STOCHASTIC DELAY DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH HURST PARAMETER $H > \frac{1}{2}$

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Abstract. We consider the Cauchy problem for a stochastic delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We prove an existence and uniqueness result for this problem, when the coefficients have enough regularity. Furthermore, if the diffusion coefficient is bounded away from zero and the coefficients are smooth functions with bounded derivatives of any order, we prove that the law of the solution admits a smooth density with respect to Lebesgue's measure on $\mathbb{R}$.
Abbreviated title: SDDE with fBm.

1. Introduction

A general theory for the stochastic differential equations (SDE) driven by a fractional Brownian motion (fBm) is not yet established and just a few results have been proved (see e.g. Nualart and Rascanu [12], Nualart and Ouknine [10] and [11], and Coutin and Qian [4]) using different approach. Actually, the same definition of the stochastic integration with respect to fBm is not yet complety established and different approach have been proposed in the last years (see, among others, Alòs and Nualart [1], Carmona and Coutin [3] and Coutin and Qian [5]). Due to the initial stage of the general theory, it could appear at least strange that one plans to consider the class of the stochastic delay differential equations (SDDE). Indeed, this equations provide in general an infinite dimensional problem, much more difficult to be solved compared to the customary SDE's. Nevertheless, they include also problems that are easier to be solved compared to the SDE's, but that are still of great interest in the applications, for example to finance

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(see the recent papers of Arriojas et al. [2] and, in a more complicate case, of Hobson and Rogers [6]).

In the present paper we shall consider the Cauchy problem for a SDDE

$$dX(t) = b(X(t))dt + \sigma(X(t-r))dB(t), \quad t \in [0,T]$$

$$X(s) = \phi(s) \quad s \in [-r, 0],$$

where $\phi \in \mathrm{C}([-r,0])$ and the noise process $\{B(t), t \geq 0\}$ represent a fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$. As a solution to this problem, we shall define a process $\{X(t), t \in [-r,T]\}$ satisfying

$$X(t) = \phi(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s-r))dB(s), \quad t \in [0,T]$$

$$X(s) = \phi(s) \quad s \in [-r, 0],$$

where $\phi \in \mathrm{C}([-r,0])$. The stochastic integral, that appear in the previous equation, will be the Stratonovich integral with respect to the fBm, recently developed by some authors (see Definition 1 of the next section). To find a solution, we shall first solve the equation within the interval $[0,r]$; then, we use this solution process as the initial data to solve the equation within the interval $[r, 2r]$, and so on. This procedure allow us to construct a solution step by step, providing at any stage its uniqueness and its regularity. With the same approach, we will be able to prove, under the customary assumption of non degeneracy of the diffusion coefficient, that the law of the solution at any time t, admits a density with respect to Lebesgue's measure on $\mathbb{R}$.

Let us give now some remarks about the way one can get these results. We have used the classical techniques of stochastic calculus combined with some special properties of fBm. The delay allows us to avoid some of the usual problems that appear working with fBm and to use classical methods as Picard's iterations.

Our paper is divided as follows: in the next section we will introduce the basic notations concerning the fractional Brownian motion and we recall some results taken basically from [9] and [1]. Section 3 is devoted to state the existence and uniqueness of our stochastic delay differential equation driven by fBm. Finally, in section 4, we obtain the smoothness of the density of such solution.

2. Fractional Brownian motion

Let us start with some basic facts about the fractional Brownian motion (fBm) and the stochastic calculus that can be developed with respect to this process.

Fix a parameter $\frac{1}{2} < H < 1$. The fractional Brownian motion of Hurst parameter H is a centered Gaussian process $B = \{B(t), t \in [0,T]\}$ with the covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1}$$

Let us assume that B is defined in a complete probability space $(\Omega, \mathcal{F}, P)$. One can show (see, for instance, Alòs and Nualart [1]) that

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)dr, \tag{2}$$

where $K(t,s)$ is the kernel defined by

$$K(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (r-s)^{H-\frac{3}{2}} r^{H-\frac{1}{2}} dr,$$

for $s < t$, where $c_H = \left[\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right]^{1/2}$ and $B(\alpha,\beta)$ is the Beta function. We assume that $K(t,s) = 0$ if $s > t$. It is worth to notice that equation (2) implies that R is nonnegative definite and, therefore, there exists a Gaussian process with this covariance.

Let us denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R(t,s).$$

One can show that

$$R(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr,$$

where $\alpha_H = H(2H-1)$. This implies

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi(r)\psi(u) du dr \tag{3}$$

for all φ and ψ in $\mathcal{E}$. The mapping $\mathbf{1}_{[0,t]} \longrightarrow B(t)$ can be extended to an isometry between $\mathcal{H}$ and the first chaos H_1 associated with B, and we denote this isometry by $\varphi \to B(\varphi)$. The elements of $\mathcal{H}$ may not be functions but distributions of negative order. Due to this reason, it is convenient to introduce the Banach space $|\mathcal{H}|$ of measurable functions φ on $[0,T]$ satisfying

$$\|\varphi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\varphi(r)| \, |\varphi(u)| \, |r-u|^{2H-2} \, dr du < \infty. \tag{4}$$

One can prove (see Pipiras and Taqqu [15]) that the space $|\mathcal{H}|$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$, that we will identify with $|\mathcal{H}|$. The continuous embedding $L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}|$ has been proved in Mémin, Mishura and Valkeila [7].

