

# Forcing with finite conditions

Sy D. Friedman\*  
Institute for Formal Logic  
University of Vienna

## Abstract

We present a generalisation to  $\omega_2$  of Baumgartner's forcing for adding a CUB subset of  $\omega_1$  with finite conditions.

The following well-known result appears in Baumgartner, Harrington, Kleinberg [2]:

*Theorem 1.* Suppose that  $X \subseteq \omega_1$ . Then the following are equivalent:

- a.  $X$  contains a CUB subset in an outer model which preserves  $\omega_1$ .
- b.  $X$  is stationary.

Proof. (a) implies (b) because any two CUB sets must intersect. Conversely, consider the forcing  $P$  whose conditions are closed, countable subsets of  $X$ , ordered by end-extension. Clearly  $P$  adds a CUB subset to  $X$ ; we must show that  $\omega_1$  is preserved.

First a general comment about  $\omega_1$ -preservation. We say that  $D$  is *pre-dense below*  $p$  iff every condition below  $p$  is compatible with an element of  $D$ . Then  $\omega_1$ -preservation is a consequence of the following:

(\*) For any  $p$  and  $D_i$ ,  $i < \omega$  with each  $D_i$  predense below  $p$ , there are  $q \leq p$  and *countable*  $d_i$ ,  $i < \omega$  with  $d_i \subseteq D_i$  and  $d_i$  predense below  $q$  for each  $i < \omega$ .

---

\*The author wishes to thank the CRM for its kind hospitality during the latter half of September, 2003, when this article was prepared. Research supported by Forschungsprojekt Nr. P16334-NO5 des österreichischen Fonds zur Förderung der wissenschaftlichen Forschung. The author wishes to thank Bill Mitchell, who independently obtained a similar result, for sharing his insights into this problem.

For if (\*) holds and  $p$  forces  $\sigma$  to be a function from  $\omega$  to  $\omega_1$ , then we can consider  $D_i = \{q \mid \text{for some } \alpha < \omega_1, q \text{ forces } \sigma(i) = \alpha\}$ ; by (\*), there is  $q \leq p$  and a countable  $\beta$  such that  $q$  forces  $\sigma(i) < \beta$  for each  $i < \omega$ , and therefore  $q$  forces that  $\sigma$  is bounded.

Now to see that  $P$  preserves  $\omega_1$ , suppose that  $\langle D_i \mid i < \omega \rangle$  is a sequence of sets which are predense below  $p$  and choose a continuous elementary chain  $\langle M_j \mid j < \omega_1 \rangle$  of countable elementary submodels of  $H_\theta$ ,  $\theta$  large, so that  $X, p, \langle D_i \mid i < \omega \rangle$  belong to  $M_0$  and  $M_j \in M_{j+1}$  for each  $j$ . As  $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$  is CUB, we can choose  $j$  so that  $\alpha = M_j \cap \omega_1 \in X$ . Then as each  $D_i \cap M_j$  is predense below  $p$  on  $P \cap M_j$ , we can choose  $p = p_0 \geq p_1 \geq \dots$  so that  $p_{i+1}$  belongs to  $M_j$  and extends an element  $r_i$  of  $D_i$  for each  $i < \omega$ , and in addition  $\bigcup_i p_i$  has supremum  $\alpha$ . Then  $q = \bigcup_i p_i \cup \{\alpha\}$  is a condition extending  $p$ , and for each  $i$ ,  $\{r_i\} \subseteq D_i$  is predense below  $q$ , proving (\*).  $\square$

Next we ask the following.

*Question.* Suppose that  $X$  is a *constructible*, stationary subset of  $\omega_1$ . Then is there a *constructible* forcing  $P$  which preserves  $\omega_1$  and adds a CUB subset to  $X$ ?

Baumgartner answered this question affirmatively.

*Theorem 2.* Let  $X$  be a stationary subset of  $\omega_1$ . Then there is a forcing  $P$  which adds a CUB subset to  $X$ , such that  $P$  preserves  $\omega_1$  and  $P$  belongs to  $L[X]$ .

*Proof.* We add a CUB subset to  $X$  *with finite conditions*, using a technique of Uri Avraham [1] (a variant of Baumgartner's original technique). A condition is a finite set  $p$  of disjoint closed intervals in  $\omega_1$  whose left endpoints belong to  $X$ . (We allow the one-point intervals  $[\alpha, \alpha]$ ,  $\alpha \in X$ .) A condition  $q$  extends  $p$  iff  $q$  contains  $p$ .

It is clear that for generic  $G$ ,  $C_G =$  the set of all left endpoints of intervals in  $\bigcup G$  is an unbounded subset of  $X$ . Each countable ordinal either belongs to some interval in  $G$  or fails to be a limit point of  $X$ ; it follows that  $C_G$  is closed. It only remains to show that  $\omega_1$  is preserved.

