# CODING INTO RAMSEY SETS

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Abstract. In [6] W. T. Gowers formulated and proved a Ramsey-type result which lies at the heart of his famous dichotomy for Banach spaces. He defines the notion of weakly Ramsey set of block sequences of an infinite dimensional Banach space and shows that every analytic set of block sequences is weakly Ramsey. We show here that Gowers' result follows quite directly from the fact that all $G_\delta$ sets are weakly Ramsey, if the Banach space does not contain $c_0$, and from the fact that all $F_{\sigma\delta}$ sets are weakly Ramsey, in the case of an arbitrary Banach space. We also show that every result obtained by the application of Gowers' theorem to an analytic set can also be obtained by applying the Theorem to a $F_{\sigma\delta}$ set (or to a $G_\delta$ set if the space does not contain $c_0$). This fact explains why the only known applications of this technique are based on very low-ranked Borel sets (open, closed, $F_\sigma$, or $G_\delta$).

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## 1. Introduction

Let $E = (E, \|\cdot\|)$ be an infinite dimensional Banach space with a fixed Schauder basis $(e_n)_n$. We write $S(E)$ to denote the unit sphere of $E$. Recall the following definitions and notations from [6]: The support of a vector $x$ is the set $\operatorname{supp} x = \{m : e^*_m(x) \neq 0\}$. Let $\Sigma(E)$ be the set of infinite sequences $((x_n, \lambda_n))_n$, where $\|x_n\| = 1$, $x_n$ has finite support, $\max\operatorname{supp} x_n < \min\operatorname{supp} x_{n+1}$, and $\lambda_n \in [0, 1]$ for every $n$. A sequence $((x_n, \lambda_n))_n$ is called a *block sequence of* $E$. Notice that $\Sigma(E) \subseteq (S(E) \times [0, 1])^{\mathbb{N}}$ is a closed subset and hence $\Sigma(E)$ is a Polish space. For a block sequence $X = ((x_n, \lambda_n))_n$, let $[X]$ be the set of *block subsequences* of $X$, i.e., $[X] = \{((y_n, \mu_n))_n \in \Sigma(E) : y_m \in \langle x_n\rangle_n \text{ for all } m\}$, where $\langle x_n\rangle_n$ denotes the linear span of $\{x_n\}_n$. The subset of $[X]$ consisting of all sequences $((y_n, \mu_n))_n$ where $\mu_n = 1$ for every $n$ is denoted by $[X]_1$.

Given a set of block sequences $\sigma$ and $X \in \Sigma(E)$, we say that $\sigma$ is *large for* $[X]$ iff $\sigma \cap [Y] \neq \emptyset$ for every $Y \in [X]$. For $\sigma$ a set of block sequences, the game $\Game_\sigma[X]$ is defined as follows: There are two players, $I$ and $II$. $I$

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starts playing $Y_0 \in [X]$, then $II$ chooses $x_0 \in Y_0$ and some $\lambda_0 \in [0,1]$. Then player $I$ plays $Y_1 \in [X]$, and $II$ chooses $y_1 \in Y_1$ with $y_1 > y_0$, and $\lambda_1 \in [0,1]$, and so on. Notice that when we write $y \in Y = ((y_n, \lambda_n))_n$ we understand that $y$ belongs to the unit sphere of the linear span of $\{y_n\}_n$. $II$ wins the game iff $((y_n, \lambda_n))_n \in \sigma$. Otherwise $I$ wins. A strategy for $II$ (in $X$) is a function $\Phi : [X]_f \times [X] \to X \times [0,1]$ satisfying that $\Phi(((y_0, \lambda_0), \dots, (y_{n-1}, \lambda_{n-1})), Y) \in Y \times [0,1]$, where $[X]_f$ denotes the set of finite block subsequences $((y_0, \mu_0), \dots, (y_n, \mu_n))_n$ of $X$, i.e., $y_i \in \langle X \rangle$, $\max \operatorname{supp} y_i < \min \operatorname{supp} y_{i+1}$, and $\mu_i \in [0,1]$ for every $0 \leq i \leq n$. A strategy $S$ for player $II$ is a winning strategy if whenever $II$ plays according to $S$, then he wins the game. The set $\sigma$ is *strategically large for* $[X]$ iff Player $II$ has a winning strategy for $\Game_\sigma[X]$. $\sigma$ is called *weakly Ramsey* iff for every $X \in \Sigma(E)$, if $\sigma$ is large for $[X]$ then for every $\Delta > 0$ there is some $Y \in [X]$ such that $\sigma_\Delta$ is strategically large for $\sigma(Y)$, where for a decreasing sequence of strictly positive reals $\Delta = (\delta_n)_n$, the $\Delta$-*expansion* $\sigma_\Delta$ *of* $\sigma$ is the set of block sequences $Y = ((y_n, \lambda_n))_n$ such that there is some $X = ((x_n, \mu_n))_n \in \sigma$ such that $\max\{\|x_n - y_n\|, |\lambda_n - \mu_n|\} \leq \delta_n$ for every $n$, which is denoted in short by $d(X,Y) \leq \Delta$. Recall the following result due to W. T. Gowers.

**Theorem 1.1.** [6] *Every analytic set of block sequences is weakly Ramsey.* □

This theorem has several consequences regarding the geometry of Banach spaces. Maybe the best known is Gowers' dichotomy: Every infinite-dimensional Banach space has a subspace which either has an unconditional basis or is hereditarily indecomposable. However, all the known consequences of Theorem 1.1 do not use the full strength of the result, but only that the $G_\delta$ sets are weakly Ramsey. One of the aims of this paper is to determine if there are applications which need the full strength of the theorem. We show that every result that is obtained by applying Theorem 1.1 to an analytic set of block sequences can also be obtained from the fact that a very simple Borel set ($G_\delta$ sets if the space does not contain $c_0$ and $F_{\sigma\delta}$ for arbitrary spaces) is weakly Ramsey. Moreover, Gowers' Theorem 1.1 itself follows quite directly from the fact that these Borel sets are weakly Ramsey, which also gives an alternative proof of the Theorem.

Let us make some comments about why this phenomenon occurs. Let $\Sigma_1(E)$ be the set of *normalized* block sequences $((x_n, 1))_n$, that we denote by $(x_n)_n$. A set $A \subseteq S(\langle e_n \rangle_n)$ is called *asymptotic* iff $A \cap S(\langle x_n \rangle_n) \neq \emptyset$ for every block sequence $(x_n)_n$ in $\Sigma(E)$. For example the sets

$$A_i = \{x \in S(\langle e_n \rangle_n) \, : \, (-1)^i e^*_{\min \operatorname{supp} x}(x) > 0\}$$

are clearly asymptotic for $i = 0, 1$. These sets are responsible for the fact that the stronger Ramsey-like property defined for $\sigma \subseteq \Sigma_1(E)$ by: "$\sigma$ is *Ramsey* iff for every normalized block sequence $X$, if $\sigma$ is large for $[X]_1$, then there is some normalized block subsequence $Y$ of $X$ such that $[Y]_1 \subseteq \sigma$" fails to be true even for $F_\sigma$ sets of block sequences, since for example the set $\sigma_0 = \{(x_n)_n \in \Sigma_1(E) : x_0 \in A_0\}$ and its complement $\sigma_1 = \{(x_n)_n \in \Sigma_1(E) : x_0 \in A_1\}$ are both $F_\sigma$ and large for every $[X]$. Notice that $(\sigma_0)_\Delta = (\sigma_1)_\Delta = \Sigma_1(E)$ for every $\Delta > 0$, which suggests that the weaker form of Ramsey property defined by: "$\sigma$ is *almost-Ramsey* iff for every $X$, if $\sigma$ is large for $[X]_1$ then for every $\Delta > 0$ there is some normalized block subsequence $Y$ of $X$ such that $[Y]_1 \subseteq \sigma_\Delta$" could be good enough. This turns out to be case for the space $c_0$ since it was shown by Gowers [6] that every analytic set of normalized block sequences of $c_0$ is almost-Ramsey. However, the almost-Ramsey property fails to be true for open sets of normalized block sequences for spaces $E$ which are not $c_0$-saturated, namely there must be some normalized block sequence $X$ with no normalized block subsequence $Y$ equivalent to $c_0$, and hence two asymptotic sets $A_0$ and $A_1$ of $Y$ which are separated, i.e., $d(A_0, A_1) = \delta$, some $\delta > 0$ (see [9]). Then the set $\sigma_0 = \{(x_n)_n \in \Sigma_1(E) : x_0 \in (A_0)_{\delta/2}\}$ is open and large for $[Y]$ but $(\sigma_0)_{(\delta/2,\delta/2,\dots)}$ does not contain $[Z]_1$ for any $Z \in [Y]_1$.

But the existence of two disjoint asymptotic sets (separated or not) is not only an obstacle to have a stronger Ramsey property but it also allows to "code information" as follows. Recall that an analytic set $\sigma$ of block sequences is the first projection of a closed set $C$ of $\Sigma(E) \times \mathcal{N}^\uparrow$ where $\mathcal{N}^\uparrow$ denotes the space of strictly increasing sequences of natural numbers. In other words, to know whether a block sequence $(x_n)_n$ is in $\sigma$ we need to know that $(((x_n, \lambda_n))_n, (k_n)_n) \in C$ for some sequence $(k_n)_n$ of integers. We are going to use the existence of two asymptotic sets to code in a single block sequence $((x_0, \lambda_0), (y_0, 1), (x_1, \lambda_1), (y_1, 1), \dots)$ both $((x_n, \lambda_n))_n$ and $(k_n)_n$ in such a way that the corresponding set $\tau$ of block sequences coding pairs in $C$ is of low Borel complexity, depending on whether the asymptotic sets are separated or not. If they are separated, then $\tau$ is a $G_\delta$ set. And, roughly speaking, all the consequences derived from the fact that $\sigma$ is weakly Ramsey (or almost Ramsey in the case of $c_0$) will follow from the fact that $\tau$ is weakly Ramsey (respectively almost Ramsey in the case of $c_0$).

A further application of our coding technique for decreasing the topological complexity of a set of block sequences, is to show that it is not possible to prove that the class of weakly Ramsey sets is closed under complements. More precisely, we show that there is a coanalytic set which fails to have the weakly Ramsey property. As it is shown in the last Section, in order to provide such a set it is necessary to use some extra set-theoretical assumptions. A similar result holds for the almost Ramsey property in the case of $c_0$.

