$$\limsup_k \|e_{\beta+1} \circ h[Z_{n+1} \cap X^n_\alpha] \cap J_k\|_{I_k} < \infty.$$

Put $X^{n+1}_\alpha = X^n_\alpha \setminus Z_{n+1}$.

CASE 2. If for some $\beta \leq \alpha$, and $h \in M_\beta \cap \mathsf{FtO}$,

$$\limsup_k \|e_{\beta+1} \circ h[X^n_\alpha \setminus X_{n+1}] \cap J_k\|_{I_k} < \infty.$$

Put $X^{n+1}_\alpha = X^n_\alpha \cap Z_{n+1}$.

In all other cases let $X^{n+1}_\alpha = X^n_\alpha \cap Z_{n+1}$.

We have to check that cases 1 and 2 are mutually exclusive. Without loss of generality we can assume that the case 1 holds and let β_0 be the smallest β such that for some $h_0 \in M_{\beta_0}$, $\limsup_k \|e_{\beta_0+1} \circ h_0[Z_{n+1} \cap X^n_\alpha] \cap J_k\|_{I_k} < \infty$.

The following lemma will complete the construction:

Lemma 14. *For every $\beta \leq \alpha$, and every $h \in M_\beta \cap \mathsf{FtO}$,*

$$\limsup_k \|e_{\beta+1} \circ h[X^n_\alpha \setminus Z_{n+1}] \cap J_k\|_{I_k} = \infty.$$

Proof. By minimality of β_0, the statement is true for $\beta < \beta_0$. It also holds for $\beta = \beta_0$ since the induction hypothesis gives us that $\limsup_k \|e_{\beta_0+1} \circ h[X^n_\alpha] \cap J_k\|_{I_k} = \infty$, for every $h \in M_{\beta_0}$.

Suppose that the Lemma is false and let $\gamma_0 > \beta_0$ be the first ordinal such that $\limsup_k \|e_{\gamma_0+1} \circ h_1[X^n_\alpha \setminus Z_\alpha] \cap J_k\|_{I_k} < \infty$ for some $h_1 \in M_{\gamma_0} \cap \mathsf{FtO}$. It means that

$$\forall^\infty k \; e_{\gamma_0+1} \circ h_1[X^n_\alpha \setminus Z_{n+1}] \cap J_k \neq J_k.$$

If $l \notin J_k \setminus e_{\gamma_0+1} \circ h_1[X^n_\alpha \setminus Z_{n+1}]$ then $h_1^{-1}(e^{-1}_{\gamma_0+1}(\{l\})) \cap X_\alpha \subseteq Z_{n+1}$. Since e_{γ_0+1} is a Cohen real over M_{β_0} there are infinitely many k such that for every $l \in J_k$, $h_1^{-1}(e^{-1}_{\gamma_0+1}(\{l\})) \cap X_\alpha$ contains an $(n_l, h_0, \beta_0, \alpha)$ witness, where $n_l \longrightarrow \infty$ when $l \to \infty$. Thus $\limsup_k \|e_{\beta_0+1} \circ h_0[X^n_\alpha \cap Z_{n+1}] \cap J_k\|_{I_k} = \infty$, which is a contradiction. □

To define $X_{\alpha+1}$ work in $M_{\alpha+1}$ and apply the limit step construction to the sequence $\{X^n_\alpha : n \in \omega\}$.

5. A construction of a nondestructible strongly non-Hausdorff ultrafilter

Now we are ready to prove Theorem 6.

Let $\mathbb{M}_{\omega_2}$ be a countable support iteration of Miller forcing (or Blass-Shelah forcing notion) of length ω_2 (see [1] for definitions). We will show that ultrafilter U_0 constructed in the proof of theorem 4 remain strongly non-Hausdorff in $\mathbf{V}^{\mathbb{M}_{\omega_2}}$.

