CANONICAL EQUIVALENCE RELATIONS ON NETS OF PS_{c_0}

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ABSTRACT. We give a list of canonical equivalence relations on discrete nets of the positive unit sphere of c_0. This generalizes results of W. T. Gowers [1] and A. D. Taylor [4].

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1. INTRODUCTION

Let FIN be the family of nonempty finite sets of positive integers. A block sequence is an infinite sequence $(x_n)_n$ of elements of FIN such that for every n, $\max x_n < \min x_{n+1}$ (usually written as $x_n < x_{n+1}$). If $(x_n)_n$ is a block sequence, $\langle (x_n)_n \rangle$ denotes the set of finite unions $x_{n_0} \cup \cdots \cup x_{n_m}$. Recall the following pigeonhole principle known as Hindman's Theorem [2]: For every coloring $c : \mathrm{FIN} \to \{0, 1, \ldots, l-1\}$ there is some block sequence $(x_n)_n$ such that c is constant on $\langle (x_n)_n \rangle$. This is equivalent to the fact that for any equivalence relation on FIN with a finite number of classes there is a block sequence $(x_n)_n$ such that $\langle (x_n)_n \rangle$ is contained in one class. However, for arbitrary infinite colorings of the form $c : \mathrm{FIN} \to \mathbb{N}$ there is no hope to find a monochromatic structure $\langle (x_n)_n \rangle$, as can be observed considering the coloring c defined by $c(s) = \min s$. Nevertheless, there are results saying that there is always a "substructure" where the coloring has a certain "canonical" form, much in the spirit of the original motivation of F.P. Ramsey [3] for discovering his famous Theorem. For example, A. D. Taylor [4] showed that any equivalence relation on FIN can be reduced to one of the following five canonical relations:

$$\min, \max, (\min, \max), =, \mathrm{FIN}^2,$$

where for two finite sets s, t $s \min t$ iff the minimum of s is equal to the minimum of t, $s \max t$ iff the maximum of s is equal to the maximum of t, $s(\min, \max)t$ iff both minimum and maximum are the same. More precisely,

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for every equivalence relation $\sim$ on FIN there is a block sequence $(x_n)_n$ such that $\sim$ restricted to $\langle (x_n)_n \rangle$ is one of the canonical relations.

Given a positive integer k, let FIN_k be the set of mappings $x : \mathbb{N} \to \{0, 1, \dots, k\}$ such that the support $\operatorname{supp} x = \{n : x(n) \neq 0\}$ is finite and k is in the range of x. Let $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ be the mapping defined by $T(x)(n) = \max\{x(n) - 1, 0\}$. A k-block sequence $(x_n)_n$ is an infinite sequence of members of FIN_k such that $\max \operatorname{supp} x_n < \min \operatorname{supp} x_{n+1}$, for every n. There is now the natural definition of $\langle (x_n)_n \rangle$ as the set of elements of FIN_k of the form $T^{i_0} x_{n_0} \vee \cdots \vee T^{i_m} x_{n_m}$ such that some $i_j = 0$, and where for $i > 0$, $T^i x$ is defined by $T^i x(n) = \max\{x(n) - i, 0\}$, $T^0 = Id$, and for $x \in \mathrm{FIN}_i$, $y \in \mathrm{FIN}_j$, $x \vee y$ is defined by $(x \vee y)(n) = \max\{x(n), y(n)\}$. W. T. Gowers' result [1] states that FIN_k has a pigeonhole principle, i.e., for every equivalence relation on FIN_k with a finite number of classes there is a k-block sequence $(x_n)_n$ such that $\langle (x_n)_n \rangle$ is contained in one class. Observe that FIN_1 is isomorphic to FIN, so this result can be naturally viewed as a generalization of Hindman's result.

The aim of this paper is to characterize any equivalence relation on FIN_k with arbitrary number of classes. More precisely, we will give a non redundant finite list $\mathcal{T}_k$ of equivalence relations such that for every equivalence relation $\sim$ on FIN_k there is a block sequence such that $\sim$ restricted to it is in $\mathcal{T}_k$. An equivalence relation belonging to the canonical non redundant list can be identified with a particular procedure of assigning invariants to vectors of FIN_k. For example, a typical member of $\mathcal{T}_k$ will be defined from the following invariants: Fix some $1 \leq i \leq k$, and let us assign to any vector s of FIN_k the first integer n for which $s(n) = i$. Let us call this invariant $\min_i s$. Another more complex example is the following. Given two integers i and l such that $1 \leq l \leq i - 1 \leq k$, let us assign to any vector s of FIN_k the set of integers n such that $\min_{i-1} s \leq n \leq \min_i s$, and $s(n) = l$.

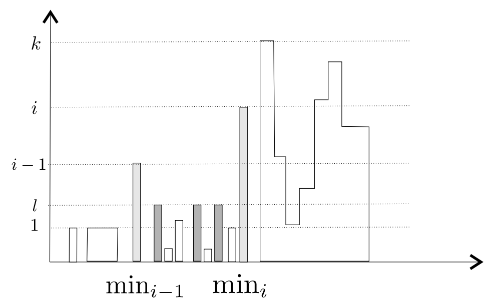


FIGURE 1. an example of invariant

Of course these assignments are not well defined for an arbitrary vector of FIN_k. Nevertheless, we will show that any k-block sequence will have a k-block subsequence for which all the vectors have all possible invariants well defined. The precise definitions will be given in Section 3.

The main idea to assign invariants to a given equivalent relation comes from Taylor's proof [4]. Let us explain how it works. Given an equivalence relation $\sim$ on FIN, let $s : [\mathrm{FIN}]^{[3]} \to \{0,1\}^4$ be the coloring defined by

$$a = (a_0, a_1, a_2) \mapsto \begin{cases} s(a)(0) = 1 & \text{iff } a_0 \sim a_1 \\ s(a)(1) = 1 & \text{iff } a_0 \cup a_1 \sim a_0 \\ s(a)(2) = 1 & \text{iff } a_0 \cup a_1 \sim a_1 \\ s(a)(3) = 1 & \text{iff } a_0 \cup a_1 \cup a_2 \sim a_0 \cup a_2, \end{cases} \tag{1}$$

where $[\mathrm{FIN}]^{[3]}$ is the set of 3-sequences of finite sets (a_0, a_1, a_2) such that $a_0 < a_1 < a_3$. Since $[\mathrm{FIN}]^{[3]}$ has a pigeonhole principle, there is a block sequence $X = (x_n)_n$ such that s is constant in $[X]^{[3]}$ with value $s_0 \in \{0,1\}^4$. Analyzing all the possible cases for s_0, it is possible to conclude which of the relations of the list $\{\min, \max, (\min, \max), =, \mathrm{FIN}^2\}$ is equal to $\sim$ restricted to X (precisely, in some block subsequence Y of X). Let us restate the fact that s is constant in a different way. Fix an alphabet of countably many variables $\{x_n\}_n$. An $\sim$-equation e is a pair $x_{i_0} \cup \cdots \cup x_{i_l} \sim x_{j_0} \cup \cdots \cup x_{j_m}$, where $0 = i_0 < \cdots < i_l$, $j_0 < \cdots < j_m$ (formally, the pair is $(x_{i_0} \cup \cdots \cup x_{i_l}, x_{j_0} \cup \cdots \cup x_{j_m})$). e is true in X iff for any sequence $a_0 < \cdots < a_{\max\{i_l, j_m\}}$ in X, $a_{i_0} \cup \cdots \cup a_{i_l} \sim a_{j_0} \cup \cdots \cup a_{j_m}$, and it is false in X iff for any sequence $a_0 < \cdots < a_{\max\{i_l, j_m\}}$ in X, $a_{i_0} \cup \cdots \cup a_{i_l} \not\sim a_{j_0} \cup \cdots \cup a_{j_m}$. e is decided in X if it is either true or false in X. So, the fact that s is constant on X is equivalent to say that the equations $x_0 \sim x_1$, $x_0 \cup x_1 \sim x_0$, $x_0 \cup x_1 \sim x_1$ and $x_0 \cup x_1 \cup x_2 \sim x_0 \cup x_2$ are all decided in X. For an arbitrary k, the type of equations to be considered is more rich. As an example, for $k = 2$, the equations

$$x_0 \cup x_1 \cup Tx_2 \sim x_1 \cup Tx_2,\ x_0 + Tx_1 \cup x_2 \sim x_0 \cup x_2,$$

need to be considered. Of course the best situation would be to start with a block sequence which decides all the equations, but this condition is too strong, since we will show that it is equivalent to be in the list $\mathcal{T}_k$. Nevertheless, we will show that for a given equivalence relation $\sim$ on FIN_k there is a k-block sequence X which decides a sufficient number of $\sim$-equations to assign a list of invariants to $\sim$, and hence to recognize $\sim$ as a member of the list $\mathcal{T}_k$. We will characterize also the members of $\mathcal{T}_k$ as the equivalence relations for which their equations are always true or always false, and we will show that the number t_k of equivalence relations in $\mathcal{T}_k$ can be described using standard arithmetic functions. For example, we show in Section 5 that this number is

$$t_k = |\mathcal{T}_k| = e^2 \left[k \left[\Gamma(k,1) - \Gamma(k+1,1) \right]^2 + \Gamma(k+1,1)^2 \right].$$

Since FIN_k is isomorphic to a net of the positive sphere of c_0, our result implies the immediate analogue for those nets. In particular, this will imply that given an equivalence relation R on PS_{c_0} and given some $\delta > 0$ there is an infinite dimensional block subspace X of c_0 and some equivalence relation R' in our finite list such that any R'-class in X will be included in some R-class in X.

This paper is organized as follows. In Section 2 we introduce FIN_k as a natural copy of a net of the positive sphere of c_0, extending some concepts coming from Banach space theory to FIN_k. We also state the Pigeonhole principle for FIN_k. Equations are defined in Section 3, together with the canonical invariants associated to a vector. We describe the vectors for which these invariants are well defined, and we show that they appear "everywhere". We define also the family $\mathcal{T}_k$. In Section 4 our main Theorem is proved, and in Section 5 we give an explicit formula to compute the cardinality of $\mathcal{T}_k$. Sections 6 and 7 deal with the finite version of our main result, and with some consequences on equivalence relations on the positive sphere of c_0, respectively.

2. First Definitions and Results

Recall that $c_0 = c_0(\mathbb{R})$ is the Banach space of sequences of real numbers tending to 0, with the sup norm defined for a vector $\vec{x} = (x_n)_n$ of c_0 by $\|\vec{x}\| = \sup_n |x_n|$. Let $(e_n)_n$ be its canonical Schauder basis, i.e., $e_n(m) = \delta_{n,m}$. The support of a vector $\vec{x} = (x_n)_n$, is $\mathrm{supp}\, \vec{x} = \{n : x_n \neq 0\}$ and let c_{00} be the linear subspace of c_0 consisting on the vectors $\vec{x} = (x_n)_n$ with finite support, i.e., only finitely many of the coordinates of x are not zero. Given two vectors $\vec{x}$ and $\vec{y}$ of c_{00} we write $\vec{x} < \vec{y}$ to denote $\max \mathrm{supp}\, \vec{x} < \min \mathrm{supp}\, \vec{y}$.

Let PS_{c_0} be the set of norm one positive vectors of c_0, i.e., the set of all vectors $\vec{x} = (x_n)_n$ such that $\|\vec{x}\| = 1$, and such that $x_n \geq 0$, for every n, and let PB_{c_0} be the set of positive vectors of the unit ball of c_0. Observe that PB_{c_0} is a lattice with respect to $(x_n)_n \vee (y_n)_n = (\max\{x_n, y_n\})_n$ and $(x_n)_n \wedge (y_n)_n = (\min\{x_n, y_n\})_n$, with $0 = (0)_n$, and $1 = (1)_n$. Notice also that PS_{c_0} is closed under the operation $\vee$, and that $x \vee y = x + y$ if x and y have disjoint support. Given $A, N \subseteq c_0$, and $\delta_0 > 0$, we say that N is a *δ_0-net* of A iff *(a)* $\|\vec{x} - \vec{y}\| \geq \delta_0$ for every $\vec{x} \neq \vec{y} \in N$ and *(b)* for every $\vec{a} \in A$ there is some $\vec{x} \in N$ such that $\|\vec{a} - \vec{x}\| \leq \delta_0$.

For a given $0 < \delta < 1$, let k be the first integer such that $1/(1+\delta)^{k-1} \le \delta$, and let $\varepsilon = 1/(1+\delta)$. Let

$$\mathcal{N}_\delta = \{x \in PB_{c_{00}} \ : \ x(i) \in \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{k-1}, 0\}\}$$
$$\mathcal{M}_\delta = \{x \in PS_{c_{00}} \ : \ x(i) \in \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{k-1}, 0\}\}.$$

Since $\varepsilon^i - \varepsilon^{i+1} = \varepsilon^i(1-\varepsilon) = \varepsilon^i(\delta/(1+\delta)) < \delta$ and $\varepsilon^{k-1} \le \delta$, it is clear that $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are ε^{k-1}-nets of PB_{c_0} and PS_{c_0}, respectively. It is also clear that $\mathcal{N}_\delta$ is a sub-lattice of PB_{c_0} with respect to $\vee$ and $\wedge$. One of the main properties of $\mathcal{N}_\delta$ is that it is closed by the scalar multiplication with ε, identifying $\varepsilon^l = 0$ for $l \ge k$ (which means that we identify the coordinates less than ε^k with 0). Also, for two $\vec{x}, \vec{y} \in \mathcal{M}_\delta$, we have that $\vec{x} \vee \varepsilon^i \vec{y}, \varepsilon^i \vec{x} \vee \vec{y} \in \mathcal{M}_\delta$, for every $0 \le i \le k-1$. Notice that $\mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \varepsilon^i \mathcal{M}_\delta$, disjoint union. Let $\Theta = \Theta_\delta : \mathcal{N}_\delta \to \{0, 1, \dots, k\}^{\mathbb{N}}$ be defined by

$$\Theta((x_m)_m)(n) = \begin{cases} k - \log_\varepsilon(x_n) & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0. \end{cases}$$

The nets $\mathcal{M}_\delta$ and $\mathcal{N}_\delta$, and the mapping Θ give a different interpretation of FIN_k as a geometrical object of PS_{c_0}. It is not difficult to show that this new definition coincides with the one given in the introduction.

Definition 2.1. Given k, let $0 < \delta = \delta(k) < 1$ be such that $1/(1+\delta)^{k-1} = \delta$, and let $\mathrm{FIN}_k = \Theta(\mathcal{M}_\delta)$, i.e., it is the set of functions $s : \mathbb{N} \to \{0, 1, \dots, k\}$ eventually 0, and with k in the range. Elements of FIN_k are called *k-vectors.*

Observe that $\Theta\text{"}\mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \Theta\text{"}\varepsilon^i \mathcal{M}_\delta$, and that $\Theta\text{"}\varepsilon^i \mathcal{M}_\delta$ is the set of all functions $s : \mathbb{N} \to \{0, 1, \dots, k\}$ eventually 0, and with $k-i$ in the range. So, $\Theta\text{"}\varepsilon^i \mathcal{M}_\delta = \mathrm{FIN}_{k-i}$. Hence, $\Theta\text{"}\mathcal{N}_\delta = \bigcup_{i=1}^{k} \mathrm{FIN}_i =: \mathrm{FIN}_{\le k}$, whose members are called *$(\le k)$-vectors.*

We can transfer the algebraic structure of $\mathcal{N}_k \subseteq c_{00}$ to $\mathrm{FIN}_{\le k}$ via Θ. In particular, for $s, t \in \mathrm{FIN}_{\le k}$, let the *support of s* be $\mathrm{supp}\ s = \{n \ : \ s(n) \neq 0\}$, we write $s < t$ to denote that $\max \mathrm{supp}\ s < \min \mathrm{supp}\ t$, define $s \vee t$ and $s \wedge t$ by

$$(s \vee t)(n) = \max\{s(n), t(n)\} \text{ and } (s \wedge t)(n) = \min\{s(n), t(n)\},$$

and transfer the multiplication by ε, that we will call T, i.e., for a $(\le k)$-vector s, let

$$T(s) = \Theta(\varepsilon \Theta^{-1}(s)) = (s-1) \vee 0.$$

Consequently, $\mathrm{FIN}_{\le k}$ is a lattice with operations $\vee$ and $\wedge$, and it is closed under T. We will use the order $\le_L$ to denote the lattice-order of $\mathrm{FIN}_{\le k}$, i.e., for $s, t \in \mathrm{FIN}_{\le k}$, we write $s \le_L t$ iff $s \wedge t = s$. Notice that $\mathrm{FIN}_i \vee \mathrm{FIN}_j = \mathrm{FIN}_{\max_{\{i,j\}}}$, and that $\mathrm{FIN}_i \wedge \mathrm{FIN}_j = \mathrm{FIN}_{\min_{\{i,j\}}}$. We will denote $s \vee t$ by $s + t$ whenever $s < t$.

Let $S : FIN_{k-1} \to FIN_k$ be an inverse map for T, defined for a $(k-1)$-vector a by

$$S(a)(n) = \begin{cases} a(n)+1 & \text{if } n \in \text{supp } a \\ 0 & \text{if not} \end{cases}$$

Let $\mathrm{FIN}_k^{[\infty]} = \{(s_n)_n \in (\mathrm{FIN}_k)^\infty \ : \ s_n < s_{n+1} \text{ for every } n \geq 0\}$ be the set of *k-block sequences*. For $n \in \mathbb{N}$, let $\mathrm{FIN}_k^{[n]} = \{(s_0, \dots, s_{n-1}) \in (\mathrm{FIN}_k)^n \ : \ s_i < s_{i+1} \text{ for every } i = 0, \dots, n-2\}$ and let $\mathrm{FIN}_k^{[<\infty]} = \bigcup_{n \geq 1} \mathrm{FIN}_k^{[n]}$ be the set of *finite k-block sequences*. For $\alpha = (s_n)_{n<N} \in \mathrm{FIN}_k^{[\leq\infty]}$ ($N \leq \infty$), let $\langle\alpha\rangle$ be the set of all k-vectors of α, i.e., $\langle\alpha\rangle = \Theta(\text{LinSpan } \Theta^{-1}\{s_n\}_n \cap \mathcal{N}_k)$, where for a subset A of c_0, LinSpan A denotes the linear span of A. Notice that $\mathrm{FIN}_k = \langle(\Theta e_n)_n\rangle$. For $i \leq k$, let $\langle\alpha\rangle_i$ be the set of i-vectors of α. For $M \leq N \leq \infty$, and $\alpha = (s_n)_{n<N}$ let $[\alpha]^{[M]}$ be the set of *k-block subsequences* of α, defined as $[\alpha]^{[M]} = \{(s_n)_{n<M} \in \mathrm{FIN}_k^{[M]} \ : \ s_n \in \langle\alpha\rangle \ (0 \leq n < N)\}$. Without loss of generality we will identify $[\alpha]^{[1]}$ with $\langle\alpha\rangle$.

Given two finite block sequences α and β, and two infinite ones A and B, we define $\alpha \preceq \beta$ if and only if $\alpha \in [\beta]^{[|\alpha|]}$, $\alpha \preceq A$ if and only if $\alpha \in [A]^{[|\alpha|]}$, and $B \preceq A$ if and only if $B \in [A]^{[\infty]}$. Notice that all these definitions come from the notion of subspace. For example, $A \in \langle B\rangle$ if and only if the space generated by $\Theta^{-1}A$ is a subspace of the space generated by $\Theta^{-1}B$.

For a k-block sequence $A = (a_i)_i$ and $a \in \langle A\rangle$, since $\langle A\rangle = \Theta(\text{LinSpan } \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_k)$, we have that $\Theta^{-1}a \in \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_k$. Therefore, $\Theta^{-1}a = \sum_{i=0}^m \varepsilon^{d_i}\Theta^{-1}a_i$, for some m, and with possibly some $d_i = 0$. This implies that $a = \Theta(\sum_{i=0}^m \varepsilon^{d_i}\Theta^{-1}a_i) = \sum_{i=0}^m \Theta(\varepsilon^{d_i}\Theta^{-1}a_i) = \sum_{i=0}^m T^{d_i}a_i$.

