AXIOMS OF GENERIC ABSOLUTENESS

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ABSTRACT. We give a unified presentation of the set-theoretic axioms of generic absoluteness, we survey the known results regarding their consistency strength and some of their consequences as well as their relationship to other kinds of set-theoretic axioms, and we provide a list of the main open problems.

1. Introduction

A common theme of Set Theory, after the discovery by Cohen in 1963 of the method of *forcing* for building models of ZFC, has been *how to get rid of forcing.* The forcing technique has proved to be, over the last 40 years, an extremely powerful and flexible tool for building models of ZFC with the most varied properties, thereby proving the independence from the ZFC axioms of a large amount of mathematical statements. Spurred by this array of independence results, it has become a challenge for Set Theory to discover new axioms that would eliminate the relativity of the truth of mathematical statements with respect to different models of ZFC obtained by forcing. The ultimate axiom of this sort would be to require that V is elementary equivalent to $V^{\mathbb{P}}$ for every forcing notion $\mathbb{P}$. But this is clearly impossible, since we can force incompatible statements, for instance, the Continuum Hypothesis and its negation. So, one has to restrict either the class of statements that are absolute between V and its generic extensions, or the class of forcing extensions, or both. But how do we choose between two forceable statements that cannot be both simultaneously true? The answer is that we do not choose, that is, we do not discriminate against statements of the same logical complexity. We want all forceable statements of the same logical complexity to be true. This is always the case for Σ_1

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statements, i.e., formulas of the language of Set Theory which, in normal form, have only existential quantifiers. But for formulas of the next higher level of complexity, the Σ_2, requiring that all those that are forceable are true, is false (both the Continuum Hypothesis and its negation are Σ_2). Thus, we must abandon the ultimate axiom and settle for weaker forms. Namely, we may require that some definable subclass W of V, which is seen as an approximation to V, is sufficiently elementarily equivalent to W as computed in some generic extensions of V. Whenever W is a substructure of W as computed in the generic extensions, then one can even require that W be an elementary substructure of the generic W. That is, we may have elements of the ground-model W as parameters. Many axioms of this sort have been studied in the literature and are currently one of the main topics in the foundations of Set Theory. These are the axioms of *generic absoluteness*, which may also be called axioms of *generic invariance*, that is, they assert that whatever can be forced is true, subject only to the restrictions that are strictly necessary for them to be consistent with the axioms of ZFC.

We give here a unified presentation of the axioms of generic absoluteness. We survey the known results regarding their consistency strength and their relationship to the axioms of large cardinals and the bounded forcing axioms. We illustrate some of their consequences, specially in Descriptive Set Theory, and we provide a list of the main open questions.

Our notation and basic definitions are standard, as in [24]. V is the class of all sets. If $\mathbb{P}$ is a forcing notion, i.e., a partial ordering, in V, then $V^{\mathbb{P}}$ is the Boolean-valued model corresponding to the Boolean completion of $\mathbb{P}$. In this survey we will only consider forcing notions that are *sets*, as opposed to *class* forcing notions. A formula of the first-order language of set theory is Σ_0 if all its quantifiers are bounded. For $n > 0$, a formula is Σ_n if, in its normal form, begins with a sequence of n blocs of unbounded quantifiers of the same kind, starting with a block of existential quantifiers, and followed by a Σ_0 formula. L is the constructible universe. For an infinite cardinal κ, $H(\kappa)$ is the set of all sets whose transitive closure has cardinality $< \kappa$. For a transitive set X, $L(X)$ is the smallest transitive model of ZF that contains all the ordinals and X. $L(\mathbb{R})$ is the smallest transitive model of ZF that contains all ordinals and all the reals. If x is a real number, $L[x]$ is the least transitive model of ZFC that contains all the ordinals and x.

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2. A general framework for axioms of generic absoluteness

An axiom of generic absoluteness is given by requiring that some definable subclass W of V is sufficiently elementarily equivalent to W, as computed in certain generic extensions of V. That is, for some class of sentences Φ and all forcing notions $\mathbb{P}$ belonging to some class of forcing notions Γ,

$$W^V \equiv_\Phi W^{V^{\mathbb{P}}}$$

read: W^V is Φ-elementarily equivalent to $W^{V^{\mathbb{P}}}$. Namely, for all $\varphi \in \Phi$ and all $\mathbb{P} \in \Gamma$,

$$W^V \models \varphi \quad \text{iff} \quad W^{V^{\mathbb{P}}} \models \varphi$$

Thus, we have three variables:

- The subclass W of V.
- The class Φ of sentences.
- The class Γ of forcing notions.

By combining these three variables we obtain the different sorts of axioms of generic absoluteness. In general, the larger the classes W, Φ, and Γ, the stronger the axiom. But there is always a trade-off between them. For instance, as we will see, in the extreme case when one of the three variables is maximal, e.g., when $W = V$, Φ is the class of all sentences, or Γ is the class of all forcing notions, then the other two must be very small for the axiom to be consistent with ZFC.

To unify our presentation we will use the following notation for axioms of generic absoluteness: $\mathcal{A}(W, \Phi, \Gamma)$ is the assertion that W is Φ-elementarily equivalent to $W^{V^{\mathbb{P}}}$ for all $\mathbb{P} \in \Gamma$.

Since we want $\mathcal{A}(W, \Phi, \Gamma)$ to be an axiom, hence a sentence in the first-order language of Set Theory, W, Φ, and Γ must be definable classes.

Notice that if $\Phi \subseteq \Phi'$ and $\Gamma \subseteq \Gamma'$, then $\mathcal{A}(W, \Phi', \Gamma')$ implies $\mathcal{A}(W, \Phi, \Gamma)$. Also notice that $\mathcal{A}(W, \Phi, \Gamma)$ is equivalent to $\mathcal{A}(W, \bar{\Phi}, \Gamma)$, where $\bar{\Phi}$ is the closure of Φ under finite Boolean combinations. Thus, for instance, $\mathcal{A}(W, \Sigma_n, \Gamma)$ is equivalent to $\mathcal{A}(W, \Pi_n, \Gamma)$.

An important case is when Φ is a class of sentences with parameters. In this case, for $\mathcal{A}(W, \Phi, \Gamma)$ to make sense, the parameters must belong to $W \cap W^{V^{\mathbb{P}}}$, for all $\mathbb{P} \in \Gamma$. In particular, if $\Phi = \Sigma_n(W)$, i.e., Φ is the class of all Σ_n sentences with parameters from W, and $W \subseteq W^{V^{\mathbb{P}}}$ for all $\mathbb{P} \in \Gamma$, then $\mathcal{A}(W, \Phi, \Gamma)$ is just the assertion that W is a Σ_n-elementary substructure of $W^{V^{\mathbb{P}}}$, for all $\mathbb{P} \in \Gamma$. As is customary, instead of $\Sigma_n(W)$ we will write $\underset{\sim}{\Sigma}_n$. If the class of parameters is some X properly contained in W, then we write $\Sigma_n(X)$ for the class of Σ_n sentences with parameters from X.

