HADAMARD-TYPE THEOREMS FOR HYPERSURFACES IN HYPERBOLIC SPACES

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Abstract. We prove that a bounded, complete hypersurface in hyperbolic space with normal curvatures greater than -1 is diffeomorphic to a sphere. The completeness condition is relaxed when the normal curvatures are bounded away from -1. The diffeomorphism is constructed via the Gauss map of some parallel hypersurface. We also give bounds for the total curvature of this parallel hypersurface.

1. Introduction

Classical Hadamard theorem [5] states that a compact oriented hypersurface M immersed into the Euclidean space $\mathbb{R}^{n+1}$ with positive sectional curvature is diffeomorphic to the sphere $\mathbb{S}^n$ via the Gauss map. Even more, it is actually embedded and it is the boundary of a convex body. The theorem remains true even for non-negative sectional curvature, as shown by Chern and Lashof [4] when $n = 2$, and do Carmo and Lima [2] for arbitrary dimension (see also [6] and [8]).

This classical result was extended to the case of hypersurfaces in a sphere $\mathbb{S}^{n+1}$ by do Carmo and Warner [3]. In this case, a natural substitute for the above condition on the sectional curvature is the requirement that the sectional curvature of M be greater than or equal to one (the curvature of the ambient space). Their proof makes an extensive use of the Beltrami maps in order to transform problems on the sphere $\mathbb{S}^{n+1}$ to problems in the Euclidean space $\mathbb{R}^{n+1}$, and then apply Euclidean results. Besides, they also

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indicate that the same method works for hypersurfaces in hyperbolic space, although they do not enter into details.

Here we obtain similar results for not necessarily convex hypersurfaces in hyperbolic space. For instance, we prove that a bounded hypersurface with normal curvatures greater or equal than -1 must be a sphere, provided that the metric induced by the ambient is complete. In case that the normal curvatures are greater than $-\lambda > -1$, it is enough that the metric $\langle A + \lambda I, A + \lambda I\rangle$ is complete (where A denotes the shape operator). Also in this case, explicit diffeomorphisms are constructed via the Gauss map of the parallel hypersurfaces at sufficiently big distance. Besides, we get bounds for the total curvature of these parallel hypersurfaces.

2. Statement of the main results

In order to set up our notation, we will consider the Minkowskian model of the hyperbolic space. Let $\mathbb{R}^{n+2}_1$ be the $(n+2)$-dimensional Minkowski space endowed with canonical coordinates $(x_0, x_1, \dots, x_{n+1})$ and the Lorentzian metric given by

$$\langle\ ,\ \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.$$

The $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$ is the simply connected Riemannian manifold with sectional curvature -1, which is realized as the hyperboloid

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+2}_1 : \langle x, x\rangle = -1, x_0 > 0\}$$

with (positive definite) induced metric from $\mathbb{R}^{n+2}_1$. By a hypersurface in $\mathbb{H}^{n+1}$ we mean an isometric immersion $\psi : M^n \to \mathbb{H}^{n+1}$ of an orientable Riemannian connected manifold M of dimension $n \geq 2$.

Since M is orientable, there exists a globally defined unit normal field ν on M, and we may assume that M is oriented by ν. By parallel transport to the origin of $\mathbb{R}^{n+2}_1$, we can regard the field ν as a map $\nu : M^n \to \mathbb{S}^{n+1}_1$, where $\mathbb{S}^{n+1}_1$ is the de Sitter space, that is, the hyperquadric in $\mathbb{R}^{n+2}_1$ given by

$$\mathbb{S}^{n+1}_1 = \{x \in \mathbb{R}^{n+2}_1 : \langle x, x\rangle = 1\}.$$

As usual, we will refer to the map ν as the Gauss map of the hypersurface. The differential of ν defines then the shape operator of M, $A : \mathcal{X}(M) \to \mathcal{X}(M)$, given by $A(X) = -d\nu(X)$, whose eigenvalues are the principal curvatures of the hypersurface.

For every real number ϱ, consider the (possibly with degenerate points) parallel hypersurface $\psi_\varrho : M^n \to \mathbb{H}^{n+1}$, which is given by

$$\psi_\varrho(p) = \overline{\exp}_{\psi(p)}(\varrho\nu(p)) = \cosh(\varrho)\psi(p) + \sinh(\varrho)\nu(p), \quad p \in M,$$

where $\overline{\exp}$ denotes the exponential map in $\mathbb{H}^{n+1}$. At points where ψ_ϱ is an immersion, its unit normal field ν_ϱ is given by

$$\nu_\varrho = \frac{\partial \psi_\varrho}{\partial \varrho} = \sinh(\varrho)\psi + \cosh(\varrho)\nu.$$

This can be seen as a well defined map $\nu_\varrho : M^n \to \mathbb{S}^{n+1}_1$ (even at degenerate points of ψ_ϱ). On the other hand, it is easy to see that for every $a \in \mathbb{H}^{n+1}$, the intersection of $\mathbb{S}^{n+1}_1$ and the hyperplane $a^\perp = \{x \in \mathbb{R}^{n+2}_1 : \langle a, x\rangle = 0\}$ defines a round n-sphere of radius one, and the projection $\Pi : \mathbb{S}^{n+1}_1 \to \mathbb{S}^n$ from $\mathbb{S}^{n+1}_1$ onto that sphere $\mathbb{S}^n = \mathbb{S}^{n+1}_1 \cap a^\perp$ is given by

$$\Pi(x) = \frac{1}{\sqrt{1 + \langle a, x\rangle^2}}(x + \langle a, x\rangle a), \quad x \in \mathbb{S}^{n+1}_1.$$