Definition 2.2. Given a k-block sequence $A = (a_n)_n$, let $C_A : \langle A\rangle \to FIN_k$ be the mapping satisfying

$$a = \sum_{n=0}^{\infty} T^{k-C_A(a)(n)} a_n, \tag{2}$$

for every k-block vector a of A. Since $\Theta^{-1}a = \sum_{n\geq 0} \varepsilon^{k-C_A(a)(n)}\Theta^{-1}a_n$, for every a, C_A is well defined. (2) is the *canonical decomposition of a in A*. Notice that $C_A(a) \in \mathrm{FIN}_k$ for every a.

For two $(\leq k)$-vectors s and t, we write $s \sqsubseteq t$ iff $t\restriction\text{supp } s = s$, i.e., if t restricted to the support of s is equal to s, and we write $s \perp t$ iff $s \wedge t = 0$, i.e., if s and t have disjoint support. Notice that if $s = \sum_{n=0}^\infty T^{k-l_n}a_n$ is the canonical decomposition of a (in arbitrary A), then $T^{k-l_n}a_n \sqsubseteq a$, for every n. Observe also that $T^{k-l_n}a_n \perp T^{k-l_{n'}}a_{n'}$, for every $n \neq n'$.

Proposition 2.3. *Fix $A = (a_n)_n$, and $a \in \langle A\rangle$. Then, for every n, if there are some $r \leq k$, and m such that $T^{k-r}a_n(m) = a(m) \neq 0$, then necessarily $C_A(s)(n) = r$ (i.e., $T^{k-r}a_n \sqsubseteq a$).* □

Recall the following W. T. Gowers' pigeonhole principle for FIN_k.

Theorem 2.4. [1] *If* FIN_k *is partitioned into finitely many pieces, then there is* $A \in \mathrm{FIN}_k^{[\infty]}$ *such that* $\langle A \rangle$ *is in only one of the pieces.*

The following is the natural higher dimensional version of this Theorem, first proved in [5] in a even more general case (for $\mathrm{FIN}_k^{[\infty]}$ and Borel colorings).

Lemma 2.5. *Suppose that* $f : \mathrm{FIN}_k^{[n]} \to \{0, \dots, l-1\}$*. Then, there is a block sequence* X *such that* f *is constant on* $[X]^{[n]}$.

Proof. The proof is by induction over n. Suppose it is true for $n-1$. Notice that this implies that the result is not only true for $FIN_k^{[n-1]}$, but for $[Z]^{[n-1]}$, for every k-block sequence Z (using the canonical isomorphism $\Lambda : \mathrm{FIN}_k \to \langle (z_n)_n \rangle$ defined by $\Lambda(\Theta e_n) = z_n$). We are going to define two sequences $\{\theta_r\}_n$ and $\{X_r\}_r$ such that:

1. $\theta_r = (a_0, \dots, a_{r-1}) \in \mathrm{FIN}_k^{[r]}$, and X_r is a k-block sequence.
2. $X_r > \theta_r$, $a_0 = \Theta e_0$, $a_r = \min X_{r+1}$, and $X_{r+1} \preceq X_r$.
3. For every $(b_0, \dots, b_{n-2}) \in [\theta_r]^{[n-1]}$, f is constant on $\{(b_0, \dots, b_{n-2}, x) : x \in X_r\}$ with value $\varepsilon((b_0, \dots, b_{n-2}), r)$.

They are constructed using Theorem 2.4. Notice that $\varepsilon((b_0, \dots, b_{n-2}), r) = \varepsilon((b_0, \dots, b_{n-2}), s)$, whenever $(b_0, \cdots {}_{n-2}) \in [\theta_r]^{[n-1]}$ and $r < s$, since $X_s \preceq X_r$. Let $X = (a_r)_r$. Define the mapping $\varepsilon : [X]^{[n-1]} \to \{0, 1, \dots, l-1\}$ by $\varepsilon(b_0, \dots, b_{n-2}) = \varepsilon((b_0, \dots, b_{n-2}), r)$, for some (any) r. By inductive hypothesis, there is some $Y \preceq X$ such that ε is constant on $[Y]^{[n-1]}$. We show that F is indeed constant on $[Y]^{[n]}$: Fix two sequences $s_i = (b_0^{(i)}, \dots, b_{n-1}^{(i)})$ of $[Y]^{[n]}$, $i = 0, 1$. Then, $\varepsilon(b_0^{(0)}, \dots, b_{n-2}^{(0)}) = \varepsilon(b_0^{(1)}, \dots, b_{n-2}^{(1)})$, and we are done, since $\varepsilon(b_0^{(i)}, \dots, b_{n-2}^{(i)}) = F(b_0^{(i)}, \dots, b_{n-1}^{(i)})$.

□

3. Equations, Staircases and Canonical equivalence Relations

Roughly speaking, terms are natural mappings that assign k-vectors to finite block sequences of k-vectors of a fixed length n, and which are defined from the operations $+$ and T^i of FIN_k. For example, the mapping that assigns to a block sequence (a_1, a_2) of k-vectors the k-vector $a_1 + Ta_2$ is a k-term which can be understood as the mapping with two variables x_1, x_2 defined by $f(x_1, x_2) = x_1 + Tx_2$.

From a fixed two k-terms f and g, both with n variables, and one equivalence relation $\sim$ on FIN_k, we can define the natural coloring $c_{f,g} : [\mathrm{FIN}_k]^n \to \{0, 1\}$ via $c_{f,g}(a_1, \dots, a_n) = 1$ if and only if $f(a_1, \dots, a_n) \sim$

$g(a_1,\dots,a_n)$. A k-equation will be $f \sim g$. Notice that the pigeonhole principle introduced in Lemma 2.5 shows that for every equation $f \sim g$ (f and g with n variables) there is some infinite block sequence A such that, either for every $(a_1,\dots,a_n)$ in $[A]^{[n]}$, $f(a_1,\dots,a_n) \sim g(a_1,\dots,a_n)$, or for all $(a_1,\dots,a_n)$ in $[A]^{[n]}$, $f(a_1,\dots,a_n) \not\sim g(a_1,\dots,a_n)$, i.e., in A the equation $f \sim g$ is either true or false. As we explained in the Introduction, Taylor proves that an equivalence relation $\sim$ on FIN is "controlled" by a list of 4 equations (precisely $x_0 \sim x_1$, $x_0 \sim x_0 + x_1$, $x_1 \sim x_0 + x_1$ and $x_0 + x_1 + x_2 \sim x_0 + x_2$). This is going to be also the case for arbitrary k, of course with a more complex list of equations.

3.1. **Terms and equations.**

Definition 3.1. Let $\mathtt{X} = \{x_n\}_{n\ge 1}$ be a countable infinite alphabet of variables. Consider the trivial map $\mathtt{x} : \mathtt{X} \to \mathbb{N}$, defined by $x_n \mapsto \mathtt{x}(x_n) = n$. A *free k-term* $\mathtt{p}$ is a map of the form $s \circ \mathtt{x}$ where s is a k-vector, i.e., it is a map $\mathtt{p} : \mathtt{X} \to \{0,\dots,k\}$ such that supp $\mathtt{p}$ is finite, and k is in the range of $\mathtt{p}$. A natural representation of $\mathtt{p}$ is

$$\mathtt{p} = \mathtt{p}(x_0,\dots,x_l) = \sum_{i=0}^{l} T^{k-m_i} x_i,$$

where $0 \le m_i \le k$, and at least one $m_i = k$. For example $T^2 x_1 + T x_2 + x_4$, and $x_1 + x_5$ are both free 3-terms. Notice that, if $\mathtt{p}$ is a free k-term, then $\mathtt{p} \circ \mathtt{x}^{-1}$ is a k-vector. A *free* $(\le k)$*-term* is $s \circ \mathtt{x}$, where s is a $(\le k)$-vector. It follows that the set of free $(\le k)$-terms is a lattice. For example

$$\mathtt{p}(x_0,\dots,x_n) \vee \mathtt{q}(x_0,\dots,x_m) = (\mathtt{p} \circ \mathtt{x}^{-1} \vee \mathtt{q} \circ \mathtt{x}^{-1}) \circ \mathtt{x}.$$

For a fixed free $(\le k)$-term $p(x_0,\dots,x_n) = \sum_{i=0}^{n} T^{k-m_i} x_i$, there are two kinds of substitutions: 1. Given a sequence of free $(\le k)$-terms $t_0,\dots,t_n$, consider the substitution of each x_i by t_i and get

$$\mathtt{p}(t_0,\dots,t_n) = \bigvee_{i=0}^{n} T^{k-m_i} t_i.$$

In the case that $\mathtt{p}$ and $t_0,\dots,t_n$ are free k-terms, then $\mathtt{p}(t_0,\dots,t_n)$ is also a free k-term.
2. For a block sequence $(a_0,\dots,a_n)$ of $(\le k)$-vectors, replace each x_i by a_i, and get

$$\mathtt{p}(a_0,\dots,a_n) = \sum_{i=0}^{n} T^{k-m_i} a_i.$$

If $\mathtt{p}$ is a free k-term, and $a_0,\dots,a_n$ are k-vectors, then the result of the substitution $\mathtt{p}(a_0,\dots,a_n)$ is a k-vector. The main reason to introduce free k-terms is the following notion of equations.

Definition 3.2. A *free k-equation* (*free equation* in short) is a pair $\{\mathtt{p}(x_0,\dots,x_n),\mathtt{q}(x_0,\dots,x_{n'})\}$ of free k-terms. Given a fixed equivalence relation $\sim$ on FIN_k, we will write the previous free equation as

$$\mathtt{p}(x_0,\dots,x_n)\sim\mathtt{q}(x_0,\dots,x_{n'}).$$

Given s_0, s_1, i_0 and i_1-vectors respectively, a free j_0-term $\mathtt{p}$, and a free j_1-term $\mathtt{q}$ satisfying that $\max\{i_l, j_l\} = k$ for $l = 0, 1$, we consider the equations of the form $s+\mathtt{p}\sim t+\mathtt{q}$ and $\mathtt{p}+s\sim\mathtt{q}+t$, called *$k$-equations* (or *equations*, if there is no confusion). Observe that now the substitutions of $(b_0,\dots,b_n)$ in the equation $s+p\sim t+q$ will be allowed only when $b_0 > s,t$, and for an equation $\mathtt{p}+s\sim\mathtt{q}+t$, provided that $b_n < s,t$.

Definition 3.3. We say that a k-equation $s+p(x_0,\dots,x_n)\sim t+q(x_0,\dots,x_n)$ (or $p(x_0,\dots,x_n)+s\sim q(x_0,\dots,x_n)+t$) *holds* (or is *true*) in A iff for every $(a_0,\dots,a_n)$ in A with $a_0 > s,t$ (resp. $a_n < s,t$), $s+p(a_0,\dots,a_n)\sim s+q(a_0,\dots,a_n)$ (resp. $p(a_0,\dots,a_n)+s\sim q(a_0,\dots,a_n)+s$). The equation $s+p(x_0,\dots,x_n)\sim t+q(x_0,\dots,x_n)$ (or $p(x_0,\dots,x_n)+s\sim q(x_0,\dots,x_n)+t$) is false in A iff for every $(a_0,\dots,a_n)$ in A with $a_0 > s,t$ (resp. $a_n < s,t$), $s+p(a_0,\dots,a_n)\not\sim s+q(a_0,\dots,a_n)$ (resp. $p(a_0,\dots,a_n)+s\not\sim q(a_0,\dots,a_n)+s$). The equation is *decided in A* iff, either is true, or false in A.

It is clear that, given a k-equation $\mathtt{p}(x_0,\dots,x_n)\sim\mathtt{q}(x_0,\dots,x_{n'})$, we can assume that $n = n'$, since we can extend the terms of the equation adding summands of the form $T^k x$ and not changing the "meaning" of the k-equation.

Some properties of equations that will be useful.

Proposition 3.4. *Suppose that all free equations with at most five variables are decided in a given block sequence A. Then,*

1. *If $x_0+T^{k-i}x_1+x_2\sim x_0+x_2$ is true in A, then $x_0+T^{k-j}x_1+x_2\sim x_0+x_2$ is true in A for every $j\le i$.*
2. *If $x_0+x_1+Tx_2\sim x_0+Tx_2$ or $Tx_0+x_1+x_2\sim Tx_0+x_2$ are true in A, then $x_0+x_1+x_2\sim x_0+x_2$ is also true in A.*
3. *If the equation $x_0+x_1+T^ix_2\sim x_0+T^ix_2$ is true in A, then the equation $x_0+x_1+T^jx_2\sim x_0+T^jx_2$ also is true in A for every $j\le i$.*
4. *If the equation $T^ix_0+x_1+x_2\sim T^ix_0+x_2$ is true in A, then the equation $T^jx_0+x_1+x_2\sim T^jx_0+x_2$ also is true in A for every $j\le i$.*
5. *If the equation $x_0+T^{k-r_1}x_1+T^{k-r_0}x_2\sim x_0+T^{k-r_0}x_2$ holds, then also the equation $x_0+T^{k-r_2}x_1+T^{k-r_0}x_2\sim x_0+T^{k-r_0}x_2$ for every $r_1>r_2$ and r_0.*

PROOF. Fix A deciding all the equations with at most five variables. 1. Fix $j < i$. Then,

$$x_0 + T^{k-i}x_1 + T^{k-j}x_2 + x_3 \sim x_0 + T^{k-i}(x_1 + T^{i-j}x_2) + x_3 \sim x_0 + x_3 \quad \text{hold in } A. \tag{3}$$

Hence,

$$x_0 + T^{k-i}x_1 + (T^{k-j}x_2 + x_3) \sim x_0 + (T^{k-j}x_2 + x_3) \text{ holds in } A, \tag{4}$$

and we are done. 2. Suppose now that $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is true in A. Then

$$x_0 + x_2 + Tx_3 \sim x_0 + Tx_3 \text{ and } x_0 + x_1 + x_2 + Tx_3 \sim x_0 + Tx_3 \quad \text{are true in } A. \tag{5}$$

Hence, $x_0 + x_1 + x_2 + Tx_3 \sim x_0 + x_2 + Tx_3$ holds in A, and therefore, $x_0+x_1+x_2 \sim x_0+x_2$ is true in A. 3. Suppose that $x_0+x_1+T^ix_2 \sim x_0+T^ix_2$ is true in A, and fix $j \geq i$. Then, $x_0 + x_1 + x_2 + T^j(x_3 + T^{i-j}x_4) \sim x_0 + x_1 + x_2 + T^jx_3 + T^ix_4 \sim x_0 + T^ix_4$ hold in A, and

$$x_0 + x_1 + T^j(x_2 + T^{i-j}x_3) \sim x_0 + x_1 + T^jx_2 + T^ix_3 \sim x_0 + T^ix_3 \quad \text{hold in } A, \tag{6}$$

which implies what we wanted. 4. is showed as 3. Let us show 5. Fix $r_1 > r_2$ and r_0 and suppose that the equation $x_0+T^{k-r_1}x_1+T^{k-r_0}x_2 \sim x_0+T^{k-r_0}x_2$ holds in A. Then, $x_0+T^{k-r_2}x_1+T^{k-r_1}x_2+T^{k-r_0}x_3 \sim x_0+T^{k-r_1}(T^{r_1-r_2}x_1+x_2)+T^{k-r_0}x_3 \sim x_0+T^{k-r_0}x_3$ and $(x_0+T^{k-r_2}x_1)+T^{k-r_1}x_2+T^{k-r_0}x_3 \sim x_0+T^{k-r_2}x_1 \sim T^{k-r_0}x_3$ holds in A. Therefore, $x_0 + T^{k-r_2}x_1 \sim T^{k-r_0}x_3 \sim x_0 + T^{k-r_0}x_3$ is true in A. □

3.2. **System of staircases and canonical equivalence relations.** To try to classify equivalence relations of FIN_k is somehow the same that to find canonical properties of a typical k-vector. For example, one of these properties can be the cardinality or the minimum of its support. Coming back to Taylor's result, he proves that these properties are the minimum, the maximum, both together, the cardinality (which corresponds to the identity) and any property that fulfills all 1-vectors (and which corresponds to the equivalence relation where any two vectors are related). For an arbitrary $k > 1$, it is expected a bigger list of canonical properties. One example is, for a given k-vector a, the first n of the support of a such that $a(n) = k$; another one is fixed $1 \leq i \leq k$, the minimum n such that $a(n) = i$, that is not always well defined, since for $i < k$, there are k-vectors where i does not appear in their range. Nevertheless, this last property seems very natural. We will introduce a type of block sequences, called systems of staircases, where these properties (and some others) are well defined, for every k-vector.

Definition 3.5. For an integer $i \in [1,k]$, let $\min_i, \max_i : FIN_k \to \mathbb{N}$ be the mappings $\min_i(s) = \min s^{-1}\{i\}$, $\max_i(s) = \max s^{-1}\{i\}$, if defined and 0, if not. A k-vector a is a *system of staircases (sos)* if and only if

1. Rang $s = \{0, 1 \dots, k\}$,
2. $\min_i a < \min_j a < \max_j a < \max_i a$, for $i < j \le k$,
3. for every $1 \le i \le k$,

$$\text{Rang } a \restriction [\min_{i-1} a, \min_i a] = \{0, \dots, i\},$$
$$\text{Rang } a \restriction [\max_i a, \max_{i-1} a] = \{0, \dots, i\},$$
$$\text{Rang } a \restriction [\min_k a, \max_k a] = \{0, \dots, k\}.$$

The following figure illustrates the previous definition.

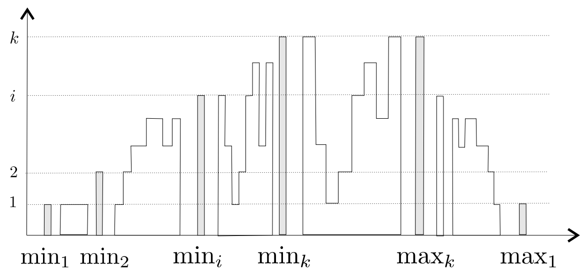


FIGURE 2. A typical sos.

A block subspace $A = (a_n)_n$ is a *system of staircases* if and only if all k-vector $a \in \langle A \rangle$ is a sos. In the next Proposition we show, among other properties, that for every k-block sequence A there is sos $B \in [A]^{[\infty]}$.

Proposition 3.6.

1. *T preserves sos, i.e., if a is a sos k-vector, then Ta is a sos $(k-1)$-vector.*
2. *$T^{k-j}a + b$, $a + T^{k-j}b$ are sos's, provided that $a < b$ are sos. Therefore, for every k-term $p(x_0, \dots, x_n)$ and every block sequence of sos $(a_0, \dots, a_n) \in [\mathrm{FIN}_k]^{[n+1]}$, the substitution $p(a_0, \dots, a_n)$ is also a sos.*
3. *$A = (a_n)_n$ is a sos if and only if a_n is a sos for every n.*
4. *If A is a sos, then any other $B \preceq A$ is also a sos.*
5. *For every A there is some $B \preceq A$ which is a sos.*

Proof. It is not difficult to prove that 1. and 2. (for the last part of 2., use induction over the complexity of the k-term p). To show 3., let us suppose that for every n, a_n is a sos, and fix $a \in \langle (a_n)_n \rangle$. Then, there is a k-term $p(x_0, \dots, x_n)$ such that $p(a_n, \dots, a_n) = a$. Therefore, by 2., a is a sos. 4. is trivial. Let us show 5. Fix $A = (a_n)_n$. For each n, let

$$c_n = \sum_{j=1}^{k} T^{k-j} a_{(2k-1)n+j-1} + \sum_{j=1}^{k-1} T^{k-(k-j)} a_{(2k-1)n+k-1+j}.$$

Notice that for each n, Rang $c_n[0, \min_k(c_n)]$ = Rang $c_n[\max_k(c_n), \infty) = \{0, \dots, k\}$. Therefore, Rang $T^{k-j} c_n[0, \min_j T^{k-j}(c_n)]$ and Rang $T^{k-j} c_n [\max_j T^{k-j}(c_n), \infty)$ are equal to $\{0, \dots, j\}$ for each $j \leq k$. For $n \geq 0$, let

$$b_n = \sum_{j=1}^{k} T^{k-j} c_{n(3k-1)+j-1} + \sum_{j=1}^{k} T^{k-j} c_{n(3k-1)+k-1+j} + \sum_{j=1}^{k-1} T^{k-(k-j)} c_{n(3k-1)+2k-1+j}.$$

It is not difficult to show that every b_n is a sos. □

Definition 3.7. An equivalence relation $\sim$ on FIN_k is *canonical in A* if and only if all equations are decided in every sos $B \in [A]$ in the same way, i.e., if and only if for every equation $p \sim q$, either for every sos $A \in [B]$, $p \sim q$ is true in A, or for every sos $A \in [B]$, $p \sim q$ is false in A. We will say that R is *canonical* if it is canonical in FIN_k.