Suppose that  $p$  is a condition and  $D_i$ ,  $i < \omega$  are predense below  $p$ . Choose a continuous elementary chain  $\langle M_j \mid j < \omega_1 \rangle$  of countable elementary submodels of  $H_\theta$ ,  $\theta$  large, so that  $X, p, \langle D_i \mid i < \omega \rangle$  belong to  $M_0$  and

$M_j \in M_{j+1}$  for each  $j$ . As  $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$  is CUB, we can choose  $j$  so that  $\alpha = M_j \cap \omega_1 \in X$ . Let  $q$  be the condition  $p \cup \{[\alpha, \alpha]\}$ . If  $r$  extends  $q$  and  $r_0 = r \upharpoonright \alpha$  then every extension  $s_0$  of  $r_0$  in  $P \cap M_j$  is compatible with  $r$ . This is because  $[\alpha, \alpha]$  belongs to  $q$ . It follows that  $d_i = D_i \cap M_j$  is predense below  $q$  for each  $i$ , as if  $r$  extends  $q$  then we can choose  $s_0 \leq r_0$  which extends a condition in  $d_i$ , and therefore since  $s_0$  is compatible with  $r$ ,  $r$  is compatible with an element of  $d_i$ . Hence  $\omega_1$  is preserved.  $\square$

A nice corollary to this result is: If  $X$  is a stationary subset of  $\omega_1$  then there is a cardinal-preserving forcing that adds a CUB subset to  $X$ . This does not follow from the proof of Theorem 1, as the forcing used there will collapse  $2^{\aleph_0}$  if CH fails.

Now we look at the situation for  $\omega_2$ . Unfortunately there is no analogue for Theorem 1.

*Theorem 3.* (See [3].) Suppose that  $0^\#$  exists. Then

$\{X \subseteq \omega_2^L \mid X \in L \text{ and } X \text{ has a CUB subset in an inner model where } \omega_2 = \omega_2^L\}$

is not constructible, and indeed has  $L$ -degree  $0^\#$ . In particular, there are  $X$  which belong to the above set but have no CUB subset in any set-generic extension of  $L$  in which  $\omega_2 = \omega_2^L$ .

However there is a nice sufficient condition for a subset of  $\omega_2$  to contain a CUB in a cardinal-preserving extension.  $X \subseteq \omega_2$  is *very stationary* iff for all  $\alpha$  in some stationary  $X_0 \subseteq X \cap \text{cof } \omega_1$ ,  $X \cap \alpha$  contains a CUB subset of  $\alpha$ . Using a variant of  $\diamond$  at  $\omega_2$  in  $L$  one can construct disjoint very stationary subsets of  $\omega_2$  in  $L$ . And one has:

*Theorem 4.* If  $X \subseteq \omega_2$  is very stationary then there is a set-forcing extension, preserving both  $\omega_1$  and  $\omega_2$ , and in which  $X$  contains a CUB subset.

*Proof.* In analogy with the proof of Theorem 1, force with closed subsets of  $X$  of ordertype less than  $\omega_2$ , ordered by end-extension. The very stationarity of  $X$  is used as in the proof of Theorem 1 to show that if  $p$  is a condition and  $D_i$ ,  $i < \omega_1$ , are predense below  $p$  then there is  $q \leq p$  extending an element of  $D_i$  for each  $i$ . It follows that no new  $\omega_1$ -sequences are added by the forcing and therefore both  $\omega_1$  and  $\omega_2$  are preserved.  $\square$

Now we pose the following question: Is there a version for  $\omega_2$  of Baumgartner's forcing (as modified by Avraham) to add a CUB with finite conditions? The next result provides a positive answer. We say that  $D_0$  witnesses that  $D \subseteq \omega_2$  is very stationary iff  $D_0 \subseteq D \cap \text{cof } \omega_1$  is stationary and  $D \cap \alpha$  contains a CUB subset of  $\alpha$  for each  $\alpha \in D_0$ .

*Theorem 5.* Assume that  $\omega_2$  is the  $L$ -successor to  $\omega_1$ ,  $D \subseteq \omega_2$  is constructible and there is a constructible witness that  $D$  is very stationary. Then there is a constructible forcing  $P$  that preserves cofinalities and adds a CUB subset of  $D$ .

Proof. Let  $D_0 \subseteq D$ ,  $D_0 \in L$  witness that  $D$  is very stationary. We may assume that successor elements of  $D$  have cofinality  $\omega$ , by replacing  $D$  with  $(D \cap \text{Lim } D) \cup \{\alpha + \omega \mid \alpha \in D\}$ .

A condition is a pair  $p = (A, S)$ , where:

1.  $A$  is a finite set of disjoint closed intervals whose left endpoints belong to  $D$  and whose left endpoints of cofinality  $\omega_1$  belong to  $D_0$ . (We allow the one-point intervals  $[\alpha, \alpha]$ ,  $\alpha \in D$ .) Let  $L$  denote the set of left endpoints of intervals in  $A$ .
2.  $S$  is a finite set of countable, constructible  $\Sigma_1$  elementary submodels  $x$  of some  $L_\beta$ ,  $\beta$  limit,  $\beta < \omega_2$ , such that:
  - 2a.  $x \cap \alpha$  is unbounded in  $\alpha$  whenever  $\alpha \in x$  has cofinality  $\omega$ .
  - 2b.  $\sup(x \cap \alpha)$  belongs to  $D$  whenever  $\alpha$  belongs to  $(x \cap D_0) \cup \{\omega_2\}$ .
  - 2c. If  $\alpha < \beta$  are adjacent elements of  $D$ ,  $\alpha$  belongs to  $x$  and  $\beta < \sup(x \cap \text{Ord})$  then  $\beta$  belongs to  $x$ .
3. For each interval  $I = [\alpha, \beta]$  in  $A$  and each  $x \in S$ :
  - 3a. If  $I$  intersects  $x$  then  $I$  belongs to  $x$ .
  - 3b. If  $I = [\alpha, \beta]$  does not intersect  $x$  and  $\alpha < \sup(x \cap \text{Ord})$  then  $\alpha_x$  belongs to  $L$ , where  $\alpha_x =$  the least ordinal  $\geq \alpha$  in  $x$ .
4. Let  $F$  be the set of all elements of  $L$  of cofinality  $\omega_1$ , together with  $\omega_2$ . For nice  $x$ , the  $F$ -height of  $x$  is the least element of  $F$  greater than  $\sup(x \cap \text{Ord})$ .
  - 4a. If  $x$  belongs to  $S$  and  $\alpha$  belongs to  $F$  then  $x \cap L_\alpha$  also belongs to  $S$ .
  - 4b. Suppose that  $x, y \in S$  have the same  $F$ -height. Then  $x \in y$ ,  $y \in x$  or  $x = y$ .

