COHOMOLOGY OF CLASSIFYING SPACES OF CENTRAL QUOTIENTS OF RANK TWO KAC-MOODY GROUPS

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The goal of this paper is to compute the mod p cohomology algebra —including the action of the Steenrod algebra and the Bockstein spectral sequence— of the classifying spaces of the quotients of (non-afine) Kac-Moody groups of rank two by finite central p-groups.

The wise have taught us (cf. *Ithaca* in [6]) that in the trips which are really worth doing what we see and learn along the trip always turns out to become more important than the final destination. We think that the present work might be an example of this. The rank two Kac-Moody groups have a large family of central subgroups which yield a rather complex series of interesting unstable algebras over the Steenrod algebra, and when we started computing these cohomology algebras we learned that in order to study them in a systematic way we needed to relate them to representation theory and to invariant theory. In this context, we believe than the relationship between rank two Kac-Moody groups, representations of the infinite dihedral group, invariant theory of pseudoreflection groups and cohomology algebras that we display here is more interesting than the particular values of each cohomology algebra.

Although our starting point is to compute the cohomology of the spaces $B(K/F)$, for K a rank two Kac-Moody group and F a central subgroup (we will be more precise after this introduction), we soon realize that we should better compute the cohomology of a larger family of spaces —the family which we call S^*— which can be viewed as a homotopy theoretic generalization (and also a p-adic completion) of the spaces $B(K/F)$. The spaces in the set S^* can be parametrized by matrices in $GL_2(\widehat{\mathbb{Z}}_p)$ plus some extra data. A better parametrization of S^* is given by a set called R^* whose elements are faithful representations of the infinite dihedral group D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$ plus some obstruction classes, modulo the action of the outer automorphisms of D_∞. Hence, our results depend on the integral p-adic representation theory of the infinite dihedral group. This theory has been developed in [2] and we would like to point out that the work in [2] was

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motivated by the present paper and, moreover, the cohomological results in the present paper helped us in shaping the results in [2].

We want to mention also that this work is part of a more general project which, roughly speaking, aims to investigate the homotopy theory of the Kac-Moody groups with the tools that have lead to the development of the homotopy theory of compact Lie groups (see, for instance, the surveys [8] and [13]). We would like to point out the papers [5] and [1] as further examples of how some of the homotopy theoretical results and techniques of compact Lie groups can be extended, with some appropriate reformulation, to Kac-Moody groups. This paper generated also our motivation for the work in [3].

The present paper is organized as follows. In section 1 we recollect the notation on Kac-Moody groups and their central quotients that we use later. In section 2 we introduce the colimit decomposition of $B(K/F)$ that we will use to compute the cohomology and we define the family of spaces S^* which contains all spaces $B(K/F)$. In section 3 we relate the spaces in S^* to representations of D_∞ and we introduce a set R^* of representation data with obstruction classes which parametrizes S^*. Moreover, we review the results on representations of D_∞ which we proved in [2], in a form which is more appropriate to our needs. Then, in the next three sections, we compute, for each data element of R^*, the mod p cohomology of the space in S^* corresponding to this data. The first of these three sections gives some generic information and the two other sections consider the case of the prime two and the case of the odd primes, respectively. We use the invariant theory of finite reflection groups as developed in [14]. In the final section, we return to Kac-Moody groups and we compute, for each quotient K/F, the representation data in R^* which gives $B(K/F)$ and we have in this way enough information to compute the mod p cohomology algebras, the Steenrod algebra actions and the Bockstein spectral sequences of all spaces, of the form $B(K/F)$, but a few cases.

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1. Rank two Kac-Moody groups and central quotients

We choose positive integers a, b such that $ab > 4$. Along this paper K will always denote the unitary form of the Kac-Moody group associated to the generalized Cartan matrix

$$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

Sometimes we write $K(a,b)$ instead of K when we want to make explicit the values of a and b used to construct K. The integers a and b can be interchanged, since the group associated to (a,b) is isomorphic to the group associated to (b,a). The case $ab<4$ gives rise to compact Lie groups while the case $ab=4$ is called the affine case and will be left aside. These infinite dimensional topological groups and their classifying spaces BK have been studied from a homotopical point of view in several works, like [10], [9], [11], [12], [5], [1]. We recall here some properties of K and BK which we will use along this work and which can be found in the references that we have just mentioned.

By construction, K comes with a standard maximal torus of rank two T_K which is a maximal connected abelian subgroup of K. Any two such subgroups are conjugated. The Weyl group W of K is an infinite dihedral group acting on the Lie algebra of T_K through reflections ω_1 and ω_2 given, in the standard basis, by the integral matrices:

$$w_1=\begin{pmatrix}-1 & b\\ 0 & 1\end{pmatrix},\quad w_2=\begin{pmatrix}1 & 0\\ a & -1\end{pmatrix}.$$

The matrices of determinant $+1$ in W form a subgroup W^+ of index two which is infinite cyclic generated by $\omega_1\omega_2$.

The cohomology of K and BK was computed by Kitchloo ([12]) using, among other tools, the existence of a Schubert calculus for the homogeneous space K/T_K.

The center of K is also well understood ([10]):

$$ZK=\begin{cases}2\mathbb{Z}/(ab-4)\times\mathbb{Z}/2, & a\equiv b\equiv 0 \pmod 2\\ \mathbb{Z}/(ab-4), & \text{otherwise.}\end{cases}$$

Hence, we have a family of central p-subgroups F of K and the purpose of this paper is to study the spaces $B(K/F)$.

Let us fix now the notation that we will use in this paper to refer to the various quotients of K by central subgroups. We denote by ν_p the p-adic valuation.

- If p is an odd prime or $p=2$ and a or b is odd then there is a unique central subgroup F of K of order p^m for any $0\le m\le \nu_p(ab-4)$. We denote K/F by $P_{p^m}K$.
- If $p=2$ and a and b are both even then the 2-primary part of the center of K is non-cyclic of the form $\mathbb{Z}/2^t\times\mathbb{Z}/2$, $t=\nu_2((ab-4)/2)$. There are several quotient groups. We denote by P_2^RK the quotient of K by the *right* subgroup of the center of order two. We denote by $P^L_{2^m}K$, $0\le m\le t$ the quotient of K by the *left* subgroup of the center of order 2^m. We denote by $P^D_{2^m}K$, $0<m\le t$ the quotient of

K by the *diagonal* subgroup of the center of order 2^m. Finally, we denote by $P^N_{2^{m+1}}K$, $0 < m \leq t$ the quotient of K by the *non-cyclic* subgroup of the center of order 2^{m+1}.

2. Colimit decompositions of BK and $B(K/F)$ and the spaces in S^*

A fundamental result in the homotopy theory of the classifying spaces of Kac-Moody groups is the following (cf. [12]). If L is any Kac-Moody group with *infinite* Weyl group and $\{P_I\}$ are the parabolic subgroups of L indexed by proper subsets I of $\{1, \ldots, \mathrm{rank}(L)\}$ then there is a homotopy equivalence

$$BL \simeq \operatorname*{hocolim}_{I} BP_I.$$

Let us give a more precise description of this homotopy colimit in the case of the rank two group $K = K(a, b)$.

There are group homomorphisms $\varphi_i \colon SU(2) \to K$, $i = 1, 2$, such that the images of φ_1 and φ_2 generate K. If D is the unit disc in $\mathbb{C}$ and we write

$$z_i(u) = \varphi_i \begin{pmatrix} u & (1 - ||u||^2)^{1/2} \\ -(1 - ||u||^2)^{1/2} & \bar{u} \end{pmatrix}$$

then K has a presentation with generators $\{z_i(u) \,|\, u \in D,\, i = 1, 2\}$ and relations

(i) $z_i(u)z_i(v) = z_i(uv)$ if $u, v \in S^1$.
(ii) $z_i(u)z_i(-\bar{u}) = z_i(-1)$ if $u \in D\backslash S^1$.
(iii) $z_i(u)z_i(v) = z_i(u')z_i(v')$ if $u, v \in D\backslash S^1$, $u \neq v$, for some unique $u' \in D\backslash S^1$ and $v' \in S^1$.
(iv) $z_i(u)z_j(v)z_i(u)^{-1} = z_j(u^{a_{ij}}v)z_j(u^{-a_{ij}})$ if $u \in S^1$, $v \in D$ and (a_{ij}) is the Cartan matrix of K.

Then BK is a homotopy push out

$$\begin{array}{ccc} BT_K & \longrightarrow & BH_1 \\ \downarrow & & \downarrow \\ BH_2 & \longrightarrow & BK \end{array}$$

where H_1 and H_2 are rang two compact Lie groups which contain the maximal torus T_K and can be described from the generators $z_i(u)$ in the following way: H_i is generated by $z_i(u)$ for $u \in D$ and $z_j(\lambda)$ for $j \neq i$ and $\lambda \in S^1$. Hence, these groups are split extensions

$$H_i = SU(2) \rtimes S^1$$

and the action of S^1 on $SU(2)$ can be read from relation (iv) above and it turns out to be

$$\lambda \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & \lambda^{-c} y \\ \lambda^{c} z & t \end{pmatrix} \tag{1}$$

where $c = b$ for H_1 and $c = a$ for H_2. The Weyl group of both H_1 and H_2 is of order two. Depending on the parity of the integer c, this group is isomorphic to either $SU(2) \times S^1$ or $U(2)$:

Proposition 2.1. *Let H be the split extension of S^1 by $SU(2)$ with action given by (1). Then*

$$H \cong \begin{cases} S^1 \times SU(2), & c \text{ even} \\ U(2), & c \text{ odd.} \end{cases}$$

Proof. If $c = 2c'$ then consider the isomorphism

$$\begin{aligned} H = SU(2) \rtimes S^1 &\xrightarrow{\psi} S^1 \times SU(2) \\ (A, \tau) &\longmapsto (\tau, A\tau^{-c'}) \end{aligned}$$

If $c = 2c' + 1$ then consider the isomorphism

$$\begin{aligned} H = SU(2) \rtimes S^1 &\xrightarrow{\psi} U(2) \\ (A, \tau) &\longmapsto A \begin{pmatrix} \tau^{-c'} & 0 \\ 0 & \tau^{c'+1} \end{pmatrix}. \end{aligned}$$ □

This proposition gives concrete descriptions of the push out diagrams for BK for any value of a and b. We have:

- $a \equiv b \equiv 0 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{ BS^1 \times BSU(2) \xleftarrow{\begin{pmatrix} 1 & 0 \\ -\frac{a}{2} & 1 \end{pmatrix}} BT_K \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & -\frac{b}{2} \end{pmatrix}} BS^1 \times BSU(2) \right\}.$$

- $a \equiv b \equiv 1 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{ BU(2) \xleftarrow{\begin{pmatrix} \frac{1-a}{2} & 1 \\ \frac{1+a}{2} & -1 \end{pmatrix}} BT_K \xrightarrow{\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2} \end{pmatrix}} BU(2) \right\}.$$

- $a \equiv 0$, $b \equiv 1 \pmod 2$. Then

$$BK \simeq \operatorname{hocolim}\left\{ BS^1 \times BSU(2) \xleftarrow{\begin{pmatrix} 1 & 0 \\ -\frac{a}{2} & 1 \end{pmatrix}} BT_K \xrightarrow{\begin{pmatrix} 1 & \frac{1-b}{2} \\ -1 & \frac{1+b}{2} \end{pmatrix}} BU(2) \right\}.$$

Here each matrix M written above an arrow means a map $B(i \circ \rho)$ where $i\colon T_K \hookrightarrow K$ is the inclusion and $\rho\colon T_K \to T_K$ is the homomorphism inducing M on the Lie algebra level.

If we want to work one prime at a time (and we will want to do so) then we can complete the above push out diagrams and obtain $BK_p^\wedge$ as the p-completion of a push out of the form $(BH)_p^\wedge \leftarrow BT_p^\wedge \to (BH')_p^\wedge$.

Notice that the distinction between the three different types of diagrams above is only important at the prime two, since $BU(2)$ and $BS^1 \times BSU(2)$ are homotopy equivalent at any odd prime. Moveover, If N denotes the normalizer of T_K in K then the natural map $BN \to BK$ is a mod p homotopy equivalence for any odd prime p (see [12]).

Notice also that the classifying spaces of the central quotients of K have also a colimit decomposition of this same form:

$$B(K/F) \simeq \operatorname{hocolim}\{B(H/F) \leftarrow B(T_K/F) \to B(H'/F)\}\,.$$

The above considerations suggest considering the family S of *all* spaces X which can be constructed out of two rank two compact Lie groups H, H' with Weyl group of order two, as a push out $BH \leftarrow BT \to BH'$. We introduce the following definition:

Definition 2.2. Choose a prime p (which will be omitted from the notation) and choose a matrix $M \in GL_2(\widehat{\mathbb{Z}}_p)$. Let T be a torus of rank two. Then:

(1) If p is odd, define $X\,(M)$ as the p-completion of the push out

$$(BS^1 \times BSU(2))_p^\wedge \xleftarrow{M} (BT)_p^\wedge \xrightarrow{\text{id}} (BS^1 \times BSU(2))_p^\wedge\,.$$

(2) If $p=2$, define $X^{k,l}\,(M)$, $k,l \in \{0,1,2\}$ as the 2-completion of the push out

$$(BH_k)_2^\wedge \xleftarrow{M} (BT)_2^\wedge \xrightarrow{\text{id}} (BH_l)_2^\wedge\,,$$

where $H_0 = S^1 \times SU(2)$, $H_1 = S^1 \times SO(3)$ and $H_2 = U(2)$.

The map called id is induced by the inclusion of the standard maximal torus and the map called M is induced by the self equivalence of $BT_p^\wedge$ given by M (in the standard basis) followed by the inclusion of the standard maximal torus.

Notice that a space obtained from a push out of the form

$$(BH)_p^\wedge \xleftarrow{M} (BT_K)_p^\wedge \xrightarrow{N} (BH')_p^\wedge$$

fits also in the above definition as $X^{k,l}\left(MN^{-1}\right)$ for some k, l. Hence, the set S of all (homotopy types of) spaces of the form $X^{k,l}\,(M)$ contains all spaces that we want to study in this paper. Notice also that if we turn around the diagram used to define $X^{k,l}(M)$ we get the diagram for $X^{l,k}\left(M^{-1}\right)$ and so both spaces are homotopy equivalent. Hence, we can assume $k \leq l$ without loss of generality.

