

Operads and the Hopf algebras of renormalisation

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Abstract

Functors from (co)operads to bialgebras relate Hopf algebras that occur in renormalisation to operads, which simplifies the proof of the Hopf algebra axioms, and induces a characterisation of the corresponding group of characters and Lie algebra of primitives of the dual in terms of the operad. In addition, it is shown that the Wick rotation formula leads to canonical algebras for one of these operads.

1 Introduction

Kreimer [10] observes that a governing principle of renormalisation is given by the antipode of a Hopf algebra. More Hopf algebras related to renormalisation have been defined since then. Apart from checking the Hopf algebra axioms, one is interested in the group of characters and the Lie algebra of primitive elements of the dual of these Hopf algebras. This paper shows that these coproducts often decompose into more elementary operations which make checking of the Hopf algebra axioms less cumbersome. Moreover, one obtains a description of the group of characters and the Lie algebra of primitive elements of the dual.

Section 3 defines functors $C \mapsto H_C$ and $C \mapsto \bar{H}_C$ from 1-reduced cooperads to connected complete Hopf algebras. This section shows that for C of finite type, the character groups $\text{Hom}_{\text{Alg}}(H_C, k)$ and $\text{Hom}_{\text{Alg}}(\bar{H}_C, k)$ of H_C and \bar{H}_C are given as explicit functors from operads to groups applied to the dual operad of C . The Lie algebra of primitive elements of the dual Hopf algebra is given by well known functors from operads to Lie algebras applied to the the dual operad of C .

Section 4 explores some examples. The bitensor algebra of a bialgebra and its Pinter Hopf algebra are obtained using the constructions of section 3 (cf. Van der Laan-Moerdijk [15]). The Hopf algebra of higher order differentials of the line, and its non-commutative version (e.g. Brouder-Frabetti [2]) are obtained from the construction applied to the operad of commutative algebras.

Section 5 is devoted to one specific example. This section constructs an operad of graphs and shows that the Connes-Kreimer Hopf algebra of graphs [6] is the Hopf algebra constructed from the suboperad of one particle irreducible graphs.

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2 Preliminaries

Throughout this article we work in the category of vector spaces over a field k of characteristic 0.

By S_n we denote the symmetric group on n letters, and by kS_n its group algebra which is the vector space spanned by the set S_n whose multiplication is the linear extension of multiplication in S_n . If S_n acts on a vector space V , the coinvariants of the group action are denoted V_{S_n} and the invariants by V^{S_n} .

For a vector space V we will denote by $TV = \bigoplus_{n \geq 0} V^{\otimes n}$ the unital tensor algebra on V with the concatenation product, and by $SV = \bigoplus_{n \geq 0} (V^{\otimes n})_{S_n}$ the unital symmetric algebra on V which is the quotient of TV by its commutator ideal $[TV, TV]$. Both SV and TV are graded algebras with respect to n . If $W = \bigoplus_n W_n$ is a graded vector space, denote by $W^* = \bigoplus_n W_n^*$ its graded dual.

2.1 Operads

A non-symmetric operad is a sequence $\{P(n)\}_{n \geq 1}$ of vector spaces together with composition maps

$$\gamma : P(n) \otimes P(m_1) \otimes \cdots \otimes P(m_n) \longrightarrow P(m_1 + \cdots + m_n),$$

and an identity element $\text{id} \in P(1)$. These structures satisfy the following axioms:

- (i). The composition satisfies the associativity relation: If we write

$$p(q_1, \dots, q_n) := \gamma(p, q_1, \dots, q_n)$$

for $p \in P(n)$, and $q_i \in P(m_i)$ for $i = 1, \dots, n$, then for any such elements p and q_i , and any $r_j \in P(k_j)$ for $j = 1, \dots, m_1 + \cdots + m_n$ and for $k_j \geq 0$ the following holds:

$$\begin{aligned} (p(q_1, \dots, q_n))(r_1, \dots, r_{m_1 + \cdots + m_n}) = \\ p(q_1(r_1, \dots, r_{m_1}), \dots, q_n(r_{m_1 + \cdots + m_{n-1} + 1}, \dots, r_{m_1 + \cdots + m_n})). \end{aligned}$$

- (ii). The identity element $\text{id} \in P(1)$ acts as a left and right identity: If $p \in P(n)$, then

$$\text{id}(p) = p = p(\text{id}, \dots, \text{id}).$$

There is a natural grading on the total space $\bigoplus_n P(n)$, defined by

$$\left(\bigoplus_n P(n)\right)^m = P(m+1).$$

A collection P is a sequence of vector spaces $\{P(n)\}_{n \geq 1}$ such that each $P(n)$ has a right S_n -module structure. An (symmetric) operad is a collection P together with an non-symmetric operad structure on the sequence of vector spaces, and

(iii) composition is equivariant with respect to the S_n -actions in the sense that

$$\begin{aligned} p\sigma(q_1, \dots, q_n) &= p(q_{\sigma(1)}, \dots, q_{\sigma(n)})\hat{\sigma} \\ p(q_1\sigma_1, \dots, q_n\sigma_n) &= p(q_1, \dots, q_n)(\sigma_1 \times \dots \times \sigma_n) \end{aligned}$$

for any $p \in P(n)$, $\sigma \in S_n$ and $q_i \in P(m_i)$, $\sigma_i \in S_{m_i}$ for $i = 1, \dots, n$, where $\hat{\sigma}$ is the permutation on $m_1 + \dots + m_n$ elements that permutes consecutive blocks of lengths m_1, m_2, \dots, m_n according to $\sigma \in S_n$.

Dually (in the sense of inverting direction of arrows in the defining diagrams), one defines (non-symmetric) cooperads. In particular, a non-symmetric cooperad C is a sequence of vector spaces $\{C(n)\}_{n \geq 1}$, together with cocomposition maps

$$\gamma^* : C(n) \longrightarrow \bigoplus_k \bigoplus_{n_1 + \dots + n_k = n} C(k) \otimes C(n_1) \otimes \dots \otimes C(n_k),$$

and a counitality $\text{id}^* : C(1) \longrightarrow k$ that satisfy the dual relations. A Cooperad is a non-symmetric cooperad together with a collection structure on C that makes γ^* an equivariant map.

For more extensive background on (co)operads read Getzler-Jones [7], and Markl-Shnider-Stasheff [11].

2.2 Graphs

A graph η consist of sets $\mathbf{v}(\eta)$ of vertices, a set $\mathbf{e}(\eta)$ of internal edges, and a set $\mathbf{l}(\eta)$ of external edges or legs; together with a map that assigns to each edge a pair of (not necessary distinct) vertices and a map that assigns to each leg a vertex. To draw a graph, draw a dot for each vertex v , and for each edge e draw a line between the two vertices assigned to it, and for each leg draw a line that in one end ends in the vertex assigned to it. If $v \in \mathbf{v}(\eta)$, denote by $\mathbf{l}(v) \subset \mathbf{e}(v) \cup \mathbf{l}(\eta)$ the set of legs and edges attached to v . Call these the legs of v . A morphism of graphs consists of morphisms of vertices, edges, and legs compatible with the structure maps.

A connected graph t is a tree if $|\mathbf{v}(t)| = |\mathbf{e}(t)| + 1$. A rooted tree is a tree together with a basepoint $r \in \mathbf{l}(t)$, the root.