Therefore for every $a \in \mathbb{H}^{n+1}$ and every $\varrho \in (-\infty, \infty)$, we may consider the map $G_\varrho : M^n \to \mathbb{S}^n$ given by

$$\begin{aligned} G_\varrho &= \Pi \circ \nu_\varrho = \frac{1}{\sqrt{1 + \langle a, \nu_\varrho\rangle^2}} (\nu_\varrho + \langle a, \nu_\varrho\rangle a) \\ &= \frac{\nu + \tanh(\varrho)\psi + \langle a, \nu + \tanh(\varrho)\psi\rangle a}{\sqrt{1 - \tanh^2(\varrho) + \langle a, \nu + \tanh(\varrho)\psi\rangle^2}}. \end{aligned}$$

Besides, observe that the resulting expression for G_ϱ extends to the case where $\varrho = \pm\infty$ because $\langle \nu \pm \psi, a\rangle \neq 0$. In fact, it holds that $\langle \nu + \psi, a\rangle < 0$ and $\langle \nu - \psi, a\rangle > 0$, so that

$$G_{-\infty} = \frac{\nu - \psi + \langle a, \nu - \psi\rangle a}{\langle a, \nu - \psi\rangle} \quad \text{and} \quad G_\infty = \frac{\nu + \psi + \langle a, \nu + \psi\rangle a}{\langle a, -\nu - \psi\rangle}.$$

In this paper we prove the following result.

Theorem 1. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a hypersurface in the hyperbolic space, $n \geq 2$, and assume that there exists a real number λ_0, with $-1 \leq \lambda_0 \leq 1$, such that the operator $A + \lambda_0 I$ is non-degenerate, where I denotes the identity in $\mathcal{X}(M)$. In particular, $A + \lambda_0 I$ defines a Riemannian metric on M by*

$$g_{\lambda_0}(X, Y) = \langle (A + \lambda_0 I)X, (A + \lambda_0 I)Y\rangle, \quad X, Y \in \mathcal{X}(M).$$

If $\psi(M) \subset \mathbb{H}^{n+1}$ is bounded and (M, g_{λ_0}) is complete (as a Riemannian manifold), then M is diffeomorphic to a sphere $\mathbb{S}^n$. Even more, after and appropriate orientation of M it follows that $A + \lambda I$ is positive definite for every $\lambda_0 \leq \lambda \leq 1$, and the map $F_\lambda : M^n \to \mathbb{S}^n = \mathbb{S}^{n+1}_1 \cap a^\perp$ given by

$$F_\lambda = G_{-\operatorname{arctanh}(\lambda)} = \frac{\nu - \lambda\psi + \langle a, \nu - \lambda\psi\rangle a}{\sqrt{1 - \lambda^2 + \langle a, \nu - \lambda\psi\rangle^2}}, \tag{1}$$

where ν is the Gauss map of M, defines a diffeomorphism between M and $\mathbb{S}^n$, for every $a \in \mathbb{H}^{n+1}$ and $\lambda_0 \leq \lambda \leq 1$.

Recall that a hypersurface in a Riemannian space form is said to be convex if its second fundamental form h is everywhere positive definite (after an appropriate orientation), $h > 0$. Making $\lambda_0 = 0$ in Theorem 1 we obtain the following consequence.

Corollary 2. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a convex hypersurface in the hyperbolic space, $n \geq 2$, such that its third fundamental form defines a complete Riemannian metric on M. If $\psi(M) \subset \mathbb{H}^{n+1}$ is bounded, then M is diffeomorphic to a sphere $\mathbb{S}^n$. Actually, for every $a \in \mathbb{H}^{n+1}$ and $0 \leq \lambda \leq 1$, the map $F_\lambda : M^n \to \mathbb{S}^n = \mathbb{S}^{n+1}_1 \cap a^\perp$ given by*

$$F_\lambda = \frac{\nu - \lambda\psi + \langle a, \nu - \lambda\psi\rangle a}{\sqrt{1 - \lambda^2 + \langle a, \nu - \lambda\psi\rangle^2}},$$

defines a diffeomorphism between M and $\mathbb{S}^n$.

Even more, as another application of Theorem 1 we can extend Corollary 2 to non-convex hypersurfaces as follows.

Theorem 3. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a complete hypersurface in the hyperbolic space such that its second fundamental form satisfies*

$$h \geq -1.$$

If $\psi(M) \subset \mathbb{H}^{n+1}$ is bounded, then M is diffeomorphic to a sphere $\mathbb{S}^n$.

The assumption on the second fundamental form is sharp in the sense that there exist examples of compact non-spherical hypersurfaces with $h \geq -1-\delta$ for every $\delta > 0$ (see Example 10).

Finally, another consequence of our Theorem 1 is the following result.

Theorem 4. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a compact hypersurface in hyperbolic space, and assume that $A + \lambda_0 I$ is positive definite for some $0 \leq \lambda_0 \leq 1$. Then for every $\lambda_0 \leq \lambda \leq 1$, it follows that*

$$\int_M \det(A + \lambda I)\ \mathrm{d}V \leq \omega_n(\cosh(r) + \lambda\sinh(r))^n,$$

where ω_n denotes the volume of a round n-sphere of radius one, and r denotes the radius of a geodesic ball in $\mathbb{H}^{n+1}$ containing the image of M. Besides, equality holds for some $\lambda_0 \leq \lambda \leq 1$ if and only if M is a geodesic sphere in $\mathbb{H}^{n+1}$ of radius r.