Hence, we have enlarged the set of spaces that we are going to consider in this paper in a way that we obtain a more general framework which will allow us a more systematic study of the classifying spaces of the central quotients of the rank two Kac-Moody groups. This larger family of spaces S is parametrized by a set M of diagrams $\{M;\{k,l\}\}$, in the following way. We define M as the set of diagrams

$$(BH_k)^\wedge_p \xleftarrow{M} (BT)^\wedge_p \xrightarrow{\text{id}} (BH_l)^\wedge_p$$

with $M \in GL_2(\widehat{\mathbb{Z}}_p)$ and $k \leq l$ in I, where $I = \{0\}$ for $p > 2$ and $I = \{0,1,2\}$ for $p = 2$. The assignment $\{M;\{k,l\}\} \mapsto X^{k,l}(M)$ gives a surjection $\mathsf{M} \twoheadrightarrow \mathsf{S}$.

We can associate to each element $\{M;\{k,l\}\} \in \mathsf{M}$ a subgroup $W < GL_2(\widehat{\mathbb{Z}}_p)$ in the following way. W is the subgroup generated by the matrices A_l and $M^{-1}A_kM$ with

$$A_0 = A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This group W is a quotient of the infinite dihedral group and so it is either finite or infinite dihedral. We denote by M^* the set of elements in M whose corresponding group is infinite dihedral, and we denote by S^* the image of M^* in S. Notice that the spaces $(BK/F)^\wedge_p$ that we want to investigate in this paper are in S^*. The surjection

$$\Phi\colon \mathsf{M}^* \longrightarrow\!\!\!\!\!\rightarrow \mathsf{S}^*$$

gives a parametrization of the set S^* in terms of p-adic matrices (with some indices, if the prime is 2). In the next section we present another parametrization of S^* which will be more appropriate for our research of the cohomology of the spaces in S^*.

3. Spaces in S^* and representations of D_∞

In this section we give another parametrization of the set S^* which is related to the integral p-adic representation theory of the infinite dihedral group and which will allow us to give a more intrinsic view of the spaces $X^{k,l}(M)$ and will be useful to organize the cohomological computations of the following sections of this paper.

We denote by R^* the set of equivalence classes of pairs (ρ, c) where ρ is a faithful representation of the infinite dihedral group D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$ and c is an *obstruction class* in $H^3(D_\infty; L)$, for L a 2-dimensional $\widehat{\mathbb{Z}}_p$-lattice with the action of D_∞ induced by ρ. The equivalence classes are taken with respect to the natural action of $\mathrm{Out}(D_\infty)$ on the set of representations and on $H^*(D_\infty; L)$.

Remark 3.1. From a different point of view, we could denote by $\mathsf{R} = \mathsf{Ext}(D_\infty, (T^2)^\wedge_p)$ the set of equivalence classes of fibrations

$$(BT^2)^\wedge_p \longrightarrow BN \longrightarrow BD_\infty\,,$$

where two fibrations are equivalent if there are homotopy equivalences f', f, and f'' such that the diagram

$$\begin{array}{ccccc} (BT^2)^\wedge_p & \longrightarrow & BN_1 & \longrightarrow & BD_\infty \\ \simeq \big\downarrow f' & & \simeq \big\downarrow f & & \simeq \big\downarrow f'' \\ (BT^2)^\wedge_p & \longrightarrow & BN_2 & \longrightarrow & BD_\infty \end{array}$$

is homotopy commutative. Such a fibration is known to be determined by a homomorphism $\rho\colon D_\infty \longrightarrow GL_2(\widehat{\mathbb{Z}}_p)$, and an extension class $c \in H^3(D_\infty; L)$, for L the 2-dimensional $\widehat{\mathbb{Z}}_p$-lattice with the given action of D_∞. Moreover, two fibrations determined by (ρ_1, c_1) and (ρ_2, c_2) respectively are equivalent if and only if ρ_1 and ρ_2 are conjugated and $c_1 = c_2$, up to the induced action of $\mathrm{Out}(D_\infty)$.

In case of Kac-Moody groups, BN would be a fibrewise p-completion of the maximal torus normalizer. In the general case it might be understood as a homotopy theoretic maximal torus normalizer. However, we not not plan here to go deeper into these considerations.

We are interested in the case where the representation ρ is faithful. Consistently with our notation in the previous section for S^* and M^*, we have thus chosen to denote by R^* the subset of equivalence classes of R determined by pairs (ρ, c) where ρ is a faithful representation of D_∞.

Recall that D_∞ is generated by two involutions and the only non trivial outer automorphism of D_∞ permutes these two involutions. A representation ρ of D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$ is given by two matrices of order two, R_1, R_2 and, according to [2], for $p > 2$ these matrices are conjugated to $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, while for $p = 2$ they are conjugated to either A_1 or $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We write $\rho \in \mathrm{Rep}_{i,j}$, $i \leq j$, to indicate that ρ is defined by two involutions conjugated to A_i and A_j, respectively. Notice that $\mathrm{Out}(D_\infty)$ identifies $\mathrm{Rep}_{i,j}$ and $\mathrm{Rep}_{j,i}$. Since $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$, we see that $H^3(W; L) \cong H^3(\mathbb{Z}/2; L) \oplus H^3(\mathbb{Z}/2; L)$ and this vanishes if $p > 2$. Hence, the obstruction class c is irrelevant in the the odd prime case and R^* is just the set $\mathrm{fRep}(D_\infty)$ of faithful representations of D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$, modulo the action of the outer automorphisms of D_∞. This set is equivalent to the set of all conjugacy classes of infinite dihedral subgroups of $GL_2(\widehat{\mathbb{Z}}_p)$.

For $p = 2$, an elementary computation shows that

$$H^3(D_\infty; L) \cong \begin{cases} 0, & \rho \in \mathrm{Rep}_{2,2} \\ \mathbb{Z}/2, & \rho \in \mathrm{Rep}_{1,2} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & \rho \in \mathrm{Rep}_{1,1}\,. \end{cases}$$

$\mathrm{Out}(D_\infty)$ acts as the identity if $\rho \in \mathrm{Rep}_{2,2} \coprod \mathrm{Rep}_{1,2}$ and permutes the two summands if $\rho \in \mathrm{Rep}_{1,1}$. We can summarize the structure of R^* in this way:

- If p is odd then the elements of R^* are the conjugacy classes of infinite dihedral subgroups $W < GL_2(\widehat{\mathbb{Z}}_p)$, i.e., $\mathsf{R}^* = \mathrm{fRep}(D_\infty)/\mathrm{Out}(D_\infty)$.
- If p is even then R^* splits as a disjoint union of four sets:
  - The set of all (ρ, c) with $\rho \in \mathrm{fRep}_{2,2} \,/\, \mathrm{Out}(D_\infty)$ and $c = 0$.
  - The set of all (ρ, c) with $\rho \in \mathrm{fRep}_{1,2}$ and $c = 0$ or $c = 1$.
  - The set of all (ρ, c) with $\rho \in \mathrm{fRep}_{1,1} \,/\, \mathrm{Out}(D_\infty)$ and $c = 0$ or $c = (1, 1)$.
  - The set of all (ρ, c) with $\rho \in \mathrm{fRep}_{1,1}$ and $c = (0, 1)$.

There is a surjection $\Psi\colon \mathsf{M}^* \twoheadrightarrow \mathsf{R}^*$ which assigns to $\{M; \{k, l\}\}$ the representation $\rho\colon D_\infty \to GL_2(\widehat{\mathbb{Z}}_p)$ given by the matrices $R_1 = A_l$, $R_2 = M^{-1} A_k M$ and the obstruction class given by

M^*	R^*	
$M; \{0, 0\}$	$[M] \in \mathrm{Rep}_{1,1}$	$c = (1, 1)$
$M; \{0, 1\}$	$[M] \in \mathrm{Rep}_{1,1}$	$c = (0, 1)$
$M; \{1, 1\}$	$[M] \in \mathrm{Rep}_{1,1}$	$c = 0$
$M; \{0, 2\}$	$[M] \in \mathrm{Rep}_{1,2}$	$c = 1$
$M; \{1, 2\}$	$[M] \in \mathrm{Rep}_{1,2}$	$c = 0$
$M; \{2, 2\}$	$[M] \in \mathrm{Rep}_{2,2}$	$c = 0$

Remark 3.2. This assignment of an obstruction class is done in a way which reflects the non-splitting of the maximal torus normalizer of $SU(2)$ in contrast to the splitting of the maximal torus normalizer of $SO(3)$.

Then, we have a diagram

$$\begin{array}{ccc} \mathsf{M}^* & \xrightarrow{\Phi} & \mathsf{S}^* \\ {\scriptstyle\Psi}\downarrow & & \\ \mathsf{R}^* & & \end{array}$$

and we claim now that Φ factors through R^*, up to homotopy. This is an easy consequence of the following result.

Proposition 3.3. *Let $\mathcal{D}$ be the subgroup of diagonal matrices in $GL_2(\widehat{\mathbb{Z}}_p)$ and let $\mathcal{Y}$ be the subgroup of matrices $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right) \in GL_2(\widehat{\mathbb{Z}}_p)$. Then:*

1. *$\mathcal{D}$ is the centralizer of $A_0 = A_1$ and $\mathcal{Y}$ is the centralizer of A_2.*
2. *$X^{k,l}(M) \simeq X^{l,k}\left(M^{-1}\right)$.*
3. *$X^{k,l}(M) \simeq X^{k,l}(AMB)$ for any A in the centralizer of A_j and B in the centralizer of A_i.*

Proof. The proof of part 1 is straightforward and part 2 has already been discussed in the preceding section. To prove part 3, notice that the corollary 3.5 in [7] shows that any diagonal matrix lifts to self equivalences of $(BS^1 \times BSU(2))^\wedge_p$ and $(BS^1 \times BSO(3))^\wedge_p$ while any matrix in $\mathcal{Y}$ lifts to a self equivalence of $BU(2)^\wedge_p$. Using this, it is not difficult to construct an equivalence between the diagram for $X^{k,l}(M)$ and the diagram for $X^{k,l}(AMB)$. □

We have thus obtained a surjection (up to homotopy)

$$\bar{\Phi}\colon \mathsf{R}^* \longrightarrow\!\!\!\!\!\rightarrow \mathsf{S}^*$$

which allows us to parametrize the homotopy types of spaces in S^* by (faithful integral p-adic) representations of D_∞, plus (for $p = 2$) some obstruction classes.

Remark 3.4. An interesting question which arises naturally in this context is whether the map $\bar{\Phi}$ is injective. The injectivity of $\bar{\Phi}$ could be interpreted as saying that the spaces in S^* and, in particular, the completions of the classifying spaces of central quotients of rank two Kac-Moody groups, are *determined by the normalizer of a maximal torus.* We are not going to consider these type of problems here.

Hence, the study of the homotopy type of the classifying spaces of the central quotients of rank two Kac-Moody groups has led us, in a natural way, to the representation theory of D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$. This theory is developed in [2] in a purely algebraic way and thus a complete description of these representations is available. We reproduce here the results of [2] with a slightly different notation which is more convenient for the applications in the next sections of this paper.

First of all, the set $\mathrm{Rep}(D_\infty)$ of representations of D_∞ in $GL_2(\widehat{\mathbb{Z}}_p)$ splits as a disjoint union

$$\mathrm{Rep}(D_\infty) = \coprod_{i,j\in\mathcal{I}} \mathrm{Rep}_{i,j}$$

where the index set $\mathcal{I}$ is $\{0,1,2,3\}$ for $p=2$ and $\{0,1,3\}$ for $p>2$. $\mathrm{Rep}_{i,j}$ contains all representations such that the two generating involutions r_1, r_2 of D_∞ are conjugated to R_i and R_j, respectively, where

$$R_0 = I, \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_3 = -I.$$

Given a matrix $M \in GL_2(\widehat{\mathbb{Z}}_p)$ and given representations σ_i, σ_j of $\mathbb{Z}/2$ given by matrices R_i, R_j, we can consider the representation $\rho \in \mathrm{Rep}_{i,j}$ given by $\rho(r_1) = R_i$ and $\rho(r_2) = M^{-1}R_jM$. This assignment yields a bijection

$$C(R_j)\backslash GL_2(\widehat{\mathbb{Z}}_p)/C(R_i) \cong \mathrm{Rep}_{i,j}$$

where $C(R)$ denotes the centralizer of R in $GL_2(\widehat{\mathbb{Z}}_p)$. After this identification, the non trivial outer automorphism of D_∞ interchanges $\mathrm{Rep}_{i,j}$ and $\mathrm{Rep}_{j,i}$ and acts on $\mathrm{Rep}_{i,i}$ by $M \mapsto M^{-1}$. To avoid trivial cases, we only need to consider $\mathrm{Rep}_{1,1}$ for $p>2$ and $\mathrm{Rep}_{1,1}$, $\mathrm{Rep}_{1,2}$ and $\mathrm{Rep}_{2,2}$ for $p=2$. Each of these sets has a description by double cosets:

$$\begin{aligned} \mathrm{Rep}_{1,1} &\cong \mathcal{D}\backslash GL_2(\widehat{\mathbb{Z}}_p)/\mathcal{D} \\ \mathrm{Rep}_{1,2} &\cong \mathcal{Y}\backslash GL_2(\widehat{\mathbb{Z}}_2)/\mathcal{D} \\ \mathrm{Rep}_{2,2} &\cong \mathcal{Y}\backslash GL_2(\widehat{\mathbb{Z}}_2)/\mathcal{Y} \end{aligned}$$

where the subgroups $\mathcal{D}$ and $\mathcal{Y}$ are as defined in 3.3. We use the notation $\overline{\mathrm{Rep}}_{i,j}$ to denote the mod p reductions of the representations in $\mathrm{Rep}_{i,j}$.[1]

Proposition 3.5 ([2]). *The functions $\Gamma_{1,1}$, δ_1 and δ_2 defined on $GL_2(\widehat{\mathbb{Z}}_p)$ by*

$$\delta_1 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \nu_p(xz),$$

$$\delta_2 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \nu_p(yt),$$

$$\Gamma_{1,1} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \frac{xt}{xt-yz} \in \widehat{\mathbb{Z}}_p,$$

