THE SUPER-CHARACTER FORMULA FOR A CLASS OF INTEGRABLE INFINITE DIMENSIONAL MODULES OF BORCHERDS-KAC-MOODY LIE SUPERALGEBRAS

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In [R1] we gave the character formula for integrable infinite dimensional highest weight modules of Borcherds-Kac-Moody (BKM) Lie superalgebras. In this short note, we give a formula for their super-character. The proof is similar to the one in [R1]. However in [R1] we omitted certain details which ought to be emphasized for otherwise one of the main arguments would not hold.

We first remind the reader of the definition of a BKM Lie superalgebra.

Let I be a finite set indexing the simple roots and S a subset of I indexing the odd simple roots. The finite dimensional Borcherds-Kac-Moody Lie superalgebras have a Cartan decomposition of Borcherds-Kac-Moody type. In other words, they have an abelian even Lie algebra subalgebra H – called a Cartan subalgebra – with bilinear form $(.,.)$ containing elements h_i, $i \in I$ such that the Cartan matrix $A = (a_{ij})$, $a_{ij} = (h_i, h_j)$ satisfies

(i) $a_{ij} \leq 0$ if $i \neq j$;

(ii) $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;

and are generated by the Lie subalgebra H and elements $e_i, f_i, i \in I$ satisfying the following defining relations:

(1) $[e_i, f_j] = \delta_{ij} h_i$;

(2) $[h, e_i] = (h, h_i)e_i$, $[h, f_i] = -(h, h_i)f_i$;

(3) $\deg e_i = 0 = \deg f_i$ if $i \notin S$, $\deg e_i = 1 = \deg f_i$ if $i \in S$;

(4) $(\mathrm{ad}\,(e_i))^{1-\frac{2a_{ij}}{a_{ii}}} e_j = 0 = (\mathrm{ad}\,(f_i))^{1-\frac{2a_{ij}}{a_{ii}}} f_j$ if $a_{ii} > 0$ and $i \neq j$;

(5) $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

We next fix some more notation, which we keep standard.

Notation. Set G to be a BKM superalgebra and H a Cartan subalgebra. The roots spaces will be written $G_\alpha = \{x \in G : [h, x] = \alpha(h)x\}$.
We will write Δ for the set of roots and respectively Δ^+, Δ_0, Δ_1, Δ_0^+, Δ_1^+ for the set of positive, even, odd, even positive, odd positive roots of

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G. We also set $\Delta_E^+ = \{\alpha \in \Delta_0^+ : \frac{1}{2}\alpha \notin \Delta\}$, $I_< = \{i \in I : a_{ii} \leq 0\}$ and $S_0 = \{i \in S : a_{ii} = 0\}$.
The Weyl group W is the group generated by the reflections induced by all the simple roots of positive norm.
For $w \in W$, set

$$\epsilon(w) = \begin{cases} 1 & \text{if } w \text{ is the product of an even number of reflections} \\ -1 & \text{otherwise} \end{cases}.$$

For any weight $\Lambda \in H^*$, $L(\Lambda)$ will denote the G-module of highest weight Λ.
Set ρ to be the Weyl vector, i.e. $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all $i \in I$.

We now state a result proved in [R2, Lemma 2.2] which we will be frequently using:

Lemma 1. *For any root $\alpha \in \Delta^+$ of norm 0, there exists $w \in W$ such that $w(\alpha)$ is a simple odd root. Furthermore for all $w \in W$, $w(\alpha)\Delta^+$.*

We are interested in integrable modules. This means that all finite type [R2, Definition 2.3] root vectors act in a locally nilpotent manner. Hence, $(\Lambda, \alpha_i) \geq 0$ whenever $a_{ii} > 0$. In this paper, we add a technical condition and so consider irreducible highest weight modules $L(\Lambda)$ for which that

$$(\Lambda, \alpha_i) \geq 0 \quad \forall i \in I. \tag{1}$$

We are mostly interested in infinite dimensional BKM superalgebras. Hence all roots of negative norm are of infinite type [R2, Corollary 2.5]. Therefore, the super-character formula we give in this paper applies to a large class of modules.