We write Σ_ω for the class of all sentences of the first-order language of set theory, and $\underset{\sim}{\Sigma}_\omega$ for the class of all such sentences with parameters in W.

If $\Phi = \Sigma_\omega$, then $\mathcal{A}(W, \Phi, \Gamma)$ should be regarded as an axiom schema, that is, as $\mathcal{A}(W, \Sigma_n, \Gamma)$ for each $n \in \omega$. Similarly if $\Phi = \utilde{\Sigma}_\omega$.

If Γ contains only one element, $\mathbb{P}$, then we will write $\mathcal{A}(W, \Phi, \mathbb{P})$, instead of $\mathcal{A}(W, \Phi, \Gamma)$. If Γ is the class of all set-forcing notions, then we just write $\mathcal{A}(W, \Phi)$.

3. Some trivial cases

Let us first consider some forms of $\mathcal{A}(W, \Phi, \Gamma)$ that are either provable or refutable in ZFC. So, as axioms, they are of no interest, since they are either trivial or false.

3.1. $\Phi \subseteq \utilde{\Sigma}_0$**:** In this case, $\mathcal{A}(W, \Phi, \Gamma)$ holds for all transitive W and all Γ such that W^V is contained in $W^{V^{\mathbb{P}}}$ for all $\mathbb{P} \in \Gamma$.

3.2. $\Phi \subseteq \Sigma_1(H(\omega_1))$**:** By the Levy-Shoenfield absoluteness theorem (see [24]), we have that $\mathcal{A}(W, \Phi, \Gamma)$ holds for every transitive model W of a weak fragment of ZF that contains the parameters of Φ, and all Γ, provided W^V is contained in $W^{V^{\mathbb{P}}}$ for all $\mathbb{P} \in \Gamma$. In particular, $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_1)$, $\mathcal{A}(H(\kappa), \Sigma_1(H(\omega_1)))$, $\kappa > \omega_1$, and $\mathcal{A}(V, \Sigma_1(H(\omega_1)))$ all hold.

3.3. $W = V$**:** A proper class W, different from V, can never be an elementary substructure of V, since otherwise, by elementarity, for every ordinal α, $V_\alpha^W = V_\alpha$, and so $W = V$. In particular, $\mathcal{A}(W, \utilde{\Sigma}_1, \{\mathbb{P}\})$ fails for any nontrivial $\mathbb{P}$, i.e., any $\mathbb{P}$ that adds some new set.

3.4. $W = L$**:** For every forcing notion $\mathbb{P}$, $L^V = L^{V^{\mathbb{P}}}$. Hence, in this case $\mathcal{A}(W, \Phi)$ holds for all Φ.

3.5. $W = H(\omega)$**:** For every forcing notion $\mathbb{P}$, $H(\omega)^V = H(\omega)^{V^{\mathbb{P}}}$. Hence, $\mathcal{A}(W, \Phi)$ is true for all Φ.

4. A natural class of axioms

A family of natural axioms of generic absoluteness is obtained when: (1) $W = H(\kappa)$ or $W = L(H(\kappa))$, for some definable uncountable cardinal κ; (2) Φ is the class of Σ_n sentences, some $n \in \omega$, or the class Σ_ω of all sentences, with or without parameters from W; and (3) Γ is a definable class of posets which contains at least a non-trivial element.

Why the $H(\kappa)$ and not the V_α, α an ordinal? On the one hand, for a regular cardinal κ, $H(\kappa)$ is a model of ZF minus the Power-set axiom, and so it satisfies Replacement, thus being a better model in the sense of forcing. For instance, if $\mathbb{P} \in H(\kappa)$, then, a filter $G \subseteq \mathbb{P}$ is generic over V iff it is generic over $H(\kappa)$. Moreover, for $\Phi = \utilde{\Sigma}_1$, which, as we will see, is one of the most relevant cases, we have that if $\kappa < \lambda$, then $\mathcal{A}(H(\lambda), \underset{\sim}{\Sigma}_1, \Gamma)$ implies $\mathcal{A}(H(\kappa), \underset{\sim}{\Sigma}_1, \Gamma)$.

Of particular interest, besides the class of all posets, are the classes of posets that have been extensively studied in the literature and form an increasing chain, namely, the ccc posets, the proper and semi-proper posets, the posets that preserve stationary subsets of ω_1, and the posets that preserve ω_1 (see [24] for the definitions). To make the notation more readable, we shall write *ccc* for the class of ccc posets, *Proper* for the class of proper posets, *Semi-proper* for the class of semi-proper posets, *Stat-pres* for the class of posets that preserve stationary subsets of ω_1, and ω_1*-pres* for the class of posets that preserve ω_1. We have:

$$\textit{ccc} \subset \textit{Proper} \subset \textit{Semi-proper} \subset \textit{Stat-pres} \subset \omega_1\textit{-pres}.$$

Of interest are also some subclasses of the ccc posets, namely, the classes of posets which are σ-centered (a poset is σ-centered if it can be partitioned into countably many classes so that each class is finite-wise compatible), the σ-linked (a poset is σ-linked if it can be partitioned into countably many classes so that each class is pair-wise compatible), the Knaster (a poset has the Knaster property if every uncountable set contains an uncountable subset of pair-wise compatible elements), and the Productive-ccc (a poset is productive-ccc if its product with any ccc poset is also ccc). We will write *Prod-ccc* and *Knaster* for the classes of productive-ccc posets and Knaster posets, respectively. These classes also form a chain:

$$\sigma\textit{-centered} \subset \sigma\textit{-linked} \subset \textit{Knaster} \subset \textit{Prod-ccc} \subset \textit{ccc}.$$

Another classification of forcing notions is obtained with regard to their definability. Both interesting and natural are the axioms $\mathcal{A}(H(\kappa), \Phi, \Gamma)$, where Γ consists of those posets in one of the classes above that are definable (Σ_n or Π_n definable, some n) in $H(\kappa)$, with or without parameters. An important example is the class of projective posets, namely, those definable in $H(\omega_1)$ with parameters. As usual, that a poset is, say, Σ_n-definable in $H(\omega_1)$, means that the set, the ordering relation, and the incompatibility relation are all Σ_n-definable in $H(\omega_1)$. Similarly for the Π_n-definable posets.