Canonical equivalence relations are those for which all the equations $p \sim q$ are decided in every sos in the same way. It is not difficult to see that all the equivalence relations of the list $\{\min, \max, (\min, \max), =, \mathrm{FIN}^2\}$ are canonical in FIN. Indeed Taylor's result is equivalent to the fact that there are no more canonical equivalence relations that the one in this list. It will be shown later that for every k, there is a finite list of canonical equivalence relations. Indeed we will give an explicit description of how canonical equivalence relations looks like. Roughly speaking, the canonical equivalence relations are the relations with a definition made out from the typical invariants of a sos. For example, two k-vectors s and t are related iff the first position n of the support of s where $s(n) = k-1$, and the corresponding for t are the same. The following definition gives the list of invariants associated to a sos that will characterize completely the canonical equivalence relations, hence the list $\mathcal{T}_k$. We will prove later that any equivalence relation belongs to $\mathcal{T}_k$ when restricted to some sos.

Definition 3.8. For a set X, a k-block sequence A, and an arbitrary map $f : \langle A \rangle \to X$, we can define the relation R_f on $\langle A \rangle$ by $sR_f t$ if and only if $f(s) = f(t)$. Whenever there is no confusion, we will use the notation sft instead of $sR_f t$. Fix now a sos A. Recall that for $i \in [1,k]$, $\min_i(s) = \min\{n : s(n) = i\}$. This mapping can be easily seen as $\min_i : \langle A \rangle \to FIN_i$ in the following natural way

$$\min_i(s)(n) = \begin{cases} i & \text{if } n = \min_i(s) \\ 0 & \text{otherwise.} \end{cases}$$

For $I \subseteq \{1, \dots, k\}$, let $\min_I : \langle A \rangle \to \mathrm{FIN}_{\max I} \subseteq \mathrm{FIN}_{\leq k}$ be defined by $\min_I(s)(n) = i$ if $n = \min_i(s)$, for $i \in I$ and 0 otherwise, i.e., $\min_I(s) = \{(\min_i(s), i) : i \in I\}$, and extended by 0 in the rest. In a similar manner, we define

$$\max_i(s)(n) = \begin{cases} i & \text{if } n = \max_i(s) \\ 0 & \text{otherwise,} \end{cases}$$

and $\max_I : FIN_k \to FIN_{\max I}$, defined by $\max_I(s) = \{(\max_i(s), i) : i \in I\}$, again extended by 0. Clearly $\min_I = \bigvee_{i \in I} \min_i$ and $\max_I = \bigvee_{i \in I} \max_i$, where $\vee$ is the sup operation on $\mathrm{FIN}_{\leq k}$ and for $f, g : \langle A \rangle \to \mathrm{FIN}_{\leq k}$, $(f \vee g)(s) = f(s) \vee g(s)$.

We need to introduce a new class of functions. For $l \leq i - 1$, let $\theta^0_{i,l}, \theta^1_{i,l} : \langle A \rangle \to FIN_l$ be the mappings defined by

$$\theta^0_{i,l}(s) = \{(n,l) : n \in (\min_{i-1}(s), \min_i(s)) \,\&\, s(n) = l\}, \text{ extended by 0, and}$$
$$\theta^1_{i,l}(s) = \{(n,l) : n \in (\max_i(s), \max_{i-1}(s)) \,\&\, s(n) = l\}, \text{ extended by 0.}$$

I.e., for a given n,

$$\theta^0_{i,l}(s)(n) = \begin{cases} l & \text{if } n \in (\min_{i-1}(s), \min_i(s)) \text{ and } s(n) = l \\ 0 & \text{otherwise, and} \end{cases}$$
$$\theta^1_{i,l}(s)(n) = \begin{cases} l & \text{if } n \in (\max_i(s), \max_{i-1}(s)) \text{ and } s(n) = l \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $k = 4$, $i = 3$, $l = 2$ and a given sos 4-vector s, $\theta^2_{3,2}(s)$ is the 2-vector such that $(\theta^2_{3,2}(s))(n) = 2$ for every n such that *(a)* $s(n) = 2$, and *(b)* n is in the interval between $\min_2(s)$ (i.e., the first m such that $s(m) = 2$) and $\min_3(s)$ (i.e., the first m such that $s(m) = 3$), and it is zero otherwise. For $1 \leq l \leq k$, let

$$\theta^2_l(s) = \{(n,l) : n \in (\min_k(s), \max_k(s)) \,\&\, s(n) = l\} \text{ extended by zero.}$$

We illustrate this new definition with another example: For $k = 4$, $l = 3$ and a sos 4-vector s, $\theta^2_3(s)$ is the 3-vector with value $l = 3$ in every element n of the support of s such that *(a)* $s(n) = 3$, and *(b)* n is in between $\min_4(s)$ and $\max_4(s)$, and 0 otherwise.

Let $\theta^0_{i,-1}, \theta^1_{i,-1}$ and θ^2_{-1} be the 0 mapping (hence, the equivalence relations associated to them are all equal to FIN^2_k). For $I=\emptyset$, the mappings $\min_I$ and $\max_I$ are simply the 0 functions, i.e., $0(s)(n)=0$ for all s and n.

Remark 3.9. 1. Sometimes we will use $\min_i, \max_i$ as a integers, instead of a i-vector, i.e., $\min_i(s)$ will denote the unique integer n such that $\min_i(s)(n)=i$.
2. Notice that we can extend one of the mappings f defined before for FIN_k to all $\mathrm{FIN}_{\le k}$ by setting $\bar{f}(s)=f(s)$, if it is well defined, and $\bar{f}(s)=0$, if not. For example, for a $(\le k)$-vector s, $\min_i(s)(n)=i$ iff $i\in \mathrm{Rang}\, s$ and n is the minimum m such that $s(m)=i$, and $\min_i(s)=0$ otherwise; and $\theta^0_{i,l}(s)$ will have the same definition, provided that the mappings $\min_{i-1}$ and $\min_i$ are well defined for s, and so on.

Proposition 3.10. *Suppose that $l\neq -1$. Then,*

1. $\sim_{\theta^0_{i,l}} \subseteq \sim_{\min_{i-1}} \cap \sim_{\min_i}$, $\sim_{\theta^1_{i,l}} \subseteq \sim_{\max_i} \cap \sim_{\max_{i-1}}$, *and* $\sim_{\theta^2_l} \subseteq \sim_{\min_k} \cap \sim_{\max_k}$.
2. $\sim_{\theta^2_l} \subseteq \sim_{\theta^2_{l+1}}$ *and* $\sim_{\theta^\varepsilon_{i,l}} \subseteq \sim_{\theta^\varepsilon_{i,l+1}}$.

Proof. We show the result for $\theta^0_{i,l}$. The other cases can be shown in a similar way. For suppose that $\theta^0_{i,l}(s)=\theta^0_{i,l}(t)$, and we work to show that $\min_{i-1}(s)=\min_{i-1}(t)$. Let n be such that $\min_{i-1}(t)(n)=i-1$. By symmetricity, it suffices to show that $s(n)=i-1$. So, let r be the unique integer such that $T^{k-C_A(t)(r)}a_r(n)=i-1$. Note that $C_A(t)(r)\ge i-1$. There are two cases:

(a) $C_A(t)(r)=i-1$. Since a_r is sos, there is $m\ge n$ such that $T^{k-C_A(t)(r)}$ $a_r(m)=l$, and hence $\theta^0_{i,l}(t)(m)=l$ and $\theta^0_{i,l}(s)(m)=l$. This implies that $C_A(s)(r)=C_A(t)(r)$, and hence $T^{k-C_A(t)(r)}a_r \sqsubseteq s$. Hence, $s(n)=T^{k-C_A(t)(r)}a_r(n)=i-1$.

(b) $C_A(t)(r)>i-1$. Then, $\theta^0_{i,l}$ is well defined for $T^{k-C_A(t)(r)}a_r$, and $\theta^0_{i,l}(T^{k-C_A(t)(r)}a_r)\sqsubseteq \theta^0_{i,l}(t)=\theta^0_{i,l}(t)$, which implies that $T^{k-C_A(t)(r)}a_r\sqsubseteq s$, and again we are done.

Let us show now 2. for θ^2_l. For suppose that $\theta^2_l(s)=\theta^2_l(t)$, i.e.,

$$\{n\in[\min_k(s),\max_k(s)]\,:\,s(n)=l\}=\{n\in[\min_k(s),\max_k(s)]\,:\,t(n)=l\}.$$

Let $n\in(\min_k(s),\max_k(s))$ be such that $s(n)=l+1$. We work to show that $t(n)=l+1$. Let r be the unique integer such that $T^{k-C_A(s)(r)}a_r(n)=l+1$. Then, $C_A(s)(r)\ge l+1$, and since a_r is a sos, $T^{k-C_A(s)(r)}a_r^{-1}\{l\}\neq\emptyset$. Moreover,

Claim. $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\}\cap(\min_k(s),\max_k(s))\neq\emptyset$.

Proof of Claim: Let r_0, r_1 be the unique integers such that $a_{r_0}(\min_k(s)) = a_{r_1}(\max_k(s)) = k$. Observe that $r_0 \le r \le r_1$. There are two cases: If $r_0 < r < r_1$, then we are done since $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap [\min_k(s), \max_k(s)] = (T^{k-C_A(s)(r)}a_r)^{-1}\{l\}$ is non empty.

Suppose that $r_0 = r$ (the case $r_1 = r$ is similar). Then, $C_A(s)(r) = k$, and $\min_k s = \min_k a_r$. So, $(a_r)^{-1}\{l\} \cap (\min_k(a_r), \max_k(s)) \neq \emptyset$, since a_r is a sos, and therefore Rang $a_r \restriction (\min_k a_r, \max_k a_r) = \{0, \dots, k\}$.

□

Notice now that for every $m \in (T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k s, \max_k s)$, $t(m) = l$, since $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k s, \max_k s) \subseteq \theta^2_l(t)$. By Proposition 2.3, $C_A(t)(r) = C_A(s)(r)$, and hence $T^{k-C_A(s)(r)}a_r \sqsubseteq t$, which implies that $t(n) = T^{k-C_A(s)(r)}a_r(n) = s(n) = l$. □

We split the family of mappings introduced in Definition 3.8 into natural sub-families as follows.

Definition 3.11. Let $\mathcal{F}_{\min} = \{\min_1, \dots, \min_k\}$, $\mathcal{F}_{\max} = \{\max_1, \dots, \max_k\}$, $\mathcal{F}_{\mathrm{mid}^\varepsilon} = \{\theta^\varepsilon_{i,l} \, : \, i \in \{1, \dots, k\} \; l \in \{1, \dots, i-1\}\}$, for $\varepsilon = 0, 1$, and $\mathcal{F}_{\mathrm{mid}} = \{\theta^2_l \, : \, l \in \{1, \dots, k\}\} \cup \{0\}$. Notice that the family $\mathcal{F} = \mathcal{F}_{\min} \cup \mathcal{F}_{\max} \cup \mathcal{F}_{\mathrm{mid}^0} \cup \mathcal{F}_{\mathrm{mid}^1} \cup \mathcal{F}_{\mathrm{mid}}$ is pairwise incompatible, i.e., $f \perp g$ for every $f \neq g \in \mathcal{F}$.

For a fixed block sequence A we say that a function $f : \langle A \rangle \to \mathrm{FIN}_{\le k}$ is *staircase* (in A) if it is in the lattice closure of $\mathcal{F}$. An equivalence relation $\sim$ in A is *staircase* (in A) iff $\sim = \sim_f$ for some staircase mapping f.

Given two mappings $f, g : \langle A \rangle \to X$ we say that f and g are equivalents (in A) $f \equiv g$ iff $\sim_f \equiv \sim_g$, i.e., if f and g define the same equivalence relation in A.

The next two propositions give some clarity to the previous definitions.

Proposition 3.12. *An equivalence relation $\sim_f$ is staircase iff there are $I_\varepsilon \subseteq \{1, \dots, k\}$, $J_\varepsilon \subseteq \{j \in I_\varepsilon \, : \, j-1 \in I_\varepsilon\}$, $(l^{(\varepsilon)}_j)_{j \in J_\varepsilon}$ with $l^{(\varepsilon)}_j \le j-1$ (for $\varepsilon = 0, 1$) and $l^{(2)}_k$ such that*

$$\sim_f = \sim_{\min_{I_0}} \cap \bigcap_{j \in J_0} \sim_{\theta^0_{j, l^{(0)}_j}} \cap \sim_{\theta^2_{l^{(2)}_k}} \cap \sim_{\max_{I_1}} \cap \bigcap_{j \in J_1} \sim_{\theta^1_{j, l^{(1)}_j}} .$$

We say that $(I_0, J_0, (l^{(0)}_j)_{j \in J_0}, I_1, J_1, (l^{(1)}_j)_{j \in J_1}, l^{(2)}_k)$ are the values of f.

Proof. This decomposition is a direct consequence of Proposition 3.10. □

Proposition 3.13. *Fix a staircase mapping f a sos $A = (a_n)_n$, and k-vectors s and t of A.*

1. *Either $f = 0$ or there is a unique sequence $f_0 < f_1 < \cdots < f_n$, $f_0 \neq 0$ such that $f \equiv \bigvee_{i=0}^n f_i$ in A.*

2. *$f(s) = f(t)$ if and only if $f_i(s) = f_i(t)$ for every $0 \le i \le n$.*
3. *$f(s) = f(t)$ iff $f(s{\restriction}\text{supp } t) = f(t)$ and $f(t{\restriction}\text{supp } s) = f(s)$*[1].
4. *Suppose that $s_0, s_1 < t_0, t_1$ are $(\le k)$-vectors of A such that $s_0 + t_0$, $s_1 + t_1$ and $s_0 + t_1$ are k-vectors. If $f(s_0 + t_0) = f(s_1 + t_1)$, then $f(s_0 + t_0) = f(s_0 + t_1)$.*

Proof. 1. is a consequence of Proposition 3.12. 2. is clear from 1. Let us check 3. Using 2., we may assume that $f \in \mathcal{F}$. There are several cases.
(a) $f = \min_i$. Suppose that $\min_i(s) = \min_i(t)$. Then, $i \in \text{Rang } s{\restriction}\text{supp } t$, and hence $\min_i(s{\restriction}\text{supp } t) = \min_i s = \min_i t = \min_i(t{\restriction}\text{supp } s)$. Suppose now that $\min_i s < \min_i t$. Then, $\min_i s < \min_i t \le \min_i(t{\restriction}\text{supp } s)$. So, $\min_i(t{\restriction}\text{supp } s) \ne \min_i s$.
(b) $f = \max_i$ is shown in the same way.
(c) $f = \theta^0_{i,l}$. Suppose that $\theta^0_{i,l}(s) = \theta^0_{i,l}(t)$. Then, by *(a)*, $\min_j s = \min_j t{\restriction}\text{supp } s$, and $\min_j t = \min_j s{\restriction}\text{supp } t$, where $j = i - 1$ or $j = i$. Fix $n \in (\min_{i-1}(s), \min_i(s))$ such that $s(n) = l$. Then, $t(n) = l$, and hence $\theta^0_{i,l}(t{\restriction}s)(n) = l$. Now suppose that $\theta^0_{i,l}(t{\restriction}s)(n) = l$. Then, $t(n) = l$, and hence $s(n) = l$.

Suppose that $\theta^0_{i,l}(s) = \theta^0_{i,l}(t{\restriction}\text{supp } s)$ and $\theta^0_{i,l}(t) = \theta^0_{i,l}(s{\restriction}\text{supp } t)$. Then, $\min_j(s) = \min_j(t)$ for $j = i - 1, i$. Fix n such that $\theta^0_{i,l}(s)(n) = l$. Then, $\theta^0_{i,l}(t{\restriction}\text{supp } s)(n) = l$, which implies that $t(n) = l$.
(d) The cases of $f = \theta^1_{i,l}$ and $f = \theta^2_l$ have a similar proof that *(c)*.

Let us check 4. Fix s_0, s_1, t_0, t_1 as in the statement, and suppose that $f(s_0 + t_0) = f(s_1 + t_1)$. Suppose that $f = \min_i$. If $\min_i(s_0 + t_0) = \min_i(s_0)$, then clearly $\min_i(s_0 + t_0) = \min_i(s_0 + t_1)$. If not we have that $\min_i(s_0 + t_0) = \min_i(t_0)$, hence by our assumptions $\min_i(s_1 + t_1) = \min_i(t_0)$. Since $s_1 < t_0$, it follows that $\min_i(s_1 + t_1) = \min_i(t_1)$ and we are done. Suppose now that $f = \max_i$. If $\max_i(s_0 + t_0) = \max_i(s_0)$, then $\max_i(s_1 + t_1) = \max_i(s_1)$ (again using now that $t_1 > s_0$), and therefore, t_1 is a $(< i)$-vector. So, $\max_i(s_0 + t_0) = \max_i(s_0 + t_1)$. If $\max_i(s_0 + t_0) = \max_i(t_0)$, then $\max_i(s_1 + t_1) = \max_i(t_1)$ and we are done. Suppose now that $f = \theta^0_{i,l}$ and suppose that $\theta^0_{i,l}(s_0 + t_0)(n) = \theta^0_{i,l}(s_1 + t_1)(n) = l$. If $s_1(n) = l$, then $s_0(l) = l$, and hence $(s_0 + t_1)(n) = l$. If $t_1(n) = l$, then clearly $(s_0 + t_1)(n) = l$. By symmetry, we are done in this case. The proof for $f = \theta^0_{i,l}$ and $f = \theta^2_l$ are rather similar.

□

Proposition 3.14. *Any staircase equivalence relation is canonical.*

[1]Notice that $s{\restriction}\text{supp } t$ is not necessarily a k-vector, but still we can apply f to it. See Remark 3.9.