Remark 2. When the BKM Lie superalgebra G is finite dimensional, because there may be roots of negative type, which would then necessarily be of finite type, the formula we give applies to a class of highest weight modules which are in general of infinite dimension: Indeed, the module $L(\Lambda)$ is integrable if and only if $\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z}_+$ for all roots α of finite type with non-trivial norm. Hence, if there is a positive root α of negative norm, then the above condition (1) implies that $(\Lambda, \alpha) \geq 0$, whereas integrability forces $(\Lambda, \alpha) \leq 0$. Therefore $(\Lambda, \alpha) = 0$. As $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$ and all $i \in I$ appear in the support of β [R2, Corollary 2.5], this implies that $(\Lambda, \alpha_i) = 0$ for all $i \in I$ and so $\Lambda = 0$. So the only integrable module the formula we give here applies to is the trivial irreducible representation of finite dimensional BKM Lie superalgebras with roots of negative norm. See [R3] for arbitrary integrable modules.

When G is finite dimensional with no roots of negative norm, G is either of type $A(m,0)$ or $B(0,n)$. In the latter case, all roots have positive norm, and our formula applies to all integrable modules, and we recover the results of [K1]. In the former case, there is a unique simple odd root α_1 of norm 0 and by definition of the Weyl vector $(\rho,\alpha_1)=0$. And so our formula applies in this case, to all typical integrable modules satisfying $(\Lambda,\alpha_1)>0$ and to all atypical modules satisfying $(\Lambda,\alpha_1)=0$.

So, when G is finite dimensional, our formula applies to highest weight modules which are not in general integrable, and in particular are infinite dimensional.

Similarly as in [K3] or [R1], the super-character of the Verma module of highest weight Λ can be computed and the next equalities follow using standard arguments.

Proposition 3. *The super-character of the highest weight G-module $L(\Lambda)$ is given by:*

$$e(\rho)\prod_{\alpha\in\Delta_0^+}(1-e(-\alpha))\operatorname{sch}L(\Lambda)=\sum_{\substack{\lambda\leq\Lambda\\ \left|\lambda+\rho\right|^2=\left|\Lambda+\rho\right|^2}}c_\lambda e(\lambda+\rho)\prod_{\alpha\in\Delta_1^+}(1-e(-\alpha)) \tag{2}$$

where $c_\lambda\in\mathbf{Z}$ and $c_\Lambda=1$.

Before proving the super-character formula, we need to see that the Weyl group W keeps the set of weights λ satisfying $c_\lambda\neq 0$ invariant as it is not completely obvious in the case when there are odd roots of norm 0.

Lemma 4. *We keep the notation of Proposition 3. For all $\lambda\in H^*$ and all $w\in W_E$, $c_{w(\lambda+\rho)-\rho}=\epsilon(w)e_\lambda$.*

Proof. To prove the Lemma, it suffices to show that for any weight $\lambda\in H^*$ satisfying $c_\lambda\neq 0$,

$$r_i(\lambda+\rho)\leq\Lambda+\rho\quad\forall a_{ii}>0. \tag{i}$$

Indeed suppose this holds. Then $\lambda+\rho=r_i(\mu+\rho)$ for some weight $\mu\in H^*$ such that $\mu\leq\Lambda$ and $\left|\mu+\rho\right|=\left|\Lambda+\rho\right|$. Hence, we may write

$$\sum_{\substack{\lambda\leq\Lambda\\ \left|\lambda+\rho\right|^2=\left|\Lambda+\rho\right|^2}}d_\lambda e(\lambda+\rho)=\sum_{\substack{\lambda\leq\Lambda\\ \left|\lambda+\rho\right|^2=\left|\Lambda+\rho\right|^2}}c_\lambda e(r_i(\lambda+\rho)).$$