Of special interest are the axioms of the form $\mathcal{A}(H(\omega_2), \Phi, \Gamma)$, both for their consequences and for being equivalent to the Bounded Forcing Axioms (see 6.4 below): Given a partial ordering $\mathbb{P}$, the *Bounded Forcing Axiom for* $\mathbb{P}$, in short $BFA(\mathbb{P})$, is the following statement:

For every collection $\{I_\alpha : \alpha < \omega_1\}$ *of maximal antichains of* $\mathbf{B} =_{df}$ *r.o.*$(\mathbb{P}) \setminus \{\mathbf{0}\}$*, each of size* $\leq \omega_1$*, there exists a filter* $G \subseteq \mathbf{B}$ *such that for every* α*,* $I_\alpha \cap G \neq \emptyset$*.*

For a class of posets Γ, $BFA(\Gamma)$ is the statement that for every $\mathbb{P} \in \Gamma$, $BFA(\mathbb{P})$.

MA_{ω_1}, Martin's axiom for ω_1, is $BFA(ccc)$. BPFA, the bounded proper forcing axiom, is $BFA(Proper)$. BSPFA, the bounded semi-proper forcing axiom, is $BFA(Semi\text{-}proper)$. Finally, BMM, the bounded Martin's maximum, is $BFA(Stat\text{-}pres)$.

Thus, in this paper we will concentrate on axioms of the form $\mathcal{A}(W, \Phi, \Gamma)$, where $W = H(\kappa)$ or $L(H(\kappa))$, where $\Phi = \Sigma_n$, $\undertilde{\Sigma}_n$, Σ_ω, or $\undertilde{\Sigma}_\omega$, and where Γ is one of the classes of forcing notions considered above.

We shall begin by looking at the smallest non-trivial W, namely, $H(\omega_1)$.

5. $W = H(\omega_1)$

For every forcing notion $\mathbb{P}$, $H(\omega_1) \subseteq H(\omega_1)^{V^{\mathbb{P}}}$. Hence, from 3.2 above, the first interesting case is when $\Phi = \Sigma_2$.

5.1. $\Phi = \Sigma_2$**:** $\mathcal{A}(H(\omega_1), \Sigma_2)$ fails in L, since saying that there exists a non-constructible real is Σ_2, hence it is not provable in ZFC. However,

Theorem 5.1. ([30], [41]) *If $X^\sharp$ exists for every set X, then $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2)$ holds.*

The following theorem follows from a result of Feng-Magidor-Woodin [29], and was proved independently by S. Friedman (see [13]).

Theorem 5.2.
(1) *$\mathcal{A}(H(\omega_1), \Sigma_2)$ is equiconsistent with ZFC.*
(2) *$\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2)$ is equiconsistent with the existence of a Σ_2-reflecting cardinal. i.e., a regular cardinal κ such that $V_\kappa \preccurlyeq_{\Sigma_2} V$.*

The proof of the Theorem above actually shows that $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2)$ implies that ω_1 is a Σ_2-reflecting cardinal in $L[x]$, for every real x. This has recently been improved by S. Friedman:

Theorem 5.3. ([17]) *$\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Stat\text{-}pres)$ implies that ω_1 is a Σ_2-reflecting cardinal in $L[x]$, for every real x. Hence, by 5.2, $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Stat\text{-}pres)$ is equiconsistent with the existence of a Σ_2-reflecting cardinal.*

In particular, Friedman shows, using R. Schindler's *faster reshaping forcing* that $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Stat\text{-}pres)$ implies that ω_1 is inaccessible in $L[x]$, for every real x.

It follows immediately from [18] and [9] that $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Semi\text{-}proper)$ does not imply that ω_1^L is countable. Actually, as S. Friedman ([17]) has recently observed, $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Semi\text{-}proper)$ is equiconsistent with ZFC. (This contradicts the statement of Theorem 2 of [13], the proof of which actually shows that if $\mathcal{A}(H(\omega_1), \undertilde{\Sigma}_2, Proper)$ holds after forcing with a certain proper poset, then either ω_1 is Mahlo in L or ω_2 is inaccessible in L.)

As we will see in section 6 below, Bounded Forcing Axioms are actually equivalent to axioms of generic $\utilde{\Sigma}_1$-absoluteness for $H(\omega_2)$ ([8], [9]). But they also imply generic $\utilde{\Sigma}_2$-absoluteness for $H(\omega_1)$. Namely,

Theorem 5.4. ([8], [32]) *MA_{ω_1} implies $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, ccc)$.*

More generally,

Theorem 5.5. ([8])

(1) *BPFA implies $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Proper)$.*
(2) *BSPFA implies $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Semi\text{-}proper)$.*
(3) *BMM implies $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Stat\text{-}pres)$.*

That the last four implications cannot be reversed can be easily seen, as S. Friedman has pointed out, by noticing that all axioms of the form $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_n, \Gamma)$ are preserved after collapsing the continuum to ω_1 by σ-closed forcing. Hence, they are all consistent with CH, and so they do not imply any of the bounded forcing axioms.

It is also worth mentioning the following surprising result of S. Friedman:

Theorem 5.6. ([17]) *Let θ be the statement that every subset of ω_1 is constructible from a real, that is, for every $X \subseteq \omega_1$ there exists $x \subseteq \omega$ with $X \in L[x]$.*

Suppose that ω_1 is not weakly-compact in $L[x]$ for some $x \subseteq \omega$. Then,

(1) *MA_{ω_1} is equivalent to $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, ccc)$ plus θ.*
(2) *BPFA is equivalent to $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Proper)$ plus θ.*
(3) *BSPFA is equivalent to $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Semi\text{-}proper)$ plus θ.*

Let us point out that while BSPFA is consistent with $\omega_1^L = \omega_1$ ([18]), BMM implies that ω_1 is weakly-compact in $L[x]$ for every $x \subseteq \omega$ (see Theorem 6.6 below).

For many of the ccc and proper forcing notions $\mathbb{P}$ associated to regularity properties of sets of reals, like Lebesgue measurability, the Baire property, or the Ramsey property, there are interesting characterizations of the axioms $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, \mathbb{P})$ in terms of these regularity properties holding for the projective classes of $\utilde{\Delta}^1_2$ and $\utilde{\Sigma}^1_2$ sets. Analogous characterizations also hold for the axioms $\mathcal{A}(H(\omega_1), \Sigma_2, \mathbb{P})$ and the Kleene classes Δ^1_2 and Σ^1_2. We state here only the parametrized forms. For instance, from [7] we have:

Let *Cohen* be the poset for adding a Cohen real, and let *Random* be the poset for adding a random real. Then,

- $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Cohen)$ is equivalent to the statement that every $\utilde{\Delta}^1_2$ set of reals has the property of Baire.
- $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Random)$ is equivalent to the statement that every $\utilde{\Delta}^1_2$ set of reals is Lebesgue measurable.

The following are due to H. Judah ([14], [26]): Let *Amoeba* be the amoeba poset for measure, let *Amoeba-category* be the amoeba poset for category, and let *Hechler* be the Hechler forcing for adding a dominating real. Then

- $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Hechler)$ and $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, \textit{Amoeba-category})$ are both equivalent to the statement that every $\utilde{\Sigma}^1_2$ set of reals has the property of Baire.
- $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Amoeba)$ is equivalent to the statement that every $\utilde{\Sigma}^1_2$ set of reals is Lebesgue measurable.