## 2. Basic definitions and results

**Definition 1.** Given a finite block sequence $s = ((x_0, \lambda_0), \dots, (x_k, \lambda_k))$ (which can be empty) and an infinite one $A = ((a_n, \mu_n))_n$, let $[s; A]$ be the set of block sequences $s^\frown(b_n)_n = ((x_0, \lambda_0), \dots, (x_k, \lambda_k), (b_0, \rho_0), (b_1, \rho_1), \dots)$ such that $x_k < b_0$ and $b_n$ is in the unit sphere $S(A)$ of $A$ for every $n$. Notice that $[\emptyset; A] = [A]$. In other words, $[s, A]$ is the set of all block sequences that result from the "concatenation" of $s$ with a block sequence of elements of $A$. $lh(s)$ denotes the length of $s$, i.e., $lh(s) = k+1$ if $s = ((x_0, \lambda_0), \dots, (x_k, \lambda_k))$. Given a finite or infinite block sequence $\alpha = ((x_n, \lambda_n))_{n \le N}$ $(N \le \infty)$ let $\min \alpha = x_0$, and $A \setminus s = ((a_n, \lambda_n))_{n \ge n_0}$ where $n_0$ is the minimal $n$ such that $a_n > x_k$.

Let the *D-topology* on $\Sigma(E)$ be that with basic open sets $[((x_0, \lambda_0), \dots, (x_k, \lambda_k)); E] = \{((x_0, \lambda_0), \dots, (x_k, \lambda_k))^\frown((a_n, \mu_n))_n \ : \ a_0 > x_k\}$. Given a finite block sequence $s$, let $c_s : \Sigma(E) \to \Sigma(E)$ be defined by $c_s(A) = s^\frown(A \setminus s)$. Notice that $c_s$ is continuous. For $\sigma \subseteq \Sigma(E)$, let $\sigma^s = c_s^{-1}\sigma$. The *N-topology* on $\Sigma(E)$ is the the topology inherited from $(S(E) \times [0,1])^{\mathbb{N}}$.

Given a set of block sequences $\sigma$ and $s < A$, we say that $\sigma$ is *(strategically) large for* $[s; A]$ iff $c_s^{-1}\sigma$ is (strategically) large for $[A]$. If $\Phi$ is an strategy for Player $II$ and $(Y_n)_n$ is an infinite run for player $I$ we denote by $\Phi * (Y_n)_n$ the block sequence

$$(\Phi(Y_0), \Phi((\Phi(Y_0)), Y_1), \Phi((\Phi(Y_0), \Phi((\Phi(Y_0)), Y_1)), Y_2), \dots)$$

**Definition 2.** Given a block sequence $X$ we denote by $\mathcal{O}_X(\sigma)$ the union of the basic $D$-open sets $[t; X]$ such that $\sigma$ is large for $[t; X]$, and by $\mathcal{O}_X^{str}(\sigma)$ the union of the basic $D$-open sets $[t; X]$ such that $\sigma$ is strategically large for $[t; X]$. For $n \in \mathbb{N}$, let $\mathcal{O}_n^{str}(\sigma)$ be the union of $[t; X]$ such that $|t| > n$ and $\sigma$ is strategically large for $[t; X]$. Notice all these sets introduced here are $D$-open subsets of $[X]$.

**Proposition 2.1.** *Let $\sigma \subseteq \Sigma$, and $\Delta > 0$. Then for every block sequence $X$,*

1. $\mathcal{O}_X(\sigma)_\Delta \subseteq \mathcal{O}_X(\sigma_\Delta)$ *and* $\mathcal{O}_X^{str}(\sigma)_\Delta \subseteq \mathcal{O}_X^{str}(\sigma_\Delta)$.
2. *For every $Y \in [X]$, if $\mathcal{O}_X^{str}(\sigma)$ is strategically large for $[Y]$, then $\sigma$ is also strategically large for $[Y]$.*

Proof. 1. is trivial. We show 2.: For suppose that for some $Y \in [X]$ $\mathcal{O}_X(\sigma)_\Delta$ is strategically large for $[Y]$, i.e., there is a winning strategy $\Phi$ for player $II$ in the game $\eth_{\mathcal{O}_X(\sigma)_\Delta}[Y]$. We sketch a winning strategy for player $II$ in the game $\eth_{\sigma_\Delta}[Y]$. Player $II$ follows $\Phi$ until he obtains some $s \in [Y]_f$

such that $\sigma$ is strategically large for $[s;X]$, and then he chooses a winning strategy $\Phi'$ in the game $\Game_\sigma[s;X]$, hence for the game $\Game_\sigma[s;Y]$. After this point he plays according to $\Phi'$ in order to reach some block sequence $Z$ such that $t^\frown Z \in \sigma$. $\square$

Recall the following result from [6].

**Proposition 2.2.** *Every $D$-open set of block sequences is weakly Ramsey.* $\square$

From this, one easily shows the following.

**Proposition 2.3.** *Suppose that $\mathcal{O}_X^{str}(\sigma)$ is large for $[X]$. Then for every $\Delta > 0$ there is some $Y \in [X]$ such that $\sigma_\Delta$ is strategically large for $[Y]$.*

Proof. If $\mathcal{O}_X^{str}(\sigma)$ is large for $[X]$, since $\mathcal{O}_X^{str}(\sigma)$ is a $D$-open subset of $[X]$, then $\mathcal{O}_X^{str}(\sigma)_\Delta$ is strategically large for $[Y]$, for some $Y \in [X]$, an by Proposition 2.1 we are done. $\square$

**Proposition 2.4.** *Let $\sigma \subseteq \Sigma(E)$. The following conditions are equivalent:*

1. *$\sigma$ is weakly Ramsey.*
2. *$\sigma \cap [X]$ is weakly Ramsey for every $X \in \Sigma(E)$.*
3. *For every block sequence $X$ there is a block subsequence $Y \in [X]$ such that $\sigma \cap [Y]$ is weakly Ramsey.*

Proof. $1 \Rightarrow 2$. Fix a block sequence $Y$ such that $\sigma \cap [X]$ is large for $[Y]$, and fix $\Delta > 0$. Then we can choose $Z \in [Y] \cap (\sigma \cap [X])$. Since $\sigma$ is large for $[Z]$, there is some $W \in [Z]$ such that $\sigma_\Delta$ is strategically large for $[W]$, or equivalently $(\sigma \cap [W])_\Delta$ is strategically large for $[W]$. But $(\sigma \cap [W])_\Delta \subseteq (\sigma \cap [X])_\Delta$, and hence $(\sigma \cap [X])_\Delta$ is strategically large for $[W]$. $2 \Rightarrow 3$ is trivial. Let us show that $3 \Rightarrow 1$: For suppose that $\sigma$ is large for $[X]$. Fix $\Delta > 0$. By assumption, we can find $Y \in [X]$ such that $\sigma \cap [Y]$ is weakly Ramsey. Since $\sigma \cap [Y]$ is large for $[Y]$, there is $Z \in [Y]$ such that $(\sigma \cap [Y])_\Delta$ is strategically large for $[W]$, hence $\sigma_\Delta$ is strategically large for $[W]$. $\square$

In the case of normalized block sequences of $c_0$ we have a stronger Ramsey-like result.

**Theorem 2.1.** [6] *Every analytic set of normalized block sequences of $c_0$ is almost-Ramsey.* $\square$

A Banach space $E$ is called *$c_0$-saturated* if every infinite dimensional closed subspace of $E$ contains a closed subspace isomorphic to $c_0$. Equivalently, every normalized block sequence $X \in \Sigma_1(E)$ contains a normalized block subsequence $(y_n)_n$ which is equivalent to the natural basis of $c_0$.

**Corollary 2.1.** *If $E$ is a $c_0$-saturated space then every analytic set of normalized block sequences of $E$ is almost-Ramsey.*

Proof. Suppose $E$ is $c_0$-saturated and $\sigma$ is an analytic subset of $\Sigma_1(E)$. Suppose that $\sigma$ is large for some $[X]$. To simplify the notation we identify a normalized block sequence $((x_n,1))_n$ with $(x_n)_n$. Since $E$ is $c_0$ we can find some block subsequence $Y=(y_n)_n$ of $X$ such that $(y_n)_n$ is $C$-equivalent to the natural basis $(v_n)_n$ of $c_0$ for some $C>0$, i.e., the natural function $h:\overline{\langle y_n\rangle}\to c_0$ extending $h(y_n)=v_n$ is an isomorphism such that $\|h\|,\|h^{-1}\|\le C$. This equivalence defines naturally a homeomorphism $H:[Y]_1\to\Sigma_1(c_0)$ defined by

$$H((z_n)_n)=\left(\frac{h(z_n)}{\|h(z_n)\|_\infty}\right)_n \tag{1}$$

Fix $\Delta=(\delta_n)_n>0$. Since $\sigma'=H(\sigma)\subseteq\Sigma_1(c_0)$ is analytic and large for $[(v_n)_n]$ we can find $W\in\Sigma_1(c_0)$ such that $[W']_1\subseteq\sigma'_{(\delta_n/C)_n}$, and hence $[H^{-1}(W)]_1\subseteq\sigma_\Delta$. □

## 3. Coding with asymptotic sets

**Definition 3.** An *asymptotic pair* of $E$ is a pair $\mathfrak{A}=\{A_0,A_1\}$ where $A_0$ and $A_1$ are disjoint asymptotic sets of $E$. We denote by $\Sigma^{\mathfrak{A}}(E)$ the set of sequences $((x_n,\lambda_n))_n$ such that $x_{2n+1}\in A_0\cup A_1$ and $\lambda_{2n+1}=1$ for every $n$. Let $\Sigma^{\mathfrak{A}}_1(E)$ be the set of block sequences $((x_n,\lambda_n))_n$ of $\Sigma^{\mathfrak{A}}(E)$ such that $x_{2n+1}\in A_1$ for infinitely many $n$'s. By definition, a sequence $((x_n,\lambda_n))_n\in\Sigma^{\mathfrak{A}}_1(E)$ naturally *codes the pair* $(((x_{2n},\lambda_{2n}))_n,(k_n)_n)$, where $\{k_n\}_n$ is the increasing enumeration of the set $\{k\,:\,x_{2k+1}\in A_1\}$. Consider the corresponding mapping $\Lambda_{\mathfrak{A}}:\Sigma^{\mathfrak{A}}_1(E)\to\Sigma(E)\times\mathcal{N}^{\uparrow}$ defined by $\Lambda_{\mathfrak{A}}(((x_n,\lambda_n))_n)=(((x_{2n},\lambda_{2n}))_n,(k_n)_n)$ for every $((x_n,\lambda_n))_n\in\Sigma^{\mathfrak{A}}_1(E)$, where $\mathcal{N}^{\uparrow}$ denotes the set of strictly increasing sequences of positive integers as a topological subspace of the Baire space $\mathcal{N}=\mathbb{N}^{\mathbb{N}}$.