2.1. **Malliavin calculus and stochastic integrals for the fBm.** In order to construct a stochastic calculus of variations with respect to the Gaussian process B, we shall follow the general approach introduced, for instance, in Nualart [8] (see also Nualart et al. [13]). Let us recall the definition of the derivative and divergence operators and some basic facts of this stochastic calculus of variations, taken mainly from Alòs and Nualart [1].

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{5}$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$. The derivative operator D of a smooth and cylindrical random variable F of the form (5) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(B(\phi_1), \ldots, B(\phi_n))\phi_j.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any $k \geq 1$ set D^k the iteration of the derivative operator. For any $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = E|F|^p + E\Big(\sum_{i=1}^k \left\|D^i F\right\|_{\mathcal{H}^{\otimes i}}^p \Big).$$

Proceeding as before, given a Hilbert space V we denote by $\mathbb{D}^{1,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$E(F\delta(u)) = E\left\langle DF, u\right\rangle_{\mathcal{H}}, \tag{6}$$

where u is a random variable in $L^2(\Omega;\mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by Dom δif the mapping $F \mapsto E\left\langle DF, u\right\rangle_{\mathcal{H}}$ is continuous in $L^2(\Omega)$. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in Dom δ.

Two basic properties of the divergence operator will follow:

i) For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$

$$E\delta(u)^2 = E\left\|u\right\|_{\mathcal{H}}^2 + E\left\langle Du, (Du)^*\right\rangle_{\mathcal{H}\otimes\mathcal{H}}, \tag{7}$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

ii) For any $u \in \mathbb{D}^{2,2}(\mathcal{H})$, $\delta(u)$ belongs to $\mathbb{D}^{1,2}$ and for any h in $\mathcal{H}$

$$\left\langle D\delta(u), h\right\rangle_{\mathcal{H}} = \delta\left(\left\langle Du, h\right\rangle_{\mathcal{H}}\right) + \left\langle u, h\right\rangle_{\mathcal{H}}. \tag{8}$$

Let us now consider the space $|\mathcal{H}| \otimes |\mathcal{H}| \subset \mathcal{H} \otimes \mathcal{H}$ of measurable functions φ on $[0,T]^2$ such that

$$\begin{aligned} &\|\varphi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \\ :&= \alpha_H^2 \int_{[0,T]^4} |\varphi(r,s)| \, |\varphi(r',s')| \, |r-r'|^{2H-2} \, |s-s'|^{2H-2} \, drdsdr'ds' < \infty. \end{aligned}$$

Let us denote by $\mathbb{D}^{1,2}(|\mathcal{H}|)$ the space of processes u such that

$$E\, \|u\|^2_{|\mathcal{H}|} + E \ \|Du\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \ < \infty. \tag{9}$$

Then $\mathbb{D}^{1,2}(|\mathcal{H}|)$ is included in $\mathbb{D}^{1,2}(\mathcal{H})$, and for a process u in $\mathbb{D}^{1,2}(|\mathcal{H}|)$ we can write

$$E\, \|u\|^2_{\mathcal{H}} = \alpha_H \int_{[0,T]^2} u(s)u(r)|r-s|^{2H-2} drds \tag{10}$$

and

$$\begin{aligned} &E \left\langle Du, (Du)^* \right\rangle_{\mathcal{H}\otimes\mathcal{H}} &(11)\\ = \quad &\alpha_H^2 \int_{[0,T]^4} D_r u(s) D_{r'} u(s') |r-s'|^{2H-2} |r'-s|^{2H-2} drdr'dsds'. \end{aligned}$$

The elements of $\mathbb{D}^{1,2}(|\mathcal{H}|)$ are stochastic processes and we will make use of the integral notation $\delta(u) = \int_0^T u(t)\delta B(t)$, and we will call this integral the Skorohod integral with respect to the fBm. Moreover, if $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ one can also define an indefinite integral process given by $X_t = \int_0^t u(s)\delta B(s)$.

Let us define now a Stratonovich type integral with respect to B. By convention we put $B(t) = 0$ if $t \notin [0,T]$. Following the approach by Russo and Vallois [16] we can give the following definition:

Definition 1. *Let $u = \{u(t), t \in [0,T]\}$ be a stochastic process with integrable trajectories. The Stratonovich integral of u with respect to B is defined as the limit in probability as ε tends to zero of*

$$(2\varepsilon)^{-1} \int_0^T u(s)(B(s+\varepsilon) - B(s-\varepsilon))ds,$$

provided this limit exists, and it is denoted by $\int_0^T u(t)dB(t)$.