In particular, making $\lambda_0 = 0$ in Theorem 4 we get the following result.

Corollary 5. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a compact convex hypersurface in hyperbolic space. Then for every $0 \leq \lambda \leq 1$, it follows that*

$$\int_M \det(A + \lambda I)\ \mathrm{d}V \leq \omega_n(\cosh(r) + \lambda \sinh(r))^n,$$

where ω_n denotes the volume of a round n-sphere of radius one, and r denotes the radius of a geodesic ball in $\mathbb{H}^{n+1}$ containing the image of M. Besides, equality holds for some $\lambda_0 \leq \lambda \leq 1$ if and only if M is a geodesic sphere in $\mathbb{H}^{n+1}$ of radius r.

It is worth noting that

$$\omega_n(\cosh(r) + \lambda \sinh(r))^n = \int_\Sigma \det(A_\Sigma + \lambda I)\ \mathrm{d}V_\Sigma,$$

where Σ is a geodesic sphere in $\mathbb{H}^{n+1}$ of radius r (see Example 12). Therefore, in particular, Corollary 5 characterizes the geodesic spheres of hyperbolic space as those hypersurfaces in $\mathbb{H}^{n+1}$ which maximize the integral $\int_M \det(A + \lambda I)\mathrm{d}V$, for every $0 \leq \lambda \leq 1$, among all the compact convex hypersurfaces in $\mathbb{H}^{n+1}$ which are bounded by that geodesic sphere.

Finally, it is worth pointing out that Theorem 4 and Corollary 5 can be rewritten in terms of the total Gauss-Kronecker curvature of the parallel hypersurfaces to ψ as follows.

Corollary 6. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a compact hypersurface in hyperbolic space, and assume that $A + \lambda_0 I$ is positive definite for some $0 \leq \lambda_0 \leq 1$. Then, for every $\varrho \leq \varrho_0 = -\mathrm{arctanh}(\lambda_0) \leq 0$, the parallel hypersurface $\psi_\varrho : M^n \to \mathbb{H}^{n+1}$ at a distance ϱ is an immersion and its total Gauss-Kronecker curvature satisfies*

$$\int_M K^\varrho \mathrm{d}V_\varrho \leq \omega_n(\cosh(r - \varrho))^n,$$

where ω_n denotes the volume of a round n-sphere of radius one, and r denotes the radius of a geodesic ball in $\mathbb{H}^{n+1}$ containing the image of M. Besides, equality holds for some $\varrho \leq \varrho_0$ if and only if M is a geodesic sphere in $\mathbb{H}^{n+1}$ of radius r.

Corollary 7. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a compact convex hypersurface in hyperbolic space. Then for every $\varrho \leq 0$ the parallel hypersurface $\psi_\varrho : M^n \to \mathbb{H}^{n+1}$ at a distance ϱ is an immersion and its total Gauss-Kronecker curvature satisfies*

$$\int_M K^\varrho \mathrm{d}V_\varrho \leq \omega_n(\cosh(r - \varrho))^n,$$

where ω_n denotes the volume of a round n-sphere of radius one, and r denotes the radius of a geodesic ball in $\mathbb{H}^{n+1}$ containing the image of M.

Besides, equality holds for some $\varrho \leq 0$ if and only if M is a geodesic sphere in $\mathbb{H}^{n+1}$ of radius r.

This follows from the fact that, when ψ_ϱ is an immersion, the determinant $\det(A - \tanh(\varrho)I)$ is, up to a factor, the Gauss-Kronecker curvature of the parallel hypersurface (see Section 5 for the details).

3. Proof of Theorem 1

A straightforward computation from the expression for F_{λ_0} given in (1) shows that

$$
\begin{aligned}
dF_{\lambda_0}(X) \quad = \quad & -\frac{1}{\sqrt{1-{\lambda_0}^2+\langle a, \nu-\lambda_0\psi\rangle^2}}(A+\lambda_0 I)(X) \\
& -\frac{(1-{\lambda_0}^2)\langle a,(A+\lambda_0 I)(X)\rangle}{(1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2)^{3/2}}a \\
& +\frac{\langle a,\nu-\lambda_0\psi\rangle\langle a,(A+\lambda_0 I)(X)\rangle}{(1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2)^{3/2}}(\nu-\lambda_0\psi)
\end{aligned} \tag{2}
$$

for every tangent vector field $X \in \mathcal{X}(M)$. This implies that

$$
\begin{aligned}
\langle dF_{\lambda_0}(X), dF_{\lambda_0}(Y)\rangle \quad = \quad & \frac{\langle (A+\lambda_0 I)(X),(A+\lambda_0 I)(Y)\rangle}{(1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2)} \\
+ \quad & (1-{\lambda_0}^2)\frac{\langle a,(A+\lambda_0 I)(X)\rangle\langle a,(A+\lambda_0 I)(Y)\rangle}{(1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2)^2}
\end{aligned}
$$

for every tangent vector fields $X, Y \in \mathcal{X}(M)$. In particular,

$$
F^*_{\lambda_0}(\langle\cdot,\cdot\rangle_{\mathbb{S}^n}) \geq \frac{1}{(1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2)}g_{\lambda_0}(\cdot,\cdot). \tag{3}
$$