Fix an asymptotic pair $\mathfrak{A}$ of $E$. It is well known that every analytic (and in particular every Borel) subset $\sigma$ of $\Sigma(E)$ is the first projection of a closed subset of $\Sigma(E)\times\mathcal{N}^{\uparrow}$. Given a subset $C\subseteq\Sigma(E)\times\mathcal{N}^{\uparrow}$ we denote the first projection of $C$ by $\sigma(C)$. Let

$$\tau_{\mathfrak{A}}(C)=\Lambda_{\mathfrak{A}}^{-1}C=\{X\in\Sigma^{\mathfrak{A}}_1(E)\,:\,\Lambda_{\mathfrak{A}}(X)\in C\} \tag{2}$$

Given a family $\mathcal{C}$ of subsets of $\Sigma(E)\times\mathcal{N}^{\uparrow}$ we denote by $\sigma(\mathcal{C})=\{\sigma(C)\,:\,C\in\mathcal{C}\}$ and $\tau_{\mathfrak{A}}(\mathcal{C})=\{\tau_{\mathfrak{A}}(C)\,:\,C\in\mathcal{C}\}$. For example, if $\mathcal{C}$ is the family of closed sets of $\Sigma(E)\times\mathcal{N}^{\uparrow}$ then $\sigma(\mathcal{C})$ is the point-class $\underset{\sim}{\Sigma}^1_1$ of analytic subsets of $\Sigma$.

**Proposition 3.1.** *For every block sequence $X$ there is some block subsequence $Y$ of $X$ such that*

$$[Y]\times\mathcal{N}^{\uparrow}\subseteq\Lambda_{\mathfrak{A}}\text{"}([X]\cap\Sigma_1^{\mathfrak{A}})=\{\Lambda_{\mathfrak{A}}(Z)\,:\,Z\in[X]\cap\Sigma_1^{\mathfrak{A}}\}\tag{3}$$

*i.e., every pair $(Z,\vec{\varepsilon})\in[Y]\times\mathcal{N}^{\uparrow}$ is coded by some block subsequence of $X$.*

Proof. Fix a block sequence $X=((x_n,\lambda_n))_n$. Let $n_0$ be the first integer $n$ such that $S(\langle x_i\rangle_{i=0}^{n-1})\cap A_0$ and $S(\langle x_i\rangle_{i=0}^{n-1})\cap A_1$ are non empty, which is well defined since $A_0$ and $A_1$ are asymptotic sets and, by definition, every element of them has finite support. Once defined $n_k$, let $n_{k+1}$ be the first integer $n>n_k$ such that $S(\langle x_i\rangle_{i=n_k+1}^{n-1})\cap A_0$ and $S(\langle x_i\rangle_{i=n_k+1}^{n-1})\cap A_1$ are non empty. Set $Y=((x_{n_k},1))_k$. We are going to show that $[Y]\times\mathcal{N}^{\uparrow}\subseteq\Lambda_{\mathfrak{A}}\text{"}[X]\cap\Sigma_1^{\mathfrak{A}}(E)$. Fix a block sequence $Z=((z_n,\lambda_n))_n\in[Y]$ and $\vec{\varepsilon}=(\varepsilon_n)_n\in\mathcal{N}^{\uparrow}$. Let $k_r$ be the minimal integer $k$ such that $z_r\in S(\langle x_{n_0},\dots,x_{n_k}\rangle)$ for every $r$. Choose

$$w_r\in\begin{cases} S(\langle x_j\rangle_{j=n_{k_r}+1}^{n_{k_r+1}-1})\cap A_1 & \text{if } r\in\{\varepsilon_n\}_n\\ S(\langle x_j\rangle_{j=n_{k_r}+1}^{n_{k_r-1}+1})\cap A_0 & \text{if } r\notin\{\varepsilon_n\}_n\end{cases}\tag{4}$$

for every $r$. Then the block sequence $W'=((z_0,\lambda_0),(w_0,1),(z_1,\lambda_1),(w_1,1),\dots)$ is in $[X]\cap\Sigma_1^{\mathfrak{A}}(E)$, and clearly $\Lambda_{\mathfrak{A}}(W')=(Z,\vec{\varepsilon})$. □

**Corollary 3.1.** *If $\sigma(C)$ is large for $[X]$ then $\tau_{\mathfrak{A}}(C)$ is also large for $[X]$.*

Proof. Suppose that $\sigma(C)$ is large for $[X]$ and fix some block subsequence $X'$ of $X$. From Proposition 3.1 there is a block subsequence $Y$ of $X'$ such that $[Y]\times\mathcal{N}^{\uparrow}\subseteq\Lambda_{\mathfrak{A}}\text{"}(\Sigma(X')\cap\Sigma_1^{\mathfrak{A}}(E))$. Since $\sigma(C)$ is large for $[X]$ there is some $Z\in[Y]\cap\sigma(C)$. Fix $\vec{\varepsilon}\in\mathcal{N}^{\uparrow}$ such that $(Z,\vec{\varepsilon})\in C$ and choose $W\in\Sigma(X')\cap\Sigma_1^{\mathfrak{A}}(E)$ such that $\Lambda_{\mathfrak{A}}(W)=(Z,\vec{\varepsilon})$. Clearly $W\in\tau_{\mathfrak{A}}(C)\cap\Sigma(X')$. □

**Proposition 3.2.** *If $\tau_{\mathfrak{A}}(C)_\Delta$ is strategically large for $[X]$ then $\sigma(C)_\Delta$ is also strategically large for $[X]$.*

Proof. Suppose that there is some $X$ such that Player $II$ has a winning strategy $\Phi$ for the game $\Game_{(\tau_{\mathfrak{A}}(C))_\Delta}[X]$. Let us describe a winning strategy $\Phi'$ for Player $II$ for the game $\Game_{\sigma(C)_\Delta}[X]$: Start the game with Player $I$ choosing $X_0\in[X]$. Then Player $II$ splits $X_0$ into two subsequences $Y_0$ and $Z_0$ and he picks $(y_0,\lambda_0)=\Phi'(X_0)=\Phi(Y_0)$. Suppose that the next choice of Player $I$ is $X_1\in[X]$. Then player $II$ splits $X_1$ into two subsequences $Y_1$ and $Z_1$ and he chooses $(y_1,\lambda_1)=\Phi'(X_0,X_1)=\Phi(Y_0,Z_0,Y_1)$, and so on. At the end of the game the block sequence

$$((y_0,\lambda_0),(y_1,\lambda_1),\dots)=\Phi'*(Y_n)_n\in(\sigma(C))_\Delta\tag{5}$$

since

$$((y_0,\lambda_0),(z_0,1),(y_1,\lambda_1),(z_1,1),\dots) = \Phi * (Y_0,Z_0,Y_1,Z_1,\dots) \in (\tau_{\mathfrak{A}}(C))_\Delta \quad (6)$$

□

Let $\mathcal{G}(E)$ be the family of weakly Ramsey subsets of $\Sigma(E)$, and let $\mathcal{G}_a(E)$ be the family of almost Ramsey subsets of $\Sigma_1(E)$.

**Corollary 3.2.** *Fix a family $\mathcal{C}$ of subsets of $\Sigma(E)\times\mathcal{N}^{\uparrow}$. Then $\tau_{\mathfrak{A}}(\mathcal{C}) \subseteq \mathcal{G}(E)$ implies that $\sigma(\mathcal{C}) \subseteq \mathcal{G}(E)$, where $\mathcal{G}(E)$ denotes the family of weakly Ramsey subsets of $\Sigma(E)$.* □

Remark 3.1. In the case of the property of being almost Ramsey for a $c_0$-saturated space $E$ we have the analogous result showing that $\tau_{\mathfrak{A}}(\mathcal{C}) \subseteq \mathcal{G}_a(E)$ implies $\sigma(\mathcal{C}) \subseteq \mathcal{G}_a(E)$. Then reason is that if $\tau_{\mathfrak{A}}(C)_\Delta$ is very large for $[(x_n)_n]_1$, then $\sigma(C)_\Delta$ is also very large for $[(x_{2n})_n]_1$.

Next we compute the Borel complexity of the set $\Sigma_1^{\mathfrak{A}}(E)$ and the mapping $\Lambda_{\mathfrak{A}}$ depending on the complexity of $\mathfrak{A}$. We will not discuss the situation for arbitrary asymptotic pairs but only for two particular cases.

**Definition 4.** Let $\mathfrak{A} = \{A_0, A_1\}$ be an asymptotic pair of $E$. Let $\delta(\mathfrak{A}) = d(A_0, A_1) = \inf\{\|a_0 - a_1\| : a_0 \in A_0, a_1 \in A_1\}$. $\mathfrak{A}$ is called *discrete* if $A_0$ and $A_1$ are $F_\sigma$ subsets of $S(E)$, i.e. countable unions of closed sets of $S(E)$, and $\mathfrak{A}$ is called *separated* if (a) $\delta(\mathfrak{A}) > 0$ and (b) $A_0$ and $A_1$ are closed subsets of $S(\langle e_n\rangle_n)$, i.e., if both are the intersection of a closed set of $E$ with $S(\langle e_n\rangle_n)$.

Remark 3.2. 1. As it was discussed in the Introduction, discrete asymptotic pairs always exist: Let

$$A_i = \{x \in S(\langle e_n\rangle_n) : (-1)^i e^*_{\min\operatorname{supp} x}(x) > 0\}$$

for $i = 0, 1$. $A_0$ and $A_1$ are $F_\sigma$ sets since

$$A_i = \bigcup_{L\in\mathbb{N},N\in\mathbb{N}} \{x \in S(E) : \text{for every l¡L} e^*_l(x) = 0 \text{ and } (-1)^i e^*_L(x) \geq \frac{1}{N}\} \quad (7)$$

which is clearly a countable union of closed sets. Separated asymptotic pairs will exist if $E$ is not $c_0$-saturated.

2. If the space $E$ does not contain an isomorphic copy of $c_0$, then for every block sequence $X \in \Sigma(E)$ there is a block subsequence $Y$ of $X$ with a separated asymptotic pair of $Y$ (see Theorem 13.17 in [3]).