PROOF. By Proposition 3.13, it suffices to show the result only for $f \in \mathcal{F}$. Fix $f \in \mathcal{F}$, set $\sim = \sim_f$ and consider an equation $p(x_0,\dots,x_n) \sim q(x_0,\dots,x_n)$ where $p(x_0,\dots,x_n) = \sum_{d=0}^n T^{k-m_d} x_d$ and $q(x_0,\dots,x_n) = \sum_{d=0}^n T^{k-u_d} x_d$. Set $p^* = p \circ \mathtt{x}^{-1}$ and $q^* = q \circ \mathtt{x}^{-1}$. So $p^*(d) = m_d$ and $q^*(d) = u_d$ for $d \le n$ and 0 for the rest. Fix two sos A and B (B can equal to A), and suppose that $p(a_0,\dots,a_n) \sim_f q(a_0,\dots,a_n)$ for some $(a_0,\dots,a_n) \in [A]^{[n+1]}$. We work to show that $p(b_0,\dots,b_n) \sim q(b_0,\dots,b_n)$ for every $(b_0,\dots,b_n) \in [B]^{[n+1]}$. There are several cases to consider depending on f.
(a) $f = \min_i$. Let d_0 be the first d such that $m_d \ge i$, and d_1 be the first d such that $u_d \ge i$. Then $\min_i(p(a_0,\dots,a_n)) = \min_i(T^{k-m_{d_0}})c_{d_0}$ and $\min_i(q(a_0,\dots,a_n)) = \min_i(T^{k-u_{d_1}})a_{d_1}$. Since $\min_i(T^{k-m_{d_0}})a_{d_0} = \min_i(T^{k-u_{d_1}})a_{d_1}$, we have that $d_0 = d_1$ (otherwise, $a_{d_0} \perp a_{d_1}$). Hence $m_{d_0} = u_{d_1}$ (notice that $T^r a \perp T^s a$ if $r \neq s$). So p and q satisfy that for every $d < d_0$, both m_d and u_d are less than i and $m_{d_0} = u_{d_0} = i$. This implies that $\min_i p(b_0,\dots,b_n) = T^{k-m_{d_0}} b_{d_0} = \min_i q(b_0,\dots,b_n)$. *(b)* $f = \max_i$ has a similar proof. *(c)* $f = \theta^0_{i,l}$. By Proposition 3.10, $\sim_{\theta^0_{i,l}} \subseteq \sim_{\min_{i-1}} \cap \sim_{\min_i}$. For $\varepsilon = 0,1$, let d_ε be the first d such that $p^*(d_j) = q^*(d_j) \ge i-1+\varepsilon$. Notice that $d_0 \le d_1$, and that

$$\theta^0_{i,l} p(a_0,\dots,a_n) = \theta^0_{i,l} \sum_{d=d_0}^{d_1} T^{k-m_d} a_d \tag{7}$$

$$\theta^0_{i,l} q(a_0,\dots,a_n) = \theta^0_{i,l} \sum_{j=d_0}^{d_1} T^{k-u_d} a_d. \tag{8}$$

We are going to show that for every $d \in [d_0, d_1]$ either m_d and u_d are both less than l or $m_d = u_d$. For suppose that $m_d \ge l$. Then $\theta^0_{i,l} T^{k-m_d} a_d \sqsubseteq \theta^0_{i,l} s = \theta^0_{i,l} t$. Since for $T^{k-u_{d'}} a_{d'} \perp T^{k-m_d} a_d$ for every $d' \neq d$, if follows that $T^{k-m_d} a_d \sqsubseteq T^{k-u_d} a_d$, which implies that $u_d = m_d$. *(d)* The cases $f = \theta^1_{i,l}$ and $f = \theta^2_k$ have a similar proof. □

Let us give now some other properties of equations for staircase equivalence relations, whose proofs are left to the reader.

Proposition 3.15. *Suppose that $\sim$ is a staircase equivalence relation with values I_0, J_0, I_1, J_1, $(l_j^{(0)})_{j \in J_0}$, $(l_j^{(1)})_{j \in J_1}$ and $l_k^{(2)}$.*

1. *Let $0 \le r_0 < r_1 \le r_2$. If $x_0 + T^{k-r_2} x_1 + T^{k-r_0} x_2 \sim x_0 + T^{k-r_0} x_2$ is true, then $r_1 \notin I_0$.*
2. *If $l_k^2 = -1$, then the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. If $l_k^2 \neq -1$, then for every $0 < l < l_k^2$ the equation $x_0 + T^{k-l} x_1 + x_2 \sim x_0 + x_2$ holds in A.*

3. *Suppose that* $i \notin I_0$*, and let* $j = \max I_0 \cap [1, i]$*. Then the equation* $T^{k-j}x_0 + T^{k-i}x_1 + x_2 \sim T^{k-j}x_0 + x_2$ *is true.*
4. *If* $l_j^{(0)} = -1$*, then the equation* $T^{k-(j-1)}x_0 + T^{k-(j-1)}x_1 + x_2 \sim T^{k-(j-1)}x_0 + x_2$ *is true.*
5. *Suppose that* $l_j^{(0)} \neq -1$*, and let* $h < l_j^{(0)}$*. Then the equation* $T^{k-(j-1)}x_0 + T^{k-h}x_1 + x_2 \sim T^{k-(j-1)}x_0 + x_2$ *is true.*
6. *Suppose that* $p(x_0, \dots, x_n)$ *is a* $(\leq k)$*-term, and suppose that* $p(x_0, \dots, x_n) + T^{k-l}x_{n+1} + x_{n+3} \sim p(x_0, \dots, x_n) + T^{k-l}x_{n+2} + x_{n+3}$ *holds. Then* $p(x_0, \dots, x_n) + T^{k-l}x_{n+1} + x_{n+2} \sim p(x_0, \dots, x_n) + x_{n+2}$ *also holds.*

The symmetrical results are also true. □

Definition 3.16. A staircase map (or an equivalence relation) is called a min-*relation* if $I_1 = \emptyset$, and a max-*relation* if $I_0 = \emptyset$.

Remark 3.17. 1. Notice Proposition 3.10 states that if $l_k^2 \neq -1$, then $k \in I_0 \cap I_1$. Hence if $\sim$ is a min-relation or a max-relation, then $l_k^2 = -1$.
2. The equation $x + s \sim x + t$ is always true if $\sim$ is a min-relation and that $s + x \sim t + x$ is true if $\sim$ is a max-relation.

4. The main Theorem

The next theorem is the main result of this paper.

Theorem 4.1. *For every* k *and every equivalence relation* $\sim$ *on* FIN_k *there is a sos* B *such that* $\sim$ *restricted to* $\langle B \rangle$ *is a staircase equivalence relation.*

Let us explain here how to show, using our approach with equations, Taylor's result about FIN. Fix an equivalence relation $\sim$ on FIN. A diagonal procedure shows that we can find a block sequence $A = (a_n)_n$ such that for every n, every $i_0, i_1, i_2, i_3, j_0, j_1, j_2, j_3 \in \{0, 1\}$ and every $s, t \in \langle A \rangle$, the equation

$$s + T^{i_0}x_0 + T^{i_1}x_1 + T^{i_2}x_2 + T^{i_3}x_3 \sim t + T^{j_0}x_0 + T^{j_1}x_1 + T^{j_2}x_2 + T^{j_3}x_3 \quad \text{is decided in } A. \tag{9}$$

For an arbitrary k, the analogue result is stated in Lemma 4.2. We consider the same cases considered in original Taylor's proof:
(a) $x_0 \sim x_1$ *holds.* Then $\sim$ is $\langle A \rangle^2$ on $\langle A \rangle$: Let $s, t \in \langle A \rangle$, pick $u > s, t$, and hence $s, t \sim u$.
(b) $x_0 \sim x_1$ *is false,* $x_0 + x_1 \sim x_0$ *is true, and* $x_0 + x_1 \sim x_1$ *is false.* Let us check that $\sim$ is $\sim_{\min}$ on $\langle A \rangle$. Fix $s, t \in \langle A \rangle$. Suppose that $s \sim_{\min} t$, and let n be the minimum such that $C_A(s)(n) = 1$. Then $s = a_n + s'$, $t = a_n + t'$, and using that $x_0 + x_1 \sim x_0$ holds, $s, t \sim a_n$. Suppose now that $s \not\sim_{\min} t$, and

suppose that $\min(s) < \min(t)$, and pick n as before. Then $s \sim a_n$, $a_n < t$, and $a_n \sim t$, a contradiction.
(c) $x_0 \sim x_1$ is false, $x_0 + x_1 \sim x_0$ is false, and $x_0 + x_1 \sim x_1$ is true. Similar proof that 2. shows that $\sim$ is $\sim_{\min}$ on $\langle A \rangle$.
(d) $x_0 \sim x_1$ is false, $x_0 + x_1 \sim x_0$ and $x_0 + x_1 \sim x_1$ are false, and $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. We show that $\sim$ is $\sim_{\min} \cap \sim_{\max}$ on $\langle A \rangle$. It is rather easy to prove that $\sim_{\min} \cap \sim_{\max} \subseteq \sim$ on $\langle A \rangle$. For the converse, suppose that $\max s \neq \max t$ and $s \sim t$. We may assume that $\max s < \max t$. Let n be the maximal integer m such that $C_A(t)(m) = 1$. Then, $t = t' + a_n$, and hence the equation $s \sim t' + x_0$ holds and hence $t' + x_0 + x_1 \sim t' + x_0$ also holds which implies that $x_0 + x_1 \sim x_0$ holds, a contradiction. Notice that this proves that if $x_0 + x_1 \sim x_0$ is false, then $\sim \subseteq \sim_{\max}$. We assume that $\max s = \max t$ but $\min s \neq \min t$. Suppose that $\min s < \min t$. We work to show that $s \not\sim t$. Suppose again that $s \sim t$ and work for a contradiction. Let n_0, n_1 be the minimum and the maximum of the support of s in A resp., and let m_0 be the minimum of the support of t i A. Then $s = a_{n_0} + s' + a_{n_1}$, $t = a_{m_0} + t' + a_{n_1}$. Using that the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true, we may assume that $s' = t' = 0$. Since $n_0 < m_1 \leq n_1$, either the equation $x_0 + x_2 \sim x_1 + x_2$ is true or the equation $x_0 + x_1 \sim x_1$ is true. But the first case implies that the equations $x_0 + x_3 \sim x_1 + x_2 + x_3$ and $x_0 + x_3 \sim x_2 + x_3$ hold and hence $x_0 \sim x_0 + x_1$ holds, a contradiction.
(e) $x_0 \sim x_1$, $x_0 + x_1 \sim x_0$, $x_0 + x_1 \sim x_1$, $x_0 + x_1 + x_2 \sim x_0 + x_2$ are false. Then $\sim$ is $=$ on $\langle A \rangle$. For suppose that $s \sim t$, and suppose that $s \neq t$. Since $x_0 + x_1 \sim x_0$ is false, then $\max s = \max t$ (see 4. above). Let n be the maximal integer $m < \max s$ such that $C_A(s)(m) \neq C_A(t)(m)$, and without loss of generality assume that $C_A(s)(n) = 1$ and $C_A(s)(n) = 0$. Then, $s = s' + a_n + s''$, and $t = t' + s''$, with $t' < a_n$. Therefore the equation $s' + x_0 + x_1 \sim t'' + x_1$ holds, which implies that $s' + x_0 + x_1 + x_2, s' + x_0 + x_2 \sim t'' + x_2$ holds, and hence the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. A contradiction.

For arbitrary k the proof is done by induction over k. It makes use of several lemmas. From now we fix an equivalence relation $\sim$ on FIN_k. Our approach is the following. By the pigeonhole principle Theorem 2.4, there is always a sos A who decides a finite class of equations. It turns out that one of the main sort of equations we are interested in are of the form $x_0 + s \sim x_0 + t$ where s and t are $(k-1)$-vectors (and the corresponding symmetric definition). The reason is that if they are decided, then we can define naturally the $(k-1)$-equivalence relation

$$s \sim_1 t \text{ iff } x_0 + s \sim x_0 + t \text{ holds},$$

and then use inductive hypothesis to detect $\sim_1$ as a $(k-1)$-staircase equivalence relation. Few more equations decided in A will force $\sim$ to be staircase.

Lemma 4.2. *There is some sos $A = (a_n)_n$ such that for every 5-tuples $\vec{i}, \vec{j} \in \{0, \dots, k\}^5$, and every $(\leq k)$-vectors s and t of B, the k-equation*

$$s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l \text{ is decided in } A.$$

Proof. Suppose defined $\theta_n = (a_0, \dots, a_n)$, $A_n = (a_i^{(n)})_i$ such that

1. $\theta_n < A_n$.
2. The equations $s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l$ are decided in A_n for every $(\leq k)$-vectors s, t of θ_n and every $\vec{i}, \vec{j} \in \{0, \dots, k\}^5$.
3. $a_{n+1} = a_0^{(n)}$ and $A_{n+1} \in [(a_i^{(n)})_{i \geq 1}]$.

Let us show that $A = (a_n)_n$ works: Fix a equation E, $s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l$, and let n be the minimum such that s, t are $(\leq k)$-vectors of θ_n. Then E is decided in A_n, which implies that it is also decided in A.

This is the construction of the sequences: For suppose defined $\theta_n < A_n = (a_i^{(n)})_i$. Let $a_{n+1} = a_0^{(n)}$, and $\theta_{n+1} = \theta_n {}^\frown (a_{n+1})$. Let $\{E_0, \dots, E_{u-1}\}$ be the (finite) list of all k-equations of the form $s + \sum_{l=0}^{4} T^{\vec{i}(l)} x_l \sim t + \sum_{l=0}^{4} T^{\vec{j}(l)} x_l$ where s and t are $(\leq k)$-vectors in $\langle \theta_{n+1} \rangle_{\leq k}$ and $\vec{i}, \vec{j} \in \{0, \dots, k\}^5$. Let $\Lambda : [(a_i^{(n)})_{i \geq 1}]^5 \to \{0, 1\}^u$ be the finite coloring defined for $j \in \{0, \dots, u-1\}$ by $\Lambda(c_0, \dots, c_4)(j) = 0$ iff E_j is true in $(a_i^{(n)})_{i \geq 1}$. By Lemma 2.5, there is $A_{n+1} \in [(a_i^{(n)})_{i \geq 1}]^\infty$ such that Λ is constant on $[A_{n+1}]^5$, which is equivalent to that all the equations considered above are decided in A_{n+1}. □

4.1. **Proof of Theorem 4.1.** The proof is by induction over k. Suppose that Theorem 4.1 holds for $k-1$.

Lemma 4.3. *There are some sos A and some staircase equivalence relations $\sim_0$ and $\sim_1$ on $\langle A \rangle_{k-1}$, such that for every $s, t \in \langle A \rangle_{k-1}$,*

$$s + x_0 \sim t + x_0 \text{ is true in } A \text{ if and only if } s \sim_0 t, \text{ and} \tag{10}$$

$$x_0 + s \sim x_0 + t \text{ is true in } A \text{ if and only if } s \sim_1 t. \tag{11}$$

Moreover, $\sim_0$ and $\sim_1$ satisfy that for two $(k-1)$-vectors s and t of A,

$$s \sim_0 t \text{ iff the } (k-1)\text{-equation } s + x \sim_0 t + x \text{ holds in } A, \text{ and} \tag{12}$$

$$s \sim_1 t \text{ iff the } (k-1)\text{-equation } x + s \sim_0 x + t \text{ holds in } A. \tag{13}$$

Proof. Let $B = (b_n)_n$ be a sos satisfying Lemma 4.2. Then for $(k-1)$-vectors s and t of B the $(k-1)$-equations $s + x_0 \sim t + x_0$ are decided in B. Define the equivalence relation on $\langle B \rangle_{k-1}$ by

$$s \sim' t \text{ iff } s + x_0 \sim t + x_0 \text{ holds in } B.$$

It is not difficult to show that $\sim'$ is an equivalence relation. By inductive hypothesis there is some $(k-1)$-block sequence $B' = (b'_n)_n \in [(Tb_n)_n]^{[\infty]}$ and some canonical equivalence relation $\sim_0$ such that $\sim'$ coincides with $\sim_0$ on B' (since all staircase equivalence relations are canonical). The k-block sequence $A = (Sb'_n)_{n\geq 1}$ satisfies what we want for $\sim_0$.

We show now (11). Suppose that $s \sim_0 t$. Then the k-equation $s+x_0 \sim t+x_0$ holds. Since the equation $s + Tx_0 + x_1 \sim t + Tx_0 + x_1$ is decided, it must be true. So, for every k-vector $b > s, t$ it follows that $s+Tb \sim_0 t+Tb$, which is equivalent to say that the $(k-1)$-equation

$$s + x_0 \sim_0 t + x_0 \text{ holds in } A. \tag{14}$$

Suppose now (14). Fix a $(k-1)$-vector $u > s, t$. Then $s+u \sim_0 t+u$, which is equivalent by definition to that the k-equation $s + u + x_0 \sim t + u + x_0$ holds. Hence $s + x_0 \sim t + x_0$ holds, which by definition is equivalent to that $s \sim_0 t$.

Let us prove the corresponding result for $\sim_1$: Fix two $(k-1)$-vectors s, t. If $s \sim_1 t$, then $x_0 + s \sim x_0 + t$. Given a $(k-1)$-vector $u < s, t$, choose a k-vector $a < u$ in $\langle (Sb'_n)_{n\geq 0}\rangle$. Then, $a + u + s \sim a + u + t$ which implies that $u+s \sim_1 u+t$, i.e., the $(k-1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Suppose now that the $(k-1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Pick $(k-1)$-vector $u < s, t$. Then, the k-equation $x_0 + u + s \sim x_0 + u + t$ holds, and hence also $x_0 + s \sim x_0 + t$ holds (since this equation is decided).

□

Roughly speaking, (12) and (13) are telling that the $(k-1)$-relation $\sim_0$ does not depend on the part of a $(k-1)$-vector before $\min_{k-1}$ and that $\sim_1$ does not depend on the part of a $(k-1)$-vector after $\max_{k-1}$. We precise the form of the staircase relations $\sim_0$ and $\sim_1$ for which we introduce the following useful notation.

Definition 4.4. For $l \leq k$, let $\max_k^l : \mathrm{FIN}_k \to \mathrm{FIN}_k$ be defined by $(\max_k^l s)(n) = s(n)$ iff $n \leq \max_k(s)$, and $s(n) \geq l$, and 0 otherwise. Notice that $\max_k^l$ is $\equiv$ to the staircase function with values $I_0 = \{l, \dots, k\}$, $J_0 = \{l+1, \dots, k\}$, for every $j \in J_0$, $l_j^{(0)} = l$, $l_k^{(2)} = l$ and $I_1 = \{k\}$. Symmetrically, we can define $\min_k^l$ by $\min_k^l(n) = s(n)$ iff $n \geq \min_k s$ and $s(n) \geq l$.

The following Lemma detects which kind of staircase equivalence relations of FIN_k are $\sim_0$ and $\sim_1$.

Proposition 4.5. *A staircase relation $\sim$ (on a fixed sos A of k-vectors) satisfies that $s \sim t$ iff $x + s \sim x + t$ holds (in A) for every k-vectors s, t, if and only if either $\sim$ is a* max*-relation or there is some* max*-relation $\sim'$ and some $l \in \{1, \dots, k\}$ such that $\sim = \sim' \cap \max_k^l$.*

The symmetrical result for $s + x \sim t + x$ is also true.

Proof. Fix a staircase relation $\sim$ with values $I_\varepsilon, J_\varepsilon, (l_j^{(\varepsilon)})_{j\in J_\varepsilon}$ ($\varepsilon=0,1$) and $l_k^{(2)}$ such that for every k-vectors s,t, $s\sim t$ iff $x+s\sim x+t$ holds. Suppose that $I_0\neq\emptyset$, since otherwise $\sim$ is a max-relation. Let $l=\min I_0$. We work to show that $I_0=\{l,l+1,\dots,k\}$, $J_0=\{l+1,\dots,k\}$, for every $j\in J_0$, $l_j^{(0)}=l$, $l_k^{(2)}=l$ and $k\in I_1$. First we show that $l_k^{(2)}\neq -1$. If not, the equation $x_0+x_1+x_2\sim x_0+x_2$ is true and hence the equation $x_1+x_2\sim x_2$ is true, which implies that $l\notin I_0$, a contradiction. If $l_k^{(2)}>l$, then the equation $x_0+T^{k-l}x_1+x_2\sim x_0+x_2$ is true and hence the equation $T^{k-l}x_1+x_2\sim x_2$ is true, which implies again that $l\notin I_0$. If $l_k^{(2)}<l$, then the equation $T^{k-l_k^{(2)}}x_0+x_1\sim x_1$ holds and hence the equation

$$x_0+T^{k-l_k^{(2)}}x_1+x_2\sim x_0+x_2 \text{ holds,} \tag{15}$$

which is contradictory with the definition of $l_k^{(2)}$.