3 Constructions on cooperads

3.1 Bialgebras

3.1 DEFINITION Let C be a non-symmetric cooperad. We denote $B_C = T(\bigoplus_n C(n))$ the tensor algebra on the total space of C . Use the cocomposition

$$\gamma^* : C(n) \longrightarrow \bigoplus_{k, n_1 + \dots + n_k = n} C(k) \otimes (C(n_1) \otimes \dots \otimes C(n_k))$$

and the natural inclusions

$$i_1 : C(k) \longrightarrow T\left(\bigoplus_m C(m)\right),$$

and

$$i_2 : C(n_1) \otimes \dots \otimes C(n_k) \longrightarrow T\left(\bigoplus_m C(m)\right)$$

to define a map $\Delta : B_C \longrightarrow B_C \otimes B_C$ on generators as $\Delta = (i_1 \otimes i_2) \circ \gamma^*$. Extend Δ as an algebra morphism. Define the algebra morphism $\varepsilon : B_C \longrightarrow k$ as the map ε which vanishes on generators of degree $\neq 0$, and satisfies $\varepsilon|_{C(1)} = \varepsilon_C$.

3.2 LEMMA *Let C be a non-symmetric cooperad.*

- (i). *Comultiplication Δ and counit ε as defined above make $C \mapsto B_C$ a functor from non-symmetric cooperads to graded bialgebras.*
- (ii). *If C is a cooperad the bialgebra structure of B_C descends to $\bar{B}_C = S(\bigoplus_n C(n)^{S_n})$ (the symmetric algebra on the total space of invariants of C), and consequently $C \mapsto \bar{B}_C$ defines a functor from cooperads to commutative graded bialgebras.*

A detailed proof can be found in Van der Laan [14].

3.2 Hopf algebras

A (co)operad P is called **1-reduced**, or **1-connected** if $P(0) = 0$, and $P(1) = k$. Let C be a 1-reduced non-symmetric cooperad. The space $\bigoplus_n C(n)$ has a base-point given by the inverse of the counit $\varepsilon : C(1) \longrightarrow k$. In the sequel we will use pointed tensor algebra and the pointed symmetric algebra

$$T_*\left(\bigoplus_n C(n)\right) = T\left(\bigoplus_n C(n)\right)/(\varepsilon^{-1}(1) - \mathbf{1})$$

$$S_*\left(\bigoplus_n C(n)\right) = S\left(\bigoplus_n C(n)\right)/(\varepsilon^{-1}(1) - \mathbf{1}),$$

where the unit in T and S is denoted by $\mathbf{1}$. In other words, T_* is the left adjoint to the forgetful functor $(A, \mu, u) \mapsto (A, u)$ from unital associative algebras to vector spaces with a non-zero base-point, and S_* the left adjoint to the forgetful functor from unital commutative algebras to vector spaces with non-zero base-point.

3.3 DEFINITION Let C be a 1-reduced non-symmetric cooperad. Let

$$H_C = T_*\left(\bigoplus_n C(n)\right)$$

the pointed tensor algebra on the total space of C , with respect to the base-point given by the inclusion of $C(1) = k$. The coalgebra structure on B_C induces maps $\Delta: H_C \rightarrow H_C \otimes H_C$, and $\varepsilon: H_C \rightarrow k$. If C is a 1-reduced cooperad, denote

$$\bar{H}_C = S_*\left(\bigoplus_n C(n)^{S_n}\right)$$

the pointed symmetric algebra on the total space of invariants of C . Lemma 3.2 now can be adapted to a pointed version.

3.4 THEOREM *Let C be a 1-reduced non-symmetric cooperad.*

- (i). *The application $C \mapsto H_C$ defines a functor from non-symmetric cooperads to graded connected Hopf algebras.*
- (ii). *If C is a 1-reduced cooperad the application $C \mapsto \bar{H}_C$ defines a functor from cooperads to commutative graded connected Hopf algebras.*

PROOF There is a natural surjection of algebras

$$T\left(\bigoplus_n C(n)\right) \rightarrow T_*\left(\bigoplus_n C(n)\right).$$

We define a coproduct on $T_*\left(\bigoplus_n C(n)\right)$ by the formula of the coproduct in Lemma 3.2. Since C is coaugmented and ε is the counit, the cocomposition γ^* respects the base point given by $\varepsilon^{-1}: k \rightarrow C(1)$, and satisfies $(\varepsilon \otimes \text{id}) \circ \gamma^* = \text{id} = (\text{id} \otimes \varepsilon) \circ \gamma^*$. This implies the bialgebra structure is well defined on the pointed tensor algebra. Functoriality is again trivial. To prove (i) it remains to check that H_C is in fact a graded connected Hopf algebra.

A bialgebra A is called **connected** if it is \mathbb{Z} -graded, concentrated in non-negative degree, and satisfies $A^0 = k \cdot 1$. The **augmentation ideal** of a connected bialgebra A is the ideal $\bigoplus_{n \geq 1} A^n$. Since the degree 0 part of H_C is $C(1) = k$, the bialgebra H_C is connected. It is well known (cf. Milnor and Moore [12]) that any graded connected bialgebra admits an antipode and is thus a Hopf algebra.

To pass to the symmetric version (ii), argue as in the proof of Lemma 3.2, and observe that \bar{H}_C is also connected. QED

3.5 REMARK Note that the construction of H_C and \bar{H}_C defines a graded bialgebra for any coaugmented cooperad C , but that C needs to be 1-reduced in order to get a graded connected Hopf algebra.

Let P be a (1-reduced) (non-symmetric) operad of finite type. Then the linear dual collection P^* is a (1-reduced) (non-symmetric) cooperad. Regarding the bialgebras and Hopf algebras associated to the cooperad P^* I will use notation

$$B_P := B_{P^*}, \quad \bar{B}_P := \bar{B}_{P^*}, \quad H_P := H_{P^*}, \quad \text{and} \quad \bar{H}_P := \bar{H}_{P^*}.$$

3.3 Groups

3.6 DEFINITION Let P be a non-symmetric operad. Let

$$G_P = \left\{ \sum_{n=1}^{\infty} p_n \quad \text{s.t.} \quad p_n \in P(n) \text{ and } p_1 = \text{id} \right\} \subset \hat{\bigoplus}_{n \geq 1} P(n),$$

where $\hat{\bigoplus}$ denotes the completed sum. Define a multiplication \circ on this set by

$$\left(\sum_n p_n \right) \circ \left(\sum_m q_m \right) = \sum_{n, m_1, \dots, m_n} \gamma(p_n, q_{m_1}, \dots, q_{m_n}).$$

If P is an operad this multiplication defines a multiplication on

$$\bar{G}_P = \left\{ \sum_{n=1}^{\infty} p_n \quad \text{s.t.} \quad p_n \in P(n)_{S_n} \text{ and } p_1 = \text{id} \right\} \subset \hat{\bigoplus}_n (P(n)_{S_n}),$$

since the composition γ is equivariant with respect to the S_n -actions.

3.7 LEMMA *Let P be a non-symmetric operad.*

- (i). *The set G_P is a group with respect to the multiplication \circ the unit element $\text{id} \in G_P$. The application $P \rightarrow G_P$ defines a functor from non- σ operads to groups.*
- (ii). *If P is an operad, the quotient \bar{G}_P of G_P is a group. The application $P \mapsto \bar{G}_P$ defines a functor from operads to groups.*

PROOF Associativity is clear from the associativity of γ . That $\text{id} \in P(1)$ is the identity with respect to composition is also obvious. To show the existence of an inverse, use a recursive construction similar to the proof of the Formal Inverse Function Theorem. QED

3.8 REMARK A group closely related to G_P and \bar{G}_P was defined independently by Chapoton [4]. He uses the group for the study of the exponential map associated to the pre-Lie operad.

