A couple of gentle stretching exercises.

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ABSTRACT: Let μ be a regular cardinal. In this paper I prove two (forcing) existence results concerning structures governed by two parameters, the cardinal μ and an ordinal ρ less than μ^{+++}. The results improve on theorems from [M*2] where the second parameter was always the cardinal μ^{++}.

Proposition: It is consistent (with ZFC) that for each $\rho < \mu^{+++}$ there is a chain in $\mathcal{P}(\mu^+)$ mod less than μ of length ρ.

Theorem: For each $\rho < \mu^{+++}$ it is consistent (with ZFC) that there is a (locally) μ-Lindelöf, μ-tight, 0-dimensional scattered space of height ρ and width μ.

The proof of the Proposition is elementary. In contrast, the proof of the Theorem uses the machinery of μ-$\mathbb{M}$-proper forcing from [M*1] (which was also used in [M*2]), combined with ideas from Martinez's paper [Mart*]. The necessary technical details are rehearsed here, albeit without proofs, for the reader's convenience.

INTRODUCTION.

'Two cardinal combinatorics' has been a convenient tag for the circle of ideas and questions in combinatorics about structures which enjoy more than one naturally defined parameter (*cf.* [K00], [M*1]). However from the start the phrase has been more heuristic than strictly descriptive, ignoring the fact

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that in seminal examples parameters are frequently ordinals or functions from ordinals into the cardinals rather than single cardinals per se.

An early instance of this phenonenon is Roitman's influential forcing construction ([R]). Recall that the Cantor-Bendixson derivative of a topological space is the space obtained by discarding its isolated points, and that the cardinal sequence of a scattered space is the sequence of sizes of the discarded sets when the Cantor-Bendixson derivative is taken iteratedly. Roitman, in what was possibly the first instance of a beyond-ZFC generic stepping up argument (see the introduction to [M*1]), forced to add a scattered space with cardinal sequence $\langle \omega \mid \alpha < \omega_1 \rangle^\frown \omega_2$. Thus one parameter of the space is the ordinal $\omega_1 + 1$ and the other the function $f : \omega_1 + 1 \longrightarrow \mathrm{Card}$ given by $f(\alpha) = \omega$ for $\alpha < \omega_1$ and $f(\omega_1) = \omega_2$. (Perhaps the echo in the slogan 'two cardinal combinatorics' of the two cardinal gap between the values of the function f here, and similar gaps in other constructions, contributed to its vogue.)

In this paper I prove two (forcing) existence results, one elementary, one less so, concerning specific types of structures governed by two parameters, a regular cardinal μ and an ordinal ρ less than μ^{+++}. The results improve on theorems from [M*2] where the second parameter was always the cardinal μ^{++}. Before stating the results let me sketch the inspiration and motivation for considering the possibility of this sort of stretching.

In a now well-known construction, derived from Roitman's, Baumgartner and Shelah ([BS]) used an auxiliary function from pairs of ordinals less than ω_2 to countable subsets of ω_2 to prove that a 'stepped up' forcing to add an (ω, ω_2)-superatomic Boolean algebra has the countable chain condition and so preserves cardinals.

Conditions in the Baumgartner-Shelah forcing, put briefly, are partial orders on finite subsets of $\omega_2 \times \omega$ which respect the partial ordering of that set by first co-ordinates, with the set of maximal common predecessors, $i\{s, t\}$, of two distinct points s, t, constrained by the auxiliary function. (The way that the constraining actually works is explained below when the details are needed.) Thus the forcing adds a certain partial order on $\omega_2 \times \omega$, for short an *admissible* partial order.

From an admissible partial order one can follow a standard procedure to extract a superatomic Boolean algebra. The details of this process are not of particular interest right now and so will not be dwelt on here. (At this introductory stage the exact definition of admissible is not crucial either, so, for the moment, I suppress this as well.)

Recently, Juan Carlos Martinez ([Mart*]) gave an interesting, non-trivial descendent of Baumgartner and Shelah's construction, allowing him to add (ω, ρ)-superatomic Boolean algebras by forcing for arbitrary ordinals $\rho < \omega_3$.

Martinez's proof employs a system of ordinal intervals as a frame to enable one to climb to ρ without needing to ascend more than ω_2 many steps in the same interval. He stretches out the auxiliary function using this system, getting an auxiliary function from pairs of ordinals less than ρ to countable subsets of ρ. (One may assume without loss of generality that $\mathrm{cf}(\rho) = \omega_2$ since truncations of admissible partial orders will be seen immediately to be admissible themselves.)