It has been shown in Alòs and Nualart [1] that a process $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, such that

$$\int_0^T \int_0^T |D_s u(t)| \, |t-s|^{2H-2} \, dsdt < \infty$$

a.s. is Stratonovich integrable and

$$\int_0^T u(s)dB(s) = \int_0^T u(s)\delta B(s) + \alpha_H \int_0^T \int_0^T D_s u(t) \, |t-s|^{2H-2} \, dsdt. \tag{12}$$

On the other hand, if the process u has a.s. λ-Hölder continuous trajectories with $\lambda > 1-H$, then the Stratonovich integral $\int_0^T u(s)dB(s)$ exists and coincides with the path-wise Riemann-Stieltjes integral.

When $pH > 1$, can also be introduced the space $\mathbb{L}_H^{1,p}$ of processes $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ such that

$$\|u\|_{p,1} = \Big[\int_0^T E(|u(s)|^p)ds + E\Big(\int_0^T \Big(\int_0^T |D_r u(s)|^{\frac{1}{H}} dr\Big)^{pH} ds\Big)\Big]^{\frac{1}{p}} < \infty.$$

It is also known (see Nualart [8]) that

$$E\Big(\sup_{t\in[0,T]} |\int_0^t u(s)dB(s)|^p\Big) \le C\|u\|_{p,1}^p,$$

where the constant $C > 0$ depends on p, H and T.

2.2. **Some useful results.** In the next section, we shall need some additional results. They provide sufficient conditions for processes $Z = \{Z(t), t \in [0,T]\}$ to belong to the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$. Let us recall once more that these results holds in the case $H > \frac{1}{2}$.

Lemma 2. *Let $Z = \{Z(t), t \in [0,T]\}$ be a stochastic process such that for any $t \in [0,T]$, $Z(t) \in \mathbb{D}^{1,2}$ and*

$$\sup_s E(|Z(s)|^2) \le c_1 \quad \text{and} \quad \sup_{r,s} E(|D_r Z(s)|^2) \le c_2.$$

Then the stochastic process Z belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and

$$E\,\|Z\|_{|\mathcal{H}|}^2 + E\ \ \|DZ\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 < c_{H,T}(c_1+c_2).$$

Proof. The proof follows the computations given in Nualart [9], pag10. For instance, we can compute

$$\begin{aligned}
&E\,\|DZ\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 \\
&= E\left(\alpha_H^2 \int_{[0,T]^4} |D_r Z(s)|\,|D_{r'}Z(s')|\,|r-s'|^{2H-2}\,|s'-r|^{2H-2}\,drdsdr'ds'\right) \\
&\le E\left(\alpha_H^2 \int_{[0,T]^4} |D_r Z(s)|^2\,|r-s'|^{2H-2}\,|s'-r|^{2H-2}\,drdsdr'ds'\right) \\
&\le\ \alpha_H^2 \left(\frac{T^{2H-1}}{H-\frac{1}{2}}\right)^2 \int_{[0,T]^2} E(|D_r Z(s)|^2)drds \\
&\le\ \alpha_H^2 \left(\frac{T^{2H-1}}{H-\frac{1}{2}}\right)^2 T^2 \sup_{r,s} E(|D_r Z(s)|^2) \\
&=\ c_{H,T}c_2.
\end{aligned}$$

■

Lemma 3. *Let $Z = \{Z(t), t \in [0,T]\}$ be a stochastic process such that for any $t \in [0,T]$, $Z(t) \in \mathbb{D}^{1,2}$ and*

$$\sup_s E(|Z(s)|^2) \leq c_1 \quad \text{and} \quad \sup_{r,s} E(|D_r Z(s)|^2) \leq c_2.$$

Then given $r > 0$ and f a deterministic continuous function, the stochastic process $V = \{V(t), t \in [0,T]\}$ defined as

$$V(t) = \begin{cases} Z(t-r), & \text{si} \quad t > r, \\ f(t), & \text{si} \quad t < r, \end{cases}$$

belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

Proof. It is clear that V belongs to $\mathbb{D}^{1,2}(\mathcal{H})$ and that

$$D_s V(t) = \begin{cases} D_s Z(t-r), & \text{si} \quad t > r, \\ 0, & \text{si} \quad t < r. \end{cases}$$

Then it is enough to apply Lemma 2.

■

Given $s = (s_1, ..., s_k) \in [0,T]^k$; we denote by $|s|$ the length of s, that means k. For a random variable $Y \in \mathbb{D}^{k,p}$ and $s \in [0,T]^k$, we denote by $D_s^k Y$ the iterative derivative $D_{s_k} D_{s_{k-1}} \cdots D_{s_1} Y$. Let $f \in C_b^{0,\infty}(\mathbb{R})$, the space of continuous functions defined on $\mathbb{R}$ infinitely differentiable with bounded derivatives. Set

$$\Gamma_s(f;Y) = \sum_{m=1}^{|s|} \sum f^{(m)}(Y) \prod_{i=1}^{m} D_{p_i}^{|p_i|} Y,$$

where the second sum extends to all partitions $p_1, \cdots, p_m$ of length m of s.

We also need the following lemma that is an extension of the result proved in Rovira and Sanz [14].