**Proposition 3.3.**

1. *Suppose that $\mathfrak{A}$ is a discrete asymptotic pair. Then $\Sigma_1^{\mathfrak{A}}(E)$ is a $F_{\sigma\delta}$-set and $\Lambda_{\mathfrak{A}}$ is a Baire class 1 function (i.e., the pre-image by $\Lambda_{\mathfrak{A}}$ of an open set is a countable union of closed sets). Hence $\tau_{\mathfrak{A}}(C)$ is a $F_{\sigma\delta}$ set for every $C \subseteq \Sigma(E) \times \mathcal{N}^{\uparrow}$.*
2. *Suppose that $\mathfrak{A}$ is a separated asymptotic pair. Then $\Sigma_1^{\mathfrak{A}}(E)$ is a $G_\delta$ subset of $\Sigma(E)$ and $\Lambda_{\mathfrak{A}}$ is continuous. Hence $\tau_{\mathfrak{A}}(C)$ is a $G_\delta$ set for every $C \subseteq \Sigma(E) \times \mathcal{N}^{\uparrow}$. Moreover $\tau_{\mathfrak{A}}(C)$ is the intersection of a $\Sigma_1^{\mathfrak{A}}(E)$ and a N-closed subset of $\Sigma(E)$.*

Proof. Notice that a block sequence $((x_n, \lambda_n))_n \in \Sigma_1^{\mathfrak{A}}(E)$ iff $x_{2n+1} \in A_0 \cup A_1$, $\lambda_{2n+1} = 1$ for every $n$, and $x_{2n+1} \in A_1$ for infinitely many $n$. Clearly this implies that $\Sigma_1^{\mathfrak{A}}(E)$ is an $F_{\sigma\delta}$-set if $\mathfrak{A}$ is a discrete asymptotic pair, and it is a $G_\delta$ set provided that $\mathfrak{A}$ is a separated asymptotic pair.

Fix a basic open set $U = B(((z_0, \lambda_0), \dots, (z_n, \lambda_n)), \varepsilon) \times \langle k_0, \dots, k_n\rangle$ of $\Sigma(E) \times \mathcal{N}^{\uparrow}$, where

$$B(((z_0, \lambda_0), \dots,(z_n, \lambda_n)), \varepsilon) = \{((x_i, \mu_i))_i \in \Sigma(E) : \max\{\|x_i - z_i\|, |\lambda_i - \mu_i|\} < \varepsilon \text{ for every } i = 0, \dots, n\}$$

and $\langle k_0, \dots, k_n\rangle$ is the set of all sequences $(\alpha_i)_i \in \mathcal{N}^{\uparrow}$ such that $\alpha_i = k_i$, for every $i = 0, \dots, n$. Then

$$\Lambda_{\mathfrak{A}}^{-1} U = \Sigma_1^{\mathfrak{A}}(E) \cap \{((x_i, \mu_i))_i \in \Sigma(E) : \text{for every } i \leq n/2 \ \max\{\|x_{2i} - z_i\|, |\mu_{2i} - \lambda_i|\} < \varepsilon \text{and } x_{2i+1} \in A_1 \text{ iff } i \in \{k_0, \dots, k_n\}\}$$

From this one can easily prove the desired conclusions. □

Recall that subsets of Polish spaces can be classified according to their topological complexity. This yields the so-called Projective (or Lusin) hierarchy of pointclasses (see for example [7]). We shall use the following standard notation: $\utilde{\Sigma}_1^1$ is the class of analytic sets, i.e., the continuous images of Borel sets. $\utilde{\Pi}_1^1$ is the class of coanalytic sets, i.e., the complements of analytic sets. $\utilde{\Sigma}_{n+1}^1$ is the class of the continuous images of $\utilde{\Pi}_n^1$ sets, and $\utilde{\Pi}_{n+1}^1$ is the class of complements of $\utilde{\Sigma}_{n+1}^1$ sets.

The following proposition provides direct proofs that certain projective subsets of $\Sigma(E)$ are weakly Ramsey, assuming that so are much simpler sets.

**Proposition 3.4.**

1. *If all $F_{\sigma\delta}$ sets are weakly Ramsey then so are all analytic subsets of $\Sigma(E)$. Moreover, if $E$ does not contain $c_0$, then if all $G_\delta$ subsets of $\Sigma(E)$ are weakly Ramsey, so are all analytic subsets of $\Sigma(E)$.*

2. *If every coanalytic subset of* $\Sigma(E)$ *is weakly Ramsey, then so is every* $\underset{\sim}{\Sigma}^1_2$ *subset* $\Sigma(E)$*. More generally, for every* $n \geq 1$*, if every* $\underset{\sim}{\Pi}^1_n$ *subset of* $\Sigma(E)$ *is weakly Ramsey, then so is every* $\underset{\sim}{\Sigma}^1_{n+1}$ *subset of* $\Sigma(E)$*.*
3. *The corresponding results for the almost Ramsey property of* $c_0$*-saturated spaces are also true.*

Proof. 1. Recall that the class of all analytic sets of $\Sigma(E)$ is exactly $\sigma(\mathcal{C}_0)$ where $\mathcal{C}_0$ denotes the class of closed subsets of $\Sigma(E) \times \mathcal{N}^{\uparrow}$. Fix a discrete asymptotic pair $\mathfrak{A}$ of $E$. Notice that by Proposition 3.3 every element of $\tau_{\mathfrak{A}}(\mathcal{C}_0)$ is a $F_{\sigma\delta}$ set. Corollary 3.2 gives the desired result.

Suppose now that $E$ does not contain an isomorphic copy of $c_0$, and fix an analytic set $\sigma = \sigma(C) \subseteq \Sigma(E)$ where $C \subseteq \Sigma(E) \times \mathcal{N}^{\uparrow}$ is closed. By Proposition 2.4, $\sigma(C)$ is weakly Ramsey iff every block sequence $X$ as a block subsequence $Y$ such that $\sigma \cap [Y]$ is weakly Ramsey. Fix a block sequence $X$ and let $Y$ be a block subsequence of $X$ with a separated asymptotic pair $\mathcal{U} = \{A_0, A_1\}$ of $Y$. Notice that $\sigma \cap [Y] = \sigma(C_Y)$, where $C_Y = C \cap ([Y] \times \mathcal{N}^{\uparrow})$ is a closed set. Then $\tau_{\mathfrak{A}}(C_Y) \subseteq [Y]$ is a $G_\delta$ set of $[Y]$ and hence of $\Sigma(E)$. By our assumption $\tau_{\mathfrak{A}}(C)$ is weakly Ramsey and by Corollary 3.2 $\sigma \cap [Y] = \sigma(C_Y)$ is also weakly Ramsey.

For 2. we use the fact that the class $\underset{\sim}{\Sigma}^1_{n+1}$ is just the class $\sigma(\underset{\sim}{\Pi}^1_n)$, and that $\tau_{\mathfrak{A}}(\underset{\sim}{\Pi}^1_n) \subseteq \underset{\sim}{\Pi}^1_n$. □

Remark 3.3. Let us make a metamathematical comment at this point. Typically, Theorem 1.1 is used as follows. We consider two properties $P$ and $Q$ of subspaces of $E$, characterized by a set $\sigma$ of block sequences, a family $\mathcal{F}$ of infinite sequences $(Y_n)_n$ of block sequences and $\Delta > 0$ so that:

1. If $X = ((x_n, \lambda_n))_n$ is such that $[X] \cap \sigma = \emptyset$, then $(x_n)_n$ has the property $P$.
2. If $X = ((x_n, \lambda_n))_n$ is such that for every $(Y_n)_n \in \mathcal{F}$ with each $Y_n \in [X]$ there exists a block sequence $((y_n, \lambda_n))_n$ with (a) $((y_n, \lambda_n))_n \in \sigma_\Delta$ and (b) $y_n \in S(Y_n)$ for every $n$, then $(x_n)_n$ has the property Q.

It is clear that Gowers' Theorem shows that if $\sigma$ is analytic then every Banach space $E$ has a subspace $X$ with the property $P$ or with the property $Q$.

Let $C$ be a closed subset of $\Sigma(E) \times \mathcal{N}^{\uparrow}$ such that $\sigma = \sigma(C)$, and let $\tau_{\mathfrak{A}} = \tau_{\mathfrak{A}}(C)$. Then Gowers' Theorem for $\tau_{\mathfrak{A}}$ also shows that there must be some $X \in \Sigma(E)$ with the property $P$ or with the property $Q$: Indeed, if $\tau_{\mathfrak{A}} \cap [((x_n, \lambda_n))_n] = \emptyset$ then by Corollary 3.1 $\sigma \cap [((x_n, \lambda_n))_n] = \emptyset$ which implies, by 1., that $(x_n)_n$ has the property $P$. Otherwise, by Gowers' Theorem for $\tau_{\mathfrak{A}}$, we can find some $X = ((x_n, \lambda_n))_n \in \Sigma(E)$ and some winning strategy $\Phi$ for Player $II$ in the game $\eth_{(\tau_{\mathfrak{A}})_\Delta}[X]$. Now given $(Y_n)_n \in \mathcal{F}$ with each

$Y_n \in [X]$ we can split $Y_{2n}$ into two pieces $Z_n$ and $W_n$ so that

$$((y_0,\lambda_0),(z_0,\mu_0),(y_1,\lambda_1),(z_1,\mu_1),\dots) = \Phi * (Z_0,W_0,Z_1,W_1,\dots) \in (\tau_{\mathfrak{A}})_\Delta$$

hence $((y_n,\lambda_n))_n \in \sigma_\Delta$ and $y_n \in S(Z_n) \subseteq S(Y_n)$ for every $n$. By 2. this implies that $(x_n)_n$ has the property $Q$.

Moreover, in general we may assume that $\tau_{\mathfrak{A}}(C)$ is a $F_{\sigma\delta}$ set. Further, if $E$ is not $c_0$-saturated then we may even assume that $\tau_{\mathfrak{A}}(C)$ is a $G_\delta$-set: First we find $X \in \Sigma(E)$ such that every subspace of $X$ is not isomorphic to $c_0$, hence there is some block subsequence $Y$ of $X$ with a separated asymptotic pair $\mathfrak{A} = \{A_0, A_1\}$ of $Y$. Next, we replace $E$ by $Y$ and we are done since $\tau_{\mathfrak{A}}(C \cap [Y] \times \mathcal{N}^\uparrow)$ is a $G_\delta$ set.