Similar results for the Mathias forcing were obtained by Halbeisen-Judah [19]. If *Mathias* is the Mathias forcing, then:

- $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Mathias)$ is equivalent to the statement that every $\utilde{\Sigma}^1_2$ set of reals is Ramsey.

It should be noted that all these forcing notions are proper and Borel, i.e., the set of conditions is a Borel set, and both the ordering and the incompatibility relation are Borel subsets of the plane. The following is a long-standing open question:

Question 5.7. *Does $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_\omega, \Gamma)$, for Γ the class of Borel ccc forcing notions, imply that every projective set of real numbers is Lebesgue measurable?*

The most general result along these lines is the following theorem of Feng-Magidor-Woodin:

Theorem 5.8. ([29]) *$\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2)$ is equivalent to the statement that every $\utilde{\Delta}^1_2$ set of reals is universally Baire.*

Let us now consider the next level of complexity of Φ.

5.2. $\Phi = \Sigma_3$**:** We start with the following version of a result from [7], which uses a Lemma of H. Woodin from [41] to the effect that if X is an uncountable sequence of reals in V and c is Cohen-generic over V, then in $V[c]$ there is no random real over $L(X, c)$.

Theorem 5.9. *Let ω_1-Random be the σ-linked forcing notion for adding ω_1 random reals.*

(1) *$\mathcal{A}(H(\omega_1), \Sigma_3, \{\omega_1\text{-}Random, Cohen\})$ implies that ω_1 is inaccessible in $L[x]$, for every real x.*
(2) *$\mathcal{A}(H(\omega_1), \utilde{\Sigma}_2, Random)$ plus $\mathcal{A}(H(\omega_1), \Sigma_3, Cohen)$ imply that ω_1 is inaccessible in $L[x]$, for every real x.*

The point (for (1)) is that if $\omega_1^{L[x]} = \omega_1$, then we may add ω_1 random reals so that, in the forcing extension, for every real y there is a random real over $L(x, y)$. By $\mathcal{A}(H(\omega_1), \Sigma_3, \omega_1\text{-}Random)$ this holds in V, and by

$\mathcal{A}(H(\omega_1), \Sigma_3, \textit{Cohen})$ it holds after adding a Cohen real c. So, in $V[c]$, we have that there is a random real over $L(x, c)$, contradicting the aforementioned result of Woodin.

The following results give the exact consistency strengths.

Theorem 5.10. ([29]) *The following are equiconsistent:*

(1) $\mathcal{A}(H(\omega_1), \Sigma_3)$.
(2) *Every set has a sharp.*

Theorem 5.11. ([13]) *The following are equiconsistent:*

(1) $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_3)$.
(2) *Every set has a sharp and there exists a Σ_2-reflecting cardinal.*

R. Schindler [33] and, independently, S. Friedman [17], have shown that if for some set X, $X^\sharp$ does not exist, then there is a forcing notion that preserves stationary subsets of ω_1 and adds a real r such that, in the generic extension, $\omega_1^{L[r]} = \omega_1$. But then, since the sentence *There exists a real x such that $\omega_1^{L[x]} = \omega_1$* is Σ_3 in $H(\omega_1)$, by $\mathcal{A}(H(\omega_1, \Sigma_3, \textit{Stat-pres})$ we have that there is such a real x in the ground model, contradicting Theorem 5.9. This shows that

Theorem 5.12. *The following are equiconsistent:*

(1) $\mathcal{A}(H(\omega_1), \Sigma_3, \textit{Stat-pres})$.
(2) *Every set has a sharp.*

Hence, from 5.10, the axiom $\mathcal{A}(H(\omega_1), \Sigma_3, \omega_1\textit{-pres})$ is also equiconsistent with the existence of the sharp of every set.

The next theorem follows from some results due to Kunen and Harrington-Shelah [20] (see [11] and [13]):

Theorem 5.13. *For all n such that $3 \leq n \leq \omega$, the following are equiconsistent:*

(1) $\mathcal{A}(H(\omega_1), \Sigma_n, Knaster)$.
(2) $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_n, ccc)$.
(3) *There exists a weakly compact cardinal.*

An argument of A. R. D. Mathias, also implicit in [25], is used in the following result to show that $\mathcal{A}(H(\omega_1), \Sigma_3, \sigma\textit{-centered})$, plus ω_1 is inaccessible in $L[x]$ for every real x, implies that ω_1 is a Mahlo cardinal in L.

Theorem 5.14. ([11]) *For all n such that $3 \leq n \leq \omega$, the following are equiconsistent:*

(1) $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_n, \sigma\textit{-centered})$.
(2) $\mathcal{A}(H(\omega_1), \utilde{\Sigma}_n, \sigma\textit{-linked})$.

(3) *There exists a Mahlo cardinal.*

Let us recall from [12] that a ccc poset $\mathbb{P}$ is *strongly-*$\underset{\sim}{\Sigma}_n$ if it is Σ_n-definable in $H(\omega_1)$ with parameters, and the predicate "x codes a maximal antichain of $\mathbb{P}$" is also Σ_n-definable in $H(\omega_1)$ with parameters. A projective poset $\mathbb{P}$ is *absolutely-ccc* if it is ccc in every inner model W of V which satisfies ZFC and contains the parameters of the definition of $\mathbb{P}$.

Let us also recall from [11] that a projective poset $\mathbb{P}$ is *strongly-proper* if for every countable transitive model N of a fragment of ZFC with the parameters of the definition of $\mathbb{P}$ in N and such that $(\mathbb{P}^N, \leq_{\mathbb{P}}^N, \perp_{\mathbb{P}}^N) \subseteq (\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}})$, and for every $p \in \mathbb{P}^N$, there is $q \leq p$ which is $(N, \mathbb{P})$-generic, i.e., if $N \models$ "A is a maximal antichain of $\mathbb{P}$", then $A \cap N$ is predense below q.

Notice that if $\mathbb{P}$ is a projective poset, $N \preceq H(\lambda)$, and the parameters of the definition of $\mathbb{P}$ are in N, then a condition q is $(N, \mathbb{P})$-generic iff it is $(\bar{N}, \mathbb{P})$-generic, where $\bar{N}$ is the transitive collapse of N. Thus, a projective strongly-proper poset is proper.