Write  $p = (A_p, S_p)$  and let  $F_p$  denote the  $F$  of 4 above.  $q$  extends  $p$  iff  $A_q$  contains  $A_p$  and  $S_q$  contains  $S_p$ .

*Claim 1.* Suppose that  $p$  belongs to  $P$ .

(a) For any CUB  $C \subseteq \omega_2$  there exists  $\alpha \in C$  of cofinality  $\omega_1$  such that  $p$  belongs to  $L_\alpha$  and  $p^*$ , obtained from  $p$  by adding the interval  $[\alpha, \alpha]$  to  $A_p$  (and otherwise leaving  $p$  unchanged), is a condition extending  $p$ .

(b) Let  $\alpha$  and  $p^*$  be as in part (a). Then if  $q^*$  extends  $p^*$  there is  $q$  extending  $p$  in  $L_\alpha$  such that every extension of  $q$  in  $L_\alpha$  is compatible with  $q^*$ .

*Proof of Claim 1:*

(a) Choose  $\alpha$  to be any  $\alpha \in C \cap D_0$  such that  $p$  belongs to  $L_\alpha$ . Property 1 is satisfied by  $p^*$  as  $\alpha$  is greater than the right endpoint of any interval in  $A_p$ . Property 2 is the same for  $p^*$  as for  $p$ . Property 3a is the same for  $p^*$  as for  $p$ , as  $\alpha$  does not belong to any element of  $S_p$ . Property 3b is the same for  $p^*$  as for  $p$ , as  $\alpha$  is greater than  $\sup x$  for any  $x \in S_p$ . And property 4 is the same for  $p^*$  as for  $p$ , as  $\alpha$  does not belong to any element of  $S_p$ .

(b) Suppose that  $q^*$  extends  $p^*$  and define  $q$  by:

$$\begin{aligned} A_q &\text{ is } A_{q^*} \cap L_\alpha, \\ S_q &\text{ is } S_{q^*} \cap L_\alpha. \end{aligned}$$

a.  $q$  is a condition in  $L_\alpha$  which extends  $p$ .

We show that  $q$  is a condition. We need only verify properties 3b and 4.

3b. Assume that  $I \cap x = \emptyset$  and the left endpoint  $\beta$  of  $I = [\beta, \gamma]$  is less than  $\sup x$ , where  $I$  belongs to  $A_{q^*} \cap L_\alpha$  and  $x$  belongs to  $S_{q^*} \cap L_\alpha$ . Then since  $q^*$  is a condition,  $\beta_x$  is the left endpoint of some interval  $J$  in  $S_{q^*}$ . But since  $[\alpha, \alpha]$  belongs to  $A_{q^*}$ , the right endpoint of  $J$  is less than  $\alpha$  and therefore  $J$  belongs to  $S_{q^*} \cap L_\alpha = S_q$ .

For property 4, first note that  $F_q = F_{q^*} \cap \alpha$ , together with  $\omega_2$ .

4a. If  $x$  is in  $S_q$  and  $\beta \in F_q$  then  $x \cap L_\beta$  is in  $S_{q^*}$  and therefore also in  $S_q = S_{q^*} \cap L_\alpha$ , since  $x \cap L_\beta$  is an element of  $L_\alpha$ .

4b. If  $x, y \in S_q$  have the same  $F_q$ -height then since they both belong to  $L_\alpha$ , they have the same  $F_{q^*}$ -height. Thus the desired conclusion follows as  $x, y \in S_{q^*}$  and  $q^*$  is a condition.

Since  $q^*$  extends  $p^*$  and therefore  $p$ , it follows that  $q$  also extends  $p$ , since  $p$  belongs to  $L_\alpha$ .

Now suppose that  $r$  is an extension of  $q$ , and  $r$  belongs to  $L_\alpha$ . We must find a common extension of  $r$  and  $q^*$ . We define  $t$  by

$$A_t = A_r \cup A_{q^*},$$

$$S_t = S_r \cup S_{q^*}.$$

b.  $t$  is a condition extending both  $r$  and  $q^*$ .

Clearly  $t$ , if a condition, extends both  $r$  and  $q^*$ . We now verify that  $t$  is a condition, by verifying properties 1-4.

1. The intervals in  $A_t$  are disjoint, as  $r$  is a condition extending  $q$ , all intervals in  $A_r$  have right endpoint less than  $\alpha$  and all intervals in  $A_{q^*}$  not in  $A_q$  have left endpoint at least  $\alpha$ .