[1]Notice that $\overline{\mathrm{Rep}}_{i,j}$ does *not* give a classification of representations in $GL_2(\mathbb{F}_p)$.

are well defined in $\mathrm{Rep}_{1,1}$ *and are a complete system of invariants. Using these invariants,* $\mathrm{Rep}_{1,1}$ *is tabulated in table 1.*

	$\overline{\mathrm{Rep}}_{1,1}$	$\mathrm{Rep}_{1,1}$	$\Gamma_{1,1}$	δ_1	δ_2
I	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}p^s & 1\\ 1 & x\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & p^s\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $s > 0,\, x \equiv 0 \pmod p$	$\frac{p^s x}{xp^s - 1}$, 0	> 0	> 0
II	$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1\\ 1 & x\end{pmatrix}$ $x \equiv 0 \pmod p$	$\frac{x}{x-1}$	0	> 0
II′	$\begin{pmatrix}0 & 1\\ 1 & x\end{pmatrix}$ $x \neq 0$	$\begin{pmatrix}p^s & 1\\ 1 & x\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ $x \not\equiv 0 \pmod p$, $s > 0$	$\frac{xp^s}{p^s x - 1}$, 0	> 0	0
III	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & p^r\\ x & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ p^s & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $x \equiv 0 \pmod p$, $r, s > 0$	$\frac{1}{1-p^r x}$, 1	> 0	> 0
IV	$\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$ $x \neq 0$	$\begin{pmatrix}1 & p^r\\ x & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ $x \not\equiv 0 \pmod p$, $r > 0$	$\frac{1}{1-p^r x}$, 1	0	> 0
V	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 1\\ x & 1\end{pmatrix}$ $x \equiv 0 \pmod p$	$\frac{1}{1-x}$	> 0	0
VI	$\begin{pmatrix}1 & 1\\ x & 1\end{pmatrix}$ $x \neq 0, 1$	$\begin{pmatrix}1 & 1\\ x & 1\end{pmatrix}$ $x \not\equiv 0, 1 \pmod p$	$\frac{1}{1-x}$	0	0

TABLE 1. **Representations of type** $(1,1)$. Classes I, II, II' correspond to $\Gamma_{1,1} \equiv 0 \pmod p$. Classes II and II' are permuted by $\mathrm{Out}(D_\infty)$. Classes III, IV, V, VI correspond to $\Gamma_{1,1} \not\equiv 0 \pmod p$. Class VI is void for $p = 2$.

Here and below, when we say that some functions form a complete system of invariants we mean that two matrices M and N are in the same double coset if and only if these functions take the same value in M and N. One can check easily that the table 1 gives a complete set of representatives for $\mathrm{Rep}_{1,1}$ without repetition. One sees also that the range of the invariants $\Gamma_{1,1}$, δ_1, δ_2 is $\widehat{\mathbb{Z}}_p \times \{0, \ldots, \infty\}^2$, subject only to the restrictions:

$$\begin{aligned} \delta_1 + \delta_2 = \infty \quad &\Rightarrow \quad \Gamma_{1,1} = 0,\, 1 \\ 0 < \delta_1 + \delta_2 < \infty \quad &\Rightarrow \quad \nu_p(\Gamma_{1,1} - 1) = \delta_1 + \delta_2 - \nu_p(\Gamma_{1,1}) \\ \delta_1 + \delta_2 = 0 \quad &\Rightarrow \quad \nu_p(\Gamma_{1,1}) = \nu_p(\Gamma_{1,1} - 1) = 0 \end{aligned}$$

The non-trivial outer automorphism of D_∞ leaves $\Gamma_{1,1}$ invariant. It also leaves δ_1 and δ_2 invariant in the types III to VI in the table and permutes δ_1 and δ_2 in the other types.

For $p = 2$ it remains to describe $\mathrm{Rep}_{1,2}$ and $\mathrm{Rep}_{2,2}$.

Proposition 3.6 ([2])**.** *The functions $\Gamma_{1,2}$ and δ_3 defined on $GL_2(\widehat{\mathbb{Z}}_2)$ by*

$$\Gamma_{1,2}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \frac{zt - xy}{xt - yz} \in \widehat{\mathbb{Z}}_2.$$

$$\delta_3\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{cases} 0 & yt \text{ even} \\ 1 & yt \text{ odd.} \end{cases}$$

are well defined on $\mathrm{Rep}_{1,2}$ and are a complete system of invariants.

	$\overline{\mathrm{Rep}}_{1,2}$	$\mathrm{Rep}_{1,2}$	$\Gamma_{1,2}$	δ_3
I	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ $z \equiv 0 \pmod 2$	z	0
II	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ $z \not\equiv 0 \pmod 2$	z	0
III	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ $y \not\equiv 0 \pmod 2$	$-y$	1

Table 2. **Representations of type** $(1, 2)$, $p = 2$.

Proposition 3.7 ([2]). *The functions* $\Gamma_{2,2}$, ε_1, $\bar\varepsilon_1$ *and* ε_2 *defined by*

$$\begin{aligned}
&\Gamma_{2,2}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \frac{x^2+t^2-y^2-z^2}{xt-yz} \in \widehat{\mathbb{Z}}_2 \\
&\varepsilon_1 = \nu_2(x+z-y-t) \\
&\bar\varepsilon_1 = \nu_2(x+z+y+t) \\
&\varepsilon_2 = \min\bigl(\nu_2(x^2+z^2-y^2-t^2), \nu_2(xz-yt)\bigr)
\end{aligned}$$

form a complete system of invariants for $\mathrm{Rep}_{2,2}$.

Any coset $[\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)]$ has a representative of the form $\left(\begin{smallmatrix} 1 & u \\ 0 & v \end{smallmatrix}\right)$ (cf. [2, Proposition 5]), but u and v are not uniquely determined.

	$\overline{\mathrm{Rep}}_{2,2}$	$\mathrm{Rep}_{2,2}$	$\Gamma_{2,2}$
I	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & v \end{pmatrix}$ $v \equiv 1 \pmod 2$	v
II	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & u \\ 0 & v \end{pmatrix}$ $u \equiv 0,\ v \equiv 1 \pmod 2$	$\frac{1-u^2+v^2}{v}$

TABLE 3. **Representations of type** $(2,2)$ $p=2$. In class II, different matrices $\left(\begin{smallmatrix} 1 & u \\ 0 & v \end{smallmatrix}\right)$ might represent the same element in $\mathrm{Rep}_{2,2}$ (see Proposition 3.7).

4. Cohomology computations. I: General

Now, for any representation data $(\rho, c) \in \mathsf{R}^*$, we want to compute the mod p cohomology of the associated space $X = \bar\Phi(\rho, c) \in \mathsf{S}^*$, including the action of the Steenrod algebra and the Bockstein spectral sequence. Since X is defined as (the mod p completion of) a push out

$$\begin{array}{ccc}
(BT^2)^\wedge_p & \xrightarrow{\mathrm{id}} & (BH_1)^\wedge_p \\
{\scriptstyle M}\downarrow & & \downarrow \\
(BH_2)^\wedge_p & \longrightarrow & X
\end{array}$$

the obvious tool to compute the cohomology of X is the Mayer-Vietoris long exact sequence associated to this push out diagram.

The first observation that we want to point out is that a knowledge of the matrix M modulo some p-th power p^n is enough to determine the structure of $H^*(X)$ up to *level* n, in the following sense.

Theorem 4.1. *Assume that* $p > 2, n \geq 1$ *or* $p = 2, n > 1$. *Let* M *and* N *be matrices such that* $M = N \mod p^n$. *Then* $H^*(X^{k,l}(M))$ *and* $H^*(X^{k,l}(N))$ *are isomorphic as algebras. Moreover, this isomorphism respects the action of the Steenrod operations* $\mathcal{P}^i$ *(*Sq^{2i} *if* $p = 2$*) and the higher Bockstein homomorphisms* $\beta_{(r)}$ *of height* $r < n$.

Proof. Let A be the mod p^n reduction of the matrices M and N. Let $T(n)$ denoted the abelian subgroup of the torus T^2 of elements of order smaller than or equal to p^n. Define the space $X(A)$ as the push out

$$\begin{array}{ccc} (BT(n))^\wedge_p & \xrightarrow{\text{id}} & (BH_1)^\wedge_p \\ {\scriptstyle A}\downarrow & & \downarrow \\ (BH_2)^\wedge_p & \longrightarrow & X(A) \end{array}$$

It is easy to verify using the Mayer-Vietoris sequence that the natural map $X(A) \to X^{k,l}(M)$ induces a monomorphism in cohomology. We may investigate this map further as follows.

Let $I \subset H^*(BT(n))$ be the ideal consisting of nilpotent elements. It is clear from the definition that I is a $H^*(X(A))$-submodule of $H^*(BT(n))$, where the $H^*(X(A))$-module structure on $H^*(BT(n))$ is given by restriction. Notice that I is invariant under the Steenrod operations $\mathcal{P}^i$ (Sq^{2i} if p = 2) and the higher Bockstein homomorphisms $\beta_{(r)}$ of height $r < n$. Let $J = \delta(I)$ be the image of I under the boundary homomorphism $\delta\colon H^*(BT(n)) \to H^{*+1}(X(A))$ in the Mayer-Vietoris sequence. It follows from standard facts that δ is a map of $H^*(X(A))$-modules. Hence J is an ideal in $H^*(X(A))$ invariant under the above operations.

Using the Mayer-Vietoris sequence, we see that the natural map $H^*(X^{k,l}(M)) \to H^*(X(A))/J$ is an isomorphism of algebras which commutes with the action of the cohomology operations mentioned earlier. The same holds for the matrix N and thus the proof is complete. □

The computations of the cohomology of the spaces in S^* is done in the following two sections of this paper: the first deals with the prime 2 and the next deals with the odd primes.

5. Cohomology computations. II: The even prime case

Throughout this section we work at the prime 2, (ρ, c) will be an element in R^* and X will denote the space $\bar\Phi(\rho, c) \in \mathsf{S}^*$. We choose a matrix $M = \left(\begin{smallmatrix}\alpha & \beta\\ \lambda & \mu\end{smallmatrix}\right) \in GL_2(\widehat{\mathbb{Z}}_2)$ and indices k, l such that $X = X^{k,l}(M)$. We consider separately the cases of $\mathrm{Rep}_{1,1}$, $\mathrm{Rep}_{1,2}$ and $\mathrm{Rep}_{2,2}$ and we denote by $\Gamma_{1,1}$, $\Gamma_{1,2}$, $\Gamma_{2,2}$, δ_1, δ_2, δ_3, ϵ_1, $\bar\epsilon_1$ and ϵ_2 the invariants defined in section 3.

5.1. **Representations of type** $\mathrm{Rep}_{1,1}$**.** According to table 1, these representations fall into five classes **I** to **V**. (Notice that $\mathrm{Out}(D_\infty)$ identifies the classes **II** and **II'**, and class **VI** is void for $p = 2$.) The obstruction class may take three different values: $c = (1,1)$, $c = (0,1)$ or $c = 0$.

Theorem 5.1. *If W is of type $\mathrm{Rep}_{1,1}$ and $c = (1,1)$ then the cohomology algebra of X and the action of the Steenrod algebra and the higher Bockstein operations are as described in table 4.*

	Cohomology ring	Steenrod squares	BSS
I	$P[x_4, y_4] \otimes E[z_5]$	$Sq^4 z_5 = 0$	$\beta_{(s)} y_4 = z_5$ $s = \min\{\delta_1, \delta_2\} + 1 \geq 2$
II	$P[x_4, y_4] \otimes E[z_5]$	$Sq^4 z_5 = 0$ $Sq^1 y_4 = z_5$	
III	$P[x_2, y_4] \otimes E[z_3]$	$Sq^2 y_4 = Sq^2 z_3 = 0$	$\beta_{(\delta_2)} x_2 = z_3$ $\beta_{(s)} x_2^2 = x_2 z_3 \Leftrightarrow \delta_2 = \min\{\delta_1, \delta_2\}$ $\beta_{(s)} y_4 = x_2 z_3 \Leftrightarrow \delta_1 = \min\{\delta_1, \delta_2\}$ $s = \min\{\delta_1, \delta_2\} + 1$
IV	$P[x_2, y_4] \otimes E[z_3]$	$Sq^2 y_4 = Sq^2 z_3 = 0$ $Sq^1 y_4 = x_2 z_3$	$\beta_{(\delta_2)} x_2 = z_3$
V	$P[x_4, y_4] \otimes E[z_5]$	$Sq^1 y_4 = z_5$ $Sq^4 z_5 = 0$	

Table 4. $\mathrm{Rep}_{1,1}$ and obstruction $c = (1,1)$.

Proof. The Mayer-Vietoris long exact sequence associated to the push out which defines X has the form

(2)
$$\dots \longrightarrow H^i(X) \longrightarrow H^i(BS^1\times BSU(2))\oplus H^i(BS^1\times BSU(2)) \xrightarrow{\bar{M}^t+j} \\ \longrightarrow H^i(BT^2) \longrightarrow H^{i+1}(X) \longrightarrow \dots$$

and it is a sequence of $H^*(X)$-modules. The map labelled $\bar{M}^t$ is induced by the transpose of the mod 2 reduction of the matrix M and j indicates the map induced by the inclusion of the standard maximal torus of $S^1\times SU(2)$.

For simplicity, we write $P_0 = H^*(BT^2) = \mathbb{F}_2[u,v]$ and we identify $H^*(BS^1\times BSU(2))$, with its canonical image $\mathbb{F}_2[u,v^2]$ in $H^*(BT^2)$. In order to distinguish between the two components in the Mayer-Vietoris sequence (2), we will write $P_1 = \mathbb{F}_2[\bar{u},\bar{v}^2]$ for the first component and $P_2 = \mathbb{F}_2[u,v^2]$ for the second one. Using this notation, and by degree reasons the sequence (2) becomes

$$(3)\quad 0 \longrightarrow H^{\text{even}}(X) \longrightarrow P_1\oplus P_2 \xrightarrow{\varphi} P_0 \longrightarrow \Sigma^{-1}H^{\text{odd}}(X) \longrightarrow 0$$

with $\varphi = \bar{M}^t + j$. In particular, $H^{\text{even}}(X) \cong \operatorname{Ker}\varphi$ and $H^{\text{odd}}(X) \cong \Sigma\operatorname{Coker}\varphi$. We distinguish between two cases.