From [7] (see 5.9 above) we have that just for $\Gamma = \{\omega_1\text{-}Random,\ Cohen\}$, the axiom $\mathcal{A}(H(\omega_1), \Sigma_3, \Gamma)$ already implies that ω_1 is inaccessible in $L[x]$, for every real x. Notice that both ω_1-*Random* and *Cohen* are Σ_1-definable in $H(\omega_1)$ ccc forcing notions. The sharpest result at the level of consistency strength of an inaccessible cardinal is the following:

Theorem 5.15 ([11], [12]). *For every $3 \leq n \leq \omega$ the following are equiconsistent (modulo ZFC):*

(1) $\mathcal{A}(H(\omega_1), \Sigma_n, \{\omega_1\text{-}Random,\ Cohen\})$.
(2) $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_n, \Gamma)$*, where Γ is the class of posets that are absolutely-ccc and* strongly-$\underset{\sim}{\Sigma}_2$.
(3) $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_n, \Gamma)$*, where Γ is the class of strongly-proper posets that are Σ_2 definable in $H(\omega_1)$ with parameters.*
(4) *There exists an inaccessible cardinal.*

This result is optimal, for there exists a, provably in ZFC, ccc poset $\mathbb{P}$ which is both Σ_2 and Π_2 definable in $H(\omega_1)$, without parameters, and for which the axiom $\mathcal{A}(H(\omega_1), \Sigma_3, \mathbb{P})$ fails if ω_1 is not a Π_1-Mahlo cardinal in L (see [12]). A regular cardinal κ is Σ_n-Mahlo (Π_n-Mahlo) if every club subset of κ that is Σ_n-definable (Π_n-definable) in $H(\kappa)$ contains an inaccessible cardinal. Every Π_1-Mahlo cardinal is an inaccessible limit of inaccessible cardinals.

The general equiconsistency result for proper forcing notions is due to R. Schindler:

Theorem 5.16. ([34], [35]) *For every $3 \leq n \leq \omega$ the following are equiconsistent:*

(1) *$\mathcal{A}(H(\omega_1), \Sigma_n$, Proper)*
(2) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_n$, Proper)*
(3) *There exists a remarkable cardinal.*

For the definition of remarkable cardinal see [34]. The proof of the Theorem actually shows that $\mathcal{A}(H(\omega_1), \Sigma_3, \textit{Proper})$ implies that ω_1 is a remarkable cardinal in L.

The following are the main open questions about the consistency strength of these axioms:

Questions 5.17. *What is the exact consistency strength of*
(1) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \omega_1$-pres)?*
(2) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3$, Stat-pres)?*
(3) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3$, Semi-proper)?*
(4) *$\mathcal{A}(H(\omega_1), \Sigma_3$, Semi-proper)?*

The next theorem generalizes 5.4 and 5.5:

Theorem 5.18. ([9]) *Assume that $x^\sharp$ exists for every real x and that the second uniform indiscernible is $< \omega_2$. Then,*
(1) *MA_{ω_1} implies $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, ccc)$.*
(2) *BPFA implies $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \textit{Proper})$.*
(3) *BSPFA implies $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3$, Semi-proper).*
(4) *BMM implies $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3$, Stat-pres).*

There are few results on the consequences of axioms of the form $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \Gamma)$ in Descriptive Set Theory. In particular there are no known equivalences with regularity properties of projective sets of reals, as in the case of the Σ_2-absoluteness axioms considered in the last section. However, we do have the following result of H. Judah (see [7]):

Theorem 5.19.
(1) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \{$Amoeba-category, Cohen$\})$ implies that every $\boldsymbol{\Delta}^1_3$ set of reals has the property of Baire.*
(2) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \{$Amoeba, Random$\})$ implies that every $\boldsymbol{\Delta}^1_3$ set of reals is Lebesgue measurable.*

J. Brendle proved a similar result for the Ramsey property:

Theorem 5.20. ([15]) *$\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, Mathias)$ implies that every $\boldsymbol{\Sigma}^1_3$ set of reals is Ramsey.*

It is also shown in [15] that $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, Mathias)$ implies that ω_1 is inaccessible in $L[x]$, for every real x.

An analogous result also holds for Hechler forcing ([16]), namely, $\mathcal{A}(H(\omega_1), \boldsymbol{\Sigma}_3, \textit{Hechler})$ implies that ω_1 is inaccessible in $L[x]$, for every real x.

The main general result is the following:

Theorem 5.21. ([29]) *$\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_3)$ implies that every $\underset{\sim}{\Delta}^1_3$ set of reals is universally Baire.*

It is also shown in [29] that the converse does not hold.

5.3. **$\Phi = \Sigma_n$, $4 \leq n \leq \omega$.** A generalization of an argument of Woodin [41], yields the following result from [21]:

Theorem 5.22. *$\mathcal{A}(H(\omega_1), \Sigma_4)$ implies that for every set X, $X^\dagger$ exists.*

The general equiconsistency result is due to Hauser-Woodin (see [21]):

Theorem 5.23. *The following are equiconsistent:*

(1) *$\mathcal{A}(H(\omega_1), \Sigma_\omega)$.*
(2) *$\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega)$.*
(3) *There are infinitely-many strong cardinals.*

A proper class of Woodin cardinals suffices to imply full absoluteness for $H(\omega_1)$ under all set-forcing extensions. Indeed, the following theorem of Woodin is a consequence of the results proved in [42] (see also [24]).

Theorem 5.24. *If there is a proper class of Woodin cardinals, then $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega)$.*

The following is a natural open question from [29]:

Question 5.25. *Does $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega)$ imply that every projective set of reals is universally Baire?*

For the class of projective posets we have the following results: recall from [12] that a regular cardinal κ is Σ_ω-Mahlo if every club subset of κ that is definable (with parameters) in $H(\kappa)$ contains an inaccessible cardinal. If κ is Mahlo, then the set of Σ_ω-Mahlo cardinals is stationary on κ (see [12]).

Theorem 5.26. ([12]) *Let Γ be the class of projective posets, i.e., posets that are definable with parameters in $H(\omega_1)$. The following are equiconsistent:*

(1) *$\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega, \Gamma \cap$ absolutely-ccc$)$.*
(2) *There exists an Σ_ω-Mahlo cardinal.*

Theorem 5.27. ([11]) *For Γ the class of projective posets, CON(ZFC + There exists an Σ_ω-Mahlo cardinal) implies CON(ZFC + $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega, \Gamma \cap$ strongly-proper$)$).*

Whether a Σ_ω-Mahlo cardinal is necessary for the consistency of $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_\omega, \Gamma \cap$ *strongly-proper*$)$, is still open.

Other open questions are the following:

Questions 5.28.