He then uses this elongated auxiliary function to prove that a modification of the Baumgartner-Shelah forcing has the countable chain condition. Apart from using conditions which have realm a finite subset of $\rho \times \omega$ rather than of $\omega_2 \times \omega$, there are two interesting changes that he makes, both of which are necessary for the chain condition argument.

First of all, he ensures (by insisting that it holds within his forcing conditions) that for each $\alpha < \rho$

$(\star)$ if $s, t \in \{\alpha\} \times \omega$ then there are *no* points u such that $u < s, t$.

This is a very interesting demand, and not one, I imagine, that Baumgartner and Shelah had in mind when defining their forcing. It certainly will not hold (by genericity) for the partial order added by their original conditions.

It is perhaps worthwhile making the trivial observation here that for cardinality reasons it is not possible to integrate such a demand into similar forcings for adding admissible partial orders on any set $\bigcup\{\{\alpha\} \times \kappa_\alpha \mid \alpha < \lambda\}$ for which there are $\alpha < \beta < \lambda$ with $\kappa_\beta < \kappa_\alpha$. (See the second clause of the definition of admissible, Definition (3), below.) One would at least have to exclude levels β of this type from the demand. It is thus unclear how to reconcile Martinez's construction with the somewhat less delicate generalization of Baumgartner and Shelah's forcing used by Bagaria ([Bag]) to add, given a sequence $\langle \kappa_\alpha \mid \alpha < \omega_2 \rangle$ with $\kappa_\alpha \in \{\omega, \omega_1\}$ for each $\alpha < \omega_2$, an admissible partial ordering on $\bigcup\{\{\alpha\} \times \kappa_\alpha \mid \alpha < \omega_2\}$ – and hence a superatomic Boolean algebra of height ω_2 and cardinal sequence $\langle \kappa_\alpha \mid \alpha < \omega_2 \rangle$.

The second change Martinez makes is to demand that if $s <_p t$ in some condition p then there must be points lying between them at levels determined by a recipe coming from the system of intervals. I outline this demand in the case of $\rho = \omega_2^2$. Obviously this is one of the simplest cases, but in fact it is also to a large extent representative of the general construction.

I follow Martinez and write $\pi(s) = \alpha$ if $s = \{\alpha\} \times \omega$. Suppose $s <_p t$, where p is a condition in the forcing, and that $\pi(s) = \omega_2\nu_0+\xi_0$ and $\pi(t) = \omega_2\nu_1+\xi_1$. If $\nu_0 = \nu_1$ there are no additional demands.

Otherwise one has $\nu_0 < \nu_1$ and the demands are as follows. If $\xi_0 = 0$ but $\xi_1 \neq 0$ then there is some u such that $s <_p u <_p t$ and $\pi(u) = \omega_2\nu_1$. If $\xi_1 = 0$ but $\xi_0 \neq 0$ then there is some u such that $s <_p u \leq_p t$ and $\pi(u) = \omega_2(\nu_0 + 1)$. And if ξ_0, $\xi_1 \neq 0$ then there are u, v such that $s <_p u \leq_p v <_p t$, $\pi(u) = \omega_2(\nu_0 + 1)$ and $\pi(v) = \omega_2\nu_1$.

Taken together these conditions give interesting information about the location of common predecessors of any pair $s, t \in \rho\times\alpha$, and hence about $i\{s,t\}$. Let $\pi(s) = \omega_2\nu_0 + \xi_0$, $\pi(t) = \omega_2\nu_1 + \xi_1$. First of all suppose that $\nu_0 = \nu_1$. If $u <_p s, t$ and $\pi(u) = \omega_2\nu + \xi$, then one has to have that $\nu = \nu_0 = \nu_1$. For otherwise, after applying the above condition if the ξ_i are non zero, one has that there are two points in $\{\omega_2\nu_0\} \times \omega$ which have a common predecessor, contradicting Martinez's innovation $(\star)$. On the other hand, if $\nu < \nu_0, \nu_1$ and $\nu_0 \neq \nu_1$ then $\xi = 0$ since otherwise there would be two elements of $\{\omega_2(\nu + 1)\} \times \omega$ with a common predecessor, again contradicting $(\star)$.

Martinez's ingenious construction raises the question of whether one can perform similar stretchings for other objects which can be added by generic stepping up.

If there is no rank or height function associated with such an object then this question seems virtually meaningless. For example, there is no inherent ordering on the branches of a Kurepa tree or on the sets in a collection of subsets of ω_1 the intersection of each pair of which is finite. So there is little point asking whether one can force to add a Kurepa tree with set of branches of order-type ρ or a such family of subsets of ω_1 of order-type ρ — one can simply order the branches of a Kurepa tree or the sets of a strongly almost disjoint family however one likes. In order to make it meaningful one has to ask the question about analogous structures with an additional ordering, but this frequently makes the construction of the structures themselves considerably harder.