• Type **I**, **II**, or **V**; that is $\beta\equiv 1 \bmod 2$. In this case $\varphi(\bar{u}) = \alpha u + v$ and $\varphi(\bar{v}^2) = \lambda u^2+\mu v^2$, while $\varphi(u) = u$ and $\varphi(v^2) = v^2$, hence, the sequence (3) gives $H^2(X) = 0$ and in degree 4 we get that $\mathbb{F}_2[\bar{u}^2+\alpha u^2+v^2, \bar{v}^2+\lambda u^2+\mu v^2]\subset\operatorname{Ker}\varphi$. In particular, there are elements $x_4, y_4 \in H^*(X^{1,1}_W;\mathbb{F}_2)$ that project to $\bar{u}^2+\alpha u^2+v^2$ and $\bar{v}^2+\lambda u^2+\mu v^2$ respectively, thus $P[x_4,y_4]$ is a subalgebra of $H^*(X)$ and (3) might be seen as an exact sequence of $P[x_4,y_4]$-modules.

After dividing out by P_2 in $P_1\oplus P_2$ and in P_0, (3) is simplified to

$$(4)\quad 0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow P[\bar{u},\bar{v}^2] \xrightarrow{[\varphi]} [v]\,\mathbb{F}_2[u^2,v^2]\oplus[uv]\,\mathbb{F}_2[u^2,v^2] \\ \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0$$

where $[\varphi](\bar{u}) = [\alpha u + v] = [v]$. It follows that $\operatorname{Ker}\varphi \cong P[x_4,y_4]$ and $\operatorname{Coker}\varphi \cong [uv]\,P[u^2,v^2] \cong [uv]\,P[x_4,y_4]$.

Therefore, we have a splitting exact sequence of $P[x_4,y_4]$-modules

$$(5)\qquad 0 \longrightarrow \Sigma[uv]\,P[x_4,y_4] \longrightarrow H^*(X) \longrightarrow P[x_4,y_4] \longrightarrow 0\,.$$

The action of the Steenrod squares on uv is easily computed in $P_0 = P[u,v]$ and it follows that the action on its class $[uv]$ in $[uv]\,P[u^2,v^2]$ is trivial. Now, if we call z_5 the image in $H^*(X)$ of the suspension of $[uv]$, we have that the Steenrod squares act trivially on z_5 and in particular that

$z_5^2 = Sq^5 z_5 = 0$, hence

$$H^*(X) \cong \mathbb{F}_2[x_4, y_4] \otimes E(z_5)\,.$$

• Type **III**, or **IV**; that is $\beta \equiv 0 \bmod 2$. In this case $\varphi(\bar{u}) = \alpha u$ and we obtain a generator of $\operatorname{Ker}\varphi$, $\bar{u} + \alpha u$, in degree two. Also, as in the previous case $\bar{v}^2 + \mu u^2 + \lambda v^2$ is a generator in degree four of $\operatorname{Ker}\varphi$, and $\mathbb{F}_2[\bar{u} + \alpha u, \bar{v}^2 + \mu u^2 + \lambda v^2] \subset \operatorname{Ker}\varphi$. Thus, we obtain now a subalgebra $P[x_2, y_4]$ of $H^*(X)$ and then (3) is an exact sequence of $P[x_2, y_4]$-modules. Now, the quotient by P_2 in (3) gives a new exact sequence

$$(6) \quad 0 \longrightarrow \operatorname{Ker}\varphi \longrightarrow P[\bar{u}, \bar{v}^2] \xrightarrow{[\varphi]} [v]\,\mathbb{F}_2[u, v^2] \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0$$

with $[\varphi] = 0$, so that $\operatorname{Ker}\varphi \cong P[x_2, y_4]$ and $\operatorname{Coker}\varphi \cong [v]\,P[u, v^2] \cong [v^2]\,P[x_2, y_4]$, as $P[x_2, y_4]$-modules. Again the Steenrod squares act trivially on the class $[v]$, so therefore if z_3 is the image in $H^*(X)$ of the suspension of $[v]$, we have

$$H^*(X) \cong P[x_2, y_4] \oplus E(z_3)\,.$$

Finally, notice that $Sq^2(y_4) = 0$ in the quotient $P[x_2, y_4]$, and then, by degree reasons it is also trivial in $H^*(X)$. The action of all other Steenrod operations, besides Sq^1, follows from the properties of the Steenrod squares, in a straightforward way.

Finally, it remains to compute, for all cases, the action of the higher Bockstein operations, including the primary Bockstein $\beta = Sq^1$.

The argument uses the Mayer-Vietoris long exact sequence with coefficients in $\widehat{\mathbb{Z}}_2$

$$(7) \quad 0 \longrightarrow H^{\text{even}}(X; \widehat{\mathbb{Z}}_2) \longrightarrow \widehat{\mathbb{Z}}_2[\bar{u}, \bar{v}^2] \oplus \widehat{\mathbb{Z}}_2[u, v^2] \xrightarrow{\varphi} \widehat{\mathbb{Z}}_2[u, v] \longrightarrow \\ \longrightarrow \Sigma^{-1} H^{\text{odd}}(X; \widehat{\mathbb{Z}}_2) \longrightarrow 0$$

in low dimensions. A first observation is that $\operatorname{Ker}\varphi$ is torsion free. $\operatorname{Coker}\varphi$ is, in each degree, a finitely generated $\widehat{\mathbb{Z}}_2$-module that can be easily classified in terms of the homomorphism φ; that is, in terms of the matrix M^t. We obtain in degree 2 $\operatorname{Coker}\varphi^{(2)} \cong \widehat{\mathbb{Z}}_2/\beta\widehat{\mathbb{Z}}_2$. Hence in case $\beta \equiv 0 \bmod 2$; that is, for types **III**, or **IV**, we have $H^3(X; \widehat{\mathbb{Z}}_2) \cong \widehat{\mathbb{Z}}_2/\beta\widehat{\mathbb{Z}}_2 \cong \widehat{\mathbb{Z}}_2/2^{\nu_2(\beta)}\widehat{\mathbb{Z}}_2$. Notice that $\nu_2(\beta)$ coincides with the invariant δ_2 in these cases (cf. table 1), hence the result $\beta_{\delta_2}(x_2) = z_3$.

Finally, we compute the cokernel of φ in degree four and we get

$$\operatorname{Coker}\varphi^{(4)} \cong \widehat{\mathbb{Z}}_2/2\gcd(\alpha\beta, \lambda\mu) \cdot \widehat{\mathbb{Z}}_2\,.$$

Again, we can check at table 1 that $\nu_2(\alpha\beta) = \delta_2$ and $\nu_2(\lambda\mu) = \delta_1$, so, if we write $s = 1 + \min\{\delta_1, \delta_2\}$, then $H^5(X; \widehat{\mathbb{Z}}_2) \cong \widehat{\mathbb{Z}}_2/2^s\widehat{\mathbb{Z}}_2$.

	Cohomology ring	Steenrod squares and BSS
I	$\mathbf{J}_1 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = z_3, \quad Sq^1 z_4 = Sq^1 z_5 = 0,$ $Sq^2 z_3 = z_2 z_3, \ Sq^2 z_4 = 0, \ Sq^2 z_5 = z_5 z_2 + z_4 z_3,$ $Sq^4 z_5 = z_4 z_3 z_2$
		$\beta_{(s)} z_4 = z_5 \Leftrightarrow \delta_2 = \min\{\delta_1, \delta_2\}$ $\beta_{(s)} z_2^2 = z_5 \Leftrightarrow \delta_1 = \min\{\delta_1, \delta_2\}$ $s = \min\{\delta_1, \delta_2\} + 2 \geq 3$
II	$\mathbf{J}_2 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = z_3, \quad Sq^1 z_4 = 0, \quad Sq^1 z_5 = z_3^2,$ $Sq^2 z_3 = z_5, \quad Sq^2 z_4 = z_3^2, \quad Sq^2 z_5 = 0,$ $Sq^4 z_5 = z_5 z_4 + z_3^3$
		$\beta_{(2)} z_2^2 = z_5 + z_3 z_2$
II′	$\mathbf{J}_1 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = z_3, \quad Sq^1 z_4 = Sq^1 z_5 = 0,$ $Sq^2 z_3 = z_2 z_3, \ Sq^2 z_4 = 0, \ Sq^2 z_5 = z_5 z_2 + z_4 z_3,$ $Sq^4 z_5 = z_4 z_3 z_2$
		$\beta_{(2)} z_4 = z_5$
III	$\mathbf{J}_3 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = Sq^1 z_3 = Sq^1 z_4 = 0, \quad Sq^1 z_5 = z_3^2,$ $Sq^2 z_3 = z_5, \quad Sq^2 z_4 = z_3^2, \quad Sq^2 z_5 = 0$ $Sq^4 z_5 = z_5 z_4 + z_3^3$
		$\beta_{(\delta_2+1)} z_2 = z_3$ $\beta_{(s)} z_4 = z_2 z_3 \Leftrightarrow \delta_1 = \min\{\delta_1, \delta_2\}$ $\beta_{(s)} z_2^2 = z_2 z_3 \Leftrightarrow \delta_2 = \min\{\delta_1, \delta_2\}$ $s = \min\{\delta_1, \delta_2\} + 2 \geq 3$
IV	$\mathbf{J}_3 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = Sq^1 z_3 = Sq^1 z_4 = 0, \quad Sq^1 z_5 = z_3^2,$ $Sq^2 z_3 = z_5, \quad Sq^2 z_4 = z_3^2, \quad Sq^2 z_5 = 0,$ $Sq^4 z_5 = z_5 z_4 + z_3^3$
		$\beta_{(\delta_2+1)} z_2 = z_3\,, \quad \beta_{(2)} z_4 = z_2 z_3$
V	$\mathbf{J}_2 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$	$Sq^1 z_2 = z_3, \quad Sq^1 z_4 = 0, \quad Sq^1 z_5 = z_3^2,$ $Sq^2 z_3 = z_5, \quad Sq^2 z_4 = z_3^2, \quad Sq^2 z_5 = 0,$ $Sq^4 z_5 = z_5 z_4 + z_3^3$
		$\beta_{(2)} z_2^2 = z_5 + z_3 z_2$

TABLE 5. $\mathrm{Rep}_{1,1}$ and obstruction $c = (0, 1)$.

	Cohomology ring	Steenrod squares and BSS
I	$\mathbf{K}_1 = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$	$Sq^1 w_2 = w_3, \quad Sq^1 \bar{w}_2 = \bar{w}_3,$ $Sq^2 w_3 = w_2 w_3, \quad Sq^2 \bar{w}_3 = \bar{w}_2\bar{w}_3$
		$\beta_{(s)} w_2^2 = w_2\bar{w}_3 \quad s = \min\{\delta_1, \delta_2\} + 2$
II	$\mathbf{K}_2 = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$	$Sq^1 w_2 = w_3, \quad Sq^1 \bar{w}_2 = \bar{w}_3,$ $Sq^2 w_3 = w_2 w_3, \quad Sq^2 \bar{w}_3 = \bar{w}_2\bar{w}_3 + w_2\bar{w}_3$
		$\beta_{(2)} \bar{w}_2^2 = \bar{w}_2 w_3$
III	$\mathbf{K}_3 = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$	$Sq^1 w_2 = w_3 + \bar{w}_3, \quad Sq^1 \bar{w}_2 = 0,$ $Sq^2 w_3 = w_2 w_3, \quad Sq^2 \bar{w}_3 = w_2\bar{w}_3$
		$\beta_{(\delta_2+1)} \bar{w}_2 = \bar{w}_3$ $\beta_{(s)} w_2^2 = \bar{w}_2 w_3 \Leftrightarrow \delta_1 = \min\{\delta_1, \delta_2\}$ $\beta_{(s)} \bar{w}_2^2 = \bar{w}_2\bar{w}_3 \Leftrightarrow \delta_2 = \min\{\delta_1, \delta_2\}$ $s = \min\{\delta_1, \delta_2\} + 2$
IV	$\mathbf{K}_4 = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$	$Sq^1 w_2 = w_3 + \bar{w}_3, \quad Sq^1 \bar{w}_2 = 0,$ $Sq^2 w_3 = w_2 w_3, \quad Sq^2 \bar{w}_3 = w_2\bar{w}_3 + \bar{w}_2\bar{w}_3$
		$\beta_{(\delta_2+1)} \bar{w}_2 = \bar{w}_3$ $\beta_{(2)} \bar{w}_2^2 = w_2 w_3$
V	$\mathbf{K}_5 = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$	$Sq^1 w_2 = w_3 + \bar{w}_3, \quad Sq^1 \bar{w}_2 = \bar{w}_3,$ $Sq^2 w_3 = w_2 w_3, \quad Sq^2 \bar{w}_3 = w_2\bar{w}_3$
		$\beta_{(2)} \bar{w}_2^2 = w_2\bar{w}_3 + \bar{w}_2\bar{w}_3$

TABLE 6. $\mathrm{Rep}_{1,1}$ and obstruction $c = 0$.

We can choose generators such that $Sq^1y_4 = z_5$ in cases **II** and **V**, and $\beta_{(s)}y_4 = z_5$, $s \geq 2$, in case **I**. In cases of type **III**, or **IV** we already have $\beta_{(\delta_2)}x_2 = z_3$, hence $\beta_{(r)}(x_2^2) = 0$ for $r \leq \delta_2$, and $\beta_{(\delta_2+1)}(x_2^2) = x_2z_3$, if these classes survive to the $\delta_2 + 1$ page of the Bockstein spectral sequence. We have then that $Sq^1y_4 = x_2z_3$ in case **IV**, and for case **III**, $s \geq 2$ and then $\beta_{(s)}x_2^2 = x_2z_3$ if $\delta_2 = \min\{\delta_1, \delta_2\}$ but $\beta_{(s)}y_4 = x_2z_3$, otherwise. □

All other calculations in this section use the same kind of arguments and we will omit the proofs. The interested reader will have no difficulty in completing the missing details.