## 4. Another proof that analytic sets are weakly Ramsey

We present an alternative proof of the fact that all analytic sets are weakly Ramsey for a space $E$ not containing $c_0$, which is based in the fact that every block sequence of $E$ has a block subsequence with a separated asymptotic pair. More precisely, we will show rather directly that if $\mathfrak{A}$ is a separated asymptotic pair and $\tau$ is an N-closed subset of $\Sigma(E)$, then $\tau \cap \Sigma_1^{\mathfrak{A}}(E)$ is weakly Ramsey. But if $C \subseteq \Sigma(E) \times \mathcal{N}^\uparrow$ is closed, then $\tau_{\mathfrak{A}}(C)$ is one these intersections, hence it follows that $\sigma(C)$ is weakly Ramsey.

Let us point out that the way in which it is usually proved in infinite Ramsey theory that a $G_\delta$ set $\sigma$ is Ramsey is to find a set $A$ for which $\sigma$ restricted to $[A]$ is clopen. However, this approach does not work here, mainly because in general $(\bigcap_n \sigma_n)_\Delta$ and $\bigcap_n (\sigma_n)_\Delta$ are different (see Remark 6.1).

**Definition 5.** Fix an asymptotic pair $\mathfrak{A} = \{A_0, A_1\}$ of some $X \in \Sigma(E)$. Let $\Sigma_{1,f}^{\mathfrak{A}}(E)$ be the set of finite block sequences $t = ((x_0,\lambda_0),\dots,(x_n,\lambda_n))$ such that (a) $x_{2i+1} \in A_0 \cup A_1$ and $\lambda_{2i+1} = 1$ for every $i \le n/2$ and (b) $x_{2i+1} \in A_1$ for some $i \le n/2$. We say that $t \in \Sigma_{1,f}^{\mathfrak{A}}(E)^{<\infty}$ is a $\mathfrak{A}$-*admissible* sequence. For $\delta > 0$ we say that $N \subseteq \Sigma_{1,f}^{\mathfrak{A}}(E)$ is a $(\mathfrak{A},\delta)$-*admissible net* iff $N$ is countable and closed under concatenation, and for every $((x_0,\lambda_0),\dots,(x_n,\lambda_n)) \in \Sigma_{1,f}^{\mathfrak{A}}(E)$ there is some $((\widetilde{x}_0,\mu_0),\dots,(\widetilde{x}_n,\mu_n)) \in N$ such that (a) $\max\{\|x_i - \widetilde{x}_i\|, |\lambda_i - \mu_i|\} \le \delta$ and $\operatorname{supp} x_i = \operatorname{supp} \widetilde{x}_i$ for every $i \le n$, and (b) $x_{2i+1} \in A_1$ iff $\widetilde{x}_{2i+1} \in A_1$ for every $i \le n/2$. $N$ is called a $\mathfrak{A}$-*admissible net* if $N$ is a $(\mathfrak{A},\delta)$-*admissible net* for every $\delta > 0$.

For a given $\mathfrak{A}$-admissible net $N$ of $X$, $\sigma$, and $\Delta > 0$, let

$$\mathcal{O}_X^N(\sigma,\Delta) = \bigcup\{[t;X] \,:\, d(t,\widetilde{t}) \le \Delta \text{ for some } \widetilde{t} \in N \text{ such that } \sigma \text{ is large for } [\widetilde{t};X]\} \quad (8)$$

Remark 4.1. 1. $\mathfrak{A}$-admissible nets exist for every $\mathfrak{A}$: Suppose that the Schauder basis $(e_n)_n$ has basic constant $C$. Fix $n$ and a finite subset $F \subseteq \mathbb{N}$, and let $M$ be such that $\#F/M \le \min\{2^{-n}, \delta(\mathfrak{A})\}$. Consider a covering $\{I_0, ..., I_L\}$ of $[-2C, 2C]$ with each $I_i$ an interval of diameter $\le 1/M$. Now for each $f : F \to \{0, ..., L\}$ such that there is some normalized vector $x$ with supp $x = F$ and such that $e_n^*(x) \in I_{f(n)}$ for every $n \in F$ we pick a normalized $x_f^{(n)}$ with this property. In addition, if there exists such an $x$ as above in $A_\varepsilon$ ($\varepsilon = 0, 1$) then we choose $x_f^{(n)} \in A_\varepsilon$. Notice that it is not possible to have two $x \in A_0$, and $x' \in A_1$ as above (because $\#F/M \le \min\{2^{-n}, \delta(\mathfrak{A})\}$). In this manner the set $\bigcup_{F \subseteq \mathbb{N} \text{ finite}} \{x_f^{(n)} : f \in F\}$ is a countable $2^{-n}$-cover of $S(\langle e_n \rangle_n)$ with the extra properties that for every $x \in S(\langle e_n \rangle_n)$ there is $x' \in \bigcup_{F \subseteq \mathbb{N} \text{ finite}} \{x_f^{(n)} : f \in F\}$ such that $\|x - x'\| \le 2^{-n}$, supp $x =$ supp $x'$, and if in addition $x \in A_\varepsilon$ ($\varepsilon = 0, 1$), then $x' \in A_\varepsilon$. From here it is easy to build a $\mathfrak{A}$-admissible net.
2. If $N$ is a $\mathfrak{A}$-admissible net, then for every $\delta$ and every $((x_0, \lambda_0), \dots, (x_n, \lambda_n))$ there is a finite $(\mathfrak{A}, \delta)$-admissible net $N' \subseteq N$ of the set of $\mathfrak{A}$-admissible block subsequences of $(x_1, \dots, x_n)$. This fact has a proof similar than 1.

**Lemma 4.1.** *Let $\mathfrak{A}$ be an asymptotic pair of some $X$, and let $N$ be a $\mathfrak{A}$-admissible net. Suppose that $\sigma = \tau \cap \Sigma_1^{\mathfrak{A}}(E)$ is large for $[s; X]$. Then for every $\Delta > 0$ and every $Y \in [X]$ there is some $\mathfrak{A}$-admissible block subsequence $t$ of $Y \setminus s$, $\widetilde{t} \in N$, and some $Z \in [Y]$ such that (a) $d(t, \widetilde{t}) \le \Delta/2$ and (b) $\sigma_{\Delta/2} \cap \Sigma_1^{\mathfrak{A}}(E)$ is large for $[s^\frown \widetilde{t}; Z]$.*

Proof. For suppose not. Let $\sigma = \tau \cap \Sigma_1^{\mathfrak{A}}(E)$ be large for $[s; X]$, fix $\Delta > 0$ and $Y \in [X]$, and suppose that for every $\mathfrak{A}$-admissible block subsequence $t$ of $Y$ there is some $Z$ such that for every $\widetilde{t} \in N$, if $d(t, \widetilde{t}) \le \Delta/2$ then there is some $W \in [Z]$ such that $\sigma_{\Delta/2} \cap \Sigma_1^{\mathfrak{A}}(E) \cap [s^\frown \widetilde{t}; W] = \emptyset$. This condition allows to find recursively for every $n$ a finite block sequence $s < s_n = ((y_0, 1), ..., (y_n, 1))$ and an infinite block sequence $Y_n$ such that for every $n$, (a) $Y_0 = Y$, (b) $y_n = \min Y_n$, $Y_{n+1} \in [Y_n \setminus y_n]$, and (c) for every $\widetilde{t} \in N_n$ such that there is some $\mathfrak{A}$-admissible block subsequence $t$ of $((y_0, \lambda_0), \dots, (y_n, \lambda_n))$ with $d(t, \widetilde{t}) \le \Delta/2$, we have $\sigma_{\Delta/2} \cap \Sigma_1^{\mathfrak{A}}(E) \cap [s^\frown \widetilde{t}; Y_{n+1}] = \emptyset$, where $N_n \subseteq N$ is a finite $(\mathfrak{A}, \delta_n/2)$- admissible net of the set of $\mathfrak{A}$-admissible block subsequences of $((y_0, \lambda_0), \dots, (y_n, \lambda_n))$.

Then, setting $Y_\infty = (\min Y_n)_n \in [Y]$, we obtain that $\sigma \cap [s; Y_\infty] = \emptyset$, since otherwise let $s^\frown((w_n, \lambda_n))_n \in \sigma \cap [s; Y_\infty]$, and let $k$ be the minimal integer $m$ such that $((w_i, \lambda_i))_{i=0}^m$ is $\mathfrak{A}$-admissible, and let $n$ be the minimal integer such that $((w_i, \lambda_i))_{i=0}^m$ is a block subsequence of $((y_i, 1))_{i=0}^n$. Let $\widetilde{t} \in N_n \subseteq N$ be such that $d(t, ((w_i, \lambda_i))_{i=0}^m) \le \Delta/2$. Then $\widetilde{W} = s^\frown \widetilde{t}^\frown ((w_i, \lambda_i))_{i>m} \in$

$\Sigma_1^{\mathfrak{A}}(E)$, $((w_i,\lambda_i))_{i>m}\in[Y_{n+1}]$, and clearly $\widetilde{W}\in\sigma_{\Delta/2}$, a contradiction with the properties of $((y_i,1))_i$. □

Given two block sequences $X$ and $Y=((y_n,\mu_n))_n$, we write $Y\in^*[X]$ iff $((y_n,\mu_n))_{n>k}\in[X]$ for some $k$.