3.9 THEOREM *Let P be a 1-reduced non-symmetric operad of finite type.*

- (i). *The group G_P is isomorphic to the group of characters of the Hopf algebra H_P .*

(ii). If P is a 1-reduced operad of finite type, then the group \bar{G}_P is isomorphic to the group of characters of the Hopf algebra \bar{H}_P .

PROOF The comultiplication on $H_P = T_*(\bigoplus_n P^*(n))$ induces a multiplication on

$$\mathrm{Hom}_{\mathrm{Alg}}(H_P, k) \cong \mathrm{Hom}_k(\bigoplus_n P^*(n), k) \cong \hat{\bigoplus}_{n \geq 2} P(n).$$

The $n \geq 2$ comes from the unitality of algebra homomorphisms: the group consists of elements of $\hat{\bigoplus}_n P(n)$ such that the coefficient for id^* equals 1.

Choose a homogeneous basis $\{p\}$ of $\bigoplus_n P(n)$ and let $\{p^*\}$ be the dual basis. For such basis elements with p of degree $n-1$ and q and r in arbitrary degree we have

$$\langle r^* | p \cdot q \rangle = \langle \gamma^*(r^*) | (p \otimes q^{\otimes n}) \rangle = \langle r^* | \gamma(p \otimes q^{\otimes n}) \rangle \quad (3.1)$$

with respect to the standard pairing $\langle \cdot | \cdot \rangle$ between P^* and P . This shows the first part of the result. To obtain the statement on coinvariants, note that

$$\mathrm{Hom}_{\mathrm{Alg}}(\bar{H}_P, k) = \hat{\bigoplus}_{n \geq 2} P(n)_{S_n},$$

and replace $\langle \cdot | \cdot \rangle$ in Equation (3.1) by the pairing of $P(n)_{S_n}$ and $(P(n)^*)^{S_n}$.
QED

3.10 REMARK Let C be a cooperad. We want to be a bit more specific on the relation between H_C and \bar{H}_C and their groups of characters. Consider the symmetrisation $S_*(\bigoplus_n C(n))$ of H_C . The formulae for the structure on H_C make $S_*(\bigoplus_n C(n))$ a Hopf algebra. Moreover, we have maps

$$H_C \longrightarrow S_*(\bigoplus_n C(n)) \longleftarrow \bar{H}_C, \quad (3.2)$$

where the left map is a surjection and the right map is an injection. A similar diagram exists for B_C and \bar{B}_C . Since the algebra k is commutative, every character of H_P factorises through its symmetrisation $S_*(\bigoplus_n P^*(n))$. The quotient map $G_P \longrightarrow \bar{G}_P$ can thus be interpreted as the map of character groups

$$G_P = \mathrm{Hom}_{\mathrm{Alg}}(S_*(\bigoplus_n P^*(n)), k) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}}(\bar{H}_P, k) = \bar{G}_P,$$

induced by the map of Hopf algebras $\bar{H}_P \longrightarrow S(\bigoplus_n P^*(n))$.

3.4 Lie algebras

Let P be a 1-reduced non-symmetric pseudo operad. The vector space $L_P = \bigoplus_{n \geq 2} P(n)$ is a Lie algebra with respect to the Lie bracket on $p \in P(n)$ and $q \in P(m)$ given by

$$[p, q] := \sum_{i=1}^n p \circ_i q - \sum_{j=1}^m q \circ_j p, \quad (3.3)$$

where $p \circ_i q := p(\text{id}^{\otimes i-1}, q, \text{id}^{\otimes n-i})$. If P is a 1-reduced pseudo operad, this Lie algebra structure descends to the quotient $\bar{L}_P = \bigoplus_n P(n)_{S_n}$ (cf. Kapranov-Manin [8]). Both Lie algebras are graded with respect to the grading $\deg(P(n)) = n-1$, and the application $P \mapsto L_P$ (resp. $P \mapsto \bar{L}_P$) defines a functor from non-symmetric operads (resp. operads) to Lie algebras.

3.11 THEOREM *Let P be a 1-reduced non-symmetric operad of finite type.*

- (i). *The Lie algebra of primitive elements of $(H_P)^*$ is the Lie algebra L_P .*
- (ii). *If P is a 1-reduced operad of finite type, the Lie algebra of primitive elements of $(\bar{H}_P)^*$ is the Lie algebra \bar{L}_P . Consequently, $(\bar{H}_P)^*$ is the universal enveloping algebra $U(\bar{L}_P)$ of the Lie algebra \bar{L}_P .*

PROOF Let P be a 1-reduced non-symmetric operad of finite type. The Hopf algebra H_P^* is the pointed ‘cofree’ coalgebra on the total space of P with the multiplication defined on $(P(k)) \otimes (P(m_1)) \otimes \cdots \otimes (P(m_k))$ as the composition γ of the non-symmetric operad, and then extended as a coalgebra homomorphism. Projected to the cogenerators $\bigoplus_n P(n)$, the multiplication reduces to the sum of the circle- i operations:

$$\sum_{i=1}^m \circ_i : P(m) \otimes P(n) \longrightarrow P(m+n-1),$$

for $m, n > 1$. The Lie bracket on the primitive elements is the commutator of this (non-associative) product. This shows the first part of the result.

Assume that P is a 1-reduced operad of finite type. The Lie algebra of primitive elements of $(\bar{H}_P)^*$ is then the symmetric quotient \bar{L}_P since we have the factorisation through $S_*(\bigoplus_n P(n))$ (cf. Remark 3.10). From the Milnor-Moore theorem [12] it follows $H_P^* = U(\bar{L}_P)$. QED

4 First examples

4.1 Formal diffeomorphisms

4.1 EXAMPLE Consider the operad Com , which has as algebras commutative associative algebras. This operad satisfies $\text{Com}(n) = k$ for $n \geq 1$. Composition is the usual identification of tensor powers of k with k itself.

We now describe the Hopf algebra H_{Com} . Denote the generator of $\text{Com}(n)$ by e_n . Thus H_{Com} is the pointed free associative algebra on generators $\{e_n\}_{n \geq 1}$ where e_1 is identified with the unit, and where e_n is of degree $n-1$, with coproduct

$$\Delta(e_n) = \sum_{k=1}^n \sum_{n_1 + \cdots + n_k = n} e_k \otimes e_{n_1} \cdots e_{n_k},$$

where we sum over n_i such that the formula makes sense (i.e. $n_i \geq 1$).

Let $\mathcal{H}^{\text{diff}}$ be the pointed free commutative algebra on variables a_i for $i \geq 0$. The base-point is $1 \mapsto a_0$. Define a bilinear pairing between the space of generators and the group of formal power series $\varphi(x)$ with coefficients in k such that $\varphi(x) \equiv x \pmod{x^2}$ by the formula

$$\langle a_n, \varphi \rangle = \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \varphi(0).$$

Define a coproduct $\Delta : \mathcal{H}^{\text{diff}} \rightarrow \mathcal{H}^{\text{diff}} \otimes \mathcal{H}^{\text{diff}}$ on generators by the formula $\langle \Delta(a_n), f \otimes g \rangle = \langle a_n, f \circ g \rangle$. These formulae define a Hopf algebra. This Hopf algebra is the **Hopf algebra of higher order differentials of the line**. The character group of $\mathcal{H}^{\text{diff}}$ is the group of power series $\varphi(x)$ with coefficients in k such that $\varphi(x) \equiv x \pmod{x^2}$, which is also called the **group of formal diffeomorphisms of the line**. The group multiplication is composition of power series.