Theorem 5.2. *If W is of type $\mathrm{Rep}_{1,1}$ and $c = (0, 1)$ then the cohomology algebra of X is*

$$H^*(X) \cong \mathbf{J} = \frac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_3^2 z_4)}$$

and the action of the Steenrod algebra and the higher Bockstein operations are as described in table 5. □

	Cohomology ring	Steenrod squares and BSS
I	$P[x_4, y_8] \otimes E[z_9]$	$Sq^4y_8 = x_4y_8,\quad Sq^4z_9 = x_4z_9,\quad Sq^8z_9 = 0$
		$\beta_{(s)}y_8 = z_9 \quad s = 1 + \nu_2(\Gamma_{1,2}) \geq 2$
II	$P[x_4, y_8] \otimes E[z_9]$	$Sq^1y_8 = z_9,\quad Sq^4y_8 = x_4y_8,\quad Sq^4z_9 = x_4z_9$ $Sq^8z_9 = 0$
III	$P[x_2, y_8]$ $\otimes$ $E[z_3, z_7]/(x_2z_3, z_3z_7)$	$Sq^1y_8 = x_2z_7,\quad Sq^2y_8 = 0,\quad Sq^2z_7 = 0$ $Sq^4y_8 = y_8x_2^2,\quad Sq^4z_7 = z_7x_2^2 + z_3y_8$
		$\beta_{(s)}x_2 = z_3,\quad s = \nu_2(\Gamma_{1,2} + 1)$ $\beta_{(2s)}x_2^3 = z_7$

TABLE 7. $\mathrm{Rep}_{1,2}$ and obstruction $c = 1$.

Notice that here we are using subscripts as in $\mathbf{J}_i$ to denote different Steenrod algebra actions on the same algebra $\mathbf{J}$.

Theorem 5.3. *If W is of type* $\mathrm{Rep}_{1,1}$ *and* $c = 0$ *then the cohomology algebra of X is*

$$H^*(X) \cong \mathbf{K} = P[w_2, w_3, \bar{w}_2, \bar{w}_3]/(w_3\bar{w}_3)$$

and the action of the Steenrod algebra and the higher Bockstein operations are as described in table 6. □

5.2. **Representations of type** $\mathrm{Rep}_{1,2}$**.** According to table 2, these representations fall into three classes **I**, **II** and **III**. The obstruction class c may take two different values: $c = 0$ or $c = 1$.

Theorem 5.4. *If W is of type* $\mathrm{Rep}_{1,2}$ *and* $c = 1$ *then the cohomology algebra of X and the action of the Steenrod algebra and the higher Bockstein operations are as described in table 7.* □

	Cohomology ring	Steenrod squares and BSS
I	$\mathbf{L}_1 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_5z_3z_2 + z_4z_3^2)}$	$Sq^1z_2 = z_3,\ Sq^1z_4 = z_5 + z_3z_2,\ Sq^1z_5 = z_3^2,$ $Sq^2z_3 = z_5,\quad Sq^2z_4 = z_4z_2,\quad Sq^2z_5 = 0$ $Sq^4z_5 = z_5z_4 + z_5z_2^2 + z_3^3 + z_3z_4z_2$
II	$\mathbf{L}_2 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_5z_3z_2 + z_4z_3^2)}$	$Sq^1z_2 = z_3,\quad Sq^1z_4 = z_5,$ $Sq^2z_3 = z_2z_3,\ Sq^2z_4 = z_4z_2,\ Sq^2z_5 = z_4z_3$ $Sq^4z_5 = z_5z_2^2 + z_4z_3z_2$
III	$\mathbf{L}_3 = \dfrac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_5z_3z_2 + z_4z_3^2)}$	$Sq^1z_4 = z_2z_3,\quad Sq^1z_5 = z_3^2,$ $Sq^2z_3 = z_5,\ Sq^2z_4 = z_2z_4 + z_3^2,\ Sq^2z_5 = 0$ $Sq^4z_5 = z_5z_4 + z_5z_2^2 + z_3^3 + z_4z_3z_2$
		$\beta_{(s)}z_2 = z_3 \quad s = \nu_2(\Gamma_{1,2} + 1) + 1 > 1$

TABLE 8. $\mathrm{Rep}_{1,2}$ and obstruction $c = 0$

Theorem 5.5. *If W is of type* $\mathrm{Rep}_{1,2}$ *and* $c = 0$ *then the cohomology algebra of X is*

$$H^*(X) \cong \mathbf{L} = \frac{P[z_2, z_3, z_4, z_5]}{(z_5^2 + z_5z_3z_2 + z_4z_3^2)}$$

and the action of the Steenrod algebra action and the higher Bockstein operations are as described in table 8. □

5.3. **Representations of type** $\mathrm{Rep}_{2,2}$**.** Here we use the classification in types **I** and **II** of table 3. We obtain the following result.

Theorem 5.6. *If W is of type $\mathrm{Rep}_{2,2}$ then the cohomology algebra of X and the action of the Steenrod algebra and the higher Bockstein operations are as described in table 9.* □

	Cohomology ring	Steenrod squares and BSS
I	$P[x_4, y_6] \otimes E[z_7]$	$Sq^2 x_4 = y_6, \quad Sq^2 y_6 = 0, \quad Sq^4 y_6 = x_4 y_6$ $Sq^4 z_7 = x_4 z_7,$
		$\beta_{(s)} y_6 = z_7 \quad s = \nu_2(\Gamma_{2,2} + 1)$
II	$P[x_2, y_4] \otimes E[z_3]$	$Sq^2 z_3 = z_2 z_3 \quad Sq^2 y_4 = x_2 y_4$
		$\beta_{(\varepsilon_1)} x_2 = z_3$ $\beta_{(\varepsilon_2)} x_2^2 = x_2 z_3 \Leftrightarrow \varepsilon_1 + \bar{\varepsilon}_1 = \varepsilon_2$ $\beta_{(\varepsilon_2)} y_4 = x_2 z_3 \Leftrightarrow \varepsilon_1 + \bar{\varepsilon}_1 \geq \varepsilon_2$

TABLE 9. $\mathrm{Rep}_{2,2}$.

6. COHOMOLOGY COMPUTATIONS. III: THE ODD PRIME CASE

Throughout this section we work at a fixed odd prime p. We choose an element in R^* an we want to compute the mod p cohomology of the space X in S^* associated to that element in R^*. As we saw in section 3, for an odd prime p, the elements in R^* are just conjugacy classes of infinite dihedral subgroups in $GL_2(\widehat{\mathbb{Z}}_p)$. Hence, we have such a subgroup which we denote by W and which can be given either by a two by two matrix $M \in GL_2(\widehat{\mathbb{Z}}_p)$ or by three invariants $\Gamma = \Gamma_{1,1}$, δ_1, δ_2. The classification of these groups is summarized in table 1. We organize the statements in this section according to the six *types* of representations of the group W, labelled **I** to **VI** in table 1. Notice that we do not need to consider type **II'**. We denote by X_W the space in S^* which corresponds to the group W.

Let us point out that the invariants Γ, δ_1, δ_2 classify all representations of D_∞, including those which are not faithful, while we are only interested in the spaces X_W for W of infinite order. The following result provides a partial answer to the question of which values of Γ, δ_1 and δ_2 yield Weyl groups of infinite order.

Proposition 6.1 ([2]). *Let $\rho\colon D_\infty \to GL_2(\widehat{\mathbb{Z}}_p)$ (p odd) be a representation with invariants Γ, δ_1 and δ_2. Assume $\Gamma \equiv 0, 1\,(p)$. Then $\rho(D_\infty)$ has finite order if and only if $\delta_1 = \delta_2 = \infty$ or $p = 3$ and $\Gamma = 3/4,\ 1/4$.* □

This proposition implies that in the case in which $\delta_1 + \delta_2 > 0$ (i.e. all types except type **VI**) we only have to eliminate the representations given by the matrices

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \text{for any prime;}$$

$$\begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}, \begin{pmatrix}1 & 3\\ -1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ -3 & 1\end{pmatrix}, \quad \text{for } p = 3.$$

Let L be the p-adic lattice of rank two with a W-action given by the inclusion of W in $GL_2(\widehat{\mathbb{Z}}_p)$ and let us denote by P the symmetric algebra on the $\mathbb{F}_p$-vector space $\operatorname{Hom}(L, \mathbb{F}_p)$. P is thus a polynomial algebra on two generators $P = \mathbb{F}_p[u, v]$ which we grade by assigning degree two to the variables u and v. P inherits an action of the group W and we are interested now in the subalgebra of invariant elements P^W. Notice that the action of W on P is dual to the representation of W on L. Moreover, if we denote by W_p the mod p reduction of W then the action of W on P factors through W_p. The group W_p is finite dihedral and throughout this section we denote by k the integer defined as

$$k = \frac{|W_p|}{2}.$$

Let us call W *exceptional* if it belongs to types **III** or **IV** and let us call it *ordinary* otherwise.

We will use tools from the invariant theory of reflection groups. A very useful reference is the beautiful book [14].

Proposition 6.2. *The ring of invariants of W is a polynomial algebra on two generators, $P^W \cong \mathbb{F}_p[x, y]$, with $\deg(x)\deg(y) = 8k$. If W is exceptional then x has degree 2 and if W is ordinary then x has degree 4.*

Proof. Obviously, P^W depends only on the group W_p and we only need to study the mod p reductions of the matrices in table 1, which are displayed in the column labelled $\overline{\mathrm{Rep}}_{1,2}$.

In the exceptional case we have the identity matrix and the matrices $A_\lambda = \left(\begin{smallmatrix}1 & 0\\ \lambda & 1\end{smallmatrix}\right)$. In the case of the identity matrix (i.e. type **III**), we have that

W_p has order two and is generated by the linear map which fixes u and sends v to $-v$. The invariants are $\mathbb{F}_p[u, v^2]$.

The representations corresponding to A_λ are those of type **IV**. One sees easily that W_p is conjugated to

$$W_p = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \right\rangle.$$

This dihedral group has order $2p$ and its invariant theory is discussed in pages 128–129 of [14]. The invariants are polynomial in degrees 2 and $4p$, namely $P^W = \mathbb{F}_p[u, (vu^{p-1} - v^p)^2]$.

Let us discuss now the case in which W is ordinary. Table 1 provides the matrices

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad C_\lambda = \begin{pmatrix} 1 & 1 \\ \lambda & 1 \end{pmatrix}, \lambda \neq 1; \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix B produces type **I**. W_p is the representation of the elementary abelian 2-group of order 4 by $u \mapsto \pm u$, $v \mapsto \pm v$. Then $P^W = \mathbb{F}_p[u^2, v^2]$.

The matrix C_0 gives the representation

$$W_p = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

This is again a dihedral group of order $2p$ and this representation is also studied in pages 128–129 of [14]. The invariants are polynomial in degrees 4 and $2p$, namely $P^W = \mathbb{F}_p[v^2, u(u^{p-1} - v^{p-1})]$. This is type **V**.

The matrix D gives a group conjugated to

$$W_p = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

It is a dihedral group of order $4p$ and has a subgroup H of index two which is of the type C_0 that we have just studied. Then $P^H = \mathbb{F}_p[x_4, y_{2p}]$ and one sees that W/H fixes x and sends y to $-y$. Then $P^W = \mathbb{F}_p[v^2, u^2(u^{p-1} - v^{p-1})^2]$. This is type **II**.

Let us consider finally the case of the matrices C_λ for $\lambda \neq 0, 1$. This is type **VI**. k is then the order of the matrix obtained multiplying the two generating reflections of W_p. This matrix is

$$\tau = \frac{1}{1-\lambda} \begin{pmatrix} 1+\lambda & 2 \\ 2\lambda & 1+\lambda \end{pmatrix}.$$

The discriminant of the characteristic polynomial of this matrix does not vanish. Hence, τ diagonalizes over $\mathbb{F}_{p^2}$ and so the order of τ is a divisor of $p^2 - 1$. This implies that the order of W_p is coprime to p and, since W_p is generated by reflections, the classical Shephard-Todd theorem ([14], 7.4.1) implies that P^W is a polynomial algebra. $\lambda u^2 - v^2$ is an invariant of degree 4 and there are no invariants of degree 2. Hence, $P^W = \mathbb{F}_p[x_4, y_{2k}]$. □

Along the proof of the previous proposition we have obtained a computation of the value of $k = |W_p|/2$:

Proposition 6.3. *The value of k is given by*

- *Type **I**: $k = 2$.*
- *Type **II**: $k = 2p$.*
- *Type **III**: $k = 1$.*
- *Types **IV** and **V**: $k = p$.*
- *Type **VI**: k is equal to the multiplicative order in $\mathbb{F}_{p^2}$ of the roots of the polynomial $X^2 - 2(2\Gamma - 1)X + 1$.* □

We determine now the mod p cohomology of the spaces X_W. As usual, we use subscripts to denote the degrees of the generators of an algebra and we denote by $E(x, y, \ldots)$ the exterior $\mathbb{F}_p$-algebra with generators $x, y, \ldots$

Theorem 6.4. *Let p be an odd prime and let W be an infinite dihedral subgroup of $GL_2(\widehat{\mathbb{Z}}_p)$. Put $k = |W_p|/2$. Then:*

1. *$H^{even}(X_W) = H^*(BT)^W$.*
2. *If W is ordinary then $H^*(X_W) \cong \mathbb{F}_p[x_4, y_{2k}] \otimes E(z_{2k+1})$.*
3. *If W is of type **III** then $H^*(X_W) \cong \mathbb{F}_p[x_2, y_4] \otimes E(z_3)$.*
4. *If W is of type **IV** then $H^{even}(X_W) \cong \mathbb{F}_p[x_2, y_{4p}]$ and we have*
$$H^*(X_W) \cong H^{even} \cdot 1 \oplus \Big(\bigoplus_{i=0}^{p-2} \frac{H^{even}}{x_2 H^{even}} \cdot z_{4i+3} \Big) \oplus H^{even} \cdot z_{4p-1}$$
as an $H^{even}(X_W)$-module.

Proof. The proof uses the Mayer-Vietoris sequence of the push out diagram $(BH_2)^\wedge_p \leftarrow BT^\wedge_p \rightarrow (BH_1)^\wedge_p$ with $H_1 = H_2 = BS^1 \times BSU(2)$. Notice that, since p is odd, $H^*(BH_i) \cong H^*(BT)^{\langle \omega_i \rangle}$ for $i = 1, 2$, where ω_1, ω_2 are the generating reflections of W. For simplicity, let us write $P = H^*(BT)$, $H = H^{\mathrm{odd}}(X_W)$. Then the even cohomology of X_W coincides with the invariants of P under the action of the Weyl group, while the odd cohomology is given by the exact sequence
$$0 \rightarrow P^W \rightarrow P^{\omega_1} \oplus P^{\omega_2} \rightarrow P \rightarrow \Sigma^{-1} H \rightarrow 0.$$
Notice that this is an exact sequence of P^W-modules. The Poincaré series of P^W can be deduced from the information provided by 6.2. Then, the above exact sequence gives the Poincaré series of $H^{\mathrm{odd}}(X_W)$.