**Lemma 4.2.** 1. *If $\sigma$ is (strategically) large for $[X]$ then $\sigma$ is (strategically) large for $[Y]$, for every $Y\in^*[X]$.*

2. *For every family $\{\sigma_n\}_n$ of sets of block sequences and every block sequence $X$ there is some $X_0\in[X]$ such that for every $n$ and every $Y\in[X_0]$, if $\sigma_n$ is (strategically) large for $[Y]$, then $\sigma_n$ is (strategically) large for $[X_0]$.*

Proof. 1. is trivial. Let us show 2.: Fix a family $\{\sigma_n\}_n$ of sets of block sequences, and $X$. We can find inductively a sequence $(X_n)_n$ of block sequences such that (a) $X_0=X$, $X_{n+1}$ is a block subsequence of $X_n\setminus\min X_n$ and (b) if $\sigma_n$ is (strategically) large for $[Z]$ for some $Z\in[X_n]$, then $\sigma_n$ is (strategically) large for $[X_{n+1}]$. Then the block sequence $Y=(\min X_n)_n$ satisfies the desired result. We find this sequence $(X_n)_n$ inductively. Suppose defined $X_n$. If there is some block subsequence $Z$ of $X_n$ such that $\sigma_n$ is (strategically) large for $[Z]$, then let $X_{n+1}\in[X_n\setminus\min X_n]$ be one of these. If not, let $X_{n+1}=X_n\setminus\min X_n$. □

**Lemma 4.3.** *Let $\tau$ be a $N$-closed subset of $\Sigma(E)$, let $\mathfrak{A}$ be a closed asymptotic pair of $X$ and suppose that $\sigma=\tau\cap\Sigma_1^{\mathfrak{A}}(E)$ is large for $[X]$. Then for every $\Delta>0$ there exists $Y\in[X]$ such that $\sigma_\Delta$ is strategically large for $[Y]$.*

Proof. Fix a $\mathfrak{A}$-admissible net $N$. We assume that $\Delta=(\delta_n)_n<\delta(\mathfrak{A})$. Set $\Delta_n=(2^n-1)/2^{n+1}\Delta$ for every $n$. Let $X_0\in[X]$ be the result of the application of 2. of Lemma 4.2 to $X$ and the family of block sequences $\{(\tau_n)^s\,:\,n\in\mathbb{N},\,s\in N\}$, and $X_1\in[X_0]$ the result of the same application to $X_0$ and the family

$$\{\mathcal{O}^N_{X_0}\left((\tau_n)^s,\left(\frac{\delta_k}{2}\right)_{k>lh(s)}\right)\,:\,n\in\mathbb{N},\,s\in N\}$$

where $\tau_n=\sigma_{\Delta_n}\cap\Sigma_1^{\mathfrak{A}}(E)$ for every $n$. Notice that $(\tau_n)_{(\Delta_{n+1}-\Delta_n)}\cap\Sigma_1^{\mathfrak{A}}(E)\subseteq\tau_{n+1}$ and $\tau_n\subseteq\sigma_{\Delta/2}$ for every $n$. We claim that $\sigma_\Delta$ is strategically large for $[X_1]$. We proceed to sketch a winning strategy for player $II$ in the game for $\eth_{\sigma_\Delta}[X_1]$: By hypothesis, $\sigma^{\emptyset}_{\Delta_0}$ is large for $[X_0]$. By Lemma 4.1, for every $Y\in[X_0]$ there is some block subsequence $t$ of $Y$ and some $Z\in[X_0]$ such that $\tau_1$ is large for $[\widetilde{t};Z]$ for some $\widetilde{t}\in N$ with $d(\widetilde{t},t)\le\Delta/4$. By the properties of $X_0$, this is equivalent to saying that $\tau_1$ is large for $[\widetilde{t};X_0]$. So, $\mathcal{O}^N_{X_0}(\tau_1,\Delta/4)$ is large for $[X_0]$, and hence there is some $Y\in[X_1]$ such that

$(\mathcal{O}^N_{X_0}(\tau_1,\Delta/4))_{\Delta/4}$ is strategically large for $[Y]$. Since $(\mathcal{O}^N_{X_0}(\tau_1,\Delta/4))_{\Delta/4}\subseteq(\mathcal{O}^N_{X_0}(\tau_1,\Delta/2))$, by the properties of $X_1$ we can fix a winning strategy $\Phi_1$ for player $II$ for the game $\Game_{\mathcal{O}^N_{X_0}(\tau_1,\Delta/2)}[X_1]$. Player $II$ will follow $\Phi_1$ until he reaches some $t_1$ for which there is $s_1\in N$ such that (a) $d(t_1,s_1)\le\Delta/2$ and (b) $\sigma_{\Delta_1}$ is strategically large for $[s_1;X_0]$. By Lemma 4.1, for every $Y\in[X_0]$ there is some block subsequence $t>s_1$ of $Y$ and some $Z\in[X_0]$ such that $(\tau_1)_{(\Delta_2-\Delta_1)\setminus s_1}\cap\Sigma^{\mathfrak{A}}_1(E)$ is large for $[s_1{}^\frown\widetilde{t};Z]$ for some $\widetilde{t}\in N$ satisfying $\widetilde{t}>s_1$ and $d(\widetilde{t},t)\le(\delta_n/4)_{n>lh(s_1)}$. Since $(\tau_1)_{(\Delta_2-\Delta_1)\setminus s_1}\cap\Sigma^{\mathfrak{A}}_1(E)\subseteq\tau_2$ and by the properties of $X_0$, we have that $\mathcal{O}^N_{X_0}(\tau_2^{s_1},(\delta_n/4)_{n>lh(s_1)})$ is large for $[X_0]$, and hence player $II$ can follow a winning strategy $\Phi_2$ for the game $\Game_{\mathcal{O}^N_{X_0}(\tau_2^{s_1},(\delta_n/2)_{n>lh(s_1)})}[X_1]$ until he reaches $t_2>t_1,s_1$ such that there exists $s_2\in N$ such that (a) $s_2>s_1$, $d(t_2,s_2)\le(\delta_n/2)_{n>lh(s_1)}$, and (b) $\tau_2$ is large for $[s_1{}^\frown s_2;X_0]$. And so on. In this way, at the end of the game Player $II$ has produced the block sequence $Z=t_1{}^\frown t_2{}^\frown\cdots{}^\frown t_n{}^\frown\cdots$ and the auxiliary block sequence $W=s_1{}^\frown s_2{}^\frown\cdots{}^\frown s_n{}^\frown\cdots$ such that (a) $d(Z,W)\le\Delta/2$, (b) $\tau_n$ is large for $[s_1{}^\frown\cdots{}^\frown s_n;X_0]$. Notice that (b) implies that $\sigma_{\Delta/2}$ is large for $[s_1{}^\frown\cdots{}^\frown s_n;X_0]$ for every $n$, which implies, using that $\tau\cap\Sigma^{\mathfrak{A}}(E)$ is $N$-closed, that there is some $\widetilde{W}\in\tau\cap\Sigma^{\mathfrak{A}}(E)$ such that $d(W,\widetilde{W})\le\Delta/2$. Since $\delta_n/2\le\delta(\mathfrak{A})$ for every $n$, we have that $\widetilde{W}\in\Sigma^{\mathfrak{A}}_1(E)$, hence $W\in\sigma_{\Delta/2}$, and then $Z\in\sigma_\Delta$ as desired. □

This gives another proof of Gowers' Theorem 2.1 for spaces not containing $c_0$.

**Corollary 4.1.** *Suppose that $E$ does not contain $c_0$. Then every analytic set is weakly Ramsey.*

Proof. Let $\sigma$ be an analytic set of $\Sigma(E)$, for $E$ not containing $c_0$, and fix $\Delta>0$. Suppose that $\sigma$ is large for $[X]$. Choose $Y\in[X]$ and a closed asymptotic pair $\mathfrak{A}$ of $Y$. Let $C\subseteq[Y]\times\mathcal{N}^{\uparrow}$ be a closed set such that $\sigma\cap[Y]=\sigma(C)$. We know that $\tau_{\mathfrak{A}}(C)=\tau\cap\Sigma^{\mathfrak{A}}_1(E)$ for some N-closed set $\tau$ of $[Y]$ and that $\tau_{\mathfrak{A}}(C)$ is large for $[Y]$. By Lemma 4.3, there is some $Z\in[Y]$ such that $\tau_{\mathfrak{A}}(C)_\Delta$ is strategically large for $[Z]$, and hence $\sigma_\Delta$ is strategically large for $[Z]$ as desired. □

## 5. Normalized block sequences

We will now study the relationship between the weakly Ramsey property for sets of arbitrary block sequences $((x_n,\lambda_n))_n$, as we have being considering up to this point, and the weakly Ramsey property for sets of normalized block sequences $((x_n,1))_n$, as it was used in [1]. For the latter, in the game $\Game_\sigma[X]$ player $II$ only chooses normalized vectors, as opposed to pairs consisting of a normalized vector and a scalar. Somehow, the notion of weakly

Ramsey for sets of arbitrary block sequences $((x_n, \lambda_n))_n$ is the "parameterized" version of the normalized notion from [1]. In can be easily shown that for sets of normalized block sequences both notions are equivalent. But in principle, the notion of weakly Ramsey set from [1] is weaker. Nevertheless, we will show that for spaces not containing $c_0$ they are equivalent.

**Proposition 5.1.** *Suppose that $\mathfrak{A}$ is a separated asymptotic pair of $E$, and suppose that $\Delta = (\delta_n)_n > 0$ is such that $\delta_n < \delta(\mathfrak{A})$ for every $n$. Then for every subset $\sigma \subseteq \Sigma(E)$ there is a subset $\tau \subseteq \Sigma_1(E)$ such that (a) if $\sigma$ is large for $[((x_n, \lambda_n))_n]$, then $\tau$ is large for $[(x_n)_n]_1$ and (b) if $\tau_{\Delta/2}$ is strategically large for $[(x_n)_n]_1$, then $\sigma_\Delta$ is strategically large for $[(x_n, 1)_n]$. Moreover, if $\sigma$ is analytic, then so is $\tau$.*