Lemma 4. *Let $\{F_n, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}^{k,p}, k \geq 1, p \geq 2$. Assume there exists $F \in \mathbb{D}^{k-1,p}$ such that $\{D^{k-1}F_n, n \geq 1\}$ converges to $D^{k-1}F$ in $L^p(\Omega; [0,T]^{\otimes(k-1)})$ as n goes to infinity and, moreover, the sequence $\{D^k F_n, n \geq 1\}$ is bounded in $L^p(\Omega; [0,T]^{\otimes(k-1)})$. Then, $F \in \mathbb{D}^{k,p}$.*

3. Existence and uniqueness

Let $B = \{B(t), t \in [0,T]\}$ be a one dimensional fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$.

Let us define the following stochastic delay differential equation (SDDE) driven by a fBm

$$X(t) = \phi(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s-r))dB(s), \quad t \in [0,T] \tag{13}$$

where $r > 0$ and $\phi \in \mathrm{C}([-r,0])$. For simplicity let us assume $T = \bar{M}r$. The stochastic integral in (13) has to be intended as the Stratonovich type integral defined before (see Definition 1).

We shall assume that b and σ are real functions that satisfy the following conditions:

Hypotheses (H) : b and σ are $\bar{M}$-times differentiable functions with bounded derivatives up to order $\bar{M}$. Moreover, σ is bounded and $|b(0)| \leq c_1$ for some constant c_1.

Theorem 5. *Under Hypotheses (H), the SDDE (13) admits a unique solution X on $[0,T]$.*

The proof of this theorem is based in the following lemmas and propositions.

Lemma 6. *Let $M = \{M(t), t \in [0,T]\}$ be a quadratic integrable stochastic process. Assume that b is a Lipschitz function defined on $\mathbb{R}$, such that $|b(0)| \leq c_1$ for some constant c_1. Then, fixed $T_1 \leq T$, the stochastic integral equation*

$$X(t) = x + \int_0^t b(X(s))ds + M(t), \quad t \in [0,T_1], \tag{14}$$

$X(t) = 0$ if $t > T_1$, admits a unique solution X on $[0,T]$.

Proof. In order to prove the existence and uniqueness, we can prove that the classical Picard-Lindelöf iterations converge to a solution of (14). Consider

$$\begin{cases} X^{(n+1)}(t) = x + \int_0^t b(X^{(n)}(s))ds + M(t), \\ X^{(0)}(t) = x + M(t), \end{cases} \tag{15}$$

for $t \in [0,T_1]$, and $X^{(n)}(t) = 0$ for any $t \geq T_1$ and all n.

Notice that we only need to deal with $t \in [0,T_1]$. We have

$$E(|X^{(1)}(t) - X^{(0)}(t)|^2) \leq K_2 E(\int_0^t (x^2 + |M(s)|^2)ds) + K_2 \leq K_2$$

uniformly in t. For a generic n we thus have

$$E(|X^{(n+1)}(t) - X^{(n)}(t)|^2) \le K_2 \int_0^t E(|X^{(n)}(s) - X^{(n-1)}(s)|^2)ds$$
$$\le K_2^{n-1} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} E(|X^{(1)}(s) - X^{(0)}(s)|^2)ds_n \dots ds_1$$
$$\le K \frac{K_2^{n-1}}{n!} \tag{16}$$

From this we can easy prove that the Picard-Lindelöf iterations converge in $L^2(\Omega)$ on $[0, T_1]$ to a solution for equation (14). A similar argument give the uniqueness.

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Let us introduce some new notation. Fixed $m \ge 1$ and $p \ge 2$, we will say that a stochastic process $Z = \{Z(t), t \in [0,T]\}$ satisfies condition $(D_{(*,m,p)})$ if $Z(t)$ belongs to $\mathbb{D}^{m,p}$ for any $t \in [0,T]$ and

$$\sup_t E(|Z(t)|^p) \le c_{1,N,p} \quad \text{and} \quad \sup_t \sup_{u,|u|=k} E(\left|D_u^k Z(t)\right|^p) \le c_{2,N,k,p},$$

for any $k \le m$ and for some constants $c_{1,N,p}, c_{2,N,k,p}$. Notice that if Z satisfies condition $(D_{(*,m,p)})$, then lemma 2 yields that Z belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

Proposition 7. *Let $M = \{M(t), t \in [0,T]\}$ be a stochastic process satisfying condition $(D_{(*,m,p)})$. Assume that b has bounded derivatives up to order m and that $|b(0)| \le c_1$ for some constant c_1.*

Then, fixed $T_1 \le T$, the stochastic integral equation

$$X(t) = x + \int_0^t b(X(s))ds + M(t), \quad t \in [0, T_1], \tag{17}$$

$X(t) = 0$ if $t > T_1$, admits a unique solution X on $[0,T]$. Moreover the stochastic process $X = \{X(t), t \in [0,T]\}$ satisfies condition $(D_{(*,m,p)})$.

Proof. The existence of a unique solution has been proved in Lemma 6, where we have proved that the Picard-Lindelöf iterations converge to a solution of (17).