Since $\psi(M) \subset \mathbb{H}^{n+1}$ is bounded, then for every $a \in \mathbb{H}^{n+1}$ there exists a constant $c > 1$ such that

$$1 \leq -\langle a, \psi\rangle \leq c$$

on M. Let us denote by $a^\top$ the component of a which is tangent to M, that is,

$$a = a^\top + \langle a,\nu\rangle\nu - \langle a,\psi\rangle\psi. \tag{4}$$

Then $\langle a, a\rangle = -1 = |a^\top|^2 + \langle a,\nu\rangle^2 - \langle a,\psi\rangle^2$ and

$$\langle a,\nu\rangle^2 = -|a^\top|^2 + \langle a,\psi\rangle^2 - 1 \leq c^2 - 1, \tag{5}$$

which means that $\langle a,\nu\rangle$ is also bounded on M. Therefore, there exists a positive constant $C > 0$ such that

$$0 < (1-{\lambda_0}^2+\langle a,\nu-\lambda_0\psi\rangle^2) \leq C \text{ on } M,$$

so that

$$F^*_{\lambda_0}(\langle,\rangle_{\mathbb{S}^n}) \geq \frac{1}{C}\, g_{\lambda_0}. \tag{6}$$

From (3) we see that F_{λ_0} is a local diffeomorphism. Since g_{λ_0} is a complete Riemannian metric on M, the same holds for the homothetic metric $g^* = C^{-1}g_{\lambda_0}$. Then, equation (6) means that the map

$$F_{\lambda_0} : (M^n, g^*) \to (\mathbb{S}^n, \langle,\rangle_{\mathbb{S}^n})$$

increases the distance. Let us recall now that if a map, from a connected complete Riemannian manifold M_1 into another connected Riemannian manifold M_2 of the same dimension, increases the distance, then it is a covering map and M_2 is complete [7, Chapter VIII, Lemma 8.1]. Hence F_{λ_0} is a covering map, but $\mathbb{S}^n$ being simply connected ($n \geq 2$) this means that F_{λ_0} is in fact a global diffeomorphism between M and the sphere $\mathbb{S}^n$. This completes the proof of the first assertion in Theorem 1.

For the proof of the second part of Theorem 1, we make use of the following well-known fact about compact hypersurfaces in hyperbolic spaces.

Lemma 8. *Every compact hypersurface $\psi : M^n \to \mathbb{H}^{n+1}$ in hyperbolic space has a point where (after an appropriate orientation of M) all the principal curvatures are greater than 1.*

For the sake of completeness, we briefly sketch the proof of Lemma 8.

Proof of Lemma 8. Since M is compact, there exists a point $p_0 \in M$ where the hyperbolic distance to the point $a \in \mathbb{H}^{n+1}$ attains its maximum. Equivalently, $u(p_0) = \max_{p\in M} u(p)$, where $u(p) = -\langle a, \psi(p)\rangle \geq 1$. In particular, the gradient of u vanishes at p_0 and its Hessian satifies $\nabla^2 u_{p_0}(v,v) \leq 0$ for every tangent vector $v \in T_{p_0}M$. This implies that $\langle a, \nu(p_0)\rangle \neq 0$ and also that

$$\nabla^2 u_{p_0}(e_i, e_i) = -\langle a, \nu(p_0)\rangle \kappa_i(p_0) + \sqrt{1 + \langle a, \nu(p_0)\rangle^2} \leq 0,$$

where $\{e_1, \ldots, e_n\}$ is a basis of principal directions at p_0. Choosing now ν such that $\langle a, \nu(p_0)\rangle > 0$, we conclude from here that

$$\kappa_i(p_0) \geq \frac{\sqrt{1 + \langle a, \nu(p_0)\rangle^2}}{\langle a, \nu(p_0)\rangle} > 1 \text{ for every } i = 1, \ldots, n.$$

□

Therefore, since M is compact we know from Lemma 8 that there exists a point $p_0 \in M$ such that $\kappa_i(p_0) > 1$ for every $i = 1, \ldots, n$. In particular, the principal curvatures all satisfy $\kappa_i(p_0) + \lambda_0 > 1 + \lambda_0 > 0$ at p_0, and since all the $\kappa_i + \lambda_0$ do not vanish on M, they must be positive on M, that is,

$$\kappa_i(p) + \lambda_0 > 0 \text{ at every point } p \in M.$$

Therefore, for every $\lambda \geq \lambda_0$ we have $\kappa_i + \lambda \geq \kappa_i + \lambda_0 > 0$ on M, which means that the operator $A + \lambda I$ is positive definite. But from the first part of the proof, this implies that $F_\lambda : M^n \to \mathbb{S}^n$ is a diffeomorphism for every $\lambda_0 \leq \lambda \leq 1$. This finihes the proof of Theorem 1.

4. Proof of Theorem 3

In the case where there exists $\delta > 0$ such that $h \geq -1 + \delta$, Theorem 3 easily follows from Theorem 1. Indeed, in that case $A + I$ is positive definite and the corresponding Riemannian metric $g_1 = \langle A + I, A + I\rangle$ satisfies

$$g_1 \geq \delta^2 \langle\ ,\ \rangle.$$

Since the original metric $\langle\ ,\ \rangle$ is complete, then g_1 is also complete and Theorem 1 directly applies.

In the weaker hypothesis $h \geq -1$, we will reduce it to the previous case using the following Lemma.

Lemma 9. *Let ψ be an immersion of a hypersurface M^n in the hyperbolic space such that its second fundamental form satisfies*

$$h \geq -1$$

and the Riemannian metric induced by ψ on M is also complete. Assume that $\psi(M)$ is bounded. Then M admits another bounded immersion $\widetilde{\psi}$ in the hyperbolic space whose second fundamental form satisfies

$$\widetilde{h} \geq -1 + \delta,$$

for some $\delta > 0$, and such that the Riemannian metric induced by $\widetilde{\psi}$ on M is also complete.