Proof. Fix a separated asymptotic pair $\mathfrak{A} = \{A_0, A_1\}$, $0 < \Delta < \delta(\mathfrak{A})$, $\sigma \subseteq \Sigma(E)$, and $X = ((x_n, \lambda_n))_n \in \Sigma(E)$. For each $n \geq 0$ let $L_n$ be the minimal integer $l$ such that $\delta_n l \geq 1$ and let $M_n$ be the minimal integer $m$ such that $2^m \geq L_n$. Set $M_{-1} = 0$, and let $h_0^{(m)} = M_{-1} + \cdots + M_{m-1} + m + 1$ and $h_1^{(m)} = M_{-1} + \cdots + M_m + M_m + m$. Let $H_m = [h_0^{(m)}, h_1^{(m)}]$, and let $\Sigma_\Delta^{\mathfrak{A}}(E)$ be the set of normalized block sequences $(x_n)_n$ such that for every $m$ we have that $x_n \in A_0 \cup A_1$ for every $n \in H_m$. Notice that for every sequence $(x_n)_n \in \Sigma_\Delta^{\mathfrak{A}}(E)$ and every $m$, the sequence $(x_i)_{i \in H_m}$ codes a sequence in $\{0,1\}^{M_m}$, hence $(x_n)_n$ naturally codes the block sequence $\Lambda_{\mathfrak{A},\Delta}((x_n)_n) = ((x_{h_0^{(m)}-1}, \lambda((x_i)_{i \in H_m})))_m$ where

$$\lambda((x_i)_{i \in H_m}) = \left( \sum_{i \in H_m} \varepsilon_i 2^{i - \min H_m} \right) \delta_m$$

and $\varepsilon_i \in \{0,1\}$ is such that $x_i \in A_{\varepsilon_i}$. Let $\tau = \Lambda_{\mathfrak{A},\Delta/2}^{-1}(\sigma_{\Delta/2})$. The proof of (a) for $\tau$ is quite similar to the proof of Corollary 3.1. Let us show (b): Suppose that $\Phi$ is a winning strategy for player $II$ in the game $\Game_{\tau_{\Delta/2}}[X]$ for some normalized block sequence $X = (x_n)_n$. We sketch a winning strategy $\Phi'$ for player $II$ in the game $\Game_{\sigma_\Delta}[((x_n, 1))_n]$. Suppose that the game starts with Player $I$ picking $X_0 = ((x_n^{(0)}, \lambda_n^{(0)}))_n$. Let

$$(y_n^{(0)})_n = \Phi * ((x_n^{(0)})_n, (x_n^{(0)})_n, \dots) \in \tau_{\Delta/2}$$

and let $(z_n^{(0)}) \in \tau \subseteq \Sigma_{\Delta/2}^{\mathfrak{A}}(E)$ be such that $d((y_n^{(0)}, z_n^{(0)}))_n \leq \Delta/2$. Let

$$\Phi'(X_0) = (y_0^{(0)}, \lambda_0) \tag{9}$$

where $\lambda_0 = \lambda((z_i^{(0)})_{i \in H_0})$. Now suppose that Player $I$ plays $X_1 = ((x_n^{(1)}, \lambda_n^{(1)}))_n$, and let

$$(y_n^{(1)})_n = \Phi * (\overbrace{(x_n^{(0)})_n, \dots, (x_n^{(0)})_n}^{(M_0+1)}, (x_n^{(1)})_n, \dots, (x_n^{(1)})_n, ..) \in \tau_\Delta \tag{10}$$

and let $(z_n^{(1)}) \in \tau$ be such that $d((y_n^{(1)}, z_n^{(1)}))_n \leq \Delta/2$. Let

$$\Phi'((y_0^{(0)}, \lambda_0), X_1) = (y_{M_0+1}^{(1)}, \lambda_1) \tag{11}$$

where $\lambda_1 = \lambda((z_i^{(1)})_{i \in H_1})$, and so on. We show that $\Phi'$ is winning, i.e.

$$\Phi' * (X_0, X_1, \dots) = ((x_n, \lambda_n))_n \in \sigma_\Delta \tag{12}$$

Notice that

$$W = (y_0^{(0)}, \dots, y_{M_0}^{(0)}, y_{M_0+1}^{(1)}, \dots, y_{M_0+M_1+1}^{(1)}, \dots) = \tag{13}$$

$$= \Phi * (\overbrace{(x_n^{(0)})_n, \dots, (x_n^{(0)})_n}^{(M_0+1)}, \overbrace{(x_n^{(1)})_n, \dots, (x_n^{(1)})_n}^{(M_1+1)}, \dots) \in \tau_\Delta \tag{14}$$

hence we can find $Z = (z_n)_n$ such that $d(Z, W) \leq \Delta$ and $Z \in \tau$, i.e. $\Lambda_{\mathfrak{A},\Delta/2}(Z) \in \sigma_{\Delta/2}$. The proof will be finished if we show that $d(((x_n, \lambda_n))_n, \Lambda_{\mathfrak{A},\Delta/2}(Z)) \leq \Delta/2$. Notice that for every $m$ and every $i \in H_m$

$$\|z_i - z_i^{(m)}\| \leq \|z_i - y_i^{(m)}\| + \|y_i^{(m)} - z_i^{(m)}\| \leq \delta_i < \delta(\mathfrak{A}) \tag{15}$$

Hence

$$\lambda((z_i)_{i \in H_m}) = \lambda((z_i^{(m)})_{i \in H_m}) = \lambda_m \tag{16}$$

for every $m$. Notice also that for every $m$

$$\|z_{h_0^{(m)}-1} - y_{h_0^{(m)}-1}\| \leq \delta_{h_0^{(m)}-1}/2 \tag{17}$$

which implies that $d(((x_n, \lambda_n))_n, \Lambda_{\mathfrak{A},\Delta/2}(Z)) \leq \Delta/2$ since $\Delta$ is decreasing. □

**Corollary 5.1.** *Suppose that $E$ does not contain $c_0$. Then for every subset $\sigma \subseteq \Sigma(E)$ there is $\tau \subseteq \Sigma_1(E)$ such that $\sigma$ is weakly Ramsey iff $\tau$ is weakly Ramsey. Moreover, if $\sigma$ is analytic, then $\tau$ is analytic.*

Proof. This is a consequence of Proposition 2.4 and the localized version of Proposition 5.1. □

## 6. Weakly Ramsey property and complements

We give an example of a $\underset{\sim}{\Sigma}_2^1$ set (i.e. a projection of a complement of an analytic set) which is not weakly Ramsey. Since, as we show in the next section, there are standard set theoretical models in which all the sets are weakly Ramsey, we are forced to use some extra assumptions to find such a set.

**Theorem 6.1.** *If there is a good $\underset{\sim}{\Sigma}^1_2$ well ordering of the reals (see* [7]*), then there is a $\underset{\sim}{\Sigma}^1_2$ non-weakly Ramsey set of block sequences in any Banach space.*

Proof. We reproduce the example of a non-weakly Ramsey set of normalized block sequences given in [1], and we sketch how, from the assumptions of the Theorem one can make it a $\underset{\sim}{\Sigma}^1_2$ set. This is done in the context of normalized block sequences, but as we mentioned at the end of the introduction to section 5, for sets of normalized block sequences the notions of weakly Ramsey for sets of normalized block sequences and the general notion of weakly Ramsey coincide.

If there is a $\underset{\sim}{\Sigma}^1_2$ well ordering $<_w$ of $\mathbb{R}$, then every real is constructible from the parameter $a \in \mathbb{R}$ used in the definition of $<_w$ (see the Theorem of Mansfield, [7]). In particular, the Continuum Hypothesis holds. Gödel was the first to observe that $V = L$ gives a $\underset{\sim}{\Sigma}^1_2$ well-ordering of the reals, a result which was later published by Kuratowski. Thus, we can start with a good $\underset{\sim}{\Sigma}^1_2$ well-ordering $<_w$ of the set of block sequences $\Sigma(E)$ of a given Banach space $E$.

For a fixed $\Delta < 1$ we want to provide a $\underset{\sim}{\Sigma}^1_2$ set of normalized block sequences $\sigma$ such that neither Player $I$ has a winning strategy for $\sigma[Y]_1$, for any normalized $Y$ (hence, $\sigma$ is large for $[Y]_1$), nor Player II has a winning strategy for $\sigma_\Delta$. The idea is to build $\sigma$ such that any strategy for Player $I$ or $II$ is winning.

As we observed in [2] ( where the game involved in the definition of strategically large is equivalent to a game where both players pick block vectors), we can code the set of pairs $(Y, S)$, where $Y$ is a normalized block sequence and $S$ is a strategy for $I$ or $II$ in $[Y]_1$, as a Borel subset of $\mathcal{N}$. Therefore, we can also fix a good $\underset{\sim}{\Sigma}^1_2$ well-ordering $<_s$ for it.

Define for any $(Y, S)$ the normalized block sequence $X(Y, S)$ to be the $<_w$ first $X$ such that:

1. If $S$ is a strategy for player $I$ in $[Y]_1$, then $X = (x_n)_n$ is in $[Y]_1$, it is coherent with $S$ (i.e., for any $n$, $x_n \in S(x_0, \dots, x_{n-1})$), and for any $(Y'S') <_s (Y, S)$, $d(X(Y', S'), X) \nleq \Delta$.
2. If $S$ is a strategy for player $II$ in $[Y]_1$, then there is an infinite run of Player $I$ $(W_n)_n$ in $Y$ such that $X = (x_n)_n = S * (W_n)_n$ ( i.e., for any $n$, $x_n = S(W_0, \dots, W_n)$, or in other words, $X$ is the result of the full game when Player $I$ plays $(W_0, W_1, W_2, \dots)$ and for any $(Y'S') <_s (Y, S)$, $d(X(Y', S'), X) \nleq \Delta$.

In [1] we have shown that there is some $X$ satisfying this condition (notice that the set of $<_s$-predecessors of $(Y, S)$ is countable, so it is not difficult to show that one can find some $X$ satisfying the assumptions).

Let $\sigma$ be the set of block sequences $X(Y,S)$, where $S$ is a strategy for $I$. The fact that $<_s$ and $<_w$ are good $\utilde{\Sigma}^1_2$ well-orderings guarantees that $\sigma$ is a $\utilde{\Sigma}^1_2$ set.

We show that player $I$ does not have a winning strategy for $\sigma[Y]_1$, for any $Y$. Fix a pair $(Y,S)$ for $I$. Player $II$ will play to produce $X(Y,X)$, which is clearly in $\sigma$. Notice that this implies that $\sigma$ is large for $[Y]_1$, for any $Y$.

We now show that player $II$ does not have a winning strategy for $\sigma_\Delta[Y]_1$, for any $Y$: Fix a pair $(Y,S)$ for $II$. Then, by definition, Player $II$ has a run $(W_n)_n$ in $[Y]_1$ such that the result of the game $S*(W_n)_n$ is $X=X(Y,S)$. We show that $X$ is not in $\sigma_\Delta$. Fix $X'$ in $\sigma$, and let $(Y',S')$ be a pair for $I$ such that $X'=X(Y',S')$. Suppose that $(Y',S')<_s(Y,S)$. Then $X$ was chosen so that $d(X,X')\not\leq\Delta$. If $(Y,S)<_s(Y',S')$, then $X'$ was chosen so that $d(X',X)\not\leq\Delta$.