For another collection of structures the question does make sense, but has the rather simple answer that it is easy to stretch out any structure of the type being thought about from having length/height ω_2 to one of length/height any ordinal $\rho < \omega_3$. A simple example where this phenomenon occurs is in the case of strong chains in $\mathcal{P}(\omega_1)$ — that is sequences $\langle B_\alpha \mid \alpha < \omega_2\rangle$ of subsets of ω_1 such that $B_\alpha \setminus B_\beta$ is finite and $B_\beta \setminus B_\alpha$ is uncountable for $\alpha < \beta < \omega_2$ — this being the most obvious way of modifying a strongly almost disjoint set by adding an order structure on it.

In fact, in this instance nothing hinges on being in the specific case 'finite/ω_1/ω_2' and one can replace ω by any regular cardinal μ. Let μ be a regular cardinal and define a sequence $\langle B_\alpha \mid \alpha < \rho \rangle$ of subsets of μ^+ to be a *strong chain (in $\mathcal{P}(\mu^+)$ mod less than μ) of length ρ* if the cardinality of $B_\alpha \setminus B_\beta$ is less than μ and the cardinality of $B_\beta \setminus B_\alpha$ is μ^+ for all $\alpha < \beta < \rho$.

Proposition 1. Let μ be a regular cardinal and suppose that there is a strong chain in $\mathcal{P}(\mu^+)$ mod less than μ of length μ^{++}. Then for every $\rho < \omega_3$ there is a strong chain in $\mathcal{P}(\mu^+)$ mod less than μ of length ρ. ▲

Proof. By induction on ρ. First of all, note that one may clearly assume that if there is a chain of length ρ then there is one $\langle A_\alpha \mid \alpha < \rho \rangle$ of the same length but such that there is some set $A \in [\mu^+]^{\mu^+}$ such that $A \cap A_\alpha = \emptyset$ for all $\alpha < \rho$. (If $\langle A'_\alpha \mid \alpha < \rho \rangle$ is the original chain, simply let $A \in [\mu^+]^{\mu^+}$ be such that $\mu^+ \setminus A \in [\mu^+]^{\mu^+}$, let $h : \mu^+ \longrightarrow \mu^+ \setminus A$ be the increasing enumeration of $\mu^+ \setminus A$ and let $A_\alpha = h``A'_\alpha$.) Thus given a strong chain in $\mathcal{P}(\mu^+)$ of length ρ it is trivial to get one of length $\rho+\rho$. Hence, as truncations of strong chains are strong chains, it suffices to prove the proposition for powers of μ^{++}.

So let ρ be a power of μ^{++} and let $\langle \rho_i \mid i < \mathrm{cf}(\rho) \rangle$ be a continuous increasing sequence of ordinals cofinal in ρ. As $\mathrm{cf}(\rho) \leq \mu^{++}$ one has a strong chain $\langle A_{\rho_i} \mid i < \mathrm{cf}(\rho) \rangle$. Now simply fill in the chain using the inductive hypothesis and the fact that $\mathrm{otp}([\rho_i, \rho_{i+1})) < \rho$. For each $i < \mathrm{cf}(\rho)$ use the fact that $B_i = A_{\rho_{i+1}} \setminus A_{\rho_i}$ has size μ^+. If $h_i : \mu^+ \longrightarrow B_i$ is the increasing enumeration of B_i and $\langle C^i_j \mid j \in (\rho_i, \rho_{i+1}) \rangle$ is a strong chain in $\mathcal{P}(\mu^+)$ modulo less than μ of length $\mathrm{otp}((\rho_i, \rho_{i+1}))$ given by the inductive hypothesis, then set $A_j = A_{\rho_i} \cup h_i``C^i_j$. ▲

Corollary 2. Let μ be a regular cardinal. It is consistent that there is a strong chain in $\mathcal{P}(\mu^+)$ mod less than μ of length ρ for each $\rho < \mu^{+++}$. ▲

Proof. The hypothesis of Theorem (1) is consistent in the case $\mu = \omega$ by [K98], via a forcing which has the countable chain condition assuming that $\square_{\omega_1}$ holds, and in the general case by [M*2, §4], via the natural μ^+-$\mathbb{M}$-proper forcing analogue of the forcing used in [K98]. ▲

Remark. Notice that one cannot replace Martinez's forcing construction of (ω, ρ)-superatomic Boolean algebras with an elementary argument of this sort. The problem that causes an analagous inductive argument to fail is that one cannot ensure that the sets $i\{s, t\}$ are finite even if this holds for the (ω, ρ')-superatomic Boolean algebras one has for $\rho' < \rho$. ▲

For a third collection of structures, including superatomic Boolean algebras of height ω_2 and varying width ([Bag]), pcf structures on ω_2 ([M*3]), and chains of length ω_2 in ${}^{\omega_1}\omega_1$ mod finite ([K00]), how one should set about stretching them out remains unclear. (These last two types of structure and strong chains in $\mathcal{P}(\mu^+)$ mod less than μ are all only known to be addable by (κ-)$\mathbb{M}$-proper forcing.)