Proof. It will be convenient to use here the Poincaré ball model of hyperbolic space $\mathbb{B}^{n+1}$. Let $\psi : M^n \to \mathbb{B}^{n+1}$ be the given immersion oriented by a unit normal vector ν. Assume $\psi(M)$ to be contained in a hyperbolic geodesic ball $\mathcal{B}(0, r)$ centered at 0 with radius r, $\mathcal{B}(0, r) = \{x \in \mathbb{R}^{n+1} : \|x\| \leq \ell\} = B(\ell)$ with $\ell = \tanh(r/2) < 1$, where $\|x\|$ stands for the Euclidean norm. Take $1 < \mu < 1/\ell$ and consider the new immersion $\widetilde{\psi} : M^n \to \mathbb{B}^{n+1}$ defined by $\widetilde{\psi}(p) = \mu \cdot \psi(p)$. Let us see that $\widetilde{\psi}$ is the required immersion. It is clear that $\widetilde{\psi}(M)$ is bounded (in the hyperbolic distance) since it is contained in the Euclidean ball centered at 0 with radius $\mu\ell$, which corresponds to the hyperbolic geodesic ball with radius $2\text{arctanh}(\mu\ell)$. As for the completeness of the metric induced by $\widetilde{\psi}$, it is enough to note that the Euclidean dilatation $x \mapsto \mu \cdot x$ also increases hyperbolic distances. Let us check then the condition on the second fundamental form $\widetilde{h}$ of $\widetilde{\psi}$.

Recall first that intersections of spheres in $\mathbb{R}^{n+1}$ with $\mathbb{B}^{n+1}$ are totally umbilical hypersurfaces. When such spheres are tangent to the boundary $\partial\mathbb{B}^{n+1}$ they are called *horospheres*, and have constant normal curvature 1 with respect to the normal vector pointing inwards. If they meet the boundary with angle α they have constant normal curvature $\pm\cos\alpha$ [1, p-184].

Now given a point $p \in M^n$, consider the family of totally umbilical hypersurfaces $S_p(\lambda)$ which are tangent to $\psi(M)$ at $\psi(p)$ and have constant normal curvature $-\lambda$, with respect to the common normal $\nu(p)$. For every $\lambda > 1$, since $h_p \geq -1 > -\lambda$, we have (using Taylor approximation) that $\psi(M)$ is locally exterior to the sphere $S_p(\lambda)$. Thus $\widetilde{\psi}(M)$ is locally exterior to the sphere $\mu S_p(\lambda)$ (which is again umbilical) for every $\lambda > 1$. That means that $\widetilde{h}_p$ is greater or equal than the constant normal curvature of $\mu S_p(\lambda)$ for every $\lambda > 1$. By continuity, $\widetilde{h}_p$ is also greater or equal than the constant normal curvature of $\mu S_p(1)$, which is strictly greater than -1. This guarantees that, for every $p \in M$ there exists $\delta_p > 0$ such that $\widetilde{h}_p \geq -1+\delta_p$. Finally, by a compactness argument we will see that there exists a uniform δ such that $\widetilde{h}_p \geq -1+\delta$ for every $p \in M$.

Since $S_p(1)$ is a horosphere meeting $B(l)$, its Euclidean center belongs to the compact region $\Omega = \{x \in \mathbb{R}^{n+1} | (1-\ell)/2 \leq \|x\| \leq (1+\ell)/2\}$. For every $y \in \Omega$, denote by $H(y)$ the horosphere with Euclidean center y Performing the dilatation $x \mapsto \mu \cdot x$, the horosphere $H(y)$ is mapped to a sphere $\mu H(y)$ meeting $\partial\mathbb{B}^{n+1}$ with some angle $\alpha(y)$. This defines a continuous function $\alpha : \Omega \to (0, \pi/2]$. Since Ω is compact, there is some $\alpha_0 > 0$ such that $\alpha(y) \geq \alpha_0$ for every $y \in \Omega$. In particular, the sphere $\mu S_p(1)$ meets $\partial\mathbb{B}^{n+1}$ with angle $\alpha_p \geq \alpha_0$, and therefore $\widetilde{h}_p \geq -\cos\alpha_p > -\cos\alpha_0 =: -1+\delta$. □

The next Example shows that the assumptions in Theorem 3 are sharp in the sense that there exist compact non-spherical hypersurfaces with $h \geq -1-\delta$ for every $\delta > 0$.

Example 10. Let e_1, e_2, e_3 be an orthonormal basis at some point $a \in \mathbb{H}^3$. Then we can define a *revolution* torus in $\mathbb{H}^3$ as follows

$$v(\theta) = R(\cos\theta e_1 + \sin\theta e_2), \qquad c(\theta) = \overline{\exp}_a v(\theta), \qquad 0 \leq \theta \leq 2\pi$$

$$\psi(\theta,\varphi) = \overline{\exp}_{c(\theta)} r((\mathrm{d}\overline{\exp}_a)_{v(\theta)}(\cos\varphi v(\theta) + \sin\varphi e_3)) \qquad 0 \leq \theta, \varphi \leq 2\pi$$

for some $R > r > 0$. In other words,

$$\begin{aligned}\psi(\theta,\varphi) = {} & \cosh r(\cosh Ra + \sinh R(\cos\theta e_1 + \sin\theta e_2) + \\ & + \sinh r(\cos\varphi(\sinh Ra + \cosh R(\cos\theta e_1 + \sin\theta e_2))) + \sin\theta e_3)\end{aligned}$$