Define the Hopf algebra $\mathcal{H}^{\text{ncdiff}}$ of **non-commutative higher order differentials of the line** (eg. Brouder-Frabeti [2]). As an algebra $\mathcal{H}^{\text{ncdiff}}$ the free unital algebra on variables a_i for $i \geq 0$, with respect to the base-point $1 \mapsto a_0$. Define the coproduct on $\mathcal{H}^{\text{ncdiff}}$ by

$$\Delta(a_n) = \sum_{k, n-k=n_1+\dots+n_{k+1}} a_k \otimes a_{n_1} \cdot \dots \cdot a_{n_{k+1}},$$

where $k, n_i \geq 0$ for $i \leq k+1$. (We got rid of the usual binomial coefficients by introducing $1 = a_0$ in the sum.) Note that the symmetric quotient of $\mathcal{H}^{\text{ncdiff}}$ is $\mathcal{H}^{\text{diff}}$.

4.2 THEOREM *The completion (w.r.t. the (arity - 1)-grading) of the Hopf algebra \bar{H}_{Com} is the Hopf algebra of higher order differentials of the line. Its character group is \bar{G}_{Com} , the group of formal diffeomorphisms of the line. The Lie algebra of primitive elements of the dual \bar{L}_{Com} is the Lie algebra of polynomial vector fields without a constant term on the line.*

PROOF To identify the group \bar{G}_{Com} with the group of power series in one variable x with coefficients in k such that $\varphi(x) \equiv x \pmod{x^2}$ and use composition of power series as multiplication. Use the isomorphism given by $e_n \mapsto x^n$. The result on the Hopf algebras now follows since a graded complete commutative Hopf algebra of finite type is completely determined by its group of characters (Quillen [13]).

At the Hopf algebra level the isomorphism $\bar{H}_{\text{Com}} \rightarrow \mathcal{H}^{\text{diff}}$ is given on the basis of generators by $e_n^* \mapsto a_{n-1}$.

The Lie algebra \bar{L}_{Com} with basis $\{e_n\}_{n \geq 1}$ satisfies the commutation relation $[e_n, e_m] = (n-m)e_{m+n-1}$. The explicit isomorphism is thus given by $e_n \mapsto x^n \partial_x$.

QED

It is clear that the map on the Hopf algebra level lifts to the non-commutative version, which gives the following Corollary.

4.3 COROLLARY *The map defined on generators by $a_i \mapsto e_{i+1}^*$ is an isomorphism of graded Hopf algebras from $\mathcal{H}^{\text{ncdiff}}$ to H_{Com} .*

4.4 REMARK One can consider Com as the endomorphism operad of the one dimensional vector space k . Analogous if one considers the 1-reduced version of the endomorphism operad End_V of a finite dimensional vector space V , one obtains the higher dimensional analogue of the higher order differentials, formal diffeomorphisms, and polynomial vector fields. (Compare Kapranov-Manin [8], where the result is stated on the level of Lie algebras.)

4.2 Associative and Lie operad

4.5 EXAMPLE To describe the structures we get from the operad Ass of associative algebra, first observe that the surjection $\text{Ass} \rightarrow \text{Com}$ becomes an isomorphism on coinvariants: $\text{Ass}(n)_{S_n} = \text{Com}(n)_{S_n}$. thus $\bar{H}_{\text{Ass}} = \bar{H}_{\text{Com}}$, $\bar{L}_{\text{Ass}} = \bar{L}_{\text{Com}}$, and $\bar{G}_{\text{Ass}} = \bar{G}_{\text{Com}}$. However, in the non-symmetrised version there are some differences.

For the group G_{Ass} we write x^σ for the element corresponding to $\sigma \in S_n$. The group G_{Ass} is then the group of formal permutation-expanded series $\sum_\sigma c_\sigma x^\sigma$, where σ runs over permutations in S_n for all n and the coefficients c_σ are in the ground field. Moreover, the trivial permutation $(1) \in S_1$ has coefficient $c_{(1)} = 1$. Composition is the linear extension in x^σ of

$$x^\sigma \circ \left(\sum_\tau c_\tau x^\tau \right) = \sum_{\tau_1, \dots, \tau_k} c_{\tau_1} \dots c_{\tau_k} x^{\hat{\sigma}(\tau_1 \times \dots \times \tau_k)},$$

where $\hat{\sigma}$ is the permutation that permutes k blocks on which the τ_i act. (Observe that $\hat{\sigma}$ thus depends on the degree of the τ_i .) The Lie algebra structure is given on $\sigma \in S_n$ and $\tau \in S_m$ by

$$[\sigma, \tau] = \sum_{i=1}^n \sigma \circ_i \tau - \sum_{j=1}^m \tau \circ_j \sigma.$$

Dually, the Hopf algebra $H_{\text{Ass}} = T_*(\bigoplus_{n \geq 1} S_n)$ with S_n in degree $n - 1$ has the coproduct

$$\Delta(\sigma) = \sum_k \sum_{\sigma = \hat{\tau}_0 \circ (\tau_1 \times \dots \times \tau_k)} \tau_0 \otimes (\tau_1, \dots, \tau_k),$$

where the sum is over all decompositions of σ as $\hat{\tau}_0 \circ (\tau_1, \dots, \tau_k)$, where $(1) \in S_1$ is identified with the unit in H_{Ass} .

4.6 LEMMA Write $\mathbf{a}_1, \dots, \mathbf{a}_n$ for the inputs. Then a basis of $\text{Lie}(n)$ is given by

$$\{[\mathbf{a}_{\sigma(1)}, [\mathbf{a}_{\sigma(2)}, \dots, [\mathbf{a}_{\sigma(n-1)}, \mathbf{a}_n] \dots]] \mid \sigma \in S_{n-1}\}, \quad (4.4)$$

where the brackets are in right-most position.

PROOF By a dimension argument ($\dim(\text{Lie}(n)) = (n-1)!$) it suffices to show that these elements span $\text{Lie}(n)$. This can be proved by induction on n , using the Jacobi identity. QED

4.7 EXAMPLE There is a natural inclusion of operads $\text{Lie} \subset \text{Ass}$ which is defined by sending the bracket $\lambda \in \text{Lie}(2)$ to the commutator $\mu - \mu^{\text{op}}$ of the associative product $\mu \in \text{Ass}(2)$. The group G_{Lie} is therefore a subgroup of G_{Ass} . To make the group more explicit it is useful to characterise the image of Lie in Ass . Write $\mathbf{a}_1, \dots, \mathbf{a}_n$ for the inputs, then we can describe the image of $[\mathbf{a}_1, [\mathbf{a}_2, \dots, [\mathbf{a}_{n-1}, \mathbf{a}_n] \dots]]$ as the sum

$$\sum_{\sigma \in Z_n} (-1)^{n-\sigma^{-1}(n)} \mathbf{a}_{\sigma(1)} \cdot \mathbf{a}_{\sigma(2)} \cdot \dots \cdot \mathbf{a}_{\sigma(n)},$$

where Z_n consists of those permutations $\sigma \in S_n$ such that for $i := \sigma^{-1}(n)$,

$$\sigma(1) < \sigma(2) < \dots < \sigma(i) \quad \text{and} \quad \sigma(i) > \sigma(i+1) > \dots > \sigma(n).$$

By the Lemma above, the images of elements in G_{Lie} are the series in G_{Ass} that are of the form

$$\sum_n \sum_{\tau \in S_{n-1}} c_{n,\tau} \sum_{\sigma \in Z_n} (-1)^{n-\sigma^{-1}(n)} x^{\sigma \circ (\tau \times (1))}.$$

4.3 The Connes-Kreimer Hopf algebra of trees

Consider the set $T(*)$ of rooted trees without external edges different from the root. A **cut** of $t \in T(*)$ is a subset of edges. A cut is **admissible** if for every vertex $v \in \mathbf{v}(t)$ the path from v to the root contains at most one edge in the cut. If c is an admissible cut of t , denote by $R^c(t)$ the connected component of the graph obtained from t by removing the edges in c , where all new external edges are removed. Denote by P^c the disjoint union of the other components as elements of $T(*)$ with their new root.