If $\alpha \in D_1^+$ has non-zero norm, then $2\alpha \in D^+$. Hence, equation (2) may be re-written as:

$$e(\rho) \prod_{\alpha \in \Delta_E^+} (1 - e(-\alpha)) \prod_{\alpha \in \Delta_0^+ - \Delta_E^+} (1 + e(-\alpha)) \operatorname{sch} L(\Lambda) = \sum_{\substack{\lambda \leq \Lambda \\ \left|\lambda+\rho\right|^2 = \left|\Lambda+\rho\right|^2}} c_\lambda e(\lambda + \rho) \prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha,\alpha)=0}} (1 - e(-\alpha)) \tag{2'}$$

Applying the reflection r_i to the left hand side of ($2'$) multiplies it by -1. Furthermore by Lemma 1,

$$r_i\Big(\prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha,\alpha)=0}} (1 - e(-\alpha))\Big) = \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha)).$$

Hence applying r_i to the right hand side of equality ($2'$), we can deduce that

$$\sum_{\substack{\lambda \leq \Lambda \\ \left|\lambda+\rho\right|^2 = \left|\Lambda+\rho\right|^2}} (d_\lambda + c_\lambda) e(\lambda + \rho) \prod_{\substack{\alpha \in \Delta_1^+ \\ (\alpha,\alpha)=0}} (1 + e(-\alpha)) = 0. \tag{ii}$$

Suppose there exists a weight λ for which $d_\lambda \neq -c_\lambda$. We may take λ to be such that the height $\Lambda - \lambda$ is minimal. Then, equation (ii) gives

$$d_\lambda + c_\lambda + \sum_\mu d_{\lambda+\mu} + c_{\lambda+\mu} = 0,$$

the sum being taken over all distinct sums μ of positive odd roots of norm 0. By minimality of the weight λ, it follows that

$$d_\lambda + c_\lambda = 0,$$

contradicting the definition of λ.

We now prove property (i). Let $\lambda \in H^*$ be a weight satisfying $c_\lambda \neq 0$ and $r_i(\lambda + \rho) \not\leq \Lambda + \rho$ for some $i \in I$ such that $a_{ii} > 0$. We may take a weight λ with this property such that the height of $\Lambda - \lambda$ is minimal. To complete the proof, we only need to show that this leads to a contradiction

Suppose that $c_\lambda + \sum_\mu c_{\lambda+\mu} \neq 0$, where μ runs over all sums of distinct positive odd roots of norm 0. Considering the left hand side of ($2'$), we then get a contradiction. Hence,

$$c_\lambda + \sum_\mu c_{\lambda+\mu} = 0,$$

where μ runs over all sums of distinct positive odd roots of norm 0. So, there is a sum μ of positive odd roots of norm 0 such that $c_{\lambda+\mu} \neq 0$. By minimality of the height of $\Lambda - \lambda$, $r_i(\lambda + \mu + \rho) \leq \Lambda + \rho$. By Lemma 1, $r_i(\mu) > 0$. So,

$$r_i(\lambda + \rho) \leq \Lambda + \rho,$$

again contradicting assumption, and proving the result. □

Let $\mu = \sum_{i \in I} k_i \alpha_i$, where $k_i \geq 0$ for all $i \in I$ and write

$$\mathrm{ht}_0(\mu) = \sum_{i \in I - S} k_i.$$

Set

$$T_\Lambda = e(\Lambda + \rho) \sum \epsilon(\mu) e(\mu)$$

with

$$\epsilon(\mu) = \begin{cases} (-1)^{\mathrm{ht}_0(\mu)}, & \text{if } \mu = \sum\limits_{\substack{i \in I_< \\ a_{ij}=0, (\Lambda, \alpha_i)=0}} k_i \alpha_i \in \mathbf{Z}_+ \Delta^+, \quad k_i \leq 1 \quad \text{unless} \quad i \in S_0 \\ 0 & \text{otherwise} \end{cases}$$

We are now ready to prove the super-character formula.