Now, choosing the orientation given by the normal vector $\nu = \partial\psi/\partial r$, it is easy to check that $\partial\psi/\partial\theta$ and $\partial\psi/\partial\varphi$ are eigenvectors of the shape operator A, at any point $\psi(\theta,\varphi)$, with eigenvalues

$$\begin{aligned}\kappa_1 &= \frac{\langle A(\frac{\partial\psi}{\partial\theta}), \frac{\partial\psi}{\partial\theta}\rangle}{\|\frac{\partial\psi}{\partial\theta}\|^2} = \frac{\langle -\mathrm{d}\nu(\frac{\partial\psi}{\partial\theta}), \frac{\partial\psi}{\partial\theta}\rangle}{\|\frac{\partial\psi}{\partial\theta}\|^2} \\ &= \frac{\sinh r \sinh R + \cosh r \cosh R \cos\varphi}{\cosh r \sinh R + \sinh r \cosh R \cos\varphi}\end{aligned}$$

and

$$\kappa_2 = \frac{\langle A(\frac{\partial\psi}{\partial\varphi}), \frac{\partial\psi}{\partial\varphi}\rangle}{\|\frac{\partial\psi}{\partial\varphi}\|^2} = \frac{\langle -\mathrm{d}\nu(\frac{\partial\psi}{\partial\varphi}), \frac{\partial\psi}{\partial\varphi}\rangle}{\|\frac{\partial\psi}{\partial\varphi}\|^2} = -\frac{\cosh r}{\sinh r}.$$

Therefore $|\kappa_1| \leq \coth(R-r)$ and choosing for instance $R = 2r$ we have that all the normal curvatures of ψ are greater or equal than $-\coth r$, which takes, for $r \in (0,\infty)$, any value smaller than -1.

5. Proof of Theorem 4

The proof of Theorem 4 makes use of the following auxiliary result.

Lemma 11. *Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a compact hypersurface in hyperbolic space, and assume that $A + \lambda I$ is non-degenerate for a real number $\lambda \in [-1,1]$. Then (after an appropriate orientation of M) the operator $A + \lambda I$ is positive definite, and $F_\lambda : M^n \to \mathbb{S}^n$ is a diffeomorphism satisfying*

$$F_\lambda^*(\mathrm{d}\sigma) = \frac{\langle a, \lambda\nu - \psi\rangle \det(A + \lambda I)}{(1 - \lambda^2 + \langle a, \nu - \lambda\psi\rangle^2)^{\frac{n+1}{2}}} \mathrm{d}V. \tag{7}$$

Here $\mathrm{d}V$ is the n-dimensional volume element of M with respect to the chosen orientation, and $\mathrm{d}\sigma$ stands for the n-dimensional volume element of the round sphere $\mathbb{S}^n = \mathbb{S}_1^{n+1} \cap a^\perp$ with respect to the orientation induced on $\mathbb{S}^n$ from the orientation of M via the diffeomorphism F_λ.

Proof. The assertion on the positive definiteness of the operator $A + \lambda I$ follows from Lemma 8 reasoning as in the proof of second part of Theorem 1. In particular, $\det(A + \lambda I) > 0$ and, by Theorem 1, we already know that F_λ is a diffeomorphism. In order to compute $F_\lambda^*(\mathrm{d}\sigma)$, observe that the Gauss map of $\mathbb{S}^n \subset \mathbb{S}_1^{n+1}$ (as a totally geodesic spacelike hypersurface of de Sitter space) is given by $\nu_{\mathbb{S}^n} = \varepsilon a$, where $\varepsilon = \pm 1$ (soon after we will find out the precise sign of $\nu_{\mathbb{S}^n}$ which is compatible with the orientation of $\mathbb{S}^n$). Therefore, for every $x \in \mathbb{S}^n$ and every $v_1, \dots, v_n \in T_x\mathbb{S}^n$ we have

$$(\mathrm{d}\sigma)_x(v_1, \dots, v_n) = \det(v_1, \dots, v_n, \varepsilon a, x),$$

where det stands for the determinant in $\mathbb{R}^{n+2}$.

Besides, using the expression for F_λ given in (1), and its differential (2) we obtain the following

$$F_\lambda^*(\mathrm{d}\sigma)(X_1,\dots,X_n) = \det(dF_\lambda(X_1),\dots,dF_\lambda(X_n),\varepsilon a, F_\lambda) = \frac{\varepsilon(-1)^n}{(1-\lambda^2+\langle a,\nu-\lambda\psi\rangle^2)^{\frac{n+1}{2}}}\det((A+\lambda I)(X_1),\dots,(A+\lambda I)(X_n),a,\nu-\lambda\psi),$$

and writing now $a = a^\top + \langle a,\nu\rangle\nu - \langle a,\psi\rangle\psi$ we conclude that

$$F_\lambda^*(\mathrm{d}\sigma)(X_1,\dots,X_n) = \frac{\varepsilon(-1)^{n+1}\langle a,\lambda\nu-\psi\rangle\det(A+\lambda I)}{(1-\lambda^2+\langle a,\nu-\lambda\psi\rangle^2)^{\frac{n+1}{2}}}\det(X_1,\dots,X_n,\nu,\psi),$$

for every tangent vector fields $X_1,\dots,X_n \in \mathcal{X}(M)$. In other words,

$$F_\lambda^*(\mathrm{d}\sigma) = \frac{\varepsilon(-1)^{n+1}\langle a,\lambda\nu-\psi\rangle\det(A+\lambda I)}{(1-\lambda^2+\langle a,\nu-\lambda\psi\rangle^2)^{\frac{n+1}{2}}}\mathrm{d}V.$$