The following list gives the relevant properties of either forcing:

(1) $\mathbb{M}_{\omega_2}$ preserves p-points (see [5] or [1] 6.2.6),
(2) $\mathbf{V}^{\mathbb{M}_{\omega_2}} \models \mathsf{NCF}$, [6] or [5].
(3) $\mathbb{M}_{\omega_2}$ preserves non-meager sets ([1], 6.3.20). In particular, for every countable elementary submodel $M \prec \mathbf{H}(\chi)$ containing $\mathbb{M}_{\omega_2}$, a condition $p \in \mathbb{M}_{\omega_2} \cap M$ and a Cohen real e over M there exists $q \geq p$ such that q is $(M, \mathbb{M}_{\omega_2})$ generic and $q \Vdash_{\mathbb{M}_{\omega_2}} e$ is Cohen over $M[\dot{G}]$.

Since a potential counterexample would involve only countably many generic reals it suffices to show that $\mathbf{V}^{\mathbb{M}_\gamma} \models$ "U_0 is strongly non-Hausdorff", for all $\gamma < \omega_1$.

Fix $\gamma < \omega_1$ and let G be a $\mathbb{M}_\gamma$-generic filter over $\mathbf{V}$. We would like to argue that the ultrafilter U_0 constructed for the sequences $\{e_\alpha : \alpha < \omega_1\}$ and $\{M_\alpha : \alpha < \omega_1\}$ is exactly the same as the one constructed for sequences $\{e_\alpha : \alpha < \omega_1\}$ and $\{M_\alpha[G] : \alpha < \omega_1\}$. Clearly by (3) for every $\alpha > \beta$, e_α is Cohen over $M_\beta[G]$. To finish the proof it would suffice (by (1)) to know that for every α, $M_\alpha \models$ "$U_0 \cap M_\alpha$ is a p-point." In order to have it we need to modify the successor step of the construction of U_0. Suppose that X_α is given and in addition $M_\alpha \models \mathsf{CH}$. By the Theorem 6, $M_\alpha \models$ there exists a strongly non-Hausdorff p-point U^α containing X_α. The rest of the construction is identical, $X_{\alpha+1}$ is defined from a sequence $\{X^n_\alpha : \alpha < \omega\} \in M_{\alpha+1}$ cofinal in the tower generating U^α.

Acknowledgements: We would like to thank Mauro Di Nasso and Marco Forti for introducing us to the problem, and many helpful comments concerning the draft versions of this paper. We would also like to thank Alan Dow for his comments on early versions of the paper.

References

[1] Tomek Bartoszynski and Haim Judah. *Set Theory: on the structure of the real line.* A.K. Peters, 1995.

[2] Viero Benci, Marco Forti, and Mauro di Nasso. Hausdorff nonstandard extensions. *Bol. Soc. Parana. Mat.*, 20:9–20, 2002.

[3] Andreas Blass. Near coherence of filters. I. Cofinal equivalence of models of arithmetic. *Notre Dame Journal of Formal Logic*, 27(4):579–591, 1986.

[4] Andreas Blass. Near coherence of filters. II. Applications to operator ideals, the Stone-Cech remainder of a half-line, order ideals of sequences, and slenderness of groups. *Transactions of the American Mathematical Society*, 300(2):557–581, 1987.
[5] Andreas Blass and Saharon Shelah. There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. *Annals of Pure and Applied Logic*, 33(3):213–243, 1987.
[6] Andreas Blass and Saharon Shelah. Near coherence of filters. III. A simplified consistency proof. *Notre Dame Journal of Formal Logic*, 30(4):530–538, 1989.
[7] Mauro di Nasso. Ultrafilter semirings and nonstandard submodels of the Stone-Cech compactification of the natural numbers. submitted for the volume "Logic and its application in algebra and geometry", Contemporary Mathematics AMS, edited by Yi Zhang.
[8] Mauro di Nasso and Marco Forti. Hausorff ultrafilters. to appear.
[9] Mauro di Nasso and Marco Forti. Topological and Nonstandard extensions. to appear.
[10] Thomas Jech. *Set Theory*. Academic Press, 1978.

Department of Mathematics and Computer Science, Boise State University, Boise, Idaho 83725 U.S.A.
E-mail address: tomek@math.boisestate.edu, http://math.boisestate.edu/~tomek

Department of Mathematics, Hebrew University, Jerusalem, Israel
E-mail address: shelah@math.huji.ac.il, http://math.rutgers.edu/~shelah/