(1) *What is the consistency strength of* $\mathcal{A}(H(\omega_1), \Sigma_\omega,\ \omega_1\text{-}pres)$*?*
(2) *What is the consistency strength of* $\mathcal{A}(H(\omega_1), \Sigma_\omega,\ Stat\text{-}pres)$*?*
(3) *What is the consistency strength of* $\mathcal{A}(H(\omega_1), \Sigma_\omega,\ Semi\text{-}proper)$*?*
The same questions for sentences with parameters, i.e., for $\Phi = \underset{\sim}{\Sigma}_\omega$*, are also open. They are also open for projective posets in each one of the corresponding classes of posets. They are also open for* $\Phi = \underset{\sim}{\Sigma}_n$*,* $n \geq 4$*.*
(4) *What is the exact consistency strength of* $\mathcal{A}(H(\omega_1), \underset{\sim}{\Sigma}_n)$ *for* $n \geq 4$*?*
K. Hauser (unpublished) showed that this consistency strength is at least that of $n-3$ *strong cardinals and at most that of* $n-3$ *strong cardinals with a* Σ_2*-reflecting cardinal above them.*

6. $W = H(\omega_2)$

The axiom $\mathcal{A}(H(\omega_2), \Sigma_2, \Gamma)$ is false for most classes of forcing notions. Even $\mathcal{A}(H(\omega_2), \Sigma_2, \sigma\text{-}centered)$ is false if CH fails. Indeed, by adding ω_1 Cohen reals, a σ-centered forcing notion, one adds a Luzin set, that is, an uncountable set of reals that intersects every meager set in at most a countable set. Notice that saying that there exists a Luzin set is a Σ_2 statement in $H(\omega_2)$. But then we may iterate in length the continuum *Amoeba-category*, a σ-centered forcing notion, so that in the generic extension every set of size ω_1 is meager. Since any iteration of *Amoeba-category* with finite support is Knaster, the argument shows that $\mathcal{A}(H(\omega_2), \Sigma_2, Knaster)$ is false. Note that the argument also shows that $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_2, \sigma\text{-}centered)$ is false, since given any set of reals in $H(\omega_2)$ we can force with *Amoeba-category* to make it meager.

6.1. $\Phi = \Sigma_1$**:** Since $\mathcal{A}(H(\omega_2), \Sigma_1)$ holds (see 3.2), the interesting case is with parameters.

Notice that $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \mathbb{P})$ implies the negation of CH, for any $\mathbb{P}$ that adds a real number.

For ccc posets, the axiom turns out to be equivalent to Martin's axiom for ω_1, as given in the following result due independently to Stavi-Väänänen [37], and [8]:

Theorem 6.1. *The following are equivalent:*
(1) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, ccc)$*.*
(2) MA_{ω_1}*.*

In fact, we have that for every ccc poset $\mathbb{P}$, $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \mathbb{P})$ is equivalent to $MA_{\omega_1}(\mathbb{P})$.

Since, as it is well-known, Martin's axiom is consistent relative to ZFC, we have as a corollary of the Theorem above that so is the axiom $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, ccc)$.

For some particular forcing notions we have nice characterizations of the corresponding axioms, for instance, from 6.1 and some basic results on the additivity of the ideals of null and meager sets of reals (see [7]), we have the following:

Theorem 6.2.
(1) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Amoeba)$ *is equivalent to the* ω_1*-additivity of the Lebesgue measure.*
(2) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Amoeba\text{-}category)$ *is equivalent to the* ω_1*-additivity of the Baire property.*

From [9] and [18], we have an exact consistency result for the classes of proper and semi-proper posets:

Theorem 6.3. *The following are equiconsistent:*
(1) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Proper)$.
(2) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Semi\text{-}proper)$.
(3) *There exists a* Σ_2*-reflecting cardinal.*

This is a consequence of the following result, which generalizes 6.1, together with the results of [18] on the consistency strength of $BPFA$ and $BSPFA$. It shows that the bounded forcing axioms are, in fact, equivalent to axioms of generic absoluteness for $H(\omega_2)$, thus revealing them as natural axioms of set Theory.

Theorem 6.4. ([9])
(1) *The following are equivalent:*
  (a) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Proper)$.
  (b) $BPFA$.
(2) *The following are equivalent:*
  (a) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Semi\text{-}proper)$.
  (b) $BSPFA$.
(3) *The following are equivalent:*
  (a) $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, Stat\text{-}pres)$.
  (b) BMM.

Asperó-Welch [6] produced a model of BSPFA where BMM fails, assuming the consistency of a large-cardinal notion slightly weaker than an ω_1-Erdös cardinal. This was later improved by Asperó [4] by constructing such a model from the optimal large cardinal assumption, namely, the existence of a Σ_2-reflecting cardinal. Also, Schindler has shown, starting from cardinals $\kappa < \lambda < \mu < \nu$ such that κ is remarkable, λ is Σ_2-reflecting, μ is Woodin, and ν is measurable, that forcing with the Levy collapse of κ to ω_1, and then forcing BPFA with a proper forcing in V_λ, one obtains a model in which BSPFA fails.

$\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \mathbb{P})$ is clearly inconsistent with ZFC for any $\mathbb{P}$ that collapses ω_1. Further, the axiom $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \omega_1\text{-}pres)$ is also inconsistent with ZFC. For if S is a stationary and co-stationary subset of ω_1, then, by Baumgartner-Harrington-Kleinberg [23], we can add a club $C \subseteq S$ while preserving ω_1. But then the axiom would imply that such a club exists in the ground model, and so the complement of S is not stationary.

A natural question is: what is the the maximal class Γ for which $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \Gamma)$ is consistent? This has been answered by D. Asperó and H. Woodin in the following ways. On the one hand, Asperó proved the following:

Theorem 6.5. [3] *Let Γ be the class of all posets $\mathbb{P}$ such that for every set X of cardinality $\aleph_1$ of stationary subsets of ω_1 there is a condition $p \in \mathbb{P}$ such that p forces that S is stationary for every $S \in X$. The axiom $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \Gamma)$ is maximal, i.e., if $\mathbb{P} \notin \Gamma$, then $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \mathbb{P})$ fails. The axiom can be forced assuming the existence of a Σ_2-reflecting cardinal which is the limit of strongly compact cardinals.*

On the other hand, Woodin [43] provides a fine analysis of the axioms of the form $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \Gamma)$, assuming the axiom $(*)$ (see [43]), plus that every set X belongs to a model with a Woodin cardinal above X, an assumption which is, consistency-wise, weaker that the existence of a proper class of Woodin cardinals, hence consistency-wise much weaker than the large-cardinal assumption in Asperó's Theorem above. Woodin shows that for every $A \in H(\omega_2)$ and every Π_1 sentence, with A as a parameter, that holds in V, there is a ω_1-sequence of stationary subsets of ω_1 such that any forcing notion that forces the negation of the sentence must destroy one of the stationary sets in the sequence. Thus, this yields a stronger form of Asperó's result, under a, consistency-wise, much weaker hypothesis. However, while Asperó's axiom can be forced assuming large cardinals, this is not known to be the case for Woodin's $(*)$ axiom.

Numerous consequences, mostly combinatorial, of the axioms BPFA, BSPFA, and BMM are known (see [5] and [40]).