2. Clear.

3. Suppose that  $I$  is an interval in  $A_t - A_r$  and  $x$  belongs to  $S_r$ . Then  $\sup(x \cap \text{Ord})$  is less than  $\alpha$  and the left endpoint of  $I$  is at least  $\alpha$ . So property 3 is vacuous in this case. Suppose that  $I$  belongs to  $A_r$  and  $x$  belongs to  $S_t - S_r$ . Then  $x \cap L_\alpha$  belongs to  $S_q \subseteq S_r$  and therefore property 3 holds for  $I$  and  $x \cap L_\alpha$ . It follows that 3a holds for  $I$  and  $x$ , since if  $I$  intersects  $x$  it must also intersect  $x \cap L_\alpha$ . And 3b holds for  $I$  and  $x$ : If  $I$  is disjoint from  $x$  and the left endpoint  $\beta$  of  $I$  is less than  $\sup(x \cap \text{Ord})$  then  $I$  is also disjoint from  $x \cap L_\alpha$  and either (i)  $\beta$  is less than  $\sup(x \cap \alpha)$ , in which case  $\beta_x = \beta_{x \cap \alpha}$  and therefore the result follows since  $r$  is a condition, or (ii)  $\beta_x = \alpha$ , in which case the result follows since  $[\alpha, \alpha]$  belongs to  $A_{q^*}$ , or (iii)  $\beta_x = \alpha_x$ , in which case the result follows since  $q^*$  is a condition. The remaining cases, where  $I$  belongs to  $A_r$  and  $x$  belongs to  $S_r$ , or where  $I$  belongs to  $A_t - A_r$  and  $x$  belongs to  $S_t - S_r$ , are immediate since  $r$  and  $q^*$  are conditions.

4a. We must show that if  $x$  belongs to  $S_t$  and  $\beta \in F_t$  then  $x \cap L_\beta$  belongs to  $S_t$ . If  $x$  belongs to  $S_r$  then either  $\beta$  is in  $F_r$ , in which case  $x \cap L_\beta$  belongs to  $S_r \subseteq S_t$  since  $r$  is a condition, or  $\beta$  is at least  $\alpha$ , in which case  $x \cap L_\beta = x \in S_r \subseteq S_t$ . If  $x$  belongs to  $S_{q^*}$  then either  $\beta$  is in  $F_{q^*}$ , in which case the result follows since  $q^*$  is a condition, or  $\beta$  is in  $F_r$ , in which case  $x \cap L_\beta = (x \cap L_\alpha) \cap L_\beta \in S_r \subseteq S_t$ , since  $x \cap L_\alpha \in S_q \subseteq S_r$ .

4b. We must show that if  $x, y \in S_t$  have the same  $F_t$ -height, then  $x \in y, y \in x$  or  $x = y$ . If  $x$  belongs to  $S_r$  then the  $F_t$  height of  $x$  is at most  $\alpha$  and therefore  $y$  also belongs to  $S_r$ ; thus  $x, y$  have the same  $F_r$ -height and the result follows since  $r$  is a condition. If  $x$  belongs to  $S_{q^*} - S_r$  then the  $F_t$ -height of  $x$  is greater than  $\alpha$ , and therefore  $y$  also belongs to  $S_{q^*}$ ; thus  $x, y$  have the same  $F_{q^*}$ -height and the desired conclusion follows since  $q^*$  is a condition.

*This completes the proof of Claim 1.*

*Claim 2.* Suppose that  $p$  belongs to  $P$ .

- (a) For any CUB  $C \subseteq P_{\omega_1}(\omega_2)$  there exists a constructible  $x \in C$  such that  $p$  belongs to  $x$  and  $p^*$ , defined by adding  $x \cap L_\alpha$  to  $S_p$  for all  $\alpha \in F_p$  (and otherwise leaving  $p$  unchanged) is a condition extending  $p$ .
- (b) Let  $x$  and  $p^*$  be as in part (a). Then if  $q^*$  extends  $p^*$  there is  $q$  in  $x$  extending  $p$  such that every extension of  $q$  in  $x$  is compatible with  $q^*$ .

*Proof of Claim 2:*

(a) Using the fact that  $\omega_2$  equals  $\omega_2^I$ , choose  $x \in C$  to be a constructible elementary submodel of  $(L_{\omega_2}, D)$  which contains  $p$ . Also require that if  $\alpha$  belongs to  $(x \cap D_0) \cup \{\omega_2\}$  then  $\sup(x \cap \alpha)$  belongs to  $D$ . The latter is possible using the fact that  $D_0$  witnesses that  $D$  is very stationary. Define  $p^*$  as in the statement of (a). Clearly  $p^*$ , if a condition, extends  $p$ . To verify that  $p^*$  is a condition we need only check properties 3 and 4.

3. As  $p$  belongs to  $x$ , each interval in  $A_p$  belongs to  $x$  and therefore the conclusion of 3a holds for  $x$ . It follows easily that 3a also holds for  $x \cap L_\alpha$  whenever  $\alpha$  belongs to  $F_p$ . 3a holds for other elements of  $S_{p^*}$  since  $p$  is a condition. Property 3b is vacuous for  $x \cap \alpha$ ,  $\alpha \in F_p$  and holds for other elements of  $S_{p^*}$  since  $p$  is a condition.

4a. This is true for  $x$  by definition of  $p^*$ , and for other elements of  $S_{p^*}$  since  $p$  is a condition.

4b. Suppose that  $y, z \in S_{p^*}$  have the same  $F_{p^*}$ -height (=  $F_p$ -height). If both  $y, z$  belong to  $S_p$  then the desired conclusion follows since  $p$  is a condition. Assume that  $y = x \cap L_\alpha$  where  $\alpha$  belongs to  $F_p$ . If  $z$  belongs to  $S_p$  then  $z$  belongs to  $x$  and since it has the same  $F_p$ -height as  $y$ , must also belong to  $L_\alpha$ ; hence  $z$  belongs to  $y$ . If  $z$  is of the form  $x \cap L_\beta$ ,  $\beta \in F_p$ , and has the same  $F_p$ -height as  $y$  then  $z = y$ , since the  $F_p$ -height of  $x \cap L_\beta$  equals  $\beta$  for any  $\beta \in F_p$ .