We show now that $I_0=\{l,\dots,k\}$. It is clear that $I_0\subseteq\{l,\dots,k\}$ since l is the minimum of I_0. Now we work to show that $\{l,\dots,k\}\subseteq I_0$. Suppose not, and set

$$j=\min\{l,\dots,k\}\setminus I_0.$$

Then the equation $T^{k-j-1}x_0+T^{k-j}x_1+x_2\sim T^{k-j-1}x_0+x_2$ is true and hence the equation $x_0+T^{k-j-1}x_1+T^{k-j}x_2+x_3\sim x_0+T^{k-j-1}x_1+x_3$ is true, which implies that also the equation $x_0+T^{k-j}x_1+x_2\sim x_0+x_2$ holds. This is contradictory with the fact that $j>l$ and that $\sim\subseteq\sim_{\theta_l^2}$. Notice that $I_0=\{l,\dots,k\}$ implies that $J_1=\{l+1,\dots,k\}$.

We show that $l_j^{(0)}=l$ for all $j\geq l+1$. Suppose that $l_j^{(0)}=-1$. This implies that the equation $T^{k-(j-1)}x_0+T^{k-(j-1)}x_1+x_2\sim T^{k-(j-1)}x_0+x_2$ holds. Again by adding one variable at the beginning of both terms and using that $j-1\geq l$ we can arrive to a contradiction with the fact that $l_k^{(2)}=l$. Suppose now that $l_j^{(0)}<l$. Then the equation $T^{k-l_j^{(0)}}x_0+x_1\sim x_1$ is true, and adding a variable we arrive to a contradiction. Suppose that $l_j^{(0)}>l$, then the equation

$$T^{k-(j-1)}x_0+T^{k-l}x_1+x_2\sim T^{k-(j-1)}x_0+x_2 \text{ is true,} \tag{16}$$

which derives to a contradiction in the same way than before. It is not difficult to check that the converse and the symmetric situation for min are also true. □

(12) and (13) and Proposition 4.5 show the following property of $\sim_0$ and $\sim_1$

Corollary 4.6. *The relation $\sim_0$ is either a* min*-relation or there is some $l\leq k-1$ and some* min*-relation R such that $\sim_0=R\cap\min_{k-1}^l$ and $\sim_1$ is* *either a* max*-relation or there is some* $l \leq k-1$ *and some* max*-relation* R *such that* $\sim_1 = R \cap \max^l_{k-1}$. □

Recall that $\sim_0$ and $\sim_1$ are both staircase equivalence relation of FIN_{k-1}. We give now the proper interpretation of both as k-relations. Suppose that $k > 1$. We know that either $\sim_1$ is a max-relation, or $\sim_1 = \max^l_{k-1} \cap R$, with R a max-relation. Let

$$\sim'_1 = \begin{cases} \sim_1 & \text{if } \sim_1 \text{ is a max-relation} \\ \theta^1_{k,l} \cap R & \text{if } I_0 \neq \emptyset. \end{cases}$$

Notice that in the second case we have that $\max_k \subseteq \sim'_1$. We do the same for $\sim_0$: It is either a min-relation or $\sim_0 = R \cap \min^l_{k-1}$. Let

$$\sim'_0 = \begin{cases} \sim_0 & \text{if } \sim_0 \text{ is a min-relation} \\ R \cap \theta^0_{k,l} & \text{if } I_1 \neq \emptyset. \end{cases}$$

In this second case we have that $\min_k \subseteq \sim'_0$. For $k = 1$, let $\sim'_0 = \sim'_1 = \mathrm{FIN}^2_1$.

We are going to show later on that $\sim \subseteq \sim'_0 \cap \sim'_1$. The next proposition tells that the extensions $\sim'_\varepsilon$ of $\sim_\varepsilon$ have similar properties, for $\varepsilon = 0, 1$.

Proposition 4.7. *Let* s *and* t *be* $(k-1)$*-vectors. Then*

1. $s + x \sim'_0 t + x$ *holds iff* $s \sim_0 t$ *iff* $s + x \sim t + x$ *holds.*
2. $x + s \sim'_1 x + t$ *holds iff* $s \sim_1 t$ *iff* $x + s \sim x + t$ *holds.*
3. $s + x_0 + x_1 \sim'_0 t + x_0 + x_2$ *holds iff* $s \sim_0 t$.
4. $x_0 + x_2 + s \sim'_1 x_1 + x_2 + t$ *holds iff* $s \sim_1 t$.

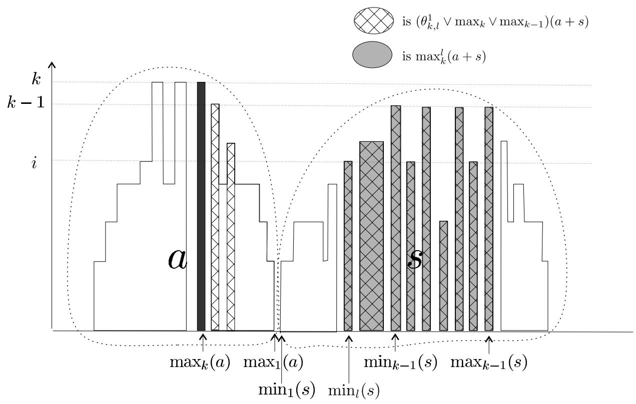


FIGURE 3. The relation between $\sim_1$ and $\sim'_1$

Proof. We show the result for $\sim_1$; for $\sim_0$ the proof is similar. If $\sim_1$ is a max relation, then there is nothing to prove. Suppose that $\sim_1 = \max^l_{k-1} \cap R$, for some $l \leq k-1$, where R is a max relation. So, $\sim_1' = \theta^{(1)}_{k,l} \cap R$ and we only have to show that

$$s \max{}^l_{k-1} t \text{ iff the equation } x + s\,\theta^{(1)}_{k,l} x + t \text{ holds,} \tag{17}$$

which is rather easy to check (see Figure 3). □

Definition 4.8. Let $D = (d_n)_n$ be a k-block sequence and a k-vector $s = \sum_{n\geq 0} T^{k-C_D(s)(n)} d_n$ of D, and let $n_0 = n_0(s)$ and $n_1 = n_1(s)$ be respectively the minimal and the maximal elements of the set of integers n such that $C_D(s)(n) = k$. We define the *first part of s in D* as the $(\leq k-1)$-vector $f_D s = \sum_{n<n_0} T^{k-C_D(s)(n)} d_n$, the *middle part of s in D* as the $(\leq k)$-vector $m_D s = \sum_{n\in(n_0,n_1)} T^{k-C_D(s)(n)} d_n$ and the *last part of s in D*, as the $(\leq k-1)$-vector $l_D s = \sum_{n>n_1} T^{k-C_D(s)(n)} d_n$. Notice the following decomposition $s = f_D s + b_{n_0} + m_D s + b_{n_1} + l_D s$.

Roughly speaking, $f_D s$ is the part of s before the occurrence of $\min_k s$, $m_D s$ is the part of s between $\min_k s$ and $\max_k s$, and $l_D s$ is the part of s after $\max_k s$. All these definitions are local, depending on a fixed sos D.

Let $\mathbb{A} = (a_n)_n$ satisfying both Lemmas 4.2 and 4.3, and let $\mathbb{B} = (b_n)_n$ be defined for every n by $b_n = Ta_{3n} + a_{3n+1} + Ta_{3n+2}$. The main reason for the definition of $\mathbb{B}$ is to guarantee that for every k-vector s of $\mathbb{B}$, $f_{\mathbb{A}} s$ and $l_{\mathbb{A}} s$ are exactly $(k-1)$-vectors. We need this because $\sim_\varepsilon$ $(\varepsilon = 0, 1)$ give information only about $(k-1)$-vectors.

From now on we work in $\mathbb{B}$, unless we say explicitly the contrary. The following Proposition tells that in $\mathbb{B}$ many equations are decided.

Proposition 4.9. *Let $p(x_1, \ldots, x_{n-1})$ and $q(x_1, \ldots, x_{n-1})$ be $(\leq k-1)$-terms. Then,*

1. *The equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ is decided in $\mathbb{B}$.*
2. *The equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ holds in $\mathbb{B}$ if and only if the equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim_1' x_0 + q(x_1, \ldots, x_{n-1})$ holds in $\mathbb{B}$.*

The symmetrical results for $\sim_0'$ are also true.

Proof. Fix two $(\leq k-1)$-terms $p = p(x_1, \ldots, x_{n-1})$, $q = q(x_1, \ldots, x_{n-1})$.
1. Fix a finite block sequence $(c_0, \ldots, c_{n-1})$ in $\mathbb{B}$. Suppose that $c_0 + p(c_1, \ldots, c_{n-1}) \sim c_0 + q(c_1, \ldots, c_{n-1})$. By definition of $\mathbb{B}$, $c_0 = c_0' + c_0''$, where c_0' is a k-vector of $\mathbb{A}$ and c_0'' is a $(k-1)$-vector of $\mathbb{A}$. Hence,

$$c_0'' + p(c_1, \ldots, c_{n-1}) \sim_1 c_0'' + q(c_1, \ldots, c_{n-1}). \tag{18}$$

Since the relation $\sim_1$ is $(k-1)$-canonical in $\mathbb{A}$, the $(k-1)$-equation

$$x_0 + p(x_1, \dots, x_{n-1}) \sim_1 x_0 + q(x_1, \dots, x_{n-1}) \text{ is true in } \mathbb{A}. \tag{19}$$

Fix $(d_0, \dots, d_{n-1})$ in $\mathbb{B}$, and set $d_0 = d_0' + d_0''$. Then,

$$d_0'' + p(d_1, \dots, d_{n-1}) \sim_1 d_0'' + q(d_1, \dots, d_{n-1}), \tag{20}$$

and hence, the equation

$$x_0 + p(d_1, \dots, d_{n-1}) \sim x_0 + q(d_1, \dots, d_{n-1}) \text{ holds in } \mathbb{A}, \tag{21}$$

which implies that $d_0 + p(d_1, \dots, d_{n-1}) \sim_1 d_0 + q(d_1, \dots, d_{n-1})$, as desired.
2. Suppose that $x_0 + p(x_1, \dots, x_{n-1}) \sim x_0 + q(x_0, \dots, x_{n-1})$ holds in $\mathbb{B}$. Then for a given block sequence $(c_0, c_1, \dots, c_{n-1})$ in $\mathbb{B}$, the equation

$$x_0 + Tc_0 + p(c_1, \dots, c_{n-1}) \sim x_0 + Tc_0 + q(c_1, \dots, c_{n-1}) \text{ holds in } \mathbb{B}. \tag{22}$$

By Proposition 4.7, (22) implies that

$$x_0 + Tc_0 + p(c_1, \dots, c_{n-1}) \sim_1' x_0 + Tc_0 + q(c_1, \dots, c_{n-1}) \text{ holds in } \mathbb{B}. \tag{23}$$

Since $\sim_1'$ is canonical, the equation

$$x_0 + Tx_1 + p(x_2, \dots, x_n) \sim_1' x_0 + Tx_1 + q(x_2, \dots, x_n) \text{ holds in } \mathbb{B}. \tag{24}$$

This implies that the equation $x_0 + p(x_1, \dots, x_{n-1}) \sim_1' x_0 + q(x_1, \dots, x_{n-1})$ holds in $\mathbb{B}$, as desired. □

Proposition 4.10. *For suppose that a, b are k-vectors of $\mathbb{B}$, s, t are $\leq (k-1)$-vectors of $\mathbb{B}$ such that $a < s$, $b < t$ and $a + s \sim_1' b + t$.*

1. *If $a, b < s, t$, then $a + s \sim a + t$.*
2. *If $l_{\mathbb{A}} a = l_{\mathbb{A}} b = 0$, and $\max_k(a) > \max_k(b)$, then $b + s \sim b + t$.*

The corresponding symmetrical results for $\sim_0'$ are also true.

Proof. Let us check 1. By Point 4. of Proposition 3.13, we have that $a + s \sim_1' a + t$. By construction of $\mathbb{B}$, $a = a' + a''$ where a' is a k-vector and a'' is a $(k-1)$-vector, both of $\mathbb{A}$. But since the relation is $\sim_1'$ is staircase, it is canonical, and hence the k-equation

$$x_0 + a'' + s \sim_1' x_0 + a'' + t \text{ holds in } \mathbb{A}. \tag{25}$$

It follows from Proposition 4.9 that $a'' + s \sim_1 a'' + t$, and hence, by definition of $\sim_1$, the k-equation

$$x_0 + a'' + s \sim x_0 + a'' + t \text{ holds in } \mathbb{A}. \tag{26}$$

Replacing in (26) x_0 by a', we obtain that $a + s \sim a + t$.
2. Since $l_{\mathbb{A}} a = l_{\mathbb{A}} b = 0$, we have that $a + s = a' + a_{n_0} + s$ and $b + t = b' + a_{m_0} + t$. Since $\max_k(a) > \max_k(b)$, it follows that $n_0 > m_0$. This together with the

fact that $a+s \sim_1' b+t$ implies that $\max_k \not\subseteq \sim_1'$ and hence, by definition, $\sim_1'$ has to be max-relation. Set $i = \max I_1(\sim_1') < k$. Then the equation

$$p(x_0,\dots,x_r) + T^{k-i'}x_{r+1} \sim_1' q(x_0,\dots,x_r) + T^{k-i'}x_{r+1} \text{ is true,} \tag{27}$$

for every terms p and q, and every $i' \ge i$. Now set

$$t = t' + T^{k-j}a_{n_0} + t''.$$

Notice that t'' is a i-vector, and s is a i'-vector for some $i' \ge i$. By (27),

$$a+s = a'+a_{n_0}+s \sim_1' b'+a_{m_0}+t'+T^{k-j}a_{n_0}+s \sim_1' b'+a_{m_0}+s = b+s. \tag{28}$$

Hence, $b+t \sim_1' b+s$, and since $b < s,t$, 1. implies that $b+s \sim b+t$.

□

The following result detects the staircase equivalence relation that will be equal to $\sim$ on $\mathbb{B}$ in terms of which equations hold or not in $\mathbb{B}$. This will end up the proof of Theorem 4.1.

Theorem 4.11.

1. *Suppose that* $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ *is true, and* $x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2$ *is false. Then* $\sim = \sim_0' \cap \sim_{\theta_l^2} \cap \sim_1'$.
2. *Suppose that* $x_0 + x_1 + x_2 \sim x_0 + x_2$ *is true.*
   (a) *If* $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ *and* $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ *are both false, then* $\sim = \sim_0' \cap \min_k \cap \max_k \cap \sim_1'$.
   (b) *If* $Tx_0+x_1+x_2 \sim Tx_0+x_2$ *is true, and* $x_0+x_1+Tx_2 \sim x_0+Tx_2$ *is false, then* $\sim = \sim_0' \cap \max_k \cap \sim_1'$.
   (c) *If* $Tx_0+x_1+x_2 \sim Tx_0+x_2$ *is false, and* $x_0+x_1+Tx_2 \sim x_0+Tx_2$ *is true, then* $\sim = \sim_0' \cap \min_k \cap \sim_1'$.
   (d) *If* $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ *and* $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ *are both true, then* $\sim = \sim_0' \cap \sim_1'$.

The proof is done in various steps. 1. is in Corollary 4.20, and 2.a., 2.b, 2.c and 2.d in Corollary 4.25, and Lemmas 4.21, 4.23 and 4.26 respectively. We start with the following proposition that gives one of the inclusions.

Proposition 4.12.

1. *If the equation* $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ *is true, then* $\sim_0' \cap \max_k \cap \sim_1' \subseteq \sim$.
2. *If the equation* $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ *is true, then* $\sim_0' \cap \min_k \cap \sim_1' \subseteq \sim$.
3. *If the equation* $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ *is true, then* $\sim_0' \cap \sim_{\theta_l^2} \cap \sim_1' \subseteq \sim$, *for every* $l \le k$,
4. *If the equation* $x_0 + x_1 + x_2 \sim x_0 + x_2$ *is true, then* $\sim_0' \cap \min_k \cap \max_k \cap \sim_1' \subseteq \sim$.

5. *If the equations* $Tx_0+x_1+x_2 \sim Tx_0+x_2$ *and* $x_0+x_1+Tx_2 \sim x_0+Tx_2$ *are both true, then* $\sim_0' \cap \sim_1' \subseteq \sim$.

PROOF. 1. Suppose that the equation $Tx_0+x_1+x_2 \sim Tx_0+x_2$ holds. Then, $Tx_0+Tx_1+x_2+x_3 \sim Tx_0+x_3$ holds, and also the equation $Tx_0+Tx_1+x_2+x_3 \sim Tx_0+x_3 \sim Tx_0+x_2+x_3$ holds. This implies that the equation

$$Tx_0+Tx_1+x_2 \sim Tx_0+x_2 \text{ is true.} \tag{29}$$

Hence the relation $\sim_0$ is a min-relation and which implies that $\sim_0'$ so is a min-relation. Set $R = \sim_0' \cap \max_k \cap \sim_1'$ and suppose that sRt. Then $\max_k s = \max_k t$. Let n be such that $\max_k s = \max_k = \max_k b_n$. Therefore, $s = s' + a_{3n+1} + s''$ and $t = t' + a_{3n+1} + t''$. It is not difficult to show that the equation

$$Tx_0 + x_1 + x_2 \, R \, Tx_0 + x_2 \text{ holds.} \tag{30}$$

So, we may assume that s' and t' are $(k-1)$-vectors of $\mathbb{A}$. Since $s \sim_1' t$, we have that $s' + a_{3n+1} + s'' \sim s' + a_{3n+1} + t''$. Since $s \sim_0' t$, we have that $s' + a_{3n+1} + t'' \sim_0' t' + a_{3n+1} + t''$, and hence $s' \sim_0 t'$, which implies that $s' + x \sim t' + x$ is true. In particular $s' + a_{3n+1} + t'' \sim t' + a_{3n+1} + t''$, i.e., $s \sim t$.

Proofs of 2., 3. and 4. are rather similar. Let us check point 5. Fix $s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s$, $t = f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t$ such that sRt, where $R = \sim_0' \cap \sim_1'$. If $m_0 = n_0$, then $sR \cap \min_k t$, and hence we are done by 2. So, suppose that $n_0 < m_0$. Since $\sim_0'$ is a min-relation and $\sim_1'$ is a max-relation, the equations $Tx_0 + x_1 + x_2 R Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 R x_0 + Tx_2$ are true. Therefore, $sRf_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$ and $tRf_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t$. Since $s \sim f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$ and $t \sim f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t$ the proof will finish if we show that

$$s_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t. \tag{31}$$

Since $s_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim_0' f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t$ and $s_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim_1' f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t$, by the last point of Proposition 4.10 (for both $\sim_0'$ and $\sim_1'$), we have that

$$s_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}t + a_{m_0} + l_{\mathbb{A}}t \text{ and } s_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t. \tag{32}$$

But $s_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t$, and we are done.

□

To show the other inclusions on Theorem 4.11 we need a new definition.

Definition 4.13. Suppose that $\sim$ is a max-relation with values $I_0 = \emptyset, I_1, J_1$ and $(l_j^{(1)})_{j \in J_1}$. For any $i \leq k-1$, define $I_1(i) = I_1 \cap [0,i]$, $J_1(i) = J_1 \cap [0,i]$, and let $\sim(i)$ be the staircase equivalence relation on

FIN_k with values $I_1(i)$, $J_1(i)$, and $(l_j^{(1)})_{j\in J_1(i)}$. Notice that

$$\sim(i+1)=\begin{cases}\sim(i) & \text{if } i+1\notin I_1\\ \max_{i+1}\cap\sim(i) & \text{if } i+1\in I_1 \text{ and } i\notin I_1\\ \max_{i+1}\cap\sim(i)\cap\theta^1_{i+1,l^{(1)}_{i+1}} & \text{if } i+1\in J_1,\end{cases}$$

and that $\sim=\sim(k)$. Notice that every $\sim(i)$ is also a staircase equivalence relations on sos of FIN_i.

Roughly speaking, $\sim(i)$ is the staircase equivalence relation with values the ones from $\sim$ who are smaller than i.