Woodin [43] showed that if one assumes the existence of a measurable cardinal, or that the non-stationary ideal on ω_1 is precipitous, then BMM implies a combinatorial principle, called ψ_{AC}, which in turn implies that there is a well-ordering of the reals in length ω_2 which is definable, with parameters, in $H(\omega_2)$, and hence $\mathfrak{c} = \aleph_2$. Further, Asperó [1] showed that if BMM holds and for some $x \in H(\omega_2)$, $x^\sharp$ does not exist, then also $\mathfrak{c} = \aleph_2$. Improving on Woodin's result, Asperó-Welch [6] showed that if there exists a ω_1-Erdös cardinal, then BMM implies ψ_{AC}. More consequences of BMM were obtained by Asperó [2], for instance, he showed that BMM implies that the dominating number is $\aleph_2$. Finally, in a truly remarkable result, Todorcevic [39] has recently shown that BMM implies that there is a well-ordering of the reals in length ω_2 which is definable, with parameters, in $H(\omega_2)$, and hence $\mathfrak{c} = \aleph_2$. For this reason alone BMM deserves a detailed study. Unfortunately, its consistency strength is not known, and, while it may even imply that every projective set is determined, it is not even known whether it implies that every projective set of reals numbers is Lebesgue measurable. A few months ago R. Schindler proved the following:

Theorem 6.6 (R. Schindler)**.** *BMM implies that for every set X there is an inner model with a strong cardinal containing X.*

Thus, in particular, BMM implies that for every set X, $X^\sharp$ exists. In his Ph. D. Thesis, G. Hjorth [22] proved that Martin's axiom plus the existence of the sharp of every real number imply that every $\underset{\sim}{\Sigma}^1_3$ set of reals is Lebesgue measurable. It follows that BMM implies that every $\underset{\sim}{\Sigma}^1_3$ set of reals is Lebesgue measurable. The upper bound on the consistency strenght of BMM is given by Woodin [43], where he shows that a model with a proper class of Woodin cardinals suffices to obtain a model of BMM.

Showing that Bounded Forcing Axioms imply that the size of the continuum is $\aleph_2$ requires some method for coding reals by ordinals less than ω_2. Two such methods were devised by Woodin and Todorcevic, respectively, in their proofs that BMM (plus a measurable cardinal in the case of Woodin) implies $\mathfrak{c} = \aleph_2$. Very recently, Justin T. Moore has discovered a new coding method which yields yet a further improvement on the aforementioned chain of results of Woodin, Asperó, Asperó-Welch, and Todorcevic:

Theorem 6.7 ([27])**.** *BPFA implies that there is a well-ordering of the reals in length ω_2 which is definable, with parameters, in $H(\omega_2)$, and hence $\mathfrak{c} = \aleph_2$.*

Questions 6.8.

(1) *Let Γ be as in Theorem 6.5. Is $\mathcal{A}(H(\omega_2), \underset{\sim}{\Sigma}_1, \Gamma)$ equivalent to the seemingly weaker BMM? (Under MA_{ω_1} there are posets in Γ that do not preserve stationary subsets of ω_1 (see* [3]*).*
(2) *Is the axiom $\mathcal{A}(H(\omega_2), \Sigma_2, \sigma\text{-linked})$ consistent?*
(3) *What is the consistency strength of BMM?*
(4) *let σ-closed $*$ ccc be the class of forcing notions consisting of an iteration of a σ-closed poset followed by a ccc poset. Such posets are proper. Does $\mathcal{A}(H(\omega_2), \Sigma_2, \sigma\text{-closed} * \text{ccc})$ imply $\mathfrak{c} = \aleph_2$?*

7. $W = H(\kappa)$, $\kappa \geq \omega_3$

The arguments from the beginning of last section show that $\mathcal{A}(H(\kappa), \underset{\sim}{\Sigma}_2,$ *σ-centered*) is false, even for the σ-centeredposets that belong to $H(\omega_3)$.

The axiom $\mathcal{A}(H(\kappa), \Sigma_1)$ holds (see 3.2). So, the only interesting case is for $\Phi = \undertilde{\Sigma}_1$.

From [8] we have that $\mathcal{A}(H(\omega_3), \undertilde{\Sigma}_1, ccc)$ is equivalent to MA_{ω_2}.

However, $\mathcal{A}(H(\kappa), \undertilde{\Sigma}_1, Proper)$ is false, all $\kappa \geq \omega_3$. Indeed, let $\mathbb{P}$ be the forcing notion for collapsing ω_2 to ω_1 with countable conditions. This is a σ-closed, hence proper, forcing notion. But $\mathcal{A}(H(\kappa), \undertilde{\Sigma}_1, \mathbb{P})$ implies that ω_2 and ω_1 have the same cardinality.

$\mathcal{A}(H(\kappa), \undertilde{\Sigma}_1, ccc)$ is easily seen to be false for $\kappa > \mathfrak{c}$. For $\kappa < \mathfrak{c}$, we have from [8] that $\mathcal{A}(H(\kappa), \undertilde{\Sigma}_1, ccc)$ is equivalent to MA_κ, and $\mathcal{A}(H(\mathfrak{c}), \undertilde{\Sigma}_1, ccc)$ is equivalent to MA.

8. $W = L(H(\omega_1))$

Since every element of $H(\omega_1)$ can be easily coded by a real number, we have that $L(H(\omega_1)) = L(\mathbb{R})$, the smallest inner model of ZF that contains all the real numbers.

In general, $L(\mathbb{R}) \not\subseteq L(\mathbb{R})^{V^{\mathbb{P}}}$. For instance, by adding ω_1 Cohen reals over L, call this model V, and then yet one more Cohen real c over V, we have that $\mathbb{R}^V \notin L(\mathbb{R})^{V[c]}$. But even if we have $L(\mathbb{R}) \subseteq L(\mathbb{R})^{V^{\mathbb{P}}}$, we cannot have $L(\mathbb{R}) \preccurlyeq_{\Sigma_1} L(\mathbb{R})^{V^{\mathbb{P}}}$ if $\mathbb{P}$ adds some real. So, the most we can hope for is $\mathcal{A}(L(\mathbb{R}), \Sigma, \Gamma)$, i.e., without parameters, or, if we want parameters, we have to restrict them to ordinals and reals. So, whenever we write $\undertilde{\Sigma}_n$ or $\undertilde{\Sigma}_\omega$, we mean $\Sigma_n(OR \cup \mathbb{R})$ and $\Sigma_\omega(OR \cup \mathbb{R})$, respectively.

Notice that $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \Gamma)$ means that for every $\mathbb{P} \in \Gamma$ there is an elementary embedding

$$j : L(\mathbb{R}) \to L(\mathbb{R})^{V^{\mathbb{P}}}$$

that is the identity on the ordinals and, of course, on the reals.

The following theorem of H. Woodin (see [43]) shows that under large cardinals the theory of $L(\mathbb{R})$, with real parameters, is generically absolute.