(b) Let  $q^*$  extend  $p^*$  and define  $q$  as follows:

$A_q$  is  $A_{q^*} \cap x$ ,  
 $S_q$  is  $S_{q^*} \cap x$

Subclaim 1.  $q$  is a condition in  $x$  extending  $p$ .

$q$ , if a condition, surely extends  $p$  since  $q^*$  extends  $p^* \leq p$  and  $p$  belongs to  $x$ . To check now that  $q$  is a condition we need only verify properties 3b and 4.

3b. Suppose that  $I$  belongs to  $A_q$ ,  $I$  is disjoint from  $y$  and the left endpoint  $\alpha$  of  $I$  is less than  $\sup y$ , where  $y$  belongs to  $S_q$ . Then  $\alpha_y$  is the left endpoint of some  $J \in A_{q^*}$  since  $q^*$  is a condition. Since  $J$  intersects  $y$ ,  $J$  must belong

to  $y$  and therefore also to  $x$ , since  $y$  belongs to  $x$ . Thus  $J$  belongs to  $A_q$ .

4a. If  $y$  belongs to  $S_q$  and  $\alpha$  belongs to  $F_q \cap \omega_2$  then  $y \cap L_\alpha$  belongs to  $S_{q^*}$  since  $q^*$  is a condition. Since both  $y$  and  $\alpha$  belong to  $x$ , we get  $y \cap L_\alpha \in S_q$ . If  $y$  belongs to  $S_q$  then  $y \cap L_{\omega_2} = y$  and therefore belongs to  $S_q$ .

4b. Suppose that  $y \in S_q$  has  $F_q$ -height  $\alpha$  and  $F_{q^*}$ -height  $\beta$ . Suppose that  $\beta$  is less than  $\sup x$ . Then either  $\beta$  equals  $\alpha$  or is the left endpoint of some interval in  $A_{q^*}$  disjoint from  $x$ . In the latter case,  $\beta_x$  is the left endpoint of some interval in  $A_{q^*} \cap x = A_q$  and therefore  $\beta_x$  belongs to  $F_q$ , since Property 2a for  $x$  implies that it must have uncountable cofinality. Thus  $\beta_x = \alpha$ . So we conclude that the  $F_{q^*}$ -height of  $y$  is the least  $\beta \in F_{q^*}$  such that either  $\beta$  is less than  $\sup x$  and  $\beta_x = \alpha$ , or  $\beta$  is greater than  $\sup x$ . Therefore the  $F_{q^*}$ -height of  $y \in S_q$  is uniquely determined by the  $F_q$ -height of  $y$ . If  $y, z \in S_q$  have the same  $F_q$ -height then they therefore also have the same  $F_{q^*}$ -height, and since  $q^*$  is a condition, either  $y \in z$ ,  $z \in y$  or  $y = z$ , as desired.

Now suppose that  $r$  in  $x$  extends  $q$ . We must find a common extension  $t$  of  $r$  and  $q^*$ . We define  $t$  by:

$$A_t = A_r \cup A_{q^*}$$

$$S_t = S_r \cup \{y \cap L_\alpha \mid y \in S_{q^*}, \alpha \in F_r\}.$$

Subclaim 2.  $t$  is a condition extending both  $r$  and  $q^*$ .

Clearly  $t$ , if a condition, extends both  $r$  and  $q^*$ . We show now that  $t$  is a condition.

1. Suppose that  $I$  is an interval in  $A_{q^*}$  but not in  $A_r$ . Then  $I$  is disjoint from  $x$ . If the left endpoint  $\alpha$  of  $I$  is at least  $\sup x$ , then  $I$  is disjoint from all intervals in  $A_r$ , since the latter belong to  $x$ . Otherwise  $\alpha_x$  is the left endpoint of some interval  $J$  in  $A_q$ . It follows that the intervals in  $A_r$  are disjoint from  $I$ , as they belong to  $x$  and are either equal to or disjoint from  $J$ . Thus  $A_t$  consists of pairwise disjoint intervals.

2a. Clear.

2b. We must show that if  $y$  belongs to  $S_t$  and  $\alpha \in (y \cap D_0) \cup \{\omega_2\}$  then  $\sup(y \cap \alpha)$  belongs to  $D$ . This is clear if  $y$  belongs to  $S_r$  since  $r$  is a condition. It also holds if  $y$  belongs to  $S_{q^*}$  since  $q^*$  is a condition. This implies the result for arbitrary  $y \in S_t$  since elements of  $F_r$  belong to  $D_0$ .

2c. Clear, since elements of  $F_r$  belong to  $D_0 \subseteq \text{Lim } D$ .