Remark 4.14. Observe that for a given $i\le k-1$, $s\sim(i)t$ iff the equation with variable x

$$x+s\restriction[\max_i(s),\max_1(s)]\sim(i)x+t\restriction[\max_i(s),\max_1(s)] \text{ holds.} \tag{33}$$

Proposition 4.15. *Fix $j'<j<j''$, and suppose that s is a j'-vector, t is a $(<j)$-vector, and a is a j''-vector such that $a+s\sim(j)T^la+t$ for some $l>0$. Then, $\sim(j)=\sim(j')$, and hence $s''\sim(j)t''$.*

Proof. Set $s'=a+s$ and $t'=T^la+t$, and suppose that $s'\sim(j)t'$. We are going to show that $I_2(j)=I_2(j')$, which will imply that $\sim(j)=\sim(j')$, as desired. We know that $s'\restriction[\max_j(s'),\max_1(s')]\sim(j)t'\restriction[\max_j(s'),\max_1(s')]$. Notice that for every $r\in[j,j')$, $\max_r(s')=\max_r(a)$, hence $\max_r(s')\neq\max_r(t')$, since a and T^la have nothing in common except 0's. This implies that $I_2(j)\subseteq[j',1]$ and hence $I_2(j)=I_2(j')$.

□

Lemma 4.16. *$\sim\subseteq\sim'_1(j)$, for every $j\le k$. In particular, $\sim\subseteq\sim'_1$.*

Proof. The proof is by induction over j. Notice that if $k=1$, then $\sim'_1=\mathrm{FIN}_1^2$ and hence there is nothing to prove. Suppose that $k>1$. Let I_1, J_1 and $(l_j^{(1)})_{j\in J_1}$ be the values of $\sim'_1$.

$\underline{j=1}$: Suppose that $1\in I_1$ (otherwise nothing to prove), i.e., $\sim'_1(1)=\sim_{\max_1}$. Suppose that $s\sim t$ but $\max_1(s)<\max_1(t)$, and let n and i be the unique integers such that

$$\max\nolimits_1 T^{k-i}a_n=\max\nolimits_1 t \text{ and } t=t'+T^{k-i}a_n. \tag{34}$$

So, $s=s'+T^{k-i'}a_n$, for some $i'<i$ and some k-vector s'. The fact that $s\sim t$ implies that the equation $s'+T^{k-i'}x_0\sim t'+T^{k-i}x_0$ holds in $\mathbb{B}$, which implies that the equation $s'+T^{k-i'}(x_0+T^{i'}x_1)\sim t'+T^{k-i}(x_0+T^{i'}x_1)$ is true. Therefore

$$s'+T^{k-i'}x_0+\sim t'+T^{k-i}x_0+T^{k-i+i'}x_1 \text{ holds,} \tag{35}$$

(35) implies that the equation

$$t' + T^{k-i}x_0 \sim t' + T^{k-i}x_0 + T^{k-i+i'}x_1 \text{ is true,} \tag{36}$$

and hence, also

$$x_0 + T^{k-i+i'}x_1 \sim x_0 \text{ is true in } \mathbb{B}. \tag{37}$$

But since $j - i + i' < k$, we have that

$$x_0 + Tx_1 \sim x_0 + Tx_1 + T^{k-i+i'}x_2 \text{ is true,} \tag{38}$$

and by Proposition 4.9, we have that

$$x_0 + Tx_1 \sim_1' x_0 + Tx_1 + T^{k-i+i'}x_2 \text{ holds,} \tag{39}$$

which is contradictory with the fact that $1 \in I_1$.
$j \curvearrowright j+1$. Suppose now that we have shown that $\sim \subseteq \sim_1'(j)$ and let us prove that $\sim \subseteq \sim_1'(j+1)$. There are two cases:
(a) $j \notin I_1$: Suppose that $j+1 \in I_1$ (otherwise, nothing to prove), and set

$$\beta = \max I_1 \cap [0, j]. \tag{40}$$

We notice that β can be 0. By definition of $\sim_1'$, we know that if $j+1 = k$ belongs to I_1, then $j = k-1$ also belongs to I_1. So, $j+1 < k$. We only need to show that $\sim \subseteq \max_{j+1}$: Suppose that $s \sim t$, and $\max_{j+1} s < \max_{j+1} t$; set $s = s' + T^{k-l}a_n + s''$, $t = t' + T^{k-l'}a_n + t''$, with $l < l'$, $l' \geq j+1$, and $(< (j+1))$-vectors s'' and t''. Observe that in the previous decomposition of s, s' needs to be a k-vector. By inductive hypothesis,

$$s' + T^{k-l}a_n + s'' \sim_1'(j) t' + T^{k-l'}a_n + t''. \tag{41}$$

Since $\sim_1'$ is a staircase equivalence relation, 4. of Proposition 3.13 gives that

$$s' + T^{k-l}a_n + s'' \sim_1'(j) s' + T^{k-l}a_n + t'' \ (t'' \text{ can be } 0), \tag{42}$$

which implies that $s' + T^{k-l}a_n + s'' \sim_1' s' + T^{k-l}a_n + t''$, and hence, by Proposition 4.10, $s' + T^{k-l}a_n + s'' \sim s' + T^{k-l}a_n + t''$. Resuming, we have that

$$s' + T^{k-l}a_n + t'' \sim t' + T^{k-l'}a_n + t'', \tag{43}$$

and hence, the equation

$$s' + T^{k-l}x_0 + T^{k-\alpha}x_1 \sim t' + T^{k-l'}x_0 + T^{k-\alpha}x_1 \text{ holds,} \tag{44}$$

where $j \geq \alpha \geq \beta$ is such that $t'' \in \mathrm{FIN}_\alpha$. Notice that since $j \notin I_1$, and $j \geq \alpha \geq \beta = \max I_1 \cap [0, \dots, j]$, the equation

$$x_0 + T^{k-r}x_1 + T^{k-\alpha}x_2 \sim_1' x_0 + T^{k-\alpha}x_2 \text{ is true,} \tag{45}$$

for all $r \leq j$. Hence,

$$x_0 + T^{k-r}x_1 + T^{k-\alpha}x_2 \sim x_0 + T^{k-\alpha}x_2 \text{ is true.} \tag{46}$$

There are two cases: If $l \le j$, then

$$s' + T^{k-\alpha}x_2 \sim s' + T^{k-l}x_1 + T^{k-\alpha}x_2 \sim t' + T^{k-l'}x_1 + T^{k-\alpha}x_2 \text{ is true,} \tag{47}$$

and hence,

$$x_0 + T^{k-l'}x_1 + T^{k-l'}x_2 + T^{k-\alpha}x_3 \sim x_0 + T^{k-l'}x_1 + T^{k-\alpha}x_3 \text{ is true,} \tag{48}$$

which implies that

$$x_0 + T^{k-(j+1)}x_1 + T^{k-\alpha}x_2 \sim x_0 + T^{k-\alpha}x_2 \text{ holds.} \tag{49}$$

By Proposition 4.9,

$$x_0 + T^{k-(j+1)}x_1 + T^{k-\alpha}x_2 \sim_1' x_0 + T^{k-\alpha}x_2 \text{ holds,} \tag{50}$$

which is a contradiction with the fact that $j+1 \in I_1$. Suppose now that $j+1 \le l < l'$. Then, the equation

$$s' + T^{k-l}(x_0 + T^{l-j}x_1) + T^{k-\alpha}x_2 \sim s' + T^{k-l}x_0 + T^{k-\alpha}x_2 \text{ holds,} \tag{51}$$

and hence,

$$t' + T^{k-l'}x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim t' + T^{k-l'}x_0 + T^{k-\alpha}x_2 \text{ holds,} \tag{52}$$

which implies that

$$x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim x_0 + T^{k-\alpha}x_2 \text{ holds.} \tag{53}$$

Notice that $i' - i > 0$, and hence, (53) is contradictory with the fact that $j+1 \in I_1$.

(b) $j \in I_1$. Suppose that $j+1 \in I_1$ (otherwise, nothing to prove). Then $\sim_1(j+1) = \sim_1(j) \cap \theta^{(1)}_{j+1,l} \cap \max_{j+1}$, where $l = l^{(1)}_{j+1}$. Suppose that $s \sim t$. By inductive hypothesis, $s \sim_1'(j)t$, and in particular $\max_j(s) = \max_j(t)$. Let $m_0 = \max\{\max_{j+1}s, \max_{j+1}t\}$. First we show that

$$(s\restriction[m_0, \max_j(s)])^{-1}(l) = (t\restriction[m_0, \max_j(s)])^{-1}(l) \tag{54}$$

(i.e., for all $n \in [m_0, \max_j(s)]$, $s(n) = l$ iff $t(n) = l$). For suppose not, and let

$$m_1 = \max\{m \in [m_0, \max_j(s)] : (s(m) = l \text{ or } t(m) = l) \text{ and } s(m) \ne t(m)\}.$$

Suppose that $s(m_1) = l$, and that $t(m_1) \ne 0$. Let n_1 be the unique n such that $T^{k-C_{\mathbb{B}}(n)}a_n(m_1) = s(m_1) = l$, and let $h = C_{\mathbb{B}}(n_1) \ge l$. So, $h' = C_{\mathbb{B}}(n_1) \ne h$, $s = s' + T^{k-h}a_{n_1} + s''$, and $t = t' + T^{k-h'}a_{n_1} + t''$, with s'', t'' both j-vectors. By definition of m_1, the equation

$$x + s'' \sim_1'(j+1)x + t'' \text{ holds,} \tag{55}$$

and hence, also both

$$x + s'' \sim_1' x + t'' \text{ and } x + s'' \sim x + t'' \text{ hold.} \tag{56}$$

So, $s'+T^{k-h}a_{n_1}+s''\sim t''+T^{k-h'}a_{n_1}+s''$, and hence, the equation

$$s'+T^{k-h}x_0+T^{k-j}x_1\sim t'+T^{k-h'}x_0+T^{k-j}x_1 \text{ holds.} \tag{57}$$

There are two cases: Suppose first that $h>h'$. Since $x_0+T^{k-r}x_1+T^{k-j}x_2\sim_1' x_0+T^{k-j}x_2$ is true, the equation $x_0+T^{k-r}x_1+T^{k-j}x_2\sim x_0+T^{k-j}x_2$ holds for every $r<l$. Since $l+h'-h<l$,

$$\begin{aligned} &s'+T^{k-h}x_0+T^{k-l}x_1+T^{k-j}x_2\sim s'+T^{k-h}(x_0+T^{h-l}x_1)+T^{k-j}x_2\sim\\ &\sim t'+T^{k-h'}(x_0+T^{h-l}x_1)+T^{k-j}x_2\sim t'+T^{k-h'}x_0+T^{k-(l+h'-h)}x_1+\\ &+T^{k-j}x_2\sim t'+T^{k-h'}x_0+T^{k-j}x_2\sim s'+T^{k-h}x_0+T^{k-j}x_2 \text{ hold.} \end{aligned} \tag{58}$$

Notice that we have used that $h\geq l$, and so T^{h-l} makes sense. Summarizing, the equation

$$s'+T^{k-h}x_0+T^{k-l}x_1+T^{k-j}x_2\sim s'+T^{k-h}x_0+T^{k-j}x_2 \text{ holds,} \tag{59}$$

and hence, the equation

$$x_0+T^{k-l}x_1+T^{k-j}x_2\sim_1' x_0+T^{k-j}x_2 \text{ holds,} \tag{60}$$

which is a contradiction with the fact that $\sim_1'\subseteq\theta^1_{j+1,l}$. Suppose now that $h<h'$. Then $h'>l$, and repeating the previous argument used for the case $h>h'$, we conclude that the equation

$$t'+T^{k-h'}x_0+T^{k-l}x_1+T^{k-j}x_2\sim t'+T^{k-h'}x_0+T^{k-j}x_2 \text{ holds,} \tag{61}$$

and hence,

$$x_0+T^{k-l}x_1+T^{k-j}x_2\sim_1' x_0+T^{k-j}x_2 \text{ holds,} \tag{62}$$

which is a contradiction.

The proof will be finished if we show that $\max_{j+1}s=\max_{j+1}t$: If not, w.l.o.g. we may assume that $\max_{j+1}s>\max_{j+1}t$. Let n_1 be such that $\max_{j-1}$s=$\max_{j-1}(T^{k-h}b_{n_1})$, where $h=C_{\mathbb{B}}(n_1)\geq j+1$. Then, $s=s'+T^{k-h}a_{n_1}+s''$, $t=t'+T^{k-h'}a_{n_1}+t''$, where $h'<h$ and s'',t'' are j-vectors. From (54), it follows that

$$x_0+s''\sim_1'(j+1)x_0+t'' \text{ holds,} \tag{63}$$

and hence,

$$s'+T^{k-h}a_{n_1}+t''\sim t'+T^{k-h'}a_{n_1}+t''. \tag{64}$$

This implies that the equation

$$s'+T^{k-h}x_0+T^{k-j}x_1\sim t'+T^{k-h'}x_0+T^{k-j}x_1 \text{ is true.} \tag{65}$$

Using a similar argument than above, we arrive to that the equation

$$s' + T^{k-h}(x_0 + T^{h-l}x_1) + T^{k-j}x_2 \sim t' + T^{k-h'}(x_0 + T^{h-l}x_1)+ \\ + T^{k-j}x_2 \text{ is true,} \quad (66)$$

and hence,

$$s' + T^{k-h}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-(l+h'-h)}x_1+ \\ +T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-j}x_2 \sim s' + T^{k-h}x_0 + T^{k-j}x_2 \text{ holds,} \quad (67)$$

which is again a contradiction, since it implies that

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim_1' x_0 + T^{k-j}x_2 \text{ holds.} \quad (68)$$

□

Proposition 4.17. *Suppose that a, b are k-vectors of $\mathbb{B}$, s, t are $(\leq (k-1))$-vectors of $\mathbb{B}$ such that $a < s$ and $b < t$, and suppose that $a + s \sim b + t$.*

1. *If $a < t$ and $b < s$, then $a + s \sim a + t$ and hence $a + t \sim b + t$.*
2. *If $a < t$ and $\max_k a < \max_k b$, then $a+s \sim a+t$ and hence $a+t \sim b+t$.*

Proof. 1. is a consequence of Proposition 4.10(1) and Lemma 4.16. Let us prove 2. For suppose that a, b, s, t are as in the statement. By Lemma 4.16, $a + s \sim_1' b + t$. Since $\max_k(a+s) < \max_k(b+t)$, we have that $\sim_1' = \sim_1$, where $\sim_1$ is a max-relation of FIN_{k-1}. This implies that $s \sim_1 t$, from which easily follows the desired result. □

Lemma 4.18. *If the equation $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is false, then $\sim \subseteq \max_k$.*

Proof. Suppose that $s \sim t$ but $\max_k s > \max_k t$. Set

$$\begin{aligned} s =& f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s \\ t =& f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t, \end{aligned}$$

where $n_1 > m_1$. Set $l_{\mathbb{A}}t = t' + T^{k-i}a_{n_1} + t''$, where $t' < T^{k-i}a_{n_1} < t''$, and $i < k$. By Proposition 4.17

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}a_{n_1} + l_{\mathbb{A}}s \sim f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{n_1} + l_{\mathbb{A}}s, \quad (69)$$

and therefore, the equation

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim f_{\mathbb{A}}s+ \\ + a_{n_0} + m_{\mathbb{A}}s + x_0 + Tx_1 \text{holds.} \quad (70)$$

Since $\sim\subseteq\sim_1'$ and $\sim_1'$ is a canonical relation, the $\sim_1'$-equation

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim_1' f_{\mathbb{A}}s + \\ + a_{n_0} + m_{\mathbb{A}}s + x_0 + Tx_1 \text{holds.} \quad (71)$$

Since $\sim_1'$ is a staircase relation, the truth of the last equation implies that $k \notin I_1(\sim_1')$, and hence $\sim_1'$ is a max-relation with $\max(I_1(\sim_1'))$ at most $k-1$. Therefore,

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim_1' \\ \sim_1' f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + Tx_1 \text{ is true,} \quad (72)$$

which implies that

$$f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + T^{k-i}x_0 + Tx_1 \sim \\ \sim f_{\mathbb{A}}t + a_{m_0} + m_{\mathbb{A}}t + a_{m_1} + t' + Tx_1 \text{is true.} \quad (73)$$

Hence, the equation

$$f_{\mathbb{A}}t+a_{m_0}+m_{\mathbb{A}}t+a_{m_1}+t'+Tx_1 \sim_1' \sim_1' f_{\mathbb{A}}s+a_{n_0}+m_{\mathbb{A}}s+x_0+Tx_1 \text{ holds,} \quad (74)$$

from which we conclude that

$$x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \text{ is true,} \quad (75)$$

a contradiction. □

Lemma 4.19. *Suppose that* $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ *is true but* $x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2$ *is false. Then* $\sim \subseteq \sim_{\theta_l^2}$*. In particular,* $\sim \subseteq \min_k \cap \max_k$*.*

Proof. Fix l as in the statement. Since we assume that the equation

$$x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2 \text{ is false,} \quad (76)$$

by Proposition 3.4(1,2), we know that

$$x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \text{ is false.} \quad (77)$$

So, by Lemma 4.18, we obtain that $\sim \subseteq \max_k$. Suppose that $s \sim t$. Take the decomposition

$$s = f_{\mathbb{B}}s + b_{n_0} + m_{\mathbb{B}}s + b_m + l_{\mathbb{B}}s$$
$$t = f_{\mathbb{B}}t + b_{n_1} + m_{\mathbb{B}}t + b_m + l_{\mathbb{B}}s,$$

where we implicitly assume that $l_{\mathbb{B}}s = l_{\mathbb{B}}t$, since $s \sim_1' t$. Observe that showing that $s\theta_l^2 t$ is the same that proving that

$$\text{for all } n \in [\min\{n_0, n_1\}, m], \text{ either } C_{\mathbb{B}}(s)(n), C_{\mathbb{B}}(t)(n) < l, \\ \text{or } C_{\mathbb{B}}(s)(n) = C_{\mathbb{B}}(t)(n). \quad (78)$$

Assume on the contrary that (78) is false, and let α be the last $n \in [\min\{n_0, n_1\}, m]$ for which

$$\max\{C_{\mathbb{B}}(s)(n), C_{\mathbb{B}}(t)(n)\} \geq l \text{ and } C_{\mathbb{B}}(s)(n) \neq C_{\mathbb{B}}(t)(n). \tag{79}$$

Set $l_0 = C_{\mathbb{B}}(s)(\alpha)$, and $l_1 = C_{\mathbb{B}}(s)(\alpha)$. Notice that $\alpha < m$. Without loss of generality, we assume that $l_1 < l_0$ (the other case has a similar proof). Set

$$s' = \sum_{n<\alpha} T^{k-C_{\mathbb{B}}(s)(n)} b_n$$
$$t' = \sum_{n<\alpha} T^{k-C_{\mathbb{B}}(t)(n)} b_n.$$

Using this notation, we have that the equation

$$s' + T^{k-l_0} x_0 + x_1 \sim t' + T^{k-l_1} x_0 + x_1 \text{ holds.} \tag{80}$$

There are two cases:
$\underline{n_0 \leq n_1}$. We first show that in this case $s' + T^{k-l_0} x_0$ is a k-term. If $n_0 = n_1$, then $\alpha > n_0$, and hence s' is a k-vector. Suppose that $n_0 < n_1$. If $\alpha > n_0$, then s' is a k-term. If $\alpha = n_0$, then $l_0 = k$, and clearly $s' + T^{k-k} x_0 = s' + x_0$ is a k-term. We consider two subcases:
(a) $l_1 < l \leq l_0$. Then, by our assumption that $x_0 + T^{k-(l-1)} x_1 + x_2 \sim x_0 + x_2$ holds, we have that

$$s' + T^{k-l_0} x_0 + T^{k-l_1} x_1 + x_2 \sim s' + T^{k-l_0} x_0 + x_2 \text{ holds.} \tag{81}$$

By (80),

$$s' + T^{k-l_0} x_0 + T^{k-l_1} x_1 + x_2 \sim t' + T^{k-l_1} x_0 + T^{k-l_1} x_1 + x_2 \sim \\ \sim s' + T^{k-l_0} x_0 + T^{k-l_0} x_1 + x_2 \text{ holds,} \tag{82}$$

which implies that the equation

$$s' + T^{k-l_0} x_0 + T^{k-l_0} x_1 + x_2 \sim s' + T^{k-l_0} x_0 + x_2 \text{ holds.} \tag{83}$$

This is contradictory with the fact that $l_0 \geq l$.
(b) $l \leq l_1 < l_0$. Then,

$$s' + T^{k-l_0} x_0 + T^{k-l_1}(T^{l_0-l}) x_1 + x_2 \sim s' + T^{k-l_0} x_0 + x_2 \text{ holds,} \tag{84}$$

and by (80),

$$s' + T^{k-l_0} x_0 + T^{k-l_1}(T^{l_0-l}) x_1 + x_2 \sim t' + T^{k-l_1} x_0 + \\ + T^{k-l_1}(T^{l_0-l}) x_1 + x_2 \sim s' + T^{k-l_0} x_0 + T^{k-l} x_1 + x_2 \text{ holds.} \tag{85}$$

Again, this derives into a contradiction.
$\underline{n_1 < n_0}$. It can be shown that $t' + T^{k-l_1} x_0$ is a k-term. We consider the same two subcases as above:

(a) $l_1 < l \leq l_0$. Then

$$t' + T^{k-l_1}x_0 + T^{k-l_1}x_1 + x_2 \sim s' + T^{k-l_1}x_0 + x_2 \text{ holds,} \tag{86}$$

and hence,

$$s' + T^{k-l_0}x_0 + T^{k-l_0}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds,} \tag{87}$$

which, by (80), implies that

$$t' + T^{k-l_1}x_0 + T^{k-l_0}x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds,} \tag{88}$$

a contradiction, since $l_0 \geq l$.
(a) $l \leq l_1 < l_0$. Then

$$t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds.} \tag{89}$$

Using that

$$t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-l})x_1 + x_2 \sim s' + T^{k-l_0}x_0 + T^{k-l}x_1 + x_2 \sim \\ \sim t' + T^{k-l_1}x_0 + T^{k-l}x_1 + x_2 \text{ holds,} \tag{90}$$

we arrive to a contradiction.