Some easy computations give us that

$$E(|X(t)|^p) \le K_p\Big(E(|\int_0^t b(X(s))ds|^p) + E(|M(t)|^p) + x^p\Big),$$

so

$$E(|X(t)|^p) \le K_p\Big(1 + \int_0^t E(|X(s)|^p)ds\Big)$$

and finally we get using a Gronwall's lemma that

$$\sup_t E(|X(t)|^p) < c_{1,N,p} < \infty.$$

In order to check that for all $k \leq m$, $X(t)$ belongs to $\mathbb{D}^{k,p}$ for all $t \in [0,T]$ and

$$\sup_t \sup_{u,|u|=k} E(\left|D_u^k X(t)\right|^p) \leq c_{2,N,k,p},$$

we will use the Picard-Lindelöf iterations defined in (15). We consider the hypothesis of induction, for $k \leq m$, $(\hat{H}_k)$:

- *(a)* for all $n \geq 0$, $X^{(n)}(t) \in \mathbb{D}^{k,p}$ for all t.
- *(b)* $D^{k-1}X^{(n)}(t)$ converges to $D^{k-1}X(t)$ in $L^p(\Omega, [0,T]^{k-1})$ when n tends to ∞.
- *(c)* $\sup_n \sup_t \sup_{u,|u|=k} E(\left|D_u^k X^{(n)}(t)\right|^p) \leq K_{p,k} < \infty$.

Notice that hypothesis $(\hat{H}_k)$ implies that $X(t) \in \mathbb{D}^{k,p}$. Observe that we only need to study the case $t \in [0,T_1]$, since when $t > T_1$ all the results are obvious.

Step 1. We prove $(\hat{H}_1)$, that is, the case $k = 1$. Since $X^{(n)}(t)$ converges, when n tends to ∞ to $X(t)$ in $L^p(\Omega)$ we know that *(b)* is true. In order to prove *(a)* and *(c)*, we will use another induction argument to check for all $n \geq 0$ the hypothesis $(\tilde{H}_n)$

- *(i)* $X^{(n)}(t) \in \mathbb{D}^{1,p}$ for all $t \in [0,T]$.
- *(ii)* $\sup_t \sup_u E(\left|D_u X^{(n)}(t)\right|^p) \leq K_{n,p,1} < \infty$.

From de definition of $X^{(0)}$ it is clear that $X^{(0)}(t) \in \mathbb{D}^{1,p}$ for all t and that for $t \in [0,T_1]$

$$D_u X^{(0)}(t) = D_u M(t).$$

So, our hypothesis of induction $(\tilde{H}_0)$ has been proved. Assume now that the hypothesis of induction $(\tilde{H}_n)$ is true. Then from the definition of $X^{(n+1)}$ it follows that $X^{(n+1)}(t) \in \mathbb{D}^{1,p}$ and for any $t \in [0,T_1]$

$$D_u X^{(n+1)}(t) = \int_0^t b'(X^{(n)}(s)) D_u X^{(n)}(s) ds + D_u M(t).$$

Moreover, using that M satisfies condition $(D_{*,m,p})$ we have

$$\begin{aligned} &\sup_u E(\left|D_u X^{(n+1)}(t)\right|^p) \\ &\quad \leq K_p \left(\int_0^t \sup_u E(\left|D_u X^{(n)}(s)\right|^p) ds + \sup_u E(|D_u M(t)|^p) \right) \\ &\quad \leq K_{n+1,p} < \infty. \end{aligned}$$

From here, $(\tilde{H}_{n+1})$ can be easily proved.

Finally, since we have the relationship

$$\sup_u E(\left|D_u X^{(n+1)}(t)\right|^p) \le K_p \int_0^t \sup_u E(\left|D_u X^{(n)}(s)\right|^p) ds + K_p,$$

iterating n times this inequality we get

$$\begin{aligned} &\sup_u E(\left|D_u X^{(n+1)}(t)\right|^p) \\ &\le (K_p)^2 \int_0^t \int_0^s \sup_u E(\left|D_u X^{(n-1)}(v)\right|^p) dv ds + K_p^2 t + K_p \\ &\le \sum_{k=0}^n (K_p)^{k+1} \frac{t^k}{k!} \\ &\le K_p \exp(K_p t). \end{aligned}$$

So we have that

$$\sup_n \sup_t \sup_u E(\left|D_u X^{(n)}(t)\right|^p) \le K_p \exp(K_p T) < \infty.$$

$(\hat{H}_1)$ has been proved.

Notice now that applying lemma 4 we have that, for all $t \in [0,T]$, $X(t) \in \mathbb{D}^{1,p}$. Moreover we can obtain that

$$D_u X(t) = \int_0^t b'(X(s)) D_u X(s) ds + D_u M(t),$$

and we easily have that

$$\sup_t \sup_u E(|D_u X(t)|^p) < \infty.$$

Step 2. Let us assume that $(\hat{H}_i)$, $i \le k \le m-1$ is true. We want to check $(\hat{H}_{k+1})$. We will prove *(a)* doing another induction over n similar to the induction done in Step 1. Let us consider, for all $n \ge 0$, the hypothesis $(\tilde{H}_n)$

: *(i)* $X^{(n)}(t) \in \mathbb{D}^{k+1,p}$ for all $t \in [0,T]$.