3a. Suppose that  $I$  is an interval in  $A_t$ ,  $y$  belongs to  $S_t$  and  $I$  intersects  $y$ . We must show that  $I$  belongs to  $y$ . First we consider the case where  $I$  belongs to  $A_r$  and  $y$  belongs to  $S_t - S_r$ . Write  $I = [\alpha, \beta]$  and  $y = z \cap L_\gamma$ , where  $z$  belongs to  $S_{q^*} - S_r$  and  $\gamma$  belongs to  $F_r$ . Let  $\beta^*$  be the least element of  $F_{q^*}$  greater than  $\alpha$ . Since we have shown that  $A_t$  consists of pairwise



disjoint intervals, it follows that  $\beta^*$  is greater than  $\beta$ . Therefore the the  $F_{q^*}$ -heights of  $x \cap L_{\beta^*}$  and  $z \cap L_{\beta^*}$  are both  $\beta^*$ , the former since  $\beta$  belongs to  $x$  and the latter since  $z$  intersects  $I = [\alpha, \beta]$ . Thus either  $z \cap L_{\beta^*}$  belongs to  $x \cap L_{\beta^*}$ ,  $x \cap L_{\beta^*}$  belongs to  $z \cap L_{\beta^*}$  or  $x \cap L_{\beta^*} = z \cap L_{\beta^*}$ . In the first case,  $I$  intersects  $y \cap L_{\beta^*} \in S_q \subseteq S_r$ , and therefore since  $r$  is a condition,  $I$  belongs to  $y \cap L_{\beta^*} \subseteq y$ , as desired. In the second and third cases,  $y$  contains  $x \cap L_{\beta^*}$  as a subset and therefore  $I$  as an element. Now consider the case where  $I$  belongs to  $A_t - A_r$  and  $y$  belongs to  $S_r$ . Then  $I$  belongs to  $A_{q^*}$  and intersects  $x$ , which belongs to  $S_{q^*}$ . Thus  $I$  belongs to  $x$ , contradicting the fact that  $I$  does not belong to  $A_q \subseteq A_r$ . The other two cases, where either  $I$  belongs to  $A_r$  and  $y$  belongs to  $S_r$ , or where  $I$  belongs to  $A_t - A_r$  and  $y$  belongs to  $S_t - S_r$ , follow since both  $r$  and  $q^*$  are conditions.

3b. Suppose that  $I = [\alpha, \beta]$  belongs to  $A_t$ ,  $y$  belongs to  $S_t$ ,  $I$  is disjoint from  $y$  and  $\alpha$  is less than  $\sup(y \cap \text{Ord})$ . We must show that  $\alpha_y$  is the left endpoint of some interval in  $A_t$ . First we consider the case where  $I$  belongs to  $A_r$  and  $y$  belongs to  $S_t - S_r$ . Write  $y = z \cap L_\gamma$  where  $z$  belongs to  $S_{q^*}$  and  $\gamma$  belongs to  $F_r$ . Let  $\beta^*$  be the least element of  $F_{q^*}$  greater than  $\beta$ . If  $\alpha_y = \beta^*$  then  $\alpha_y$  is the left endpoint of some interval in  $A_{q^*}$  and we are done. If  $\alpha_y > \beta^*$ , then let  $J$  be the interval of  $A_{q^*}$  with left endpoint  $\beta^*$ . Since  $q^*$  is a condition and  $\alpha_y = \alpha_z$ ,  $J$  is not an element of  $z$  and therefore is disjoint from  $z$ . Since  $q^*$  is a condition,  $\alpha_y = (\beta^*)_z$  is the left endpoint of some interval of  $A_{q^*}$ , as desired. Finally, if  $\beta < \alpha_y < \beta^*$ , it follows that  $x \cap L_{\beta^*}$  and  $z \cap L_{\beta^*}$  both have  $F_{q^*}$ -height  $\beta^*$ , and therefore  $x \cap L_{\beta^*} \in z \cap L_{\beta^*}$ ,  $z \cap L_{\beta^*} \in x \cap L_{\beta^*}$  or  $x \cap L_{\beta^*} = z \cap L_{\beta^*}$ . The first and third of these possibilities contradict our hypothesis that  $I \in x$  is disjoint from  $y$ . In the second possibility,  $z \cap L_{\beta^*}$  belongs to  $S_q$  and since  $\alpha_y = \alpha_z$  is less than  $\beta^*$ , we have that  $\alpha_y$  is the left endpoint of some interval in  $A_r$  since  $r$  is a condition. Now we consider the case where  $I$  belongs to  $A_t - A_r$  and  $y$  belongs to  $S_r$ . Thus  $I$  belongs to  $A_{q^*}$  and must be disjoint from  $x$ , else it would belong to  $x$  and therefore to  $A_r$ . As  $\alpha$  is less than  $\sup x$ , it follows that  $\alpha_x$  is the left endpoint of some interval in  $A_{q^*}$ , which in fact belongs to  $A_r$ . If  $\alpha_y = \alpha_x$  then we are done. Otherwise  $\alpha_y$  equals  $(\alpha_x)_y$ , which must be the left endpoint of an interval in  $A_r$ , since  $r$  is a condition. The remaining two cases, where either  $I$  belongs to  $A_r$  and  $y$  belongs to  $S_r$ , or where  $I$  belongs to  $A_t - A_r$  and  $y$  belongs to  $S_t - S_r$ , follow since both  $r$  and  $q^*$  are conditions.