Theorem 5. *Let $L(\Lambda)$ be an integrable G-module satisfying $(\Lambda, \alpha_i) \geq 0$ for all $i \in I$. Set*

$$R = e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))^{-1}$$

The super-character formula for for the module $L(\Lambda)$ is:

$$\mathrm{sch}\, L(\Lambda) = e(\rho) \sum_{w \in W} \epsilon(w) w(T'_\Lambda) {L'}^{-1}.$$

Proof. Let Λ be a weight satisfying inequalities (1). From Proposition 3, we get

$$e(\rho) \mathrm{ch}\, L(\Lambda) L = \sum_{\substack{\lambda \leq \Lambda \\ \left|\lambda - \rho\right|^2 = \left|\Lambda - \rho\right|^2}} c_\lambda e(\lambda + \rho). \qquad (i)$$

Lemma 4 gives $c_\lambda = \epsilon(w) c_\mu$ if $w(\lambda + \rho) = \mu + \rho$ for some $w \in W$. Let λ be such that $c_\lambda \neq 0$. Hence, for all $w \in W$, $c_{w(\lambda+\rho)-\rho} \neq 0$ and so $w(\lambda + \rho) \leq \Lambda + \rho$. Let $\mu \in \{w(\lambda + \rho) - \rho | w \in W\}$ be such that $\mathrm{ht}(\Lambda - \mu)$

is minimal. Then $(\mu+\rho,\alpha_i)\ge 0$ for all $i\in I$ such that $a_{ii}>0$, and $|\mu+\rho|^2=|\Lambda+\rho|^2$. Setting $\mu=\Lambda-\sum_{i\in I}x_i\alpha_i$, $x_i\in\mathbf{Z}_+$, the latter equation gives

$$\sum_{i\in I}x_i(\Lambda+\mu+2\rho,\alpha_i)=0. \qquad (ii)$$

From the above and the definition of ρ and Λ, $(\Lambda+\mu+2\rho,\alpha_i)>0$ for all $i\in I$ such that $a_{ii}>0$. Next, consider $i\in I$ such that $a_{ii}\le 0$. Now, $(\Lambda+\mu+2\rho,\alpha_i)=(\Lambda+\mu+\alpha_i,\alpha_i)$. Suppose that $x_i\ne 0$. Then $\mu+\alpha_i=\Lambda-\sum_{i\in I}y_i\alpha_i$ and $y_i\ge 0$ for all $i\in I$, and so $(\Lambda+\mu+2\rho,\alpha_i)\ge 0$. Hence, equation (ii) forces $x_i\ne 0$ only if $a_{ii}\le 0$ and $(\Lambda+\mu+2\rho,\alpha_i)=0$. Therefore, for all i such that $x_i\ne 0$,

$$(\Lambda,\alpha_i)=0,\quad i\ne j\Rightarrow(\alpha_i,\alpha_j)=0\quad x_i\ge 2\Rightarrow(\alpha_i,\alpha_i)=0. \qquad (iii)$$

We need to compute the sum T_1 of all terms in the right hand side of (i) for which $(\lambda+\rho,\alpha_i)\ge 0$ for all $i\in I$ with $a_{ii}>0$. Equation (2) can be re-written as

$$e(\rho)\mathrm{ch}\,L(\Lambda)\prod_{\alpha\in\Delta_0^+}(1-e(-\alpha))^{\mathrm{mult}\,\alpha}=\sum_{w\in W}w(T)\prod_{\alpha\in\Delta_1^+}(1-e(-\alpha))^{\mathrm{mult}\,\alpha}. \quad (iv)$$

Let $c_\lambda e(\lambda+\rho)$ be a term in T_1. Set

$$\lambda=\Lambda-\alpha-\beta,$$

where $\alpha=\sum_{i\in I-S}x_i\alpha_i$ and $\beta=\sum_{i\in S}z_i\alpha_i$, $x_i,z_i\in\mathbf{Z}_+$. For $x_i\ne 0$ and $z_i\ne 0$, α_i satisfies conditions (iii).