Observe now that $\langle a,\lambda\nu-\psi\rangle > 0$ for every $\lambda\in[-1,1]$. Since the orientation of $\mathbb{S}^n$ is the one induced by the diffeomorphism F_λ from the orientation of M, it must be $F_\lambda^*(\mathrm{d}\sigma) = \mu\mathrm{d}V$ for a positive function μ. Therefore, $\varepsilon = (-1)^{n+1}$ and we obtain (7). This finishes the proof of Lemma 11. □

Theorem 4 characterizes the geodesic spheres of hyperbolic space as those hypersurfaces in $\mathbb{H}^{n+1}$ which maximize the integral $\int_M \det(A+\lambda I)\mathrm{d}V$ for $0\leq\lambda\leq 1$ among all the compact convex hypersurfaces in $\mathbb{H}^{n+1}$ which are bounded by that geodesic sphere. For that reason, before proving the theorem it will be interesting to study briefly those geodesic spheres and compute that integral.

Example 12 (Geodesic spheres)**.** For a given point $a\in\mathbb{H}^{n+1}$, the geodesic sphere of radius $r>0$ centered at a is the subset

$$\Sigma(a,r) = \{x\in\mathbb{H}^{n+1} : \langle a,x\rangle = -\cosh(r)\}.$$

As is well-known, $\Sigma(a,r)$ is a compact convex hypersurface in $\mathbb{H}^{n+1}$ with Gauss map (in the orientation of Lemma 11)

$$\nu_\Sigma(p) = \frac{1}{\sinh(r)}(a-\cosh(r)p).$$

Actually, with this orientation its shape operator is given by

$$A_\Sigma(X) = -d\nu_\Sigma(X) = \coth(r)X \quad \text{for every } X\in\mathcal{X}(M).$$

That is, $\Sigma(a,r)$ is a totally umbilical hypersurface with $A=\coth(r)I$. Moreover, its is not difficult to see that $\Sigma(a,r)$ is a round n-sphere of radius

$\sinh(r)$. In particular,

$$\begin{aligned}\int_\Sigma \det(A_\Sigma + \lambda I) dV_\Sigma &= (\coth(r) + \lambda)^n \mathrm{vol}(\mathbb{S}^n(\sinh(r))) \\ &= (\cosh(r) + \lambda \sinh(r))^n \omega_n,\end{aligned}$$

where $\omega_n = \mathrm{vol}(\mathbb{S}^n)$.

Proof of Theorem 4. Assume that $\psi(M)$ is contained in a geodesic ball in $\mathbb{H}^{n+1}$ of radius $r > 0$ centered at a point $a \in \mathbb{H}^{n+1}$. This means that

$$1 \leq -\langle a, \psi \rangle \leq \cosh(r) \quad \text{on } M, \tag{8}$$

and using (5) one gets

$$-\sinh(r) \leq \langle a, \nu \rangle \leq \sinh(r) \quad \text{on } M. \tag{9}$$

By Theorem 1 we know that $F_\lambda : M^n \to \mathbb{S}^n = \mathbb{S}^{n+1}_1 \cap a^\perp$ is a diffeomorphism for every $\lambda_0 \leq \lambda \leq 1$. Using now Lemma 11 we obtain that

$$\begin{aligned}\omega_n &= \mathrm{vol}(\mathbb{S}^n) = \int_{\mathbb{S}^n} \mathrm{d}\sigma = \int_M F_\lambda^*(\mathrm{d}\sigma) \\ &= \int_M \frac{\langle a, \lambda\nu - \psi \rangle}{(1 - \lambda^2 + \langle a, \nu - \lambda\psi \rangle^2)^{\frac{n+1}{2}}} \det(A + \lambda I) \mathrm{d}V.\end{aligned} \tag{10}$$

When $\lambda < 1$ we take $\varrho = -\mathrm{arctanh}(\lambda) \leq 0$, and recalling

$$\psi_\varrho = \cosh(\varrho)\psi + \sinh(\varrho)\nu \quad \text{and} \quad \nu_\lambda = \sinh(\varrho)\psi + \cosh(\varrho)\nu,$$

equation (10) becomes

$$\omega_n = \cosh^n(\varrho) \int_M \frac{\langle a, -\psi_\varrho \rangle}{(1 + \langle a, \nu_\varrho \rangle^2)^{\frac{n+1}{2}}} \det(A + \lambda I) \mathrm{d}V, \tag{11}$$

where $\langle a, -\psi_\varrho \rangle > 0$. Observe now that $\{\nu_\varrho, \psi_\varrho\}$ constitutes an orthonormal basis of a Lorentzian 2-plane in the Minkowski space $\mathbb{R}^{n+2}_1$, and the point $a \in \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ can be decomposed as

$$a = v_\varrho + \langle a, \nu_\varrho \rangle \nu_\varrho - \langle a, \psi_\varrho \rangle \psi_\varrho,$$

where v_ϱ is the component of a in the spacelike n-plane orthogonal to $\mathrm{span}\{\nu_\varrho, \psi_\varrho\}$. In particular,

$$\langle a, -\psi_\varrho \rangle = \sqrt{1 + \langle a, \nu_\varrho \rangle^2 + |v_\varrho|^2},$$