The set $T(*)$ generates a commutative Hopf algebra \mathcal{H}_R . As an algebra it is the symmetric algebra on $T(*)$. Comultiplication Δ is defined on generators t , and extended as an algebra homomorphism:

$$\Delta(t) = \sum_c R^c(t) \otimes P^c(t), \quad (4.5)$$

where the sum is over admissible cuts. The counit $\varepsilon : \mathcal{H}_R \rightarrow k$ takes value 1 on the empty tree and 0 on other trees. The Hopf algebra \mathcal{H}_R is the **Hopf algebra of**

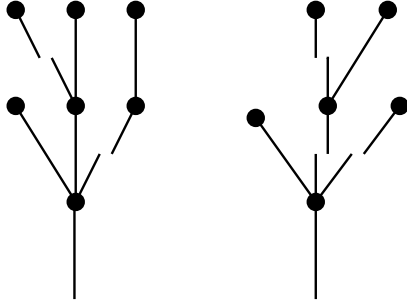


FIGURE 1: An admissible cut (on the left) and a non-admissible cut (on the right). Construct $R^c(t)$ as the connected component containing the root with all upward pointing external edges removed.

rooted trees, introduced by Kreimer [10] and further studied in Connes-Kreimer [5].

The Lie algebra of primitive elements of \mathcal{H}_R^* is isomorphic to the Lie algebra \mathcal{L}_R spanned by rooted trees without external edges other than the root, and the bracket $[s, t] = s \bullet t - t \bullet s$, where

$$t \bullet s = \sum_{v \in \mathbf{v}(t)} t \circ_v s,$$

and $t \circ_v s$ is the rooted tree consisting of the connected subtrees t and s and one edge that connects the root of s to the vertex $v \in \mathbf{v}(t)$.

4.8 EXAMPLE Let M be the collection $M(n) = k$ for $n \geq 2$ and $M(n) = 0$ otherwise. Recall the free operad satisfies $TM(n) = \bigoplus_{t \in T_2(\mathbf{n})} k$, where $T_2(\mathbf{n})$ is the set of rooted trees with a pointed bijection $\varphi : \mathbf{l}(t) \rightarrow \{0, \dots, n\}$ (i.e. the root leg is mapped to 0) where every vertex has at least 3 external edges. Thus TM is a graded operad of finite type with respect to the grading by $|\mathbf{v}(t)|$, the number of vertices. The symmetric group action permutes the labels on external edges.

4.9 PROPOSITION *There exists an inclusion $\mathcal{H}_R \rightarrow \widehat{H_{TM}}$, of the Connes-Kreimer Hopf algebra into the completion (w.r.t. the grading by external edges) of H_{TM} . This inclusion is graded with respect to the grading by the number of vertices.*

PROOF It suffices to show that there is a surjection φ of Lie algebras $\bar{L}_{TM} \rightarrow \mathcal{L}_R$ onto the Lie algebra of primitive elements of \mathcal{H}_R^* .

A vertex $v \in \mathbf{v}(t)$ of a tree t is called **saturated** if it has no external edges other than the root attached to it. Denote by J the ideal in TM spanned by trees with a saturated vertex. The map φ of Lie algebras we define factorises through

$\bar{L}_{TM/J}$. Thus it suffices to define $\varphi : L_{TM} \rightarrow \mathcal{L}_R$ on trees without a saturated vertex. On such a tree t define

$$\varphi(t) = \tilde{t} \cdot \prod_{v \in \mathbf{v}(t)} i_t(v)!,$$

where $i_t(v)$ is the number of external legs other than the root of t that are attached to vertex v and \tilde{t} is the tree t with all external edges other than the root omitted. Recall that both Lie algebras are pre-Lie algebras. To check that this is a Lie algebra homomorphism, write

$$\begin{aligned} \varphi(t) \bullet \varphi(s) &= \sum_{u \in \mathbf{v}(t)} \widetilde{t \circ_u s} \cdot \prod_{v \in \mathbf{v}(t)} \prod_{w \in \mathbf{v}(s)} i_t(v)! i_s(w)! \\ &= \sum_{u \in \mathbf{v}(t)} \widetilde{t \circ_u s} \cdot i_{t \circ_u s}(u) \cdot \prod_{v \in \mathbf{v}(t \circ_u s)} i_{t \circ_u s}(v)! \\ &= \sum_{i=1}^{|\mathbf{I}(t)|-1} \widetilde{t \circ_i s} \cdot \prod_{v \in \mathbf{v}(t \circ_u s)} i_{t \circ_u s}(v)! \\ &= \varphi(t \bullet s). \end{aligned}$$

To understand the second equality, note that $i_s(w) = i_{t \circ_u s}(w)$ for all $w \in \mathbf{v}(s)$, that $i_t(v) = i_{t \circ_u s}(v)$ if $v \neq u$, and that $i_t(u) = i_{t \circ_u s}(u) + 1$. The third equality rewrites the sum over vertices as a sum over external edges other than the root.

QED

4.4 The double symmetric algebra construction

The results in this section first appeared in Van der Laan and Moerdijk [15]. We start with a lemma. This lemma is not new, but we decided to include a sketch of the proof, since this point has led to some confusion in earlier drafts.

4.10 LEMMA (BERGER-MOERDIJK [1]) *Let A be a bialgebra.*

- (i). *The vector spaces $C_A(n) = A^{\otimes n}$ (for $n \geq 1$) form a coaugmented non-symmetric cooperad with as coidentity the counit $\varepsilon : A \rightarrow k$ and the comultiplication γ^* defined on summands by the diagram*

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\gamma^*} & A^{\otimes k} \otimes (A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k}) \\ \Delta \downarrow & & \uparrow (\mu_1 \otimes \dots \otimes \mu_k) \otimes \text{id} \\ A^{\otimes n} \otimes A^{\otimes n} & \xlongequal{\quad} & (A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k}) \otimes (A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k}), \end{array}$$

where Δ is the coproduct of $A^{\otimes n}$, and $\mu_i : A^{\otimes n_i} \rightarrow A$ is the multiplication of the algebra A . A coaugmentation for this non-symmetric cooperad is given by the unit of A .

(ii). The collection C_A with the S_n -action by permuting tensor factors is a coaugmented cooperad with respect to the same structure maps if A is commutative.