4a. We must show that if  $y$  belongs to  $S_t$  and  $\alpha$  belongs to  $F_t$  then  $y \cap L_\alpha$  belongs to  $S_t$ . This is clear if  $y$  belongs to  $S_r$  and  $\alpha$  belongs to  $F_r$ , or if  $y$  belongs to  $F_{q^*}$  and  $\alpha$  belongs to  $F_{q^*}$ , since  $r$  and  $q^*$  are conditions. This is also true if  $y$  belongs to  $S_{q^*}$  and  $\alpha$  belongs to  $F_r$ , by definition of  $S_t$ . And we may assume that  $y$  belongs to  $S_r \cup S_{q^*}$ . So we need only

check the case where  $y$  belongs to  $S_r$ ,  $\alpha$  belongs to  $F_{q^*}$  and  $\alpha$  is less than  $\sup(y \cap \text{Ord})$ . If  $\alpha$  belongs to  $x$  then it also belongs to  $F_r$  so we are done since  $r$  is a condition. Otherwise  $\alpha_x$  is defined and belongs to  $F_r$ . So since  $r$  is a condition,  $y \cap L_\alpha = y \cap L_{\alpha_x}$  belongs to  $S_r$ .

4b. We must show that if  $y, z \in S_t$  have the same  $F_t$ -height then either  $y \in z$ ,  $z \in y$  or  $y = z$ . Note that  $y, z$  also have the same  $F_r$ -height and the same  $F_{q^*}$ -height. If  $y, z$  both belong to  $S_r$  then the desired conclusion follows since  $r$  is a condition. Suppose that  $y, z$  are of the form  $y^* \cap L_\alpha, z^* \cap L_\beta$ , respectively, where  $y^*, z^*$  belong to  $S_{q^*}$  and  $\alpha, \beta \in F_r$ . We may assume that  $\alpha, \beta$  are the  $F_r$ -heights of  $y, z$ , respectively, and therefore  $\alpha = \beta$ . Let  $\alpha^*$  be the common  $F_{q^*}$ -height of  $y, z$ . Then  $y^* \cap L_{\alpha^*}, z^* \cap L_{\alpha^*}$  have  $F_{q^*}$ -height  $\alpha^*$  and therefore since  $q^*$  is a condition, we have  $y^* \cap L_{\alpha^*} \in z^* \cap L_{\alpha^*}$ ,  $y^* \cap L_{\alpha^*} = z^* \cap L_{\alpha^*}$  or  $z^* \cap L_{\alpha^*} \in y^* \cap L_{\alpha^*}$ . In the second case, we get  $y = z$ . In the first case  $y$  belongs to  $z^* \cap L_{\alpha^*}$  since it is an initial segment of  $y^* \cap L_{\alpha^*}$ ; as  $y \in L_\alpha$  we get  $y \in z^* \cap L_\alpha = z$ . The third case is identical to the first case, with  $y$  and  $z$  switched. Finally assume that  $y$  belongs to  $S_r$  and  $z = z^* \cap L_\alpha$  where  $z^* \in S_{q^*}$  and  $\alpha \in F_r$ . We may assume that  $\alpha$  is the  $F_r$ -height of  $z$ , which is also the  $F_r$ -height of  $y$ . Let  $\alpha^*$  be the common  $F_{q^*}$ -height of  $y$  and  $z$ . Then  $\alpha^*$  is also the  $F_{q^*}$ -height of  $x \cap L_{\alpha^*}$ , since  $x$  contains  $y$ , and is the  $F_{q^*}$ -height of  $z^* \cap L_{\alpha^*}$ . Since  $q^*$  is a condition, we have  $z^* \cap L_{\alpha^*} \in x \cap L_{\alpha^*}$ ,  $x \cap L_{\alpha^*} \in z^* \cap L_{\alpha^*}$  or  $z^* \cap L_{\alpha^*} = x \cap L_{\alpha^*}$ . In the first case,  $(z^* \cap L_{\alpha^*}) \cap L_\alpha = z$  belongs to  $S_r$ , so we are done since  $r$  is a condition. In the second and third cases we have  $y \in z^* \cap L_\alpha = z$ .

*This completes the proof of Claim 2.*

Claim 1 implies that  $\omega_2$  is preserved. Claim 2 implies that  $\omega_1$  is preserved. As  $P$  has cardinality  $\omega_2$ , all cofinalities are preserved.

*Claim 3.* Let  $G$  be  $P$ -generic and  $C_G = \{\gamma \mid \gamma \text{ is a left endpoint of some interval in } \cup\{A_p \mid p \in G\}\}$ . Then  $C_G$  is a CUB subset of  $D$ .

*Proof of Claim 3.*

It is clear that  $C_G$  is unbounded. We show that  $C_G$  is closed.

Suppose that  $p$  is a condition and for the sake of contradiction,  $p \Vdash (\alpha \in \text{Lim } C_G \text{ and } \alpha \notin C_G)$ . We may assume that for each  $y \in S_p$ ,  $\alpha_y$  is either a left endpoint of some interval in  $A_p$  or is forced by  $p$  to not belong to  $C_G$ . For  $q \leq p$ ,  $\alpha$  does not belong to any interval in  $A_q$ , else  $q$  forces either that  $\alpha$  belongs to  $C_G$  or is not the limit of elements of  $C_G$ .

Suppose that  $y$  belongs to  $S_p$ ,  $\alpha$  is not in  $\text{Lim } y$  but  $\alpha$  is less than  $\text{sup}(y \cap \text{Ord})$ . Then  $\alpha_y$  must be a left endpoint of some interval in  $A_p$ , else  $p$  forces that  $\alpha$  is not a limit point of  $C_G$ . Let  $\beta$  be the least element of  $F_p$  greater than  $\alpha$  and consider  $S = \{y \in S_p \mid \alpha \leq \text{sup}(y \cap \text{Ord}) < \beta\}$ . Then the elements of  $S$  form an  $\in$ -chain.