and

$$\begin{aligned} \frac{\langle a, -\psi_\varrho\rangle}{(1+\langle a,\nu_\varrho\rangle^2)^{\frac{n+1}{2}}} &= \frac{\sqrt{1+\langle a,\nu_\varrho\rangle^2+|v_\varrho|^2}}{(1+\langle a,\nu_\varrho\rangle^2)^{\frac{n+1}{2}}} \geq \frac{1}{(1+\langle a,\nu_\varrho\rangle^2)^{\frac{n}{2}}} \\ &\geq \frac{1}{(1+\langle a,\nu_\varrho\rangle^2+|v_\varrho|^2)^{\frac{n}{2}}} = \frac{1}{\langle a,-\psi_\varrho\rangle^n} \\ &\geq \frac{1}{(\cosh(r)\cosh(\varrho)-\sinh(r)\sinh(\varrho))^n}, \end{aligned}$$

taking into account that, from (8) and (9), one has

$$\begin{aligned} \langle a,-\psi_\varrho\rangle &= \cosh(\varrho)\langle a,-\psi\rangle - \sinh(\varrho)\langle a,\nu\rangle \\ &\leq \cosh(r)\cosh(\varrho)-\sinh(r)\sinh(\varrho). \end{aligned}$$

Besides, the equality holds if and only if $\langle a,-\psi\rangle=\cosh(r)$ is constant and, hence, M is a geodesic sphere of radius r.

Using the previous inequalities in (11) we get

$$\begin{aligned} \int_M \det(A+\lambda I)\mathrm{d}V &\leq \omega_n\frac{(\cosh(r)\cosh(\varrho)-\sinh(r)\sinh(\varrho))^n}{\cosh^n(\varrho)} \\ &= \omega_n(\cosh(r)+\lambda\sinh(r))^n. \end{aligned}$$

□

Finally, Corollary 6 (and Corollary 7) follows easily from Theorem 4 by the following observation. Let $\psi : M^n \to \mathbb{H}^{n+1}$ be a hypersurface in hyperbolic space $\mathbb{H}^{n+1}$ and consider, for every real number ϱ, the (possibly with degenerate points) parallel hypersurface $\psi_\varrho : M^n \to \mathbb{H}^{n+1}$ at a distance ϱ, given by

$$\psi_\varrho(p) = \cosh(\varrho)\psi(p) + \sinh(\varrho)\nu(p), \quad p \in M.$$

A direct calculation gives

$$(d\psi_\varrho)_p(v) = d\psi_p(\cosh(\varrho)v - \sinh(\varrho)A_p(v)). \tag{12}$$

Therefore, ψ_ϱ is an immersion for $\varrho \neq 0$ if and only if $A - \coth(\varrho)I$ is non-degenerate on M. In particular, if we assume that $A+\lambda_0 I$ is positive definite for some $0 \leq \lambda_0 \leq 1$, then it follows that ψ_ϱ is an immersion for every $\varrho < 0$. In that case, it is not difficult to see that

$$\nu_\varrho = \sinh(\varrho)\psi + \cosh(\varrho)\nu \tag{13}$$

is a unit normal field for ψ_ϱ and the corresponding volume element is given by

$$\mathrm{d}V_\varrho = (\cosh(\varrho))^n P(-\tanh(\varrho))\mathrm{d}V, \tag{14}$$

where

$$P(T) = \prod_{i=1}^{n}(1+\kappa_i T) = \sum_{i=0}^{n}\binom{n}{i} H_i T^i.$$

Here $H_0 = 1$ and, for $1 \leq i \leq n$, H_i denotes the i-th mean curvature of the immersion ψ. In particular, $H_1 = H$ is the mean curvature of ψ, $H_n = K$ is its Gauss-Kronecker curvature and H_2 defines a geometric quantity which is related to the intrinsic scalar curvature S of the hypersurface, because $S = \mathrm{trace(Ric)} = n(n-1)(-1+H_2)$. On the other hand, differentiating (13) and using (12) we have that

$$\sinh(\varrho)v - \cosh(\varrho)A_p(v) = -\cosh(\varrho)(A_\varrho)_p(v) + \sinh(\varrho)A_p((A_\varrho)_p(v)),$$

for any $p \in M$ and $v \in T_pM$, where A_ϱ is the shape operator of ψ_ϱ associated to ν_ϱ. This implies that if $\{e_1, \ldots, e_n\}$ is a basis of principal directions at the point p for the immersion ψ with principal curvatures $\kappa_1(p), \ldots, \kappa_n(p)$, then $\{e_1, \ldots, e_n\}$ is also a basis of principal directions at p for the immersion ψ_ϱ with corresponding principal curvatures

$$\kappa_i^\varrho(p) = \frac{\kappa_i(p) - \tanh(\varrho)}{1 - \kappa_i(p)\tanh(\varrho)}.$$

In particular, the Gauss-Kronecker curvature of the parallel hypersurface ψ_ϱ is given by

$$K^\varrho = H_n^\varrho = \frac{\det(A - \tanh(\varrho)I)}{P(-\tanh(\varrho))}. \tag{15}$$

Then, from (14) and (15) we conclude that

$$K^\varrho \mathrm{d}V_\varrho = (\cosh(\varrho))^n \det(A - \tanh(\varrho)I)\mathrm{d}V,$$

and

$$\int_M K^\varrho \mathrm{d}V_\varrho = (\cosh(\varrho))^n \int_M \det(A - \tanh(\varrho)I)\mathrm{d}V.$$

Theorem 4 does the rest.

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