PROOF Coassociativity follows directly from the fact that μ is a coalgebra morphism. Coidentity and coaugmentation are also direct from the bialgebra structure on A and $A^{\otimes n}$. Then it remains to consider the compatibility with the $S_k \times (\bigoplus_{n_1+\dots+n_k=n} (S_{n_1} \times \dots \times S_{n_k}))$ -action for

$$\gamma^* : C_A(n) \longrightarrow \bigoplus_{n_1+\dots+n_k=n} C_A(k) \otimes C_A(n_1) \otimes \dots \otimes C_A(n_k).$$

In Sweedler's notation one can write the cocomposition γ^* of C_A on a generator $(x_1, \dots, x_n) \in C_A(n)$ as

$$\gamma^*(x_1, \dots, x_n) = \sum \sum (x'_1 \star \dots \star x'_{n_1}, \dots, x'_{n-n_k+1} \star \dots \star x'_n) \otimes ((x''_1, \dots, x''_{n_1}) \otimes \dots \otimes (x''_{n-n_k+1}, \dots, x''_n)),$$

where the first sum is over all k and all partitions $n = n_1 + \dots + n_k$, and the second sum is the sum of the Sweedler notation, and where \star denotes the product of A . We study the action on the right hand side of this formula. The compatibility with the S_k -action is satisfied since the action permutes both the tensor factors in $A^{\otimes k}$, and the factors $A^{\otimes n_1}$ up to $A^{\otimes n_k}$. For compatibility with the S_{n_i} -action, consider $x'_{n_1+\dots+n_{i-1}+1} \star \dots \star x'_{n_1+\dots+n_i} \otimes (x''_{n_1+\dots+n_{i-1}+1}, \dots, x''_{n_1+\dots+n_i})$. The S_{n_i} -action only permutes elements in the right hand side. Thus, to obtain equivariance we should have that $x'_{n_1+\dots+n_{i-1}+1} \star \dots \star x'_{n_1+\dots+n_i}$ is invariant under permutation of the indices. In other words, the product needs to be commutative. QED

Recall the bialgebras $T(\bar{T}'(A))$ and $S(\bar{S}'(A))$ (Brouder-Schmitt [3]), where T (resp. S) is the unital free associative (resp. commutative) algebra functor, and \bar{T}' (resp. \bar{S}') is the non-unital free associative (resp. commutative) coalgebra functor. Comparing the equation for γ^* in Sweedler's notation with the Brouder-Schmitt formulae makes the following result a tautology.

4.11 THEOREM (VAN DER LAAN-MOERDIJK [15]) *Let A be a bialgebra. The bialgebra $T(\bar{T}'(A))$ is isomorphic to the opposite bialgebra of B_{C_A} . The bialgebra $S(\bar{S}'(A))$ is isomorphic to the opposite bialgebra of \bar{B}_{C_A} .*

Let C be a coaugmented cooperad. Then the collection $C^{>1}$ defined by $C^{>1}(1) = k$ and $C^{>1}(n) = C(n)$ for $n > 1$ is a 1-reduced cooperad with cocomposition induced by cocomposition in C through the surjection $C \longrightarrow C^{>1}$.

4.12 COROLLARY (VAN DER LAAN-MOERDIJK [15]) *The Pinter Hopf algebra associated to $T(\bar{T}'(A))$ (cf. Brouder and Schmitt [3] for terminology) is isomorphic to the opposite Hopf algebra of $H_{C_A^{>1}}$, and the Pinter Hopf algebra associated to $S(\bar{S}'(A))$ is isomorphic to the opposite Hopf algebra of $\bar{H}_{C_A^{>1}}$.*

5 Operads of graphs

The examples treated in the previous section treat Hopf algebras based on well known operads. However, a given Hopf algebra can also lead to a new operad structure. The operads of graphs to which this section is devoted are examples of this phenomenon.

5.1 The operad Γ

In this section a labelled graph will be assumed to be a graph η without edges from a vertex to itself, together with a numbering of the vertices (i.e. a bijection $\mathbf{v}(\eta) \rightarrow \{1, \dots, |\mathbf{v}(\eta)|\}$). If $k \leq |\mathbf{v}(\eta)|$ denote by $\mathbf{l}_\eta(k)$ the set of legs of the vertex numbered k in the labelled graph η . The restriction to labelled graphs without self-loops is not necessary, at this point, but it catalyses some of the arguments.

Define $\text{Graph}(n)$ as the groupoid of labelled graphs η such that $|\mathbf{v}(\eta)| = n$ with isomorphisms of labelled graphs (compatible with the numbering of vertices) as maps.

5.1 DEFINITION Let $\Gamma(1) = k$ and $\Gamma(n) = \text{colim}(\text{Graph}(n))$ for $n \geq 2$. We define the structure of an operad on the collection Γ defined by the $\Gamma(n)$ with the S_n -action on the numbers of the vertices.

Let η and ζ be two labelled graphs and let $k \leq |\mathbf{v}(\eta)|$. Assume that $|\mathbf{l}_\eta(k)| = |\mathbf{l}(\zeta)|$. For each bijection $b : \mathbf{l}_\eta(k) \rightarrow \mathbf{l}(\zeta)$ define $\eta \circ_b \zeta$ as the labelled graph defined by replacing vertex k in η by the labelled graph ζ , and connecting the legs of ζ to the remaining part of η according to bijection b . The linear ordering of the vertices is obtained from the linear ordering on the vertices of η upon replacing vertex k by the linear order on vertices of ζ . For η, ζ and k as above define the circle- k operation of the operad as

$$\eta \circ_k \zeta = \begin{cases} \sum_{[b]} \eta \circ_b \zeta & \text{if } |\mathbf{l}_\eta(k)| = |\mathbf{l}(\zeta)| \\ 0 & \text{otherwise,} \end{cases} \quad (5.6)$$

where the sum is over equivalence classes of bijections $b : \mathbf{l}_\eta(k) \rightarrow \mathbf{l}(\zeta)$ with respect to the equivalence relation $b \sim b'$ iff $\eta \circ_b \zeta \cong \eta \circ_{b'} \zeta$.

5.2 LEMMA *The \circ_k -operations defined above, make the collection Γ an operad*

PROOF Since the identities with respect to the \circ_k -operations are trivial it remains to prove associativity. Consider $(\eta \circ_k \zeta) \circ_l \theta$ for η, ζ, θ labelled graphs in Γ . If l is a vertex of η in $\eta \circ_k \zeta$, the labelled graphs ζ and θ are plugged into different vertices of η . Thus we can write

$$\begin{aligned} (\eta \circ_k \zeta) \circ_l \theta &= \sum_{[b], [b']} (\eta \circ_b \zeta) \circ_{b'} \theta \\ &= \sum_{[b], [b']} (\eta \circ_{b'} \theta) \circ_b \zeta, \end{aligned}$$

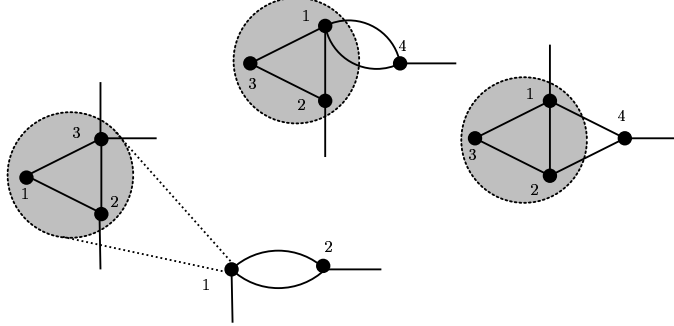


FIGURE 2: The composition \circ_1 in the operad Γ . The graph on the left is included in the two-vertex graphs. The result is the sum of the two other graphs.

which is sufficient to prove associativity in this case. Suppose on the contrary that l is a vertex of ζ in $\eta \circ_k \zeta$. Then

$$\begin{aligned} (\eta \circ_k \zeta) \circ_l \theta &= \sum_{[b]} (\eta \circ_b (\sum_{[b']} \zeta \circ_{b'} \theta)) \\ &= \sum_{[b], [b']} \eta \circ_b (\zeta \circ_{b'} \theta). \end{aligned}$$

Regarding the second equality note that it is clear we can move the sum over $[b']$ out since the automorphisms of labelled graphs have to preserve the numbering of the vertices. QED

5.2 The Connes-Kreimer Hopf algebra of graphs

Connes-Kreimer [6] introduces a Hopf algebra of graphs (without numbered vertices) associated to the φ^3 theory in six dimensions. We define the Hopf algebra \mathcal{H}_{φ^3} , which is a slightly simplified version.