Notice that for the case of $E=c_0$, $\sigma$ is a Non-Ramsey set since it is not weakly Ramsey. $\square$

**Corollary 6.1.** *If there is a good $\utilde{\Sigma}^1_2$ well-ordering of the reals, then $\mathcal{G}$ is not closed under complements.*

Proof. Fix $E$ and a discrete asymptotic pair $\mathfrak{A}$, and let $\sigma$ be a $\utilde{\Sigma}^1_2$ set which is not weakly Ramsey. Choose a coanalytic set $C\subseteq\Sigma\times\widetilde{\mathcal{N}^\uparrow}$ such that $\sigma=\sigma(C)$. Therefore $\tau_{\mathfrak{A}}(C)=\Lambda_{\mathfrak{A}}^{-1}C$ is a coanalytic subset of $\Sigma_1^{\mathfrak{A}}$, and hence it is also a coanalytic set of $\Sigma(\mathfrak{A})$. By Corollary 3.2, $\tau(C)_{\mathfrak{A}}$ is not weakly Ramsey, but its complement $\tau_{\mathfrak{A}}(C)^c$ is analytic, hence weakly Ramsey. $\square$

Remark 6.1. 1. A set $\sigma$ is called *completely weakly Ramsey* iff for every $s<A$, and every $\Delta$, if $\sigma_\Delta$ is large for $[s;A]$, then for every $\Gamma$ there is some $B\in[A]$ such that $\sigma_{\Delta+\Gamma}$ is strategically large for $[s;B]$. Let $\mathcal{G}_c$ be the family of completely Ramsey sets. Notice that if $\sigma$ is completely Ramsey, then so are $\sigma\cap[s;A]$ and $\sigma_\Delta$ for every $s<A$ and every $\Delta>0$. It is not difficult to show that the family of completely weakly Ramsey sets is closed under countable unions. However, for intersections the situation is rather different. It is shown in [6] (Corollary 8.2) that from a non weakly Ramsey set and a separated asymptotic pair for $E$ it is possible to find two completely weakly Ramsey sets $\sigma_0$ and $\sigma_1$ such that $\sigma_0\cap\sigma_1$ is not completely weakly Ramsey.
2. It can be shown in a similar manner that the same hypothesis of Corollary 6.1 implies that the family of almost Ramsey sets of normalized block sequences of $c_0$-saturated spaces is not closed under complements.

## 7. All sets of block sequences can be weakly Ramsey

In this section we extend a result from [2] on projective sets of block sequences, using a Large-cardinal hypothesis. Recall that the Solovay model ([8]) in which all sets of reals are Lebesgue measurable is obtained by taking the constructible closure $L(\mathbb{R})$ of the reals of the Boolean-valued extension $V^{\mathcal{B}}$, where $\mathcal{B}$ is the regular-open algebra of the Tychonov product $\prod_{\alpha<\kappa} B(\alpha)$ for some inaccessible cardinal $\kappa$ (here $B(\alpha)$ is the metric space of all infinite sequences of ordinals $< \alpha$). Under a suitable Large-cardinal assumption the constructible closure $L(\mathbb{R})$ of the reals of our universe is elementarily equivalent to a Solovay model ([11]). This will be our assumption throughout this section, where we prove the following.

**Theorem 7.1.** *Under a suitable large-cardinal assumption every set of infinite block sequences that belongs to $L(\mathbb{R})$ is weakly Ramsey.*

Proof. Consider a set $\sigma$ of infinite block sequences such that $\sigma \in L(\mathbb{R})$. Then $\sigma = \{X : L(\mathbb{R}) \models \phi(X, a, \alpha)\}$ for some formula $\phi$ in the formal language of set theory and parameters $a \in \mathbb{R}$ and $\alpha$ an ordinal number. Since we may assume $L(\mathbb{R})$ is a Solovay model, there is a canonical decomposition

$$\sigma = \bigcup_{\alpha<\aleph_1} \sigma_\alpha \tag{18}$$

where each $\sigma_\alpha$ is an analytic set of block sequences. This decomposition may be described as follows: Suppose $\mathcal{B}$ is a complete Boolean algebra which is the completion of a Borel partial ordering that satisfies axiom A (see [1]). The Boolean-valued extension $L(\mathbb{R})^{V^{\mathcal{B}}}$ is still a Solovay model which has its own versions $\sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{B}}_\alpha$ $(\alpha < \aleph_1)$ of the sets $\sigma$ and $\sigma_\alpha$ $(\alpha < \aleph_1)$. Moreover,

$$\sigma^{\mathcal{B}} = \bigcup_{\alpha<\aleph_1} \sigma^{\mathcal{B}}_\alpha \tag{19}$$

We refer the reader to [4] for details. Since $L(\mathbb{R})^{V^{\mathcal{B}}}$ is obtained by forcing with a Borel partial ordering that preserves $\aleph_1$ (see [1]), it is also a Solovay model and so $L(\mathbb{R})$ and $L(\mathbb{R})^{V^{\mathcal{B}}}$ are elementarily equivalent (see [2]). Therefore, any sentence about block sequences true in $L(\mathbb{R})^{V^{\mathcal{B}}}$ will be also true in $L(\mathbb{R})$.

It is not difficult to show (see [1]) that for every $\Delta > 0$ there is some $\Gamma < \Delta$ such that if $d(X, X') \leq \Gamma$, then $[X] \subseteq [X']_\Delta$. Pick one of these $\Gamma$ for $\Delta/2$. Let $[s; [A]_{(\delta_i)_{i>lh(s)}}]$ be the set of block sequences of the form $s^\frown B$, with $B \in [A]_\Delta$. For every $\alpha < \aleph_1$, let $F_\alpha$ be the family of those $[s; [A]_\Gamma]$ such that for every finite block subsequence $t$ of $s$, either $[t; A] \cap (\sigma_\alpha)_{\Delta/2} = \emptyset$ or $(\sigma_\alpha)_\Delta$ is strategically large for $[t; A]$. Let $D_\alpha$ be the union of members of $F_\alpha$.

Using Gowers' Theorem for analytic sets, one can show that $D_\alpha \cap [s;A] \neq \emptyset$ for every $[s;A]$. Recall the poset $P = P(\Delta, Y)$ introduced in [1], there for normalized block sequences, in order to diagonalize all $D_\alpha$ ($\alpha < \aleph_1$): Elements of $P$ are $[s;A]$ in $Y$ such that $s$ is in a fixed $\Delta$-net of $[Y]$. The order in $P$ is defined by $[s;A] \leq [t;B]$ iff

1. $t$ is an initial segment of $s$,
2. $A \in [B]$, and
3. $s \setminus t \in \wr B \wr^{<\infty}_{(\delta_i)_{i>|t|}}$

where given $A$ and $\Delta > 0$, we define

$$\begin{aligned}\wr A \wr^{<\infty}_{\Delta} = \{ & ((x_0,\lambda_0),\dots,(x_n,\lambda_n)) \ : \text{ there is a block subsequence} \\ & ((\tilde{x}_0,\mu_0),\dots,(\tilde{x}_n,\mu_n)) \text{ of } A \text{ such that} \\ & \max\{\|x_i,\tilde{x}_i\|, |\lambda_i - \mu_i|\} \leq \delta_i \text{ and supp } x_i = \text{supp } \tilde{x}_i \ (i=0,\dots,n)\}\end{aligned}$$

Let $\mathcal{B}$ be the complete Boolean algebra generated by $P = P(\Gamma, (e_n)_n)$. This poset $P$ is Borel and satisfies axiom A (see [1] for details). And forcing with $P$ gives a block sequence $X_\infty \in \bigcap_{\alpha<\aleph_1} D_\alpha$ in $L(\mathbb{R})^{V^{\mathcal{B}}}$.

**Claim 1.** *Work in $L(\mathbb{R})^{V^{\mathcal{B}}}$. For every $X \in [X_\infty]$ the following dichotomy holds: Either there is some finite block subsequence $s$ of $X$ and some $\alpha < \aleph_1$ such that $(\sigma_\alpha)_{2\Delta}$ is strategically large for $[s;X_\infty]$, or $[X] \cap \sigma_\alpha = \emptyset$ for every $\alpha < \aleph_1$ (i.e. $\sigma \cap [X] = \emptyset$).*

*Proof of Claim:* Suppose that there is some $\alpha$ such that $[X] \cap \sigma_\alpha \neq \emptyset$, and choose $((z_n,\lambda_n))_n \in [X] \cap \sigma_\alpha$. Since $X_\infty \in D_\alpha$, we can find $[s;[A]_\Gamma] \in F_\alpha$ such that $X_\infty \in [s;[A]_\Gamma]$. So, $X_\infty = s^\frown W$, with $W \in [A]_\Gamma$. Let $n$ be the maximal $k$ such that $((z_0,\lambda_0),\dots,(z_k,\lambda_k))$ is a block subsequence of $s$. Therefore $((z_i,\lambda_i))_{i>n} \in [W]$ and hence $d(((z_i,\lambda_i))_{i>n}, A') \leq (\delta_i/2)_{i>lh(s)}$ for some $A' \in [A]$, because of our choice of $\Gamma$. So, $[((z_0,\lambda_0),\dots,(z_n,\lambda_n));A] \cap (\sigma_\alpha)_{\Delta/2} \neq \emptyset$ and hence $(\sigma_\alpha)_\Delta$ is strategically large for $[((z_0,\lambda_0),\dots,(z_n,\lambda_n)); A]$. But $X_\infty = s^\frown W \in [s;[A]_{\Delta/2}]$, and hence $(\sigma_\alpha)_{3\Delta/2}$ is also strategically large for $[s;Y_\infty]$. □

So, there are two cases: If there is some $X \in [X_\infty]$ such that $[X] \cap \sigma_\alpha = \emptyset$ for every $\alpha$, then $[X] \cap \sigma = \emptyset$. Since $L(\mathbb{R})$ and $L(\mathbb{R})^{V^{\mathcal{B}}}$ are elementarily equivalent it must be true in $L(\mathbb{R})$ that $[X] \cap \sigma = \emptyset$ for some $X$.

Otherwise, for every $X \in [X_\infty]$ there is some finite block subsequence $s$ of $X$ and some $\alpha$ such that $(\sigma_\alpha)_{2\Delta}$ is strategically large for $[s;X_\infty]$, hence $\sigma_{2\Delta}$ is strategically large for $[s;X_\infty]$. This fact implies that the set $\mathcal{O}^{str}_{X_\infty}(\sigma_{2\Delta})$ is large. By Proposition 2.3, $\sigma_{3\Delta}$ is strategically large for $[Y]$, for some $Y \in [X_\infty]$. By elementary equivalence, this must be true in $L(\mathbb{R})$, and we are done. □

REMARK 7.1. The analogous result for almost Ramsey sets of normalized block sequences of $c_0$-saturated spaces is also true, which can be shown by the previous proof replacing "strategically large" by "very large".

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