: *(ii)* $\sup_t \sup_{u,|u|=k+1} E(\left|D_u^{k+1} X^{(n)}(t)\right|^p) \le K_{n,p,1} < \infty$.

Since for all t, $M(t) \in \mathbb{D}^{k+1,p}$, from the definition of $X^{(0)}$ it is clear that, for all t, $X^{(0)}(t) \in \mathbb{D}^{k+1,p}$ and that for $u, |u| = k+1$

$$D_u^{k+1} X^{(0)}(t) = D_u^{k+1} M(t),$$

for any $t \in [0, T_1]$. Then $(\tilde{H}_0)$ is true. Assuming now that is true until n, from the definition of $X^{(n+1)}$ it follows that, for all t, $X^{(n+1)}(t) \in \mathbb{D}^{k+1,p}$ and for $u, |u| = k+1$

$$D_u^{k+1} X^{(n+1)}(t) = \int_0^t \Gamma_u(b; X^{(n)}(s))ds + D_u^{k+1} M(t),$$

for any $t \in [0, T_1]$. The proof of *(b)* can be obtained easily from the expressions of $D_u^k X^{(n)}(t)$ and $D_u^k X(t)$. Finally to prove *(c)* set

$$\Delta_u(b; X^{(n)}(s)) = \Gamma_u(b; X^{(n)}(s)) - b^{(k+1)}(X^{(n)}(s)) D_u^{k+1} X^{(n)}(s).$$

Notice that using by the hypothesis of induction

$$\sup_{u,|u|=k+1} \sup_s E(|\Delta_u(b; X^{(n)}(s))|^p) \leq K_p.$$

Then

$$\begin{aligned} D_u^{k+1} X^{(n+1)}(t) &= \int_0^t \Delta_u(b; X^{(n)}(s))ds \\ &\quad + D_u^{k+1} M(t) + \int_0^t b^{(k+1)}(X^{(n)}(s)) D_u^{k+1} X^{(n)}(s)ds. \end{aligned}$$

Repeating the same calculations we did in the proof of $(\hat{H}_1)$ we can finish the proof of $(\hat{H}_{k+1})$.

Notice now that applying again lemma 4 we have that, for all t, $X(t) \in \mathbb{D}^{k+1,p}$ and we easily have that

$$\sup_t \sup_{u,|u|=k+1} E(\left|D_u^{k+1} X(t)\right|^p) \leq K_p.$$

■

The following Proposition studies the behavior of the stochastic integral.

Proposition 8. *Let $Y = \{Y(t), t \in [0,T]\}$ be a stochastic process satisfying condition $(D_{*,m+1,p})$. Then the stochastic Stratonovich integral*

$$M(t) := \int_0^t Y(s)dB(s), \quad t \in [0,T], \tag{18}$$

is well defined and the stochastic process $M = \{M(t), t \in [0,T]\}$ satisfies condition $(D_{*,m,p})$.

Proof. Clearly, Y is Stratonovich integrable, but the Stratonovich integral and the divergence operator do not coincide and we have

$$\int_0^t Y(s)dB(s) = \delta(Y\mathbf{1}_{[0,t]}) + \alpha_H \int_0^t \left(\int_0^T D_v Y(s) |s-v|^{2H-2} dv \right) ds.$$

Then for instance

$$E(|M(t)|^p) = E\left(\left|\int_0^t Y(s)dB(s)\right|^p\right) \tag{19}$$

$$\leq K_p E\left(\left|\delta(Y\mathbf{1}_{[0,t]})\right|^p\right) + K_p E\left(\left|\alpha_H \int_0^t \left(\int_0^T D_v Y(s)|s-v|^{2H-2}dv\right) ds\right|^p\right)$$

$$\leq K_p \|Y\|_{p,1}^p + K_p c_H \sup_{s,v} |D_v Y(s)|^p \left|\int_0^T \left(\int_0^T |s-v|^{2H-2}dv\right) ds\right|^p$$

$$\leq K_p < \infty.$$

So we have that

$$\sup_t E\left(|M(t)|^p\right) \leq K_p.$$

On the other hand, using an induction argument it is easy to check that for any $k \leq m$, $M(t) \in \mathbb{D}^{k,p}$ for any t and

$$D^k_{(u_1,\ldots,u_k)} M(t) = \sum_{i=1}^k D^{k-1}_{(u_1,\ldots,\hat{u}_i,\ldots,u_k)} Y(u_i)\mathbf{1}_{[0,t]}(u_i) + \delta(D^k_u Y\mathbf{1}_{[0,t]}),$$

where $(u_1,\ldots,\hat{u}_i,\ldots,u_k)$ denotes the point u without the component u_i.

Repeating the argument we have used in (19) we obtain that for all $k \leq m$

$$\sup_t \sup_{u,|u|=k} E(\left|D^k_u M(t)\right|^p) \leq K_p.$$

■

Proof of Theorem. To prove that equation (13) admits a unique solution on $[0,T]$, we shall first prove the result on $t \in [0,r]$. Then by induction, we shall prove that if equation (13) admits a unique solution on $[0,Nr]$, then we can extend this solution to the interval $[0,(N+1)r]$ and that this extension is unique.