A **1 particle irreducible graph** η is a connected graph η such that η has at least two external edges, such that each vertex $k \in \mathbf{v}(\eta)$ satisfies $|\mathbf{I}_\eta(k)| \in \{2, 3\}$, and such that the graphs are still connected after removing one edge. This terminology comes from the physics. Let \mathcal{H}_{φ^3} as an algebra be the unital free commutative algebra on the vector space spanned by 1 particle irreducible graphs modulo the numbering of vertices $\bar{\eta}$. The coproduct on a generator η is given by

$$\Delta(\bar{\eta}) = \sum_{\bar{\zeta} \subset \bar{\eta}} \bar{\eta}/\bar{\zeta} \otimes \bar{\zeta},$$

where the sum is over subgraphs $\bar{\zeta}$ such that $\bar{\zeta}$ is a direct union of generators and if $v \in \mathbf{v}(\bar{\zeta})$, then $\mathbf{I}_{\bar{\zeta}}(v) = \mathbf{I}_{\eta}(v)$. The direct union of generators ζ in the right tensor factor is interpreted as the product of generators. The counit is given by the usual augmentation of the unital free commutative algebra.

5.3 DEFINITION Let Γ_{PI} be the suboperad of Γ spanned by the 1 particle irreducible labelled graphs. It is easily checked that this is indeed a suboperad.

5.4 THEOREM *The Hopf algebras $\bar{H}(\Gamma_{PI})$ and \mathcal{H}_{φ^3} are isomorphic.*

PROOF By the Milnor-Moore Theorem, it suffices to compare the Lie algebras of primitive elements of the dual Hopf algebras. In both cases the Lie algebra of primitive elements is spanned by 1 particle irreducible graphs without numbering of the vertices.

If $\eta \in \Gamma_{PI}(n)$, denote by $\bar{\eta} \in \Gamma(n)_{S_n}$ the graph η obtained by forgetting the numbering of vertices. In $P(\bar{H}(\Gamma_{PI})^*)$, the Lie bracket (cf. Theorem 3.11) is given by

$$[\bar{\eta}, \bar{\zeta}] = \sum_{v \in \mathbf{v}(\bar{\eta})} \sum_{[b]} \bar{\eta} \circ_b \bar{\zeta} - \sum_{w \in \mathbf{v}(\bar{\zeta})} \sum_{[b']} \bar{\zeta} \circ_{b'} \bar{\eta}$$

where the sums in the first line are over equivalence classes of bijections $b : \mathbf{I}_{\bar{\eta}}(v) \rightarrow \mathbf{I}(\bar{\zeta})$ for $v \in \mathbf{v}(\bar{\eta})$ and $b' : \mathbf{I}_{\bar{\zeta}}(w) \rightarrow \mathbf{I}(\bar{\eta})$ for $w \in \mathbf{v}(\bar{\zeta})$. Theorem 2 in Connes-Kreimer [6] states that this Lie algebra is isomorphic to the Lie algebra of primitive elements in \mathcal{H}_{φ^3} . QED

5.5 REMARK A remark on the isomorphism is in place. Connes-Kreimer [6] uses the notation $\bar{\eta} \circ_v \bar{\zeta} := \sum_{v \in \mathbf{v}(\bar{\eta})} \sum_{[b]} \bar{\eta} \circ_b \bar{\zeta}$, but do not state explicitly that they sum over equivalence classes of bijections. The linear space $P(\mathcal{H}_{\varphi^3}^*)$ of primitive elements has a natural basis spanned by graphs (dual to the basis of the generators of \mathcal{H}_{φ^3} given above).

The Connes-Kreimer isomorphism $P(\mathcal{H}_{\varphi^3}^*) \rightarrow \bar{L}_{\Gamma_{PI}}$ mentioned in the proof of Theorem 5.4 is given by

$$\bar{\eta} \mapsto S(\bar{\eta}) \cdot \bar{\eta},$$

where $S(\bar{\eta})$ is the symmetry factor of $\bar{\eta}$ which takes the form

$$S(\bar{\eta}) = \frac{|\text{Aut}(\bar{\eta})|}{|\text{Aut}(\eta)|},$$

where $\text{Aut}(\eta)$ is the isomorphism group of η (i.e. automorphisms need to preserve the numbering of vertices), and $\text{Aut}(\bar{\eta})$ is the automorphism group of $\bar{\eta}$ (i.e. the automorphism need not preserve the numbering of the vertices). In other words, $S(\bar{\eta})$ is the cardinality of the automorphism group of $\bar{\eta}$ divided by the subgroup of automorphisms that induce the identity map on the vertices of $\bar{\eta}$. Note that $S(\bar{\eta})$ is indeed independent of the representative η of $\bar{\eta}$.

5.6 COROLLARY *The group of characters of \mathcal{H}_{φ^3} is isomorphic to $\tilde{G}_{\Gamma_{PI}}$. It is the group of formal series of connected 1 particle irreducible graphs $\sum_{\bar{\zeta}} c_{\bar{\zeta}} x^{\bar{\zeta}}$ with $c_{\eta} \in k$ and coefficient $c_{\bar{\zeta}} = 1$ for each 1-vertex graphs $\bar{\zeta}$. The composition is the linear extension of*

$$x^{\bar{\eta}} \cdot \sum_{\bar{\zeta}} d_{\bar{\zeta}} x^{\bar{\zeta}} = \sum_{\bar{\zeta}_1, \dots, \bar{\zeta}_{|v(\bar{\eta})|}} d_{\zeta_1} \cdot \dots \cdot d_{\zeta_{|v(\bar{\eta})|}} x^{\gamma(\bar{\eta}; \bar{\zeta}_1, \dots, \bar{\zeta}_{|v(\bar{\eta})|})},$$

in terms of the composition of graphs in the operad Γ_{PI} , where the one vertex trees are identified with the unit in the algebra.

5.7 REMARK In order to reproduce the Hopf algebra originally defined by Connes and Kreimer exactly, one needs to introduce a second colour for each vertex with two legs, and one needs to label external edges by elements of a certain space of distributions.

5.3 The operad $\tilde{\Gamma}$

5.8 DEFINITION The definition of the operad Γ above is not the only natural choice. Let $\tilde{\Gamma}(n) = \Gamma(n)$ as an S_n -module for $n \in \mathbb{N}$. Define \circ_k -operations on $\tilde{\Gamma}$ as

$$\eta \circ_k \zeta = \begin{cases} \sum_b \eta \circ_b \zeta & \text{if } |\mathbf{l}_{\eta}(k)| = |\mathbf{l}(\zeta)| \\ 0 & \text{otherwise,} \end{cases}, \quad (5.7)$$

where b runs over all bijections $b : \mathbf{l}(\zeta) \rightarrow \mathbf{l}_{\eta}(k)$, instead of equivalence classes (as in Formula (5.6)).

5.9 PROPOSITION *The structure defined above makes $\tilde{\Gamma}$ an operad isomorphic to Γ .*

PROOF Checking the operad axioms for $\tilde{\Gamma}$ is not difficult, the argument is analogous to the argument for Γ .