Assume first that  $y \cap \alpha$  is cofinal in  $\alpha$  for some  $y \in S$ , and let  $y_0$  be the  $\in$ -least such. Note that if  $\alpha_{y_0}$  is a left endpoint of some interval in  $A_p$  then  $\alpha$  must belong to  $D$ , by requirement 2b on conditions. We show that we can extend  $p$  to force either that  $\alpha$  belongs to  $C_G$  or that  $\alpha$  is not a limit point of  $C_G$ , achieving the desired contradiction. Note that  $D \cap y_0 \cap \alpha$  must be cofinal in  $\alpha$ , as there are cofinally many  $\gamma < \alpha$  which are forced by extensions of  $p$  into  $C_G$  and for any such  $\gamma \notin y_0$ ,  $\gamma_{y_0}$  belongs to  $D_0 \subseteq D$ . By property 2c for  $y_0$ ,  $D \cap y_0 \cap \alpha \cap \text{cof } \omega$  is also cofinal in  $\alpha$ .

If  $\alpha_{y_0}$  is not the left endpoint of some interval in  $A_p$ , then let  $\gamma$  be an element of  $D \cap y_0 \cap \alpha \cap \text{cof } \omega$  greater than the right endpoint of any interval of  $A_p$  with left endpoint less than  $\alpha$ , and larger than  $\text{sup}(y \cap \alpha)$  for all  $y \in S_p$  with  $\text{sup}(y \cap \alpha) < \alpha$ . We claim a condition results when the interval  $I = [\gamma, \alpha_{y_0}]$  is added to  $p$ :  $I$  is disjoint from intervals of  $A_p$  with left endpoint less than  $\alpha$  by choice of  $\gamma$ . And it is disjoint from intervals of  $A_p$  with left endpoint greater than  $\alpha$  since by assumption  $\alpha_{y_0}$  is not the left endpoint of an interval of  $A_p$ , and therefore neither is any ordinal between  $\alpha$  and  $\alpha_{y_0}$ .  $I$  does not intersect any  $y \in S_p$  with  $\text{sup}(y \cap \beta) < \alpha$  by choice of  $\gamma$ .  $I$  does not intersect any  $y \in S_p$  with  $\text{sup}(y \cap \alpha) < \alpha \leq \text{sup}(y \cap \beta)$ , as for such  $y$  we have  $\alpha_y > \alpha_{y_0}$  since  $\alpha_y$  must be a left endpoint of some interval of  $A_p$  and  $\alpha_{y_0}$  is not. Any other  $y \in S_p$  contains  $y_0$  and therefore also the interval  $I$  as an element. For those  $y \in S_p$  disjoint from  $I$  with  $\gamma < \text{sup}(y \cap \text{Ord})$ ,  $\gamma_y = \alpha_y$  is a left endpoint of an interval of  $A_p$ .

If  $\alpha_{y_0}$  is the left endpoint of some interval in  $A_p$ , then let  $I = [\alpha, \alpha]$ . We claim that a condition results when we add  $I$  to  $p$ : Of course  $I$  is disjoint from all intervals of  $A_p$  since  $\alpha$  does not belong to any such interval. Trivially, if  $I$  intersects  $y \in S_p$  then it belongs to  $y$ . If  $I$  is disjoint from  $y \in S_p$  and  $\alpha < \text{sup}(y \cap \text{Ord})$ , then  $\alpha_y \geq \alpha_{y_0}$ , as otherwise  $y \cap L_\beta \in S_\beta$ ,  $y_0 \in y \cap L_\beta$  and therefore  $\alpha = \text{sup}(y_0 \cap \alpha_{y_0}) \in y$ , against our hypothesis. So  $\alpha_y$  must be a left endpoint of some interval of  $A_p$ , else  $\alpha_{y_0}$  could not be.

Lastly, we treat the case where  $y \cap \alpha$  is not cofinal in  $\alpha$  for all  $y \in S$ . In this case we choose  $I = [\gamma, \alpha]$ , where  $\gamma$  is an element of  $D \cap \alpha \cap \text{cof } \omega$  greater than the right endpoint of any interval of  $A_p$  with left endpoint less than  $\alpha$ , and larger than  $\text{sup}(y \cap \alpha)$  for all  $y \in S_p$  with  $\text{sup}(y \cap \alpha) < \alpha$ . We claim

that a condition results when we add  $I$  to  $p$ :  $I$  is disjoint from all intervals in  $A_p$  by choice of  $\gamma$ .  $I$  is disjoint from each  $y \in S_p$ , as  $y \cap \alpha$  is contained in  $\gamma$  by choice of  $\gamma$  and the case hypothesis, and if  $\alpha$  were to belong to  $y$ , we would have  $\alpha = \gamma_y$  and therefore  $p$  would force that  $\alpha$  is not a limit of elements of  $C_G$ , since  $\alpha$  is not the left endpoint of any interval of  $A_q$  for all  $q \leq p$ . If  $y$  belongs to  $S_p$  and  $\gamma < \sup(y \cap \text{Ord})$  then  $\alpha_y$  must be the left endpoint of an interval of  $A_p$ , as we observed earlier.

*This completes the proof of Theorem 5.*

## References

- [1] Abraham, U. and Shelah, S. *Forcing closed unbounded sets*, Journal of Symbolic Logic, Vol. 48, pp. 643–657, 1983.
- [2] Baumgartner J., Harrington L. and Kleinberg E. *Adding a closed unbounded set*, Journal of Symbolic Logic, Vol. 41, pp. 481–482, 1976.
- [3] Friedman, S. *Cardinal-preserving extensions*, to appear, Journal of Symbolic Logic.