Assume first that W is ordinary. Then one sees easily that the Poincaré series of H is the same as the the Poincaré series of the free P^W-module $z_{2k+1}P^W$. Let us prove that H is indeed a free P^W-module. By [14] 6.1.1, it is enough to prove that $\mathrm{Tor}_1^{P^W}(\mathbb{F}_p, H) = 0$. By [14] 6.7.11, the above exact sequence is a free resolution of H as a P^W-module, hence

$$\mathrm{Tor}_1^{P^W}(\mathbb{F}_p, H) = \frac{\mathrm{Ker}\left(\mathbb{F}_p \otimes_{P^W} (P^{\omega_1} \oplus P^{\omega_2}) \to \mathbb{F}_p \otimes_{P^W} P\right)}{\mathrm{Im}\left(\mathbb{F}_p \otimes_{P^W} P^W \to \mathbb{F}_p \otimes_{P^W} (P^{\omega_1} \oplus P^{\omega_2})\right)}.$$

Now, since p is odd, we can use the averaging map $x \mapsto (x + \omega_i \cdot x)/2$ and we see that $\mathbb{F}_p \otimes_{P^W} P^{\omega_i} = (\mathbb{F}_p \otimes_{P^W} P)^{\omega_i}$ and the diagonal inclusion $P^W \to P^{\omega_1} \oplus P^{\omega_2}$ has a section. Hence, $\mathbb{F}_p \otimes_{P^W} P^W \to \mathbb{F}_p \otimes_{P^W} (P^{\omega_1} \oplus P^{\omega_2})$ is a monomorphism and

$$\mathrm{Ker}\left(\mathbb{F}_p \otimes_{P^W} (P^{\omega_1} \oplus P^{\omega_2}) \to \mathbb{F}_p \otimes_{P^W} P\right) = (\mathbb{F}_p \otimes_{P^W} P)^W.$$

If the order of W_p is prime to p then it is well known ([14], 7.5.2) that $\mathbb{F}_p \otimes_{P^W} P$ is the regular representation of W_p. Hence $(\mathbb{F}_p \otimes_{P^W} P)^W = \mathbb{F}_p$ and $\mathrm{Tor}_1^{P^W}(\mathbb{F}_p, H) = 0$.

If the order or W_p is *not* prime to p then $\mathbb{F}_p \otimes_{P^W} P$ is not the regular representation ([14], p. 221). When this happens, we have seen in the proof of 6.2 concrete descriptions of the invariants P^W. From these descriptions, it is not difficult to write down explicitly the coinvariants $\mathbb{F}_p \otimes_{P^W} P$ and see that there are no invariants of positive degree.

We have to consider now the case in which there is an invariant in degree 2, i.e. the case in which W is exceptional. In this case, we have also seen in the proof of 6.2 concrete descriptions of the ring of invariants P^W. If $\delta_1 \neq 0$ then it is very easy to compute the cokernel directly. If $\delta_1 = 0$ then the above proof breaks down because H is not a free P^W-module anymore. However, one can also compute directly H as a cokernel, at least as a P^W-module. We have $P^W = \mathbb{F}_p[u, (vu^{p-1} - v^p)^2]$ and we want to compute the cokernel H of

$$\varphi\colon \mathbb{F}_p[u, v^2] \oplus \mathbb{F}_p[u, (u+v)^2] \to \mathbb{F}_p[u, v]$$

as a P^W-module. φ is a linear map between free P^W-modules. We can take basis in the following way: $1, v^2, \ldots, v^{2p-2}$ is a basis of $\mathbb{F}_p[u, v^2]$; $1, (u+v)^2, \ldots, (u+v)^{2p-2}$ is a basis of $\mathbb{F}_p[u, (u+v)^2]$ and $1, v, \ldots v^{2p-1}$ is a basis of $\mathbb{F}_p[u, v]$. Then, if we study the matrix of φ in these basis, we see easily that $z_{4i+3} = \delta(v^{2i+1})$ for $i = 0, \ldots, p-1$ generate H as a P^W-module (δ denotes the connecting homomorphism of the Mayer-Vietoris exact sequence). Also, we see that $uv^{2i+1} \in \mathrm{Im}\,\varphi$ for $i = 0, \ldots, p-2$. This gives an epimorphism

$$\Big(\bigoplus_{i=0}^{p-2} \frac{H^{\mathrm{even}}}{x_2 H^{\mathrm{even}}} \cdot z_{4i+3}\Big) \oplus H^{\mathrm{even}} \cdot z_{4p-1} \twoheadrightarrow H.$$

Then, a computation of the Poincaré series which is left to the reader proves that this is an isomorphism. □

The results of these last two propositions are displayed in the column "cohomology rings" of table 10.

We want to compute now the action of the mod p Steenrod powers on $H^*(X_W)$ and the differentials in the mod p Bockstein spectral sequence. Since the even dimensional part of $H^*(X_W)$ coincides with the invariants of the mod p Weyl group W_p, we only need to describe the values of the Steenrod powers on the odd dimensional generators. If x and y are in $H^*(X_W)$ the notation $\beta_{(r)}(x) = y$ means that $\beta_{(i)}(x) = 0$ for $i < r$ (i.e. x survives to the E_r-page of the Bockstein spectral sequence) while $\beta_{(r)}(x) = y + \operatorname{Im}\beta_{(r-1)}$. We consider each type **I** to **VI** separately.

• Type **I**. In this case
$$H^*(X_W) \cong \mathbb{F}_p[x_4, y_4] \otimes E(z_5).$$
The *integral* representation is given by a matrix $\begin{pmatrix} p^{\delta_1} & 1 \\ 1 & \lambda \end{pmatrix}$ with $\nu_p(\lambda) = \delta_2$ (cf. table 1). The integral cohomology of BH_1 and BH_2 is given by $\widehat{\mathbb{Z}}_p[p^{\delta_1}u + v, (u + \lambda v)^2]$ and $\widehat{\mathbb{Z}}_p[u, v^2]$ respectively. As an integral lift of the class z_5 we can take the element $\delta(uv)$. Then, in a similar way as we did above, one sees that there are Bockstein relations $\beta_{(\delta_1)}(y_4) = z_5$ and $\beta_{(\delta_2)}(x_4) = z_5$. The action of the Steenrod powers on z_5 is given by the relation $\mathcal{P}^1(z_5) = (x_4^{\frac{p-1}{2}} + y_4^{\frac{p-1}{2}})z_5$.

• Type **II**. We have
$$H^*(X_W) \cong \mathbb{F}_p[x_4, y_{4p}] \otimes E(z_{4p+1}).$$
The *integral* representation is given by a matrix $\begin{pmatrix} 1 & 1 \\ 1 & \lambda \end{pmatrix}$ with $\lambda = \frac{\Gamma_{1,1}}{\Gamma_{1,1}-1}$ and $\nu_p(\lambda) = \delta_2$ (cf. table 1). The integral cohomology of BH_1 and BH_2 is given by $\widehat{\mathbb{Z}}_p[u + v, (u + \lambda v)^2)]$ and $\widehat{\mathbb{Z}}_p[u, v^2]$ respectively. Then,

(a) Analyzing the map
$$\varphi_p \colon \mathbb{F}_p[u, v^2] \oplus \mathbb{F}_p[u + v, u^2] \to \mathbb{F}_p[u, v]$$
in degrees $4p$ and $2(3p-1)$ we see that $uv^{2p-1} \notin \operatorname{Im}\varphi_p$ while $2u^pv^{2p-1} - uv^{3p-2} \in \operatorname{Im}\varphi_p$.

(b) Analyzing the map
$$\varphi_{p^2} \colon \mathbb{Z}/p^2[u, v^2] \oplus \mathbb{Z}/p^2[u + v, u^2] \to \mathbb{Z}/p^2[u, v]$$
in degree $4p$ (notice that $\lambda^2 \equiv 0\,(p^2)$) we see that $puv^{2p-1} \in \operatorname{Im}\varphi_{p^2}$, except for the case in which $p = 3$ and $\lambda \equiv 3\,(9)$.

From these facts, we can conclude that we can take as generator z_{4p+1} any non trivial multiple of $\delta(uv^{2p-1})$. Also, z_{4p+1} has to be in the image of a primary Bockstein (except for $p = 3$ and $\lambda \equiv 3\,(9)$) and we can define

$z_{4p+1} = \beta(y_{4p})$. The identity $\mathcal{P}^1(uv^{2p-1}) = u^p v^{2p-1} - uv^{3p-2}$ shows that $\mathcal{P}^1(z_{4p+1}) = -x^{(p-1)/2}z$ and the Adem relation $\mathcal{P}^1\beta\mathcal{P}^{p-1} = -\beta\mathcal{P}^p + \mathcal{P}^p\beta$ determines the value of $\mathcal{P}^p(z_{4p+1})$. The exceptional case in which $p = 3$ and $\lambda \equiv 3\,(9)$ can be investigated directly. We obtain that there is a Bockstein of order *two* joining y_{12} and z_{13} and also $\mathcal{P}^3(z_{13}) = (y_{12} + x_4^3)z_{13}$. The description of $H^*(X_W)$ is then complete.

• Type **III**. In this case

$$H^*(X_W) \cong \mathbb{F}_p[x_2, y_4] \otimes E(z_3).$$

The *integral* representation is given by a matrix $\begin{pmatrix} 1 & p^{\delta_2} \\ \lambda & 1 \end{pmatrix}$ with $\nu_p(\lambda) = \delta_1$ (cf. table 1). The integral cohomology of BH_1 and BH_2 is given by $\widehat{\mathbb{Z}}_p[u + p^{\delta_2}v, (\lambda u + v)^2]$ and $\widehat{\mathbb{Z}}_p[u, v^2]$ respectively. As an integral lift of the class z_3 we van take the element $\delta(v)$. Then, notice that $p^{\delta_2}v \in \operatorname{Im}\{P^{\omega_1} \oplus P^{\omega_2} \to P\}$. This produces a relation $\beta_{(\delta_2)}(x_2) = z_3$. On the other side, one sees that $p^{\delta_1}uv \in \operatorname{Im}\{P^{\omega_1} \oplus P^{\omega_2} \to P\}$ and so there is a relation $\beta_{(\delta_1)}(y_4) = x_2 z_3$.

The action of the Steenrod powers on z_3 is given by the relation $\mathcal{P}^1(z_3) = y_4^{\frac{p-1}{2}} z_3$. This follows from $z_3 = \delta(v)$ and the fact that the Steenrod powers commute with the connecting homomorphism.

• Type **IV**. We will not compute the action of the Steenrod algebra nor the Bockstein spectral sequence for the spaces of this (quite weird) type.

• Type **V**. This case is quite similar to the case in type **II**. We have

$$H^*(X_W) \cong \mathbb{F}_p[x_4, y_{2p}] \otimes E(z_{2p+1}).$$

The *integral* representation is given by a matrix $\begin{pmatrix} 1 & 1 \\ \lambda & 1 \end{pmatrix}$ with $\lambda = \frac{\Gamma_{1,1}-1}{\Gamma_{1,1}}$ and $\nu_p(\lambda) = \delta_1$ (cf. table 1). The integral cohomology of BH_1 and BH_2 is given by $\widehat{\mathbb{Z}}_p[u + v, (\lambda u + v)^2)]$ and $\widehat{\mathbb{Z}}_p[u, v^2]$ respectively. We leave as an exercise to the reader to check that the following holds:

(a) Analyzing the map

$$\varphi_p \colon \mathbb{F}_p[u, v^2] \oplus \mathbb{F}_p[u + v, v^2] \to \mathbb{F}_p[u, v]$$

in degrees $2p$ and $2(2p-1)$ we see that $u^{p-1}v \notin \operatorname{Im}\varphi_p$ while $u^{p-1}v^p - u^{2p-2}v \in \operatorname{Im}\varphi_p$.

(b) Analyzing the map

$$\varphi_{p^2} \colon \mathbb{Z}/p^2[u, v^2] \oplus \mathbb{Z}/p^2[u + v, v^2] \to \mathbb{Z}/p^2[u, v]$$

in degree $2p$ (notice that $\lambda^2 \equiv 0\,(p^2)$) we see that $pu^{p-1}v \in \operatorname{Im}\varphi_{p^2}$, except for the case in which $p = 3$ and $\lambda \equiv 6\,(9)$.

From these facts, we can conclude that we can take as generator z_{2p+1} any non trivial multiple of $\delta(u^{p-1}v)$. Also, z_{2p+1} has to be in the image of a primary Bockstein (except for $p = 3$ and $\lambda \equiv 6\,(9)$) and we can define $z_{2p+1} = \beta(y_{2p})$. The identity $\mathcal{P}^1(u^{p-1}v) = u^{p-1}v^p - u^{2p-2}v$ shows that $\mathcal{P}^1(z_{2p+1}) = 0$ and the Adem relation $\mathcal{P}^1\beta\mathcal{P}^{p-1} = -\beta\mathcal{P}^p + \mathcal{P}^p\beta$ yields $\mathcal{P}^p(z_{2p+1}) = x_4^{p(p-1)/2} z_{2p+1}$.

The exceptional case in which $p = 3$ and $\lambda \equiv 6\,(9)$ can be investigated directly. We obtain that there is a Bockstein of order *two* joining y_6 and z_7 and also $\mathcal{P}^3(z_7) = x_4^3 z_7$.

In this way we have a complete description of the algebra $H^*(X_W)$, including the Steenrod operations and the Bockstein spectral sequence.