Theorem 8.1. *If there is a proper class of Woodin cardinals, then* $\mathcal{A}(L(\mathbb{R}), \Sigma_\omega(\mathbb{R}))$ *holds.*

Woodin and, independently, Steel, have shown that the consistency strength of $\mathcal{A}(L(\mathbb{R}), \Sigma_\omega(\mathbb{R}))$ is roughly that of the existence of infinitely-many Woodin cardinals (see [38]). They have also shown that, actually, $\mathcal{A}(L(\mathbb{R}), \Sigma_\omega(\mathbb{R}))$ implies that the Axiom of Determinacy, AD, holds in $L(\mathbb{R})$.

It follows from results of Shelah and Woodin that $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_1,$ *Semi-proper*) is false, assuming the existence of large cardinals. For if, for instance, there is a supercompact cardinal, then Shelah has shown that one can force, by a semi-proper forcing notion that collapses the supercompact cardinal to ω_2, the non-stationary ideal on ω_1 to be saturated (see [24]) (a Woodin cardinal actually suffices for this (Shelah [36])). Since, in addition, in this model every set has a sharp, it follows from results of Woodin [43] that $L(\mathbb{R})$ computes correctly ω_2. Let $\alpha = \omega_2^V$. Then the Σ_2 sentence with α as a parameter, which states that α is a cardinal greater or equal than ω_2, is true in the $L(\mathbb{R})$ of the ground model V, but false in the $L(\mathbb{R})$ of the generic extension.

However, I. Neeman and J. Zapletal [31] have shown that $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \mathit{Proper})$ follows from large cardinals. Namely,

Theorem 8.2. *If δ is a weakly-compact Woodin cardinal, then $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \mathbb{P})$ holds for every proper poset $\mathbb{P} \in V_\delta$. Hence, $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \mathit{Proper})$ follows from the existence of a proper class of weakly-compact Woodin cardinals.*

In fact, Woodin [43] shows that the axiom $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \Gamma)$ holds for a class Γ which is larger that *Proper*, assuming only the existence of a proper class of Woodin cardinals, and gives some applications.

The consistency strength of $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \mathit{Proper})$ is rather low. Namely, as a consequence of [35] one has that:

Theorem 8.3. *The following are equiconsistent:*

(1) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \mathit{Proper})$
(2) *There exists a remarkable cardinal.*

For ccc forcing notions we have the following exact equiconsistency results. The first one follows from work of Kunen and Harrington-Shelah [20] (see [11]).

Theorem 8.4. *The following are equiconsistent:*

(1) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, Knaster)$.
(2) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, ccc)$.
(3) *There exists a weakly compact cardinal.*

For σ-linked forcing notions a Mahlo cardinal suffices:

Theorem 8.5. ([11]) *The following are equiconsistent:*

(1) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \sigma\text{-centered})$.
(2) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \sigma\text{-linked})$.
(3) *There exists a Mahlo cardinal.*

And when restricting to projective forcing notions we have the following:

Theorem 8.6. ([12]) *Let Γ be the class of projective posets. The following are equiconsistent:*

(1) $\mathcal{A}(L(\mathbb{R}), \undertilde{\Sigma}_\omega, \Gamma \cap absolutely - ccc)$.
(2) *There exists an Σ_ω-Mahlo cardinal.*

Theorem 8.7. ([11]) *For Γ the class of projective posets, CON(ZFC + There exists an Σ_ω-Mahlo cardinal) implies CON(ZFC + $\mathcal{A}(L(\mathbb{R}), \underset{\sim}{\Sigma}_\omega, \Gamma \cap$ strongly-proper)).*

It is open whether a Σ_ω-Mahlo cardinal is necessary for the consistency of $\mathcal{A}(L(\mathbb{R}), \underset{\sim}{\Sigma}_\omega, \Gamma \cap \textit{strongly-proper})$.

We finally look at the general projective ccc case:

Let us recall from [10] that if κ is a cardinal and $n \in \omega$, we say that κ is Σ_n-*weakly compact* (Σ_n-w.c., for short) iff κ is inaccessible and for every $R \subseteq V_\kappa$ which is definable by a Σ_n formula (with parameters) over V_κ and every Π^1_1 sentence Φ, if

$$\langle V_\kappa, \in, R\rangle \models \Phi$$

then there is $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \Phi$$

That is, κ *reflects* Π^1_1 *sentences with* Σ_n *predicates.*

Also, recall from [28] that a cardinal κ is Σ_ω-*weakly compact* (Σ_ω-w.c., for short), iff κ is Σ_n-w.c. for every $n \in \omega$.

Theorem 8.8 ([10])**.** *Let Γ_n be the class of ccc posets that are Σ_n or Π_n definable in $H(\omega_1)$ with parameters. The following are equiconsistent:*

1. $\mathcal{A}(L(\mathbb{R}), \underset{\sim}{\Sigma}_\omega, \Gamma_n)$
2. *There exists a Σ_n-w.c. cardinal.*

Corollary 8.9 ([10])**.** *Let Γ be the class of projective ccc forcing notions. The following are equiconsistent:*

1. $\mathcal{A}(L(\mathbb{R}), \underset{\sim}{\Sigma}_\omega, \Gamma)$
2. *There exists a Σ_ω-w.c. cardinal.*

9. $W = L(H(\omega_2))$

Notice that for every sentence φ, $H(\omega_2) \models \varphi$ iff $L(H(\omega_2))$ satisfies the Σ_2 sentence $\exists x(x = H(\omega_2) \wedge x \models \varphi)$. Thus, if Φ contains Σ_2, then $\mathcal{A}(L(H(\omega_2)), \Phi, \Gamma)$ implies $\mathcal{A}(H(\omega_2), \Phi, \Gamma)$. Hence, the negative results for $\mathcal{A}(H(\omega_2), \Phi, \Gamma)$ from section 6 apply also to $\mathcal{A}(L(H(\omega_2)), \Phi, \Gamma)$. For instance, $\mathcal{A}(L(H(\omega_2)), \Sigma_2, \textit{ccc})$ and $\mathcal{A}(L(H(\omega_2)), \underset{\sim}{\Sigma}_2, \sigma\textit{-centered})$ are false.

Also, we have that $\mathcal{A}(L(H(\omega_2)), \underset{\sim}{\Sigma}_1, \textit{Proper})$, is false. For if we collapse ω_2 to ω_1 by σ-closed (hence proper) forcing, then in the $H(\omega_2)$ of the generic extension there is an injection of ω_2^V into ω_1. But saying that there is such an injection is Σ_1 in the parameters ω_2^V and ω_1. Hence, the axiom implies that such an injection exists in V, which is impossible.

Similar considerations apply to the case $W = L(H(\omega_3))$, etc.

Question 9.1. *Is $\mathcal{A}(L(H(\omega_2)), \underset{\sim}{\Sigma}_1, \textit{ccc})$ consistent?*

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