Claim: If $e(w(\lambda+\rho)+\gamma)$ is a term in T_1 for $0\ne\gamma=\sum_{i\in S}z_i\alpha_i$, $z_i\in\mathbf{Z}_+$ and $w\in W$, then $w(\lambda+\rho)=\lambda+\rho$.

From the above, $(\lambda+\rho,\alpha_i)\ge 0$ for all $i\in I$. Therefore

$$w(\lambda+\rho)=\lambda+\rho-\sum_{i,a_{ii}>0}c_ih_i \qquad (v),$$

where $c_i\in\mathbf{Z}_+$. Since $e(w(\lambda+\rho)+\beta)$ is a term in T_1, $\sum_{i\in I-S}(x_i+c_i)\alpha_i+\sum_{i\in S}(y_i+z_i)\alpha_i$ must satisfy conditions (iii). Hence $c_i\ne 0\Rightarrow a_{ii}\le 0$, and so from (v) we can deduce that $w(\lambda+\rho)=\lambda+\rho$.

We now show by induction on $\mathrm{ht}(\beta)$ that $c_\lambda=\epsilon(\alpha)$, where $\epsilon(\alpha)=(-1)^n$ if α is the sum of n distinct pairwise perpendicular simple even roots α_i perpendicular to Λ, and $\epsilon(\alpha)=0$ otherwise.

Suppose that $\mathrm{ht}(\beta) = 0$. Then, there must be a term on the left hand side of (iv) equal to $c_\lambda e(\lambda+\rho)$. However, $\mathrm{ch}\, L(\Lambda)$ does not have λ as a term since $(\Lambda, \alpha_i) = 0$ for all $i \in I$ such that $x_i \neq 0$ [K2, Proposition 11.2]. Hence, $\lambda = \Lambda - \sum_{\mu_i \in \Delta^+} \mu_i$ and the roots μ_i in the sum are all distinct. The support of a root being connected [K2, §1], without loss of generality $\mu_i = \alpha_i$ and it follows that $x_i \leq 1$ for all $i \in I$ and so we get the desired answer for c_λ.

Next assume that $\mathrm{ht}(\beta) > 0$ and that the result holds for all weights λ with β of smaller height. Then, no term on the left hand side of (iv) equals $c_\lambda e(\lambda+\rho)$. Since β is the sum of mutually orthogonal simple roots and if any appear more than once, it has norm 0, no sub-sum of β is a root. Hence the above Claim gives

$$c_\lambda e(\lambda) + \sum_{s=1}^{r} (-1)^s c_{\lambda+\alpha_{i_1}+\dots\alpha_{i_s}} e(\lambda+\alpha_{i_1}+\dots\alpha_{i_s}) e(-\alpha_{i_1})\dots e(-\alpha_{i_s}) = 0,$$

where r is the number of distinct simple roots α_i with $z_i \neq 0$ since in each term, the α_i's must be distinct. It follows by induction that

$$c_\lambda + \epsilon(\alpha) \sum_{s=1}^{r} \binom{t}{s} (-1)^s = 0,$$

giving the desired answer for c_λ. □

The super-character formula for the trivial module leads to one of the most important formulae associated to a BKM superalgebra:

Theorem 3.4. *For any BKM superalgebra G, the super-denominator formula is*

$$\frac{\prod_{\alpha\in\Delta_0^+}(1-e(-\alpha))^{mult\alpha} - o(\alpha)}{\prod_{\alpha\in\Delta_1^+}(1-e(-\alpha))^{(o\alpha)}} = e(\rho)\sum_{w\in W}\epsilon(w)w(T').$$

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