• Type **VI**. First of all, there is a Bockstein of height r which connects y_{2k} and z_{2k+1} where the integer r is given by the following lemma:

Lemma 6.5. *The integer r such that $\beta_{(r)}(y_{2k}) = z_{2k+1}$ is given by*

- *r is such that $|W_p| = |W_{p^2}| = \cdots = |W_{p^r}| < |W_{p^{r+1}}|$.*
- *Let (λ_n) be the sequence defined recursively by $\lambda_0 = 0$, $\lambda_1 = 1$,*

$$\begin{aligned} \lambda_{2n} &= \lambda_{2n-1} - \lambda_{2n-2} \\ \lambda_{2n+1} &= 4\Gamma\lambda_{2n} - \lambda_{2n-1}. \end{aligned}$$

Then r is the p-adic valuation of λ_k and λ_i is prime to p for $i < k$.

Proof. To prove the second part of the lemma, notice that the order of the finite dihedral group W_{p^r} is equal to twice the order in $GL_2(\mathbb{Z}/p^r\mathbb{Z})$ of the matrix

$$\tau = \begin{pmatrix} 2\Gamma - 1 & 2\Gamma \\ 2\Gamma - 2 & 2\Gamma - 1 \end{pmatrix}$$

which is the product of the two generating reflections of W. In particular, k is the smallest integer such that $\tau^k \equiv I\,(p)$ and r is the largest integer such that $\tau^k \equiv I\,(p^r)$. The linear transformation

$$\begin{pmatrix} 0 & 1 \\ 1/2\Gamma & -1 \end{pmatrix}$$

transforms τ into the matrix A associated to the iterative system which produces the sequence $\{\lambda_i\}$ defined above. Then, an easy induction proof shows that

$$A^j = \begin{pmatrix} \lambda_{2j+1} & -4\Gamma\lambda_{2j} \\ \lambda_{2j} & -\lambda_{2j-1} \end{pmatrix}.$$

If $A^j \equiv I\,(p^n)$, then $\lambda_{2j} \equiv 0\,(p^n)$ and $\lambda_{2j-1} \equiv 1\,(p^n)$. Consider the sequence $\lambda_0, \lambda_1, \ldots, \lambda_{2j-1}, \lambda_{2j}$. If we compute the value of the central term λ_j starting from both extremes of the sequence, we obtain $\lambda_j \equiv -\lambda_j\,(p^n)$ and so $\lambda_j \equiv 0\,(p^n)$.

Conversely, if $\lambda_i \equiv 0\,(p^n)$ and i is odd, $i = 2j+1$, then

$$A^j \equiv \begin{pmatrix} 0 & -4\Gamma\lambda_{2j} \\ \lambda_{2j} & -4\Gamma\lambda_{2j} \end{pmatrix}.$$

Since A has determinant equal to one, we obtain $4\Gamma\lambda_{2j}^2 \equiv 1\,(p^n)$. Then, a computation shows that $A^i \equiv I\,(p^n)$. If i is even, $i = 2j$, then we have

$$A^j \equiv \begin{pmatrix} -\lambda_{2j+1} & 0 \\ 0 & -\lambda_{2j+1} \end{pmatrix}$$

with $\lambda_{2j+1} \equiv \pm 1\,(p^n)$. If $\lambda_{2j+1} \equiv -1\,(p^n)$ then the same argument as above shows that $\lambda_j \equiv 0\,(p^n)$ too. Hence, without loss of generality we can assume that $\lambda_{2j+1} \equiv 1\,(p^n)$ and so $A^{2j} \equiv I\,(p^n)$.

From this relationship between the order of the matrix A and the p-adic valuation of the elements in the sequence $\{\lambda_i\}$ the second part of the proposition follows easily.

While we have been obtaining all the results in this paper in a pure algebraic and homotopy theoretic way, the proof of the first part of the proposition will be done using the geometric structure of the Kac-Moody groups. Let M be a matrix which yields the space X_W. According to proposition 4.1, the cohomology algebra of X_W and the Bockstein spectral sequence will not change if we choose any other matrix $M' \equiv M\,(p^N)$, for some large N. The matrix M is characterized by its invariant $\Gamma_{1,1}$ which is any p-adic integer $\not\equiv 0, 1\,(p)$. On the other side, if we start with a Kac-Moody group $K(a,b)$, we obtain a space of the form $X(M')$ where M' has invariant $\Gamma_{1,1} = ab/4$ (see proposition 7.2). This shows that it is possible to find values of a and b such that the Kac-Moody group $K = K(a,b)$ has the property that $BK_p^\wedge \simeq X(M')$ with $M' \equiv M\,(p^N)$ for some large N. Hence, the action of the higher Bockstein operations on X_W should be the same as the action on BK. This action was computed by Kitchloo ([12]), using the geometric structure of the homogeneous space K/T_K (i.e. the Schubert calculus) and it agrees with the values given in this proposition. □

Using what we have seen in the proof of the above proposition, we can produce now concrete descriptions of the generators $x_4, y_{2k} \in H^*(X_W)$ in terms of the generators $u, v \in H^2(BT)$. Let $\{\lambda_i\}$ denote the sequence defined recursively in the previous proposition. Then we can choose

$$x_4 = u^2 - 4\Gamma uv + 4\Gamma v^2$$

$$y_{2k} = \prod_{j=1}^{k} (\lambda_{2j}u - \lambda_{2j-1}v)$$

and this allows us, in principle, to compute the action of the Steenrod algebra on the even dimensional part of $H^*(X_W)$.

	$\overline{\mathrm{Rep}}_{1,1}$	Cohomology ring	Steenrod powers	BSS
I	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\mathbb{F}_p[x_4, y_4] \otimes E(z_5)$	$\mathcal{P}^1(z) = (x^{(p-1)/2} + y^{(p-1)/2})z$	$\beta_{(\delta_1)}(y) = z$ $\beta_{(\delta_2)}(x) = z$
II	$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$	$\mathbb{F}_p[x_4, y_{4p}] \otimes E(z_{4p+1})$	$\mathcal{P}^1(z) = -x^{(p-1)/2}z$ $\mathcal{P}^p(z) = \mathcal{P}^1\beta\mathcal{P}^{p-1}(y) + \beta\mathcal{P}^p(y)$	$\beta(y) = z$
			except $p = 3$, $\frac{\Gamma_{1,1}}{\Gamma_{1,1-1}} \equiv 3 \bmod 9$:	$\beta_{(2)}(y) = z$
III	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\mathbb{F}_p[x_2, y_4] \otimes E(z_3)$	$\mathcal{P}^1(z) = y^{(p-1/2)}z$	$\beta_{(\delta_1)}(y) = xz$ $\beta_{(\delta_2)}(x) = z$
IV	$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	–		
V	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$	$\mathbb{F}_p[x_4, y_{2p}] \otimes E(z_{2p+1})$	$\mathcal{P}^1(z) = 0$ $\mathcal{P}^p(z) = x^{p(p-1)/2}z$	$\beta(y) = z$
			except $p = 3$, $\frac{\Gamma_{1,1-1}}{\Gamma_{1,1}} \equiv 6 \bmod 9$:	$\beta_{(2)}(y) = z$
VI	$\begin{pmatrix}1 & 1\\ x & 1\end{pmatrix}$ $x \neq 0, 1$	$\mathbb{F}_p[x_4, y_{2k}] \otimes E(z_{2k+1})$		$\beta_{(r)}(y) = z$ (see 6.5)

Table 10. **Odd primary cohomology rings.** The degrees of the generators of the cohomology rings are specified by subscripts (which are omitted in columns four and five).

Let us see now how we can compute the action of the Steenrod powers on the odd dimensional generator z_{2k+1}. If we write $H = H^{\mathrm{odd}}(X_W)$, then $H = z_{2k+1}P^W$ and the Steenrod algebra action can be described by means of the Mayer-Vietoris exact sequence (see Theorem 6.4)

$$0 \to P^W \to P^{\omega_1} \oplus P^{\omega_2} \to P \to \Sigma^{-1} H \to 0\,,$$

thus, the action of the Steenrod algebra on the class z_{2k+1} is determined if we are able to obtain a representative $J \in P$ for this class. We will show that the jacobian

$$J = \det \begin{pmatrix} \frac{\partial x_4}{\partial u} & \frac{\partial x_4}{\partial v} \\ \frac{\partial y_{2k}}{\partial u} & \frac{\partial y_{2k}}{\partial v} \end{pmatrix}$$

is a representative for z_{2k+1}. This is a non trivial element of P, relative invariant to the determinant; that is, for any $g \in W_p$, $g(J) = \det(g) \cdot J$. This implies that J, does not belong to the image of $P^{\omega_1} \oplus P^{\omega_2} \to P$. Assume otherwise that J can be written as $J = p_1 + p_2$, with p_1 invariant by w_1 and p_2 invariant by w_2. Apply w_1 to the equality $J = p_1 + p_2$ and combine to get $2J = p_2 - w_1 p_2$. Now apply w_2: $-2J = p_2 - (w_2 w_1)p_2$. Observe that J is invariant by $w_2 w_1$, and therefore we can obtain, inductively $-2nJ = p_2 - (w_2 w_1)^n p_2$. Hence $-2kJ = 0$, and since k is coprime to p, this contradicts the fact that $J \neq 0$.

We have shown that J represents a non trivial element in H. It has to be z_{2k+1}, up to a unit of $\mathbb{F}_p$, by degree reasons.

Remark 6.6. Notice that $z_{2k+1} P^W \cong H$ is a Thom module in the sense of [4]. For any linear character $\chi\colon W \to \mathbb{F}_p^*$, the relative invariants

$$P^W_\chi = \left\{\, x \in P \;\middle|\; w(x) = \chi(w) x \,\right\}$$

form a Thom module over P^W; that is, a P^W-$\mathcal{U}$-module which is free of rank one as P^W-module [4]. The relative invariants of any non modular pseudoreflection group with respect to the determinant are computed in [14], p. 227 (notice that in our case we have $\det = \det^{-1}$): $P^W_{\det^{-1}} = J \cdot P^W$, where $J = \det \frac{\partial f_i}{\partial t_j}$, if we have $P^W = \mathbb{F}_p[f_1, \ldots, f_d]$. With this notation, the above argument shows that the composition $P^W_{\det} \subset P \to \Sigma^{-1} H$ is an isomorphism of Thom modules.

7. Back to Kac-Moody groups

In section 2 we have included the family of the p-completions of the classifying spaces of the central quotients of the rank two Kac-Moody groups in a larger family which we called S^*. In section 3, we have seen how the spaces in S^* are parametrized by a set R^* which is closely related to the representations of the infinite dihedral group. Then, in the next sections, we have computed the cohomology of all spaces in S^*, as a function of the corresponding representations. In this final section we go back to Kac-Moody groups and we will describe, for each space of the form $B(K/F)$, its parameters as an element of S^* or R^* and so we will obtain in this way a complete description of the cohomology of the classifying spaces of the central quotients of the rank two Kac-Moody groups.

We start with a lemma which identifies some central quotients compact Lie groups of rank two.

Lemma 7.1. *We have the following Lie group isomorphisms:*

(1) $\begin{pmatrix} p^m & 0 \\ 0 & p^n \end{pmatrix} : \dfrac{T}{\langle (e^{2\pi i/p^m},1),\,(1,e^{2\pi i/p^n})\rangle} \xrightarrow{\cong} T.$

(2) $\begin{pmatrix} p^m & -1 \\ 0 & 1 \end{pmatrix} : \dfrac{T}{\langle (e^{2\pi i/p^m},e^{2\pi i/p^m})\rangle} \xrightarrow{\cong} T.$

(3) $\begin{pmatrix} 2^{m-1} & 1 \\ 2^{m-1} & -1 \end{pmatrix} : \dfrac{S^1\times SU(2)}{\langle (e^{2\pi i/2^m},-I)\rangle} \xrightarrow{\cong} U(2).$

(4) $\begin{pmatrix} 2^{m-1} & 2^{m-1} \\ 1 & -1 \end{pmatrix} : \dfrac{U(2)}{\langle e^{2\pi i/2^m} I\rangle} \xrightarrow{\cong} S^1\times SO(3).$

(5) $\begin{pmatrix} 2^m & 0 \\ 0 & 2 \end{pmatrix} : \dfrac{S^1\times SU(2)}{\langle (e^{2\pi i/2^m},1),\,(1,-I)\rangle} \xrightarrow{\cong} S^1\times SO(3).$

Proof. Of course, the matrices that appear in the lemma represent the linear maps induced by each isomorphism on the Lie algebra of the maximal torus. The isomorphisms in (1), (2) and (5) are obvious and we only need to discuss (3) and (4).

Notice that if A is a matrix in $U(2)$ then $A/\sqrt{\det A}$ has an indeterminacy in $SU(2)$ but it is well defined in $SO(3) = SU(2)/\pm I$. This fact allows us to define an epimorphism $\varphi\colon U(2)\to S^1\times SO(3)$ by $A\mapsto(\det A^{2^{m-1}}, A/\sqrt{\det A})$. The kernel of φ is a central subgroup in $U(2)$ generated by $e^{2\pi i/2^m}I$ and one can easily check that φ is given, on the maximal torus level, by the matrix $\begin{pmatrix} 2^{m-1} & 2^{m-1} \\ 1 & -1 \end{pmatrix}$. This proves claim (4) in the lemma.

We also have an epimorphism $\psi\colon S^1\times SU(2)\to U(2)$ given by $(\lambda,A)\mapsto \lambda^{2^{m-1}}A$, which describes $U(2)$ as a quotient of $S^1\times SU(2)$ by the central subgroup of $S^1\times SU(2)$ generated by $(e^{2\pi i/2^m},-I)$. Again, on the maximal torus level, one sees easily that ψ is given by $\begin{pmatrix} 2^{m-1} & 1 \\ 2^{m-1} & -1 \end{pmatrix}$ and we have proven claim (3) in the lemma. □

We can now represent the classifying spaces of the central quotients of K as spaces of the form $X^{k,l}(M)$. We do first the case of an odd prime.