For a graph η , denote by $\text{Aut}(\eta)$ the automorphisms of η that preserve the numbering of the vertices. Define $\varphi(\eta) = |\text{Aut}(\eta)| \cdot \eta$. This is an isomorphism of collections from Γ to $\tilde{\Gamma}$ (in characteristic 0). To see that it commutes with the \circ_i -operations, one needs for a bijection $b : \mathbf{l}_{\eta}(i) \rightarrow \mathbf{l}(\zeta)$ the equality

$$|\text{Aut}(\eta \circ_b \zeta)| \cdot |\{b' : \mathbf{l}_{\eta}(i) \rightarrow \mathbf{l}(\zeta) \text{ s.t. } [b'] = [b]\}| = |\text{Aut}(\eta)| \cdot |\text{Aut}(\zeta)|.$$

It suffices to construct a bijection

$$\psi : \text{Aut}(\eta) \times \text{Aut}(\zeta) \rightarrow \coprod_{b' \sim b} \text{Iso}(\eta \circ_b \zeta, \eta \circ_{b'} \zeta).$$

Given $(\tau, \sigma) \in \text{Aut}(\eta) \times \text{Aut}(\zeta)$, construct $\psi(\tau, \sigma)$ as follows. There are natural inclusions of $\zeta - \mathbf{l}(\zeta) \hookrightarrow \eta \circ_{b'} \zeta$ and $\eta - \mathbf{l}_{\eta}(i) \hookrightarrow \eta \circ_{b'} \zeta$ for every b' . Let $b' = \sigma|_{\zeta} \circ b$, and define $\psi(\tau, \sigma) : \eta \circ_b \zeta \rightarrow \eta \circ_{b'} \zeta$ by $\psi(\tau, \sigma)|_{\zeta - \mathbf{l}(\zeta)} := \sigma|_{\zeta - \mathbf{l}(\zeta)}$ and

$\psi(\tau, \sigma)|_{\eta-1(i)} = \tau|_{\eta-1(i)}$ on legs that are not glued. Denote the leg obtained from glueing $l \in \mathbf{l}_\eta(i)$ to $b(l) \in \mathbf{l}(\zeta)$ by $\{l, b(l)\}$. On such legs define $\psi(\tau, \sigma)(\{l, b(l)\}) := \{\tau(l), b'(\tau(l))\}$. It remains to see that ψ is a bijection.

If $\psi(\tau, \sigma) = \psi(\xi, \chi)$, then obviously $\tau|_{\eta-1(i)} = \xi|_{\eta-1(i)}$, and $\sigma|_{\zeta-1(\zeta)} = \chi|_{\zeta-1(\zeta)}$. Moreover, $\sigma|_{\mathbf{l}(\zeta)} = b' \circ b^{-1} = \chi|_{\mathbf{l}(\zeta)}$, and since $\{\tau(l), b'(\tau(l))\} = \{\xi(l), b'(\xi(l))\}$ it follows that ψ is an injection.

Given $\theta : \eta \circ_b \zeta \rightarrow \eta \circ_{b'} \zeta$, restriction to $\eta - \mathbf{l}_\eta(i)$ and $\zeta - \mathbf{l}(\zeta)$ determine $(\tau, \sigma) \in \text{Aut}(\eta) \times \text{Aut}(\zeta)$ except for glued legs. On these, let $\sigma|_{\mathbf{l}(\zeta)} = b' \circ b^{-1}$, and if $\theta(\{l, b(l)\}) = \{l', b'(l')\}$ define $\tau(l) = l'$. Then $\psi(\tau, \sigma) = \theta$ shows ψ is surjective.

QED

5.4 The Wick algebra

5.10 DEFINITION Let V be a finite dimensional vector space together with a quadratic form $b \in (V \otimes V)^*$. Let η be a graph with n vertices. For $i = 1, \dots, n$ denote $k_i = |\mathbf{l}_\eta(i)|$, the number of legs of vertex i , and write $S'^n V = \bigotimes_{i=1}^n S'^{k_i} V$. For an edge $e \in \mathbf{e}(\eta)$ with vertices $j_1, j_2 \in \mathbf{v}(\eta)$ let $\eta - e$ denote the graph with edge e omitted, and define

$$b(e) : S'^n \rightarrow S'^{(\eta-e)}$$

by applying b to one tensor factor in $S'^{k_{j_1}} V$ and one tensor factor in $S'^{k_{j_2}} V$. If $e' \neq e$ is an other edge of η , then $e' \in \mathbf{e}(\eta - e)$ implies that $b(e') \circ b(e) : S'^n \rightarrow S'^{(\eta-e-e')}$ is well defined and independent of the order of e and e' .

Let η be a graph with n vertices and edges $\mathbf{e}(\eta) = \{e_1, \dots, e_m\}$. For $i = 1, \dots, n$ denote by $k_i = |\mathbf{l}_\eta(i)|$ the number of legs of the vertex i , and by $l_i = |\mathbf{e}_\eta(i)|$ the number of legs of the vertex i that are part of an edge. Define $\tau^\eta : S'^n \rightarrow S'^{|\mathbf{l}(\eta)|}$ as

$$\tau^\eta = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sigma \circ b(e_m) \circ \dots \circ b(e_1).$$

That is, first we contract tensor factors corresponding to all edges, and on the result in $\bigotimes_{i=1}^n S'^{l_i} V$ we apply the sum of all permutations to symmetrise the remaining tensor factors, which means that we end up in $S'^{(l_1 + \dots + l_n)} V$. The result is again independent of the order of the edges. Extend τ^η as zero to a map

$$\tau^\eta : (S'V)^{\otimes n} \rightarrow S'V.$$

5.11 THEOREM Let V be a vector space with a quadratic form $b \in (V \otimes V)^*$. The symmetric coalgebra $S'V$ enjoys a Γ -algebra structure with respect to the maps

$$\gamma(\eta, p_1, \dots, p_{|\mathbf{v}(\eta)|}) = \frac{1}{|\text{Aut}(\eta)|} \tau^\eta(p_1, \dots, p_{|\mathbf{v}(\eta)|}).$$

PROOF It suffices to show that $\tilde{\gamma}(\eta, p_1, \dots, p_n) = \tau^\eta(p_1, \dots, p_n)$ defines a $\tilde{\Gamma}$ -algebra structure. Since compatibility with the symmetric group action is obvious, it remains to check that $\tau^\eta \circ_1 \tau^\zeta = \tau^{\eta \circ_1 \zeta}$. Let $\mathbf{e}(\eta) = \{e_1, \dots, e_n\}$ and $\mathbf{e}(\zeta) = \{e_{n+1}, \dots, e_{n+m}\}$. For any bijection $b : \mathbf{l}(\zeta) \rightarrow \mathbf{l}_\eta(1)$ we can naturally write $\eta \circ_b \zeta = \{e_1, \dots, e_{n+m}\}$. Then

$$\tau^{\eta \circ_1 \zeta} = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sum_{b: \mathbf{l}(\zeta) \rightarrow \mathbf{l}_\eta(1)} \sigma \circ b(e_{n+m}) \circ \dots \circ b(e_1),$$

and

$$\tau^\eta \circ_1 \tau^\zeta = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sum_{\tau \in S_{|\mathbf{l}(\zeta)|}} \sigma \circ b(e_n) \circ \dots \circ b(e_1) \circ \tau \circ b(e_{n+m}) \circ \dots \circ b(e_{n+1}).$$

It is not hard to see that summing over τ and summing over b have the same effect. QED

5.12 REMARK Theorem 5.11 is suggested by the Wick rotation formula from quantum field theory (cf. Kazhdan [9]). In particular by the asymptotic series for the functional integral of a QFT in a neighbourhood of a free QFT.

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