Actually our hypothesis of induction, for $N \leq \bar{M}$, is the following:

(H_N) The equation

$$X(t) = \phi(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s-r))dB(s), \qquad t \in [0,Nr],$$

$X(t) = 0$ *if* $t > Nr$, *has an unique solution. Moreover for all* $p \geq 2$, $X(t)$ *satisfies condition* $(D_{(*,\bar{M}-N,p)})$.

Notice that for simplicity we omit the dependence on N of the solution X. Notice also that in each step we loose one degree of regularity.

Check (H_1). Let $t \in [0, r]$; equation (13) can be written in the following easy form

$$X(t) = \phi(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(\phi(s-r))dB(s). \tag{20}$$

Let us define the process

$$M(t) := \int_0^t \sigma(\phi(s-r))\mathbf{1}_{\{t<r\}}dB(s),$$

for $t \in [0, T]$. Since ϕ is deterministic, it is immediate to see that $\sigma(\phi(\cdot - r)) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ and that $D\sigma(\phi(s-r)) = 0$. For this reason the stochastic integral in (20) is well defined and coincides with the divergence operator, due to (12). Again since ϕ is a deterministic continuous function we have that for all $k \geq 1, p \geq 2$, $M(t) \in \mathbb{D}^{k,p}$ and

$$D_u M(t) = \sigma(\phi(u-r))\mathbf{1}_{\{u<t<r\}},$$

and

$$D^k M(t) = 0, \quad \text{when } k \geq 2.$$

Then, for all $k \geq 1, p \geq 2$ we have that

$$\sup_t E(|M(t)|^p) \leq \|\sigma(\phi(.-r))\|_{p,1}^p \leq c_{3,1,p}$$

and

$$\sup_t \sup_{u,|u|=k} E(\left|D_u^k M(t)\right|^p) \leq c_{4,1,k,p}, \tag{21}$$

for some constants $c_{3,1,p}, c_{4,1,k,p}$. So, we have proved that M satisfies condition $(D_{(*,k,p)})$ for any $k \geq 1$.

From Proposition 7 we have that there exist a unique solution X and that this solution satisfies condition $(D_{(*,\bar{M}-1,p)})$.

Induction Assume that (H_N) is true until N with $N < \bar{M}$. We want to check (H_{N+1}).

Consider the stochastic process $\{Z(t), t \in [0, T]\}$ defined as

$$Z(t) = \begin{cases} \varphi(t-r), & \text{si} \quad t \leq r, \\ X(t-r), & \text{si} \quad r < t \leq (N+1)r, \\ 0, & \text{si} \quad t > (N+1)r, \end{cases}$$

where here X is the solution obtained in (H_N). Set now $Y(t) = \sigma(Z(t))$.

Then our problem became for $t \in [0, (N+1)r]$

$$X(t) = \phi(0) + \int_0^t b(X(s))ds + \int_0^t Y(s)dB(s). \tag{22}$$

Let us define the process

$$M(t) := \int_0^t Y(s)\mathbf{1}_{\{t<(N+1)r\}}dB(s), \quad t \in [0,T].$$

and prove that the Stratonovich integral is well defined. To this aim, we shall need to prove that (as pointed out in the previous chapter):

(1) $Y \in \mathbb{D}^{1,2}(|\mathcal{H}|\,)$;

(2) $\int_0^T \int_0^T |D_u Y(s)||s-u|^{2H-2}dsdu < \infty$

These two conditions can be obtained from Lemma 3 and the following facts: σ is a bounded function with bounded derivatives, Z is in $\mathbb{D}^{1,2}(|\mathcal{H}|)$, $\sup_t \sup_u E(|D_u Z(t)|^p) \leq c_{2,1,p}$, and

$$D_u Y(t) = \sigma'(Z(t))D_u Z(t).$$

On the other hand, using that the stochastic process Z satisfies condition $(D_{(*,\bar{M}-N,p)})$ and that σ has derivatives up to order $\bar{M}$, it is clear that for any $t \in [0,T]$, $Y(t) \in \mathbb{D}^{k,p}$ for all $k \leq \bar{M} - N$ and

$$D_u Y(t) = \sigma'(Z(t))D_u Z(t)$$

and still more generally, for $k \leq \bar{M} - N$

$$D_u^k Y(t) = \Gamma_u(\sigma, Z(t)).$$

Furthermore, Y will also satisfy condition $(D_{(*,\bar{M}-N,p)})$.

Applying Proposition 8 we get that M satisfies condition $(D_{(*,\bar{M}-N-1,p)})$. Finally, using Proposition 7, as we did in Step 1, we finish the proof of this Theorem.