Proposition 7.2. *Let p be an odd prime and $0\le m\le \nu_p(ab-4)$. Write $P_{p^0}K = K$. We have*

$$BP_{p^m}K^\wedge_p \simeq X\begin{pmatrix} b/2 & p^m \\ (4-ab)/4p^m & -a/2 \end{pmatrix}.$$

Proof. The case $m = 0$ is trivial. For $m > 0$ we have

$$B((S^1 \times SU(2))/MF)^\wedge_p \xleftarrow{M} (BT/F)^\wedge_p \xrightarrow{\text{id}} B((S^1 \times SU(2))/F)^\wedge_p$$

where $M = \begin{pmatrix} b/2 & 1 \\ (4-ab)/4 & -a/2 \end{pmatrix}$ and F is the cyclic subgroup of T generated by $(e^{2\pi i/p^m}, 1)$. Since p^m divides $4 - ab$, we see that MF is the cyclic subgroup of T generated by $(e^{2\pi i/p^m}, 1)$. Now the diagram above is equivalent to the diagram for $X(N)$ for

$$N = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1/p^m & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b/2 & p^m \\ (4-ab)/4p^m & -a/2 \end{pmatrix}. \quad \square$$

From the above description we can immediately recover the representation associated to the space $BP_{p^m}K(a,b)^\wedge_p$ as the one given by the invariants

$$\Gamma_{1,1} = \frac{ab}{4}, \quad \delta_1 = \nu_p(4-ab) + \nu_p(b) - m, \quad \delta_2 = \nu_p(a) + m.$$

In particular, this shows that, for p odd, any representation of D_∞ whose invariants are such that $4 < 4\Gamma_{1,1} \in \mathbb{Z}$ and $\delta_1 + \delta_2 < \infty$ appears as a space of the form $BP_{p^m}K(a,b)^\wedge_p$ for some a, b, m.

Let us consider now the case of the even prime. This case is more involved since we have to take into account the parities of a and b. Moreover, in some cases there are central 2-subgroups which are not cyclic. Let us distinguish three cases.

- $a \equiv b \equiv 1 \pmod 2$. In this case, there are no central 2-subgroups and we only need to consider the case of the simply connected group K. We have

$$BK^\wedge_2 \simeq X^{2,2}\begin{pmatrix} \frac{(1-a)(1+b)}{4} + 1 & \frac{(1-a)(b-1)}{4} + 1 \\ \frac{(1+a)(1+b)}{4} - 1 & \frac{(1+a)(b-1)}{4} - 1 \end{pmatrix}.$$

- $a \equiv 0$ $b \equiv 1 \pmod 2$. In this case, for each m such that 2^m divides $ab - 4$ there is only one central subgroup of K of order 2^m and it is cyclic. To simplify the notation, let us write $a = 2a'$, $b = 2b' + 1$. Then we have

$$BK^\wedge_2 \simeq X^{0,2}\begin{pmatrix} 1 + b' & b' \\ 1 - a' - a'b' & 1 - a'b' \end{pmatrix}.$$

The matrices corresponding to the groups $P_{2^m}K$ are given by:

Proposition 7.3. *Let $a = 2a'$, $b = 2b' + 1$. Then*

$$BP_{2^m}K_2^\wedge \simeq \begin{cases} X^{0,1}\begin{pmatrix} b & 2^{m-1} \\ \frac{4-ab}{2^{m+1}} & -\frac{a'}{2} \end{pmatrix} & 0 < m < \nu_2(ab-4); \\ & \\ X^{1,2}\begin{pmatrix} \frac{a'}{2} + 2^{m-2} & \frac{a'}{2} - 2^{m-2} \\ \frac{4-ab}{2^{m+1}} - \frac{b}{2} & \frac{4-ab}{2^{m+1}} + \frac{b}{2} \end{pmatrix} & m = \nu_2(ab-4). \end{cases}$$

Proof. Notice that the matrices above are in $GL_2(\widehat{\mathbb{Z}}_2)$ because $\nu_2(ab-4) = 1$ when a' is odd. Let us recall that BK is a push out

$$B(S^1 \times SU(2)) \xleftarrow{M} BT \xrightarrow{\text{id}} BU(2)$$

with $M = \begin{pmatrix} 1+b' & b' \\ 1-a'-a'b' & 1-a'b' \end{pmatrix}$ and we have to divide out by a central cyclic subgroup F of order 2^m. This subgroup will be generated by the element $(e^{2\pi i/2^m}, e^{2\pi i/2^m})$ in T and $U(2)$. To compute the image of this element in $S^1 \times SU(2)$ we need to apply the matrix M to the vector $(1/2^m, 1/2^m)$. We get $(b/2^m, (4-ab)/2^{m+1})$ and two possibilities arise. If $m < \nu_2(ab-4)$ then $(4-ab)/2^{m+1}$ is an integer and the image of F by M is the subgroup generated by $(e^{2\pi i/2^m}, 1)$ in $S^1 \times SU(2)$. If $m = \nu_2(ab-4)$ then $(4-ab)/2^{m+1}$ is half an integer and so the image of F by M is generated by $(e^{2\pi i/2^m}, -I)$ in $S^1 \times SU(2)$. In either case, we have to transform a diagram

$$(*) \qquad \frac{S^1 \times SU(2)}{\langle (e^{2\pi i/2^m}, \pm 1)\rangle} \xleftarrow{M} \frac{T}{\langle (e^{2\pi i/2^m}, e^{2\pi i/2^m})\rangle} \xrightarrow{\text{id}} \frac{U(2)}{\langle e^{2\pi i/2^m} I\rangle}$$

in a diagram like the one used to define the spaces $X^{k,l}(N)$ for some k, l and N. Apply the isomorphisms of lemma 7.1. We see that the right hand side of the diagram $(*)$ is equivalent to $A\colon T \to S^1 \times SO(3)$ with

$$A = \begin{pmatrix} 2^{m-1} & 2^{m-1} \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/2^m & 1/2^m \\ 0 & 1 \end{pmatrix}.$$

If $m < \nu_2(ab-4)$ then the left hand side of $(*)$ is equivalent to $B\colon T \to S^1 \times SU(2)$ with

$$B = \begin{pmatrix} 2^m & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+b' & b' \\ 1-a'-a'b' & 1-a'b' \end{pmatrix}\begin{pmatrix} 1/2^m & 1/2^m \\ 0 & 1 \end{pmatrix}.$$

If $m = \nu_2(ab-4)$ then the left hand side of $(*)$ is isomorphic to $U(2)$. In either case, we can easily check the matrices in the proposition. $\square$

• $a \equiv b \equiv 0 \pmod 2$. In this case we need to take into account the various groups $P^L_{2^m}K$, P^R_2K, $P^D_{2^m}K$ and $P^N_{2^{m+1}}K$ for $0 < m \leq \nu_2((ab-4)/2)$. Put $a = 2a'$, $b = 2b'$. One checks immediately that

$$BK_2^\wedge \simeq X^{0,0}\begin{pmatrix} b' & 1 \\ 1-a'b' & -a' \end{pmatrix}.$$

Proposition 7.4. *Let $a = 2a'$, $b = 2b'$ and let $0 < m \leq \nu_2((ab-4)/2)$. Then*

(1) *If b' is even then $m = 1$ and*

$$BP^L_2K \simeq X^{0,1}\begin{pmatrix} 2a' & 1 \\ 1-a'b' & -b'/2 \end{pmatrix}.$$

(2) *If b' is odd then*

$$BP^L_{2^m}K_2^\wedge \simeq \begin{cases} X^{0,0}\begin{pmatrix} b' & 2^m \\ (1-a'b')/2^m & -a' \end{pmatrix} & 0 < m < \nu_2((ab-4)/2); \\ \\ X^{0,2}\begin{pmatrix} a'+2^{m-1} & a'-2^{m-1} \\ \frac{1-a'b'}{2^m}-\frac{b'}{2} & \frac{1-a'b'}{2^m}+\frac{b'}{2} \end{pmatrix} & m = \nu_2((ab-4)/2). \end{cases}$$

Proof. The proof of this is similar to the proofs of 7.2 and 7.3. $BP^L_{2^m}$ is given by a diagram which is obtained from the diagram of Lie groups

$$\frac{S^1 \times SU(2)}{\langle M(e^{2\pi i/2^m},1)\rangle} \xleftarrow{M} \frac{T}{\langle (e^{2\pi i/2^m},1)\rangle} \xrightarrow{\text{id}} \frac{S^1 \times SU(2)}{\langle (e^{2\pi i/2^m},1)\rangle}$$

with $M = \begin{pmatrix} b' & 1 \\ 1-a'b' & -a' \end{pmatrix}$. When we apply the matrix M to the vector $(1/2^m, 0)$ we obtain $(b'/2^m, (1-a'b')/2^m)$ and we have to distinguish the cases of b' even and b' odd and also the cases when $m < \nu_2(2-a'b)$ and $m = \nu_2(2-a'b)$. Finally, we need to use the identifications given by lemma 7.1. We leave the details to the reader. □

The computations for the other central quotients P^R_2K, $P^D_{2^m}K$ and $P^N_{2^{m+1}}K$ can be done using the same ideas and we will omit the proofs.

Proposition 7.5. *Let $a = 2a'$, $b = 2b'$. Then*

(1) *If a' is even then*

$$BP^R_2K_2^\wedge \simeq X^{0,1}\begin{pmatrix} 2b' & 1 \\ 1-a'b' & -a'/2 \end{pmatrix}.$$

(2) *If a' is odd then*

$$BP^R_2K_2^\wedge \simeq X^{1,2}\begin{pmatrix} (a'+1)/2 & (a'-1)/2 \\ 1-b'-a'b' & 1+b'-a'b' \end{pmatrix}. \quad \square$$

Proposition 7.6. *Let $a = 2a'$, $b = 2b'$ and $1 \le m \le \nu_2(2 - a'b)$. Then*

(1) *If $a'b'$ is odd, then*

- $BP_2^D K_2^\wedge \simeq X^{1,2}\begin{pmatrix} (b'+1)/2 & (b'-1)/2 \\ 1-a'b'-a' & 1-a'b'+a' \end{pmatrix}$.
- $BP_{2^m}^D K_2^\wedge \simeq$
  $$X^{2,2}\begin{pmatrix} \frac{1-a'b'}{2^m} + \frac{b'-a'}{2} + 2^{m-2} & \frac{1-a'b'}{2^m} + \frac{b'+a'}{2} - 2^{m-2} \\ \frac{a'b'-1}{2^m} + \frac{b'+a'}{2} + 2^{m-2} & \frac{a'b'-1}{2^m} + \frac{b'-a'}{2} - 2^{m-2} \end{pmatrix},$$
  if $1 < m < \nu_2(2 - a'b)$.
- $BP_{2^m}^D K_2^\wedge \simeq X^{0,2}\begin{pmatrix} b' + 2^{m-1} & b' - 2^{m-1} \\ \frac{1-a'b'}{2^m} - \frac{a'}{2} & \frac{1-a'b'}{2^m} + \frac{a'}{2} \end{pmatrix}$,
  if $m = \nu_2(2 - a'b)$.

(2) *If a' and b' are both even, then $m = 1$ and*
$$BP_2^D K_2^\wedge \simeq X^{2,2}\begin{pmatrix} \frac{b'-a'b'-a'}{2} + 1 & \frac{b'-a'b'+a'}{2} \\ \frac{b'+a'b'+a'}{2} & \frac{b'+a'b'-a'}{2} - 1 \end{pmatrix}.$$

(3) *If a' is even and b' is odd, then $m = 1$ and*
$$BP_2^D K_2^\wedge \simeq X^{1,2}\begin{pmatrix} (b'+1)/2 & (b'-1)/2 \\ 1-a'b'-a' & 1-a'b'+a' \end{pmatrix}.$$

(4) *If a' is odd and b' is even, then $m = 1$ and*
$$BP_{2^m}^D K_2^\wedge \simeq X^{0,2}\begin{pmatrix} b'+1 & b'-1 \\ (1-a'b'-a')/2 & (1-a'b'+a')/2 \end{pmatrix}. \quad \square$$

Proposition 7.7. *Let $a = 2a'$, $b = 2b'$ and $1 \le m \le \nu_2(2 - a'b)$. Then*
$$BP_{2^{m+1}}^N K \simeq X^{1,1}\begin{pmatrix} b' & 2^{m-1} \\ (1-a'b')/2^{m-1} & -a' \end{pmatrix}. \quad \square$$

This completes the description of the spaces $B(K/F)_p^\wedge$ as spaces in S^*. From the above computations it is straightforward to determine, for each central quotient of each rank two Kac-Moody group, the values of their parameters in R^* and then to use the computations in the preceding sections to determine the cohomology of its classifying space. In the case of the prime 2, the description of the invariants associated to each space has not the simple form that it has for the odd primes, except for the case in which the representation turns out to be in $\mathrm{Rep}_{1,1}$, when we always have $\Gamma_{1,1} = ab/4$.

The table 11 displays a part of the information which we have obtained in the above propositions in a more user-friendly way.

$p > 2$				
$\mathrm{Rep}_{1,1}$ $\Gamma_{1,1} = \frac{ab}{4}$ $\delta_1 = \nu_p(4 - ab) + \nu_p(b) - m$ $\delta_2 = \nu_p(a) + m$			$P_{p^m} K$	
$p = 2$				
$\mathrm{Rep}_{1,1}$ $\Gamma_{1,1} = ab/4$	$c{=}0$			$P^N_{2m+1} K$ $m \leq \nu_2(2 - \frac{ab}{2})$
	$c{=}(0,1)$	$P_{2^m} K$ $m < \nu_2(ab - 4)$ $a \equiv 0,\ b \equiv 1\,(2)$	$P^R_2 K$ $a \equiv 0\,(4)$	$P^L_2 K$ $b \equiv 0\,(4)$
	$c{=}(1,1)$	K $a \equiv b \equiv 0\,(2)$		$P^L_{2^m} K$ $m < \nu_2(\frac{ab-4}{2})$ $b \equiv 2\,(4)$
$\mathrm{Rep}_{1,2}$	$c{=}0$	$P_{2^m} K$ $m = \nu_2(ab - 4)$ $a \equiv 0,\ b \equiv 1\,(2)$	$P^R_2 K$ $a \equiv 2\,(4)$	$P^D_2 K$ $b \equiv 2\,(4)$
	$c{=}1$	K $a \equiv 0,\ b \equiv 1\,(2)$	$P^L_{2^m} K$ $m = \nu_2(\frac{ab-4}{2})$ $b \equiv 2\,(4)$	$P^D_{2^m} K$ $1 < m = \nu_2(\frac{ab-4}{2})$ or $a \equiv 2,\ b \equiv 0\,(4)$
$\mathrm{Rep}_{2,2}$	$c{=}0$	K $a \equiv b \equiv 1\,(2)$		$P^D_{2^m} K$ $1 < m < \nu_2(\frac{ab-4}{2})$ or $a \equiv b \equiv 0\,(4)$

TABLE 11.

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