□

Corollary 4.20. *Suppose that* $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ *is true, but* $x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2$ *is false. Then,* $\sim = \sim_0' \cap \sim_{\theta_l^2} \cap \sim_1'$.

Proof. By Proposition 4.12, $\sim_0' \cap \sim_{\theta_l^2} \cap \sim_1' \subseteq \sim$. We only need to show that $\sim \subseteq \sim_0'$. For suppose that $s \sim t$, and use the decomposition

$$s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}s$$
$$t = f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t.$$

Since $\max_k s = \max_k t$, we have that $m_0 = m_1$, and since $s \sim_1' t$, by Proposition 4.7(4), we may assume that $l_{\mathbb{A}}s \sim_1 l_{\mathbb{A}}t$. By Lemma 4.19, $s \sim_{\theta_2^l} t$, and using that the equations $x_0 + T^{k-j}x_1 + x_2 \sim x_0 + x_2$ are true for all $j < l$, we may also assume that $n_0 = n_1$ and $m_{\mathbb{A}}s = m_{\mathbb{A}}t$. Therefore, the equation $f_{\mathbb{A}}s + x_0 \sim f_{\mathbb{A}}t + x_0$ holds. By definition of $\sim_0$, we have that $f_{\mathbb{A}}s \sim_0 f_{\mathbb{A}}t$, and by Proposition 4.7(3), $s \sim_0' t$, as desired. □

Lemma 4.21. *Suppose that* $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ *is true, and* $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ *is false. Then,* $\sim = \sim_0' \cap \max_k \cap \sim_1'$.

Proof. We only need to show that $\sim \subseteq \sim_0'$. For suppose that $s \sim t$. We may assume the following decompositions of s and t

$$s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_m + l_{\mathbb{A}}s$$
$$t = f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s.$$

Notice that, since $Tx_0+x_1+x_2\sim Tx_0+x_2$ is true, we have that $x_0+x_1+x_2\sim x_0+x_2$ is true. Hence, we may assume that $m_{\mathbb{A}}s=m_{\mathbb{A}}t=0$. Notice also that, since

$$Tx_0+x_1+x_2\sim Tx_0+x_2 \text{ is true,} \tag{91}$$

and since $f_{\mathbb{A}}s$ and $f_{\mathbb{A}}t$ are $(k-1)$-vectors (this is why we use the decompositions of vectors of $\mathbb{B}$ in $\mathbb{A}$), we have that

$$s\sim f_{\mathbb{A}}s+a_{n_2}+l_{\mathbb{A}}s \text{ and } t\sim f_{\mathbb{A}}t+a_{n_2}+l_{\mathbb{A}}s. \tag{92}$$

This implies that $f_{\mathbb{A}}s\sim_0 f_{\mathbb{A}}t$, and, by Proposition 4.7(1,3),

$$s\sim_0' f_{\mathbb{A}}s+a_{n_2}+l_{\mathbb{A}}s\sim_0' f_{\mathbb{A}}t+a_{n_2}+l_{\mathbb{A}}t\sim_0' t, \tag{93}$$

as desired. □

Proposition 4.22. *Suppose that $x_0+x_1+Tx_2\sim x_0+Tx_2$ holds, and suppose that $Tx_0+x_1+x_2\sim Tx_0+x_2$ is false. Then, $\sim\subseteq \min_k$.*

Proof. Suppose that $s\sim t$. Take the decomposition

$$\begin{aligned} s=&f_{\mathbb{A}}s+a_{n_0}+m_{\mathbb{A}}s+a_{m_0}+l_{\mathbb{A}}s\\ t=&f_{\mathbb{A}}t+a_{n_1}+m_{\mathbb{A}}t+a_{m_1}+l_{\mathbb{A}}t. \end{aligned}$$

Suppose that $n_0\neq n_1$, and w.l.o.g. assume that $n_0<n_1$. Since $x_0+x_1+Tx_2\sim x_0+Tx_2$ holds, we have that

$$f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}s\sim f_{\mathbb{A}}t+a_{n_1}+l_{\mathbb{A}}t. \tag{94}$$

By Proposition 4.17(2), we have that

$$f_{\mathbb{A}}s+a_{n_0}+l_{\mathbb{A}}t\sim f_{\mathbb{A}}t+a_{n_1}+l_{\mathbb{A}}t, \tag{95}$$

and hence (since $l_{\mathbb{A}}t$ is a $(k-1)$-vector), the equation

$$f_{\mathbb{A}}s+x_0+Tx_2\sim f_{\mathbb{A}}t+x_1+Tx_2 \text{ holds.} \tag{96}$$

This implies that

$$f_{\mathbb{A}}s+x_0+x_1+Tx_3\sim f_{\mathbb{A}}t+x_2+Tx_3\sim f_{\mathbb{A}}s+x_1+Tx_3 \text{ holds,} \tag{97}$$

which implies that the equation

$$f_{\mathbb{A}}s+x_0+x_2\sim f_{\mathbb{A}}s+x_2 \text{ is true.} \tag{98}$$

Since $f_{\mathbb{A}}s$ is a $(k-1)$-vector, we have that

$$Tx_0+x_1+x_2\sim Tx_0+x_2 \text{ is true,} \tag{99}$$

a contradiction.

□

Lemma 4.23. *Suppose that $x_0+x_1+Tx_2\sim x_0+Tx_2$ is true, and $Tx_0+x_1+x_2\sim Tx_0+x_2$ is false. Then, $\sim=\sim_0'\cap\min_k\cap\sim_1'$.*

Proof. By Proposition 4.12, we have that $\sim_0' \cap \min_k \cap \sim_1' \subseteq \sim$. Let us show that $\sim \subseteq \sim_0' \cap \min_k \cap \sim_1'$. By Previous Proposition 4.22 and Lemma 4.16, we have that $\sim \subseteq \min_k \cap \sim_1'$. So, we only need to show that $\sim \subseteq \sim_0'$. For suppose that $s \sim t$ with

$$s = f_{\mathbb{A}} s + a_{n_0} + m_{\mathbb{A}} s + a_{n_1} + l_{\mathbb{A}} s$$
$$t = f_{\mathbb{A}} t + a_{n_0} + m_{\mathbb{A}} t + a_{m_1} + l_{\mathbb{A}} t.$$

Since the equation $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is true, we have that

$$f_{\mathbb{A}} s + a_{n_0} + l_{\mathbb{A}} s \sim f_{\mathbb{A}} t + a_{n_0} + l_{\mathbb{A}} t, \tag{100}$$

and, by Proposition 4.17,

$$f_{\mathbb{A}} s + a_{n_0} + l_{\mathbb{A}} s \sim f_{\mathbb{A}} t + a_{n_0} + l_{\mathbb{A}} s, \tag{101}$$

which easily leads to that $s \sim_0' t$. □

Lemma 4.24. *Suppose that $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true , and that $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ and $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ are both false. Then, $\sim \subseteq \min_k \cap \max_k$.*

Proof. By Lemma 4.18, we know that $\sim \subseteq \max_k$, and by Lemma 4.16, $\sim \subseteq \sim_1'$. So, we only need to show that $\sim \subseteq \min_k$. For suppose that $s \sim t$, set

$$s = f_{\mathbb{A}} s + a_{n_0} + m_{\mathbb{A}} s + a_m + l_{\mathbb{A}} s$$
$$t = f_{\mathbb{A}} t + a_{n_1} + m_{\mathbb{A}} t + a_m + l_{\mathbb{A}} t.$$

Suppose on the contrary that $n_0 < n_1$. There are two cases: $\underline{n_1 = m}$. Hence, $n_0 < m$ and

$$s \sim f_{\mathbb{A}} s + a_{n_0} + a_m + l_{\mathbb{A}} s \text{ and } t = f_{\mathbb{A}} t + a_m + l_{\mathbb{A}} t. \tag{102}$$

By Proposition 4.17,

$$f_{\mathbb{A}} s + a_{n_0} + a_m + l_{\mathbb{A}} s \sim f_{\mathbb{A}} t + a_m + l_{\mathbb{A}} s, \tag{103}$$

which implies that the equation

$$f_{\mathbb{A}} s + x_0 + x_1 \sim f_{\mathbb{A}} t + x_1 \text{ is true}, \tag{104}$$

a contradiction, since $f_{\mathbb{A}} s$ is a $(k-1)$-vector. $\underline{n_1 < m}$. Then, by our assumptions, and Proposition 4.17,

$$f_{\mathbb{A}} s + a_{n_0} + a_m + l_{\mathbb{A}} s \sim f_{\mathbb{A}} t + a_{n_1} + a_m + l_{\mathbb{A}} s. \tag{105}$$

Hence, the equation

$$f_{\mathbb{A}} s + x_0 + x_2 \sim f_{\mathbb{A}} t + x_1 + x_2 \text{ is true}, \tag{106}$$

which easily derives to that $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ must be true, a contradiction.

□

Corollary 4.25. *Suppose that $x_1+x_2+x_3 \sim x_1+x_3$ is true , and that $x_1+x_2+Tx_3 \sim x_1+Tx_3$ and $Tx_1+x_2+x_3 \sim Tx_1+x_3$ are both false. Then, $\sim = \sim_0' \cap \min_k \cap \max_k \cap \sim_1'$.*

Proof. By Proposition 4.12, $\sim_0' \cap \min_k \cap \max_k \cap \sim_1' \subseteq \sim$. Let us show the converse. By Lemma 4.24, we have that $\sim \subseteq \min_k \cap \max_k$. It remains to show that $\sim \subseteq \sim_0'$. For suppose that $s \sim t$, where $s = f_{\mathbb{A}}s + a_n + m_{\mathbb{A}}s + a_m + l_{\mathbb{A}}s$ and $t = f_{\mathbb{A}}t + a_n + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s$ (we may assume that $l_{\mathbb{A}}s = l_{\mathbb{A}}t$, since $\max_k(s) = \max_k(t)$). There are two cases: $\underline{n < m}$. Then, $f_{\mathbb{A}}s + a_n + a_m + l_{\mathbb{A}}s \sim f_{\mathbb{A}}t + a_n + m_{\mathbb{A}}t + a_m + l_{\mathbb{A}}s$ which directly implies that $s \sim_0' t$. The proof for $\underline{n_0 = m}$ is quite similar. □

Lemma 4.26. *Suppose that $Tx_0+x_1+x_2 \sim Tx_0+x_2$ and $x_0+x_1+Tx_2 \sim x_0+Tx_2$ are both true. Then, $\sim = \sim_0' \cap \sim_1'$.*

Proof. It is enough to show that $\sim \subseteq \sim_0'$. For suppose that $s \sim t$, with $s = f_{\mathbb{A}}s + a_{n_0} + m_{\mathbb{A}}s + a_{m_0} + l_{\mathbb{A}}s$ and $t = f_{\mathbb{A}}t + a_{n_1} + m_{\mathbb{A}}t + a_{m_1} + l_{\mathbb{A}}t$. By our assumption over the equations, we may assume that $s = f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$, and $t = f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t$. W.l.o.g. we assume that $n_0 \le n_1$, and hence, by Proposition 4.17,

$$f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t \sim f_{\mathbb{A}}t + a_{n_1} + l_{\mathbb{A}}t. \tag{107}$$

Cases: $\underline{n_0 = n_1}$. By definition of $\sim_0'$, (107) implies that

$$f_{\mathbb{A}}t + a_{n_0} + l_{\mathbb{A}}t \sim_0' f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t, \tag{108}$$

but trivially $f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}t \sim_0' f_{\mathbb{A}}s + a_{n_0} + l_{\mathbb{A}}s$, and we are done. $\underline{n_0 < n_1}$. Then,

$$f_{\mathbb{A}}s + x_0 + Tx_2 \sim f_{\mathbb{A}}t + x_1 + Tx_2 \text{ is true,} \tag{109}$$

which easily derives to that

$$f_{\mathbb{A}}s + x_1 + Tx_3 \sim f_{\mathbb{A}}t + x_1 + Tx_3 \text{ is true,} \tag{110}$$

which implies that $s \sim_0' t$. □

Corollary 4.27. *Every equivalence relation on FIN_k is canonical in some sos.* □

This corollary has the following local version.

Corollary 4.28. *For every block sequence A and every equivalence relation $\sim$ on $\langle A\rangle$ there is a sos $B \in [A]^{[\infty]}$ on which $\sim$ is canonical.*

Proof. Fix the canonical isomorphism $\Lambda : \mathrm{FIN}_k \to \langle A\rangle$ (i.e., the extension of $\Theta e_n \mapsto a_n$). It is not difficult to show the following facts:

1. $B = (b_n)_n$ is a sos iff $FB = (Fb_n)_n$ is a sos.
2. For every canonical equivalence relation $\sim_{can}$, every sos B, and $s, t \in \langle B\rangle$, $s \sim_{can} t$ iff $F^{-1}s \sim_{can} F^{-1}t$.

We define $\sim'$ on FIN_k by $s \sim' t$ iff $Fs \sim Ft$. Find a canonical equivalence relation $\sim_{can}$ and a sos B such that $\sim$ and $\sim_{can}$ are the same on $\langle B\rangle$. Let $C = FB$, which is a sos. Then $\sim$ and $\sim_{can}$ are the same in $\langle C\rangle$: $s \sim_{can} t$ iff $F^{-1}s \sim_{can} F^{-1}t$ iff $F^{-1}s \sim' F^{-1}t$ iff $s \sim t$. □

Corollary 4.29. *Every canonical equivalence relation is a staircase equivalence relation.*

PROOF. Notice that, since $\sim$ is canonical in A, $\mathbb{A} = A$ works for both Lemmas 4.2 and 4.3. Hence, $\sim$ is a staircase equivalence relation in $\mathbb{B} = (Ta_{3n} + a_{3n+1} + Ta_{3n+2})_n$. Let $\sim'$ be this staircase relation, which is equal to $\sim$ when restricted to $\mathbb{B}$. We work to show that $\sim$ and $\sim'$ are not only equal in $\mathbb{B}$, but also in A. Fix s and t in A, and take their canonical decompositions in A

$$s = \sum_{n\geq 0} T^{k-C_A(s)(n)} a_n \text{ and } t = \sum_{n\geq 0} T^{k-C_A(t)(n)} a_n.$$

Suppose first that $s \sim t$. Since $\sim$ is canonical, the equation

$$\sum_{n\geq 0} T^{k-C_A(s)(n)} x_n \sim \sum_{n\geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } A, \tag{111}$$

and hence, also holds in $\mathbb{B}$, i.e.,

$$\sum_{n\geq 0} T^{k-C_A(s)(n)} x_n \sim' \sum_{n\geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } \mathbb{B}. \tag{112}$$

But since $\sim'$ is staircase, it is canonical (Proposition 3.14), and hence, equation (112) also it holds in A, and in particular, $s \sim' t$.

Suppose now that $s \sim' t$. Since $\sim'$ is canonical in any sos, the equation

$$\sum_{n\geq 0} T^{k-C_A(s)(n)} x_n \sim' \sum_{n\geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } A, \tag{113}$$

hence, also in $\mathbb{B}$. By definition, $\sim'$ is equal to $\sim$ restricted to $\mathbb{B}$, and hence

$$\sum_{n\geq 0} T^{k-C_A(s)(n)} x_n \sim \sum_{n\geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } \mathbb{B}. \tag{114}$$

Since $\sim$ is canonical, the equation (114) holds in A, and in particular, $s \sim t$. □

5. COUNTING

Recall that $e_n(1) = \sum_{j=0}^{n} \frac{1}{j!}$ is the exponential sum-function and that $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. Notice that $\Gamma(n,1) = (n-1)!e^{-1}e_{n-1}(1)$.

Let $\mathcal{A}_k$, $\mathcal{B}_k$ be the set of min-relations and max-relations respectively, and set $a_k = |\mathcal{A}_k|$ and $b_k = |\mathcal{B}_k|$. Let $\mathcal{C}_k \subseteq \mathcal{A}_k$ be the set of min-relations R such that $k \notin I_0(R)$, and let $\mathcal{D}_k \subseteq \mathcal{B}_k$ be the set of max-relations R such that $k \notin I_1(R)$. Set $c_k = |\mathcal{C}_k|$ and $d_k = |\mathcal{D}_k|$. Notice that

1. $c_k = a_{k-1}$,
2. $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{R \cap \sim_{\min_k} \,:\, R \in \mathcal{C}_{k-1}\} \cup \{R \cap \sim_{\min_k} \cap \sim_{\theta^0_{k,l}} \,:\, l = -1$ or $l = 1, \dots, k-1,\ R \in \mathcal{A}_{k-1} \setminus \mathcal{C}_{k-1}\}$. So, $a_k = a_{k-1} + c_{k-1} + k(a_{k-1} - c_{k-1})$.