Remark 9. *Notice that in the proof of the existence and uniqueness of solution to our stochastic differential equation we need to study the Malliavin derivatives until order $\bar{M}$ of our solution, in order to define the stochastic integrals appearing in our Picard iterations. To do this study, we assume that the coefficients σ and b have bounded derivatives up to order $\bar{M}$, that are not the usual assumptions in this case when the stochastic differential equation is driven by a standard Brownian motion. As a by-product of our method, we can prove easily that if the coefficients σ and b have bounded derivatives of any order, the solution $X(t)$ belongs to $\mathbb{D}^{\infty}$. Moreover assuming the non-degeneracy on σ we can also have the smoothness of the density.*

4. Regularity of the density

Let us consider now another set of hypotheses $(\hat{\mathrm{H}})$.

Hypotheses $(\hat{\mathrm{H}})$: b and σ are real functions with bounded derivatives of any order. Moreover, σ is bounded and $|b(0)| \leq c_1$ for some constant c_1.

Theorem 10. *Assume Hypotheses* ($\hat{\mathrm{H}}$). *Then if there exists a positive constant* c_0 *such that* $|\sigma(x)| > c_0$ *for all* x, *for any* $t \in [0,T]$ *the solution of the SDDE (13)* $X(t)$ *has an infinitely differentiable density with respect to Lebesgue's measure on* $\mathbb{R}$.

Proof. Fixed $t \in [0,T]$ and using the Malliavin's criterion for the existence of a smooth density we have to check two things:

(1) $X(t) \in \mathbb{D}^\infty$,
(2) $(\int_0^T |D_u X(t)|^2 du)^{-1} \in \bigcap_{p\geq 1} L^p(\Omega)$.

Following the same steps of Theorem 5, we can prove that for any $t \in [0,T]$, $X(t) \in \mathbb{D}^\infty$. In order to prove the second condition it is enough to check that for any $p \geq 1$ there exists $\varepsilon_0 > 0$ such that

$$P\Big(\int_0^T |D_u X(t)|^2 du \leq \varepsilon \Big) \leq \varepsilon^p,$$

for all $\varepsilon \leq \varepsilon_0$.

From equation (13) we can write

$$\begin{aligned} X(t) &= \phi(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s-r))\delta B(s) \\ &\quad + \alpha_H \int_0^t \Big(\int_0^T D_v \sigma(X(s-r))|s-v|^{2H-2} dv \Big) ds. \end{aligned}$$

So, for any $u \leq t - r$ we obtain

$$\begin{aligned} D_u X(t) &= \int_u^t b'(X(s)) D_u X(s) ds + \sigma(X(u-r)) \\ &\quad + \int_{u+r}^t \sigma'(X(s-r)) D_u X(s-r) \delta B(s) \\ &\quad + \alpha_H \int_{u+r}^t \Big(\int_0^{s-r} D_u D_v \sigma(X(s-r))|s-v|^{2H-2} dv \Big) ds, \end{aligned}$$

and when $u \in (t-r, t)$ we have

$$D_u X(t) = \int_u^t b'(X(s)) D_u X(s) ds + \sigma(X(u-r)).$$

Then, using a Gronwall's inequality we have that

$$E\Big(\sup_{u\in(t-r,t), s\in(t-r,t)} |D_u X(s)|^q \Big) \leq K_q.$$

On the other hand, we can write

$$\begin{aligned} &P\big(\int_0^T |D_uX(t)|^2 du \le \varepsilon\big) \\ &\le P\big(\int_{t-\varepsilon^\alpha}^t |\int_u^t b'(X(s))D_uX(s)ds + \sigma(X(u-r))|^2 du \le \varepsilon\big) \\ &\le p_{1,\varepsilon} + p_{2,\varepsilon}, \end{aligned}$$

with

$$\begin{aligned} p_{1,\varepsilon} &= P\big(\int_{t-\varepsilon^\alpha}^t |\int_u^t b'(X(s))D_uX(s)ds + \sigma(X(u-r))|^2 du \le \varepsilon, \\ &\qquad \sup_{u\in(t-\varepsilon^\alpha,t)} \int_{t-\varepsilon^\alpha}^t |b'(X(s))D_uX(s)|ds \le \varepsilon^\beta\big) \\ p_{2,\varepsilon} &= P\big(\sup_{u\in(t-\varepsilon^\alpha,t)} \int_{t-\varepsilon^\alpha}^t |b'(X(s))D_uX(s)|ds > \varepsilon^\beta\big). \end{aligned}$$

Since $|\sigma(x)| > c_0$ for all x, when $\alpha < 1$ we clearly have that $p_{1,\varepsilon} = 0$. On the other hand, using Chebyshev's inequality, for any $q > 1$

$$\begin{aligned} p_{2,\varepsilon} &\le \frac{1}{\varepsilon^{\beta q}} E\big(\sup_{u\in(t-\varepsilon^\alpha,t)} |\int_{t-\varepsilon^\alpha}^t |b'(X(s))D_uX(s)|ds|^q\big) \\ &\le \varepsilon^{(\alpha-\beta)q} K E\big(\sup_{u\in(t-\varepsilon^\alpha,t), s\in(t-\varepsilon^\alpha,t)} |D_uX(s)|^q\big). \end{aligned}$$

So, choosing $\beta < \alpha < 1$ the proof is complete.

■

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