Hence,

$$a_k = (k+1)a_{k-1} - (k-1)a_{k-2},\ a_0 = 1,\ a_1 = 2 \tag{115}$$

By standard methods, we conclude that

$$a_k = \frac{e\ (1+k)\ k!\Gamma(1+k,1)}{\Gamma(2+k)} = k! e_k(1) \tag{116}$$

Now let $\mathcal{T}_k$ be the set of canonical equivalence relations of FIN_k and $t_k = |\mathcal{T}_k|$. Then,

$$\mathcal{T}_k = (\{R \cap S \,:\, R \in \mathcal{A}_k, S \in \mathcal{B}_k\} \setminus \{R \cap S \,:\, R \in \mathcal{A}_k \setminus \mathcal{C}_k, S \in \mathcal{B}_k \setminus \mathcal{D}_k\}) \cup$$
$$\cup \{R \cap S \cap \sim_{\theta^2_l} \,:\, R \in \mathcal{A}_k \setminus \mathcal{C}_k, S \in \mathcal{B}_k \setminus \mathcal{D}_k, l = -1 \text{ or } l = 1, \dots, k\}. \tag{117}$$

Hence,

$$t_k = a_k^2 - (a_k - c_k)^2 + (k+1)(a_k - c_k)^2 = k(a_k - a_{k-1})^2 + a_k^2 \tag{118}$$

and from (116) and (118), we obtain that

$$t_k = (k! e_k(1))^2 + k\,(k! e_k(1) - (k-1)! e_{k-1}(1))^2 \tag{119}$$

or, equivalently,

$$t_k = e^2 \left[k\,[\Gamma(k,1) - \Gamma(k+1,1)]^2 + \Gamma(k+1,1)^2 \right] \tag{120}$$

This is a table with the first numbers t_k:

k	0	1	2	3	4	5	6
t_k	1	5	43	619	13829	446881	19790815

Remark 5.1. Let us say that a canonical equivalence relation R is *linked free* iff $I_0(R)$ and $I_1(R)$ have no consecutive members and $k \notin I_0(R) \cap I_1(R)$. The number l_k of linked free canonical equivalence relations of FIN_k is the Fibonacci number F_{2k+2} for $2k+2$, since F_{l+2} is the number of subsets of $\{1, 2, \dots, l\}$ with no consecutive elements, and since R is linked free iff the set $I_0(R) \cup \{2k+1-i \,:\, i \in I_1(R)\} \subseteq \{1, 2, \dots, 2k\}$ has no consecutive numbers.

6. THE FINITE VERSION

Theorem 6.1. *For every m there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \dots, e_n\rangle$ there is some sos $(a_0, \dots, a_m) \preceq (e_0, \dots, e_n)$ such that $\sim$ is a staircase equivalence relation in $\langle a_0, \dots, a_m\rangle$.*

PROOF. Suppose not. Then, there is some m such that for every n there is some equivalence relation $\sim_n$ on $\langle e_0, \dots, e_n\rangle$ which is not a staircase relation when restricted to any sos $(a_0, \dots, a_m)$ of $(e_i)_{i=0}^n$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and define the equivalence relation $\sim$ on FIN_k by $s \sim t$ if and only if $\{n : sR_nt\} \in \mathcal{U}$, where $R_n = \sim_n \cup \{(a, a) : a \in \mathrm{FIN}_k\}$ is an equivalence relation on FIN_k. It is easy to show that $\sim$ is an equivalence relation. By Theorem 4.1, there is some sos $A = (a_n)_n$ on which $\sim$ is a staircase equivalence relation, say $\sim_{can}$. Choose n large enough such that:

1. $(a_0, \dots, a_m) \preceq (e_i)_{i=0}^n$
2. For $s, t \in \langle a_0, \dots, a_m\rangle$, $s \sim t$ iff $s \sim_n t$.

This can be done: For every pair $s, t \in \langle a_0, \dots, a_m\rangle$, let

$$A_{s,t} = \begin{cases} \{n : s \sim_n t\} \in \mathcal{U} & \text{if } s \sim t \\ \{n : s \not\sim_n t\} \in \mathcal{U} & \text{if } s \not\sim t \end{cases} \tag{121}$$

Let $n = \min \bigcap_{s,t \in \langle a_0, \dots, a_m\rangle} A_{s,t}$. Then $\sim_n$ is $\sim$ restricted to $(a_0, \dots, a_m)$, and hence is a staircase equivalence relation, a contradiction. □

Corollary 6.2. *For every m there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \dots, e_n\rangle$ there is some sos $(a_0, \dots, a_m) \preceq (e_0, \dots, e_n)$ such that $\sim$ is a canonical equivalence relation on $\langle a_0, \dots, a_m\rangle$.* □

Corollary 6.3. *For every m there is some $n = n(m)$ such that for every $(b_0, \dots, b_n)$ and every equivalence relation $\sim$ on $\langle b_0, \dots, b_n\rangle$ there is some sos $(a_0, \dots, a_m) \preceq (b_0, \dots, b_n)$ such that $\sim$ is a staircase equivalence relation when restricted to $\langle a_0, \dots, a_m\rangle$.*

PROOF. Let $n = n(m)$ be given by Theorem 6.1. Fix $b_0, \dots, b_n$, and an equivalence relation $\sim$. Let F be the canonical isomorphism between $\langle e_0, \dots, e_n\rangle$ and $\langle b_0, \dots, b_n\rangle$. Define $\sim'$ on $\langle e_i\rangle_{i=1}^n$ via F, i.e., $s \sim' t$ if and only if $F(s) \sim F(t)$. Fix a sos $(c_i)_{i=0}^m \preceq (e_i)_{i=1}^n$ and a staircase equivalence relation $\sim_{can}$ such that $s \sim' t$ if and only if $s \sim_{can} t$, for $s, t \in \langle c_i\rangle_{i=0}^m$. Let $b_i = Fc_i$, for $i = 0, \dots, m$. Observe that $(b_0, \dots, b_m)$ is a sos since sos is preserved under isomorphisms, and that $\sim_{can}$ is well defined on $(b_0, \dots, b_m)$. Since $\sim_{can}$ is staircase, $s \sim_{can} t$ if and only if $F^{-1}s \sim_{can} F^{-1}t$ for all $s, t \in \langle b_0, \dots, b_m\rangle$. Hence, $s \sim_{can} t$ iff $F^{-1}s \sim_{can} F^{-1}t$ iff $F^{-1}s \sim' F^{-1}t$ iff $s \sim t$. Therefore $\sim_{can}$ and $\sim$ coincide on $\langle b_0, \dots, b_m\rangle$.

□

7. Canonical relations and continuous maps on PS_{c_0}

Our result over equivalence relations on FIN_k gives some consequences about equivalence relations on PS_{c_0}. Let us start with some natural definitions.

For a fixed $\delta > 0$, let k be the first integer such that $1/(1+\delta)^{k-1} < \delta$, and set $\delta_i = (1+\delta)^{i-k}$, for $0 \le i \le k$. For $0 \le i \le k+1$, let

$$\gamma_i(\delta) = \begin{cases} \frac{\delta_{i-1}+\delta_i}{2} = \frac{\varepsilon^{k-i}(\varepsilon+1)}{2} & \text{if } 1 \le i \le k \\ 0 & \text{if } i = 0 \\ \delta_k = 1 & \text{if } i = k+1 \end{cases}$$

and for $0 \le i \le k$, let

$$I_i^{(\delta)} = \begin{cases} [\gamma_i(\delta), \gamma_{i+1}(\delta)) & \text{if } 0 \le i < k \\ [\gamma_k(\delta), \gamma_{k+1}(\delta) = 1] & \text{if } i = k \end{cases}$$

Observe that $\delta_i \in I_i^{(\delta)}$ for every $0 \le i \le k$, and that $[0,1] = \bigcup_{i=0}^k I_i^{(\delta)}$, disjoint union.

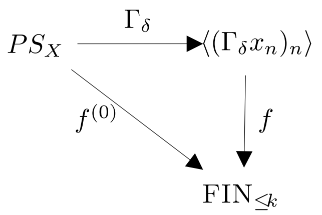
For $x = (x_m)_m \in PB_{c_0}$ and $n \in \mathbb{N}$, let $\Gamma_n^{(\delta)}(x)$ be the unique $0 \le i \le k$ such that $x_n \in I_i^{(\delta)}$, and define $\Gamma_\delta : PB_{c_0} \to \mathrm{FIN}_{\le k}$ by $\Gamma_\delta(x) = (\Gamma_n^{(\delta)}(x))_n$. Notice that $\Gamma_\delta(PS_{c_0}) \subseteq \mathrm{FIN}_k$. A vector $x \in PS_{c_0}$ is called a *δ-sos* iff $\Gamma_\delta x$ is a sos. A block sequence $(x_n)_n$ of vectors of PS_{c_0} is called a *δ-sos* iff every $x \in PS_X$ is a δ-sos. The next proposition is not difficult to prove.

Proposition 7.1. *Fix $\rho \in [0,1]$, $x, y \in PB_{c_0}$, and a k-vector s of FIN_k. Let i be the unique integer such that $\rho \in I_i^{(\delta)}$. Then,*

1. $\Gamma_\delta(x+y) = \Gamma_\delta(x) + \Gamma_\delta(y)$ *and* $\Gamma_\delta(\rho e_n) = T^{k-i}\Gamma_\delta(e_n) = T^{k-i}(\Theta_\delta^{-1} e_n)$.
2. $\Gamma_\delta(\rho\Theta_\delta^{-1} x) = T^{k-i}\Gamma_\delta(\Theta_\delta^{-1} x)$. *Therefore, if $(a_n)_n$ is a sos k-block sequence, then $(\Theta_\delta^{-1} a_n)_n$ is a δ-sos.*

□

Definition 7.2. Given a staircase mapping f of FIN_k, we consider the following two extensions to an arbitrary δ-sos $X = (x_n)_n$. The fist one is $f^{(0)} : PS_X \to \mathrm{FIN}_{\le k}$, closing the following diagram:



The second one is $f^{(1)} : PS_X \to PB_{c_0}$, defined by $f^{(1)}(x)(n) = x(n)$ iff $f^{(0)}x(n) \neq 0$.

Proposition 7.3. *Fix a staircase f, and some δ-sos X.*

1. $(f \odot g)^{(i)} = f^{(i)} \odot g^{(i)}$, *for $i = 0, 1$ and $\odot$ equal to $\vee$ or $\wedge$.*
2. $f^{(1)}$ *is a Baire class 1 function.*
3. *If $f^{(1)}x = f^{(1)}y$, then $f^{(0)}x = f^{(0)}y$ for every $x, y \in PS_X$.*
4. $\|\Theta_\delta^{-1} f^{(0)}x - f^{(1)}x\| \leq \delta$ *for every $x \in PS_X$.*
5. *For every k-vector $a \in \langle(\Gamma_\delta x_n)_n\rangle$, $f^{(1)}\Theta_\delta^{-1}a = f^{(0)}\Theta_\delta^{-1}a = fa$. Therefore $f^{(1)}\Theta_\delta^{-1}a = f^{(1)}\Theta_\delta^{-1}b$ iff $f^{(0)}\Theta_\delta^{-1}a = f^{(0)}\Theta_\delta^{-1}b$, for every k-vectors $a, b \in \langle(\Gamma_\delta x_n)_n\rangle$.*
6. *For every $x \in PS_X$ there is some k-vector $\bar{x}$ such that* $\|x - f^{(0)}\Theta_\delta^{-1}\bar{x}\| \leq \delta$ *and* $f^{(0)}x = f^{(0)}\Theta_\delta^{-1}\bar{x}$.

Proof. 1. is rather easy to check it. Let us show 2. Suppose that f is a staircase mapping. Then f in the algebraic closure of $\mathcal{F}$ (see Definition 3.11), i.e., there is a finite list $f_0, \dots, f_n \in \mathcal{F}$ such that $f = f_1 \odot_1 f_2 \odot_2 f_3 \odot_3 \cdots \odot_{n_1} f_n$, where $\odot_i$ is either $\vee$ or $\wedge$ for every $i = 1, \dots, n-1$. By point 1., $f^{(1)} = f_1^{(1)} \odot_1 f_2^{(1)} \odot_2 f_3^{(1)} \odot_3 \cdots \odot_{n_1} f_n^{(1)}$. Since for every point $x \in PS_X$, the support of $f_i^{(1)}(x)$ is finite, we may assume that $f \in \mathcal{F}$. We give the proof for the case $f = \min_i$. The other cases can be shown in a similar way. For $l > 0$ we define the following perturbations of the intervals $I_i^{(\delta)}$, let

$$I_{i,l}^{(\delta)} = \begin{cases} (\gamma_i(\delta) - \frac{1}{l}, \gamma_{i+1}(\delta)) & \text{if } i < k \\ (\gamma_k(\delta) - \frac{1}{l}, 1] & \text{if } i = k \end{cases}$$

These are open intervals of PS_{c_0}. For each l, let $f_l : PS_X \to PB_X$ be defined for $n \in \mathbb{N}$ as follows,

$$f_l(x)(n) = \begin{cases} x(n) & \text{if } x(n) \in I_{i,l}^{(\delta)} \text{ and for all } m < n \ x(n) \in [0, \gamma_i(\delta)) \\ 0 & \text{if not} \end{cases}$$

Let us check that f_l is continuous, and that $f_l \to_l f$. For suppose that $x_r \to_r x$, with $x_r, x \in PS_X$. Let n be the unique integer such that $f_l(x)(n) = x(n) > 0$, i.e., $x(n) \in I_{i,l}^{(\delta)}$ and $x(m) \in [0, \gamma_i(\delta))$ for every $m < n$. Since both sets are open, there must be some r' such that $x_{r''}(n) \in I_{i,l}^{(\delta)}$ and $x_{r''}(m) \in [0, \gamma_i(\delta))$, for every $r'' > r'$ and every $m < n$. Therefore, for all $r'' > r'$, $f_l(x_{r''}) = f_l x$. Let us check now that $f_l \to f$. Fix x, and we work to show that $f_l(x) \to f(x)$. Again, Let n be the unique integer such that $f_l(x)(n) = x(n) > 0$. Let l' be such that $x(m) \in [0, \gamma_i(\delta) - 1/l)$ for every $m < n$. Then $f_{l''}x(m) = 0$ and $f_{l''}x(n) = x(n)$, for every $l'' \geq l'$ and every $m < n$. Also, $f_{l''}x(m) = 0$ for every $m > n$. All this implies that $f_{l''}(x) = f(x)$.

It is not difficult to show the rest of the points. □

For an equivalence relation R, and $x \in PS_{c_0}$, the *R-equivalence class* of x is denoted by $[x]_R$.

Proposition 7.4. *Fix $\delta > 0$, a staircase equivalence relation R_f, and a k-block sequence $A = (a_n)_n$, where $k = k(\delta)$. Set $X = (x_n = \Theta_\delta^{-1} a_n)_n$ and $R = R_{f^{(1)}}$.*

1. *For every $x \in PS_X$ there is a k-vector $\bar{x}$ of A such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}]_R)_\delta$.*
2. *For every $x, y \in PS_X$, if $(x, y) \in R$, then $(x, z) \in R$, for every $x \wedge y \leq_L z \leq_L x \vee y$.*

Proof. 1. Fix $x \in PS_X$, and let $\bar{x}$ be a k-vector of A such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $f^{(0)}x = f^{(0)}\Theta_\delta^{-1}\bar{x}$. Set $x' = \Theta_\delta^{-1}\bar{x}$. We work to show that $[x]_R \subseteq ([x']_R)_\delta$. For suppose that $y \in [x]_R \cap PS_X$. Then $f^{(1)}x = f^{(1)}y$, and hence $f^{(0)}y = f^{(0)}x = f^{(0)}x'$. Let $\bar{y}$ be a k-vector of A such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f^{(0)}y = f^{(0)}\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f^{(0)}x' = f^{(0)}y'$, which implies that $f^{(1)}x' = f^{(1)}y'$, i.e., $y' \in [x']_R$ and hence $y \in ([x']_R)_\delta$.
2. By Proposition 3.13, we may assume that $f \in \mathcal{F}$. Again, we give a proof for the case $f = \min_i$, since the other cases can be shown in a similar way. Suppose that $(x, y) \in R_{f^{(1)}}$, and fix $z \in PS_X$ with $x \wedge y \leq_L z \leq_L x \vee y$. Let n be the unique integer such that $f^{(1)}x(n) = x(n) = y(n) = f^{(1)}y(n) > 0$. Then $x(m), y(m) \in [0, \gamma_i(\delta))$ for every $m < n$. Therefore, $z(n) = x(n) = y(n)$ and $z(m) \in [0, \gamma_i(\delta))$ for every $m < n$. This implies that $f^{(1)}(z) = f^{(1)}(x)$.

□

Definition 7.5. A *δ-staircase equivalence relation* is $R_{f^{(1)}}$ for some staircase f.

The next result is the interpretation of Theorem 4.1 in terms of equivalence relations of PS_X.

Proposition 7.6. *Let R be an equivalence relation on PS_X. Then for every $\delta > 0$ there is some δ-sos X and some δ-staircase equivalence relation $\widetilde{R}$ such that:*

1. *R and $\widetilde{R}$ coincide in a ε-net of PS_X for some $\varepsilon < \delta$.*
2. *For every $\widetilde{R}$-class α on PS_X there is a R-class β on PS_X such that $\alpha \subseteq \beta_\delta$.*

Proof. Fix δ, and let $k = k(\delta)$. Define $\bar{R}$ on FIN_k via Θ_δ. Then there is some sos k-block sequence $A = (a_n)_n$ and some staircase equivalence

relation R_f such that $\bar{R}$ and R_f coincide on $\langle A\rangle$. Set $\widetilde{R} = R_{f^{(1)}}$ and $X = (x_n)_n$, where $x_n == \Theta_\delta^{-1} a_n$ for every n.
1. For $\varepsilon = (1+\delta)^{k-1}$, $N = \Theta_\delta^{-1}(\langle (a_n)_n\rangle)$ is a ε-net of $PS(X)$ satisfying our requirements.
2. For a fixed $x \in PS_X$ choose some k-vector $\bar{x}$ of A such that $\|x - x'\| \leq \delta$ and $f^{(0)}x = f^{(0)}x'$, where $x' = \Theta_\delta^{-1}\bar{x}$. We work to show that $[x]_{\widetilde{R}} \subseteq ([x']_R)_\delta$. Suppose that $y \in PS_X$ is such that $f^{(1)}x = f^{(1)}y$. Pick some k-vector $\bar{y}$ of A such that $\|y - y'\| \leq \delta$ and $f^{(0)}y = f^{(0)}y'$ where $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f^{(0)}x = f^{(0)}y$ and hence $f^{(0)}x' = f^{(0)}y'$, which implies that $f^{(1)}x' = f^{(1)}y'$. Therefore, $y' \in [x']_R$. □

In the case of equivalence relations with some additional properties, we have the following stronger result.

Proposition 7.7. *Fix $\delta, \gamma > 0$, set $k = k(\delta)$, and suppose that R is an equivalence relation on PS_{c_0} such that*

1. *for every $x, y \in PS_{c_0}$ and every $z \in PS_{c_0}$ with $x \wedge y \leq_L z \leq_L x \vee y$, if $(x, y) \in R$, then satisfies that $(x, z) \in R$, and*
2. *for every sos k-block sequence $B = (b_n)_n$ and every $x \in PS_{(\Theta_\delta^{-1} b_n)_n}$ there is some k-vector $\bar{x}$ of B such that $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}]_R)_\gamma$.*

Then, there is some δ-sos X and some δ-staircase equivalence relation $\widetilde{R}$ such that (a) *for every R-equivalent classes α in PS_X, there is a $\widetilde{R}$-equivalent class β in PS_X such that $\alpha \subseteq \beta_{\delta+\gamma}$, and* (b) *for every $\widetilde{R}$-equivalence class β there is a R-equivalence class α such that $\beta \subseteq (\alpha)_\delta$.*

Proof. Define $\bar{R}$ on FIN_k via Θ_δ. Then, there is some sos $A = (a_n)$ and some staircase equivalence relation R_f such that $\bar{R}$ is R_f on $\langle A\rangle$. Let $\widetilde{R} = R_{f^{(1)}}$, and $X = (x_n)_n$, where $x_n = \Theta_\delta^{-1} a_n$ for every n. *(b)* is shown in Proposition 7.6. Let us show *(a)*. Fix $x \in PS_X$, and choose a k-vector $\bar{x}$ of A such that $[x]_R \subseteq ([x']_R)_\gamma$ where $x' = \Theta_\delta^{-1}\bar{x}$. Let us show that $[x']_R \subseteq ([x']_{\widetilde{R}})_\delta$ on PS_X. Fix $y \in [x']_R$. Then, there is some k-vector $\bar{y}$ of A such that $x' \wedge y \leq_L y' \leq_L x' \vee y$ and $\|y - y'\| \leq \delta$, where $y' = \Theta_\delta^{-1}\bar{y}$. Hence, $y' \in [x']_R$, and therefore, $y' \in [x']_{R'}$. □

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