However, in this paper I give a common generalisation of the results of [Mart*] and [M*2, §3], and prove that it is possible for any to force to add a (μ, ρ)-admissible partial order for any regular cardinal μ and any $\rho < \mu^{+++}$. This demonstrates that it is possible combine some non-trivial stretching out with μ^+-$\mathbb{M}$-proper forcing.

Definition 3. Let μ be a regular cardinal and ρ an ordinal. For $s = (\sigma, \gamma) \in \rho \times \mu$ set $\pi(s) = \sigma$. A (μ, ρ)*-admissible partial ordering satisfying* $(\star)$ is a partial ordering $\leq$ on $\rho \times \mu$ such that the following properties hold.

$\forall s,\, t \in \rho \times \mu \;\big(s \leq t \longrightarrow \pi(s) \leq \pi(t)\big)$

i.e. $\leq$ respects the partial ordering by first co-ordinates,

$\forall t \in \rho \times \mu \;\forall \sigma < \pi(t)\; \overline{\overline{\{\, s \leq t \mid \pi(s) = \sigma \,\}}} = \mu$

i.e. every element has μ many predecessors at every level below its own,

$\forall s,\, t \in \rho \times \mu \;\exists i\{s,\, t\} \in [\rho \times \mu]^{<\mu} \;\big(\forall u \in i\{s,\, t\}\; u \leq s,\, t \;\;\&$

$\qquad \forall v \leq s,\, t\; \exists u \in i\{s,\, t\}\; v \leq u \leq s,\, t\big)$

i.e. every pair of elements has a set of maximal common predecessors of size less than μ,

$\forall s,\, t \in \rho \times \mu \;\big(\pi(s) = \pi(t) \longrightarrow \neg \exists u \in \rho \times \mu\; (u \leq s,\, t)\big) \qquad (\star)$

i.e. each pair of elements from the same level have *no* common predecessors at all.

The bulk of remainder of the paper will be devoted to the following theorem.

Theorem 4. Let μ be a regular cardinal and $\rho < \mu^{+3}$. Suppose $2^{<\mu} \leq \mu^+$ and there is a stationary $(\mu^+, 1)$-simplified morass. Then there is a cardinal preserving forcing which adds a (ρ, μ)-admissible partial ordering satisfying $(\star)$. ▲

Note, first of all, that it suffices for the theorem to restrict attention to ρ of cofinality μ^{++} since truncations of admissible partial orderings satisfying $(\star)$ are still admissible partial orderings satisfying $(\star)$.

The forcing is a generalisation of (a modification of) the forcing of [M*2,§3].

Cardinals are preserved by the forcing because it is μ-closed, μ^+-$\mathbb{M}$-proper and has the μ^{++}-chain condition. The technology of [M*1] will be employed in a similar style that used in [M*2] in order to show that the forcing is μ^+-$\mathbb{M}$-proper. In addition, modifications in the style of [Mart*] are required. Moreover, in contrast with the proofs for the forcings in [M*2], the proof that the forcing has the μ^{++}-chain condition requires a substantial amount of work.

Consequently, I remind the reader of the material on decompositions of ordinals into ordinal intervals from [Mart*] in §0 and on μ^+-$\mathbb{M}$-proper forcing in §§1 and 2. The reader familiar with [Mart*] can safely skip §0 after glancing at Definition (0.9), while the reader familiar with [M*2] or [M*1] may safely skip §§1 and 2.

At the close of the paper I show how the following result is a consequence of Theorem (4).

Theorem 5. For any regular cardinal μ and $\rho < \mu^{+++}$ it is consistent (with ZFC) that there is a (locally) μ-Lindelöf, μ-tight, 0-dimensional scattered space of height ρ and width μ. ▲

In Theorem (5) one does not get ω-Lindelöf, that is compact, scattered spaces because the sizes of the sets of maximal common predecessors in the (μ, ρ)-admissible partial orders are only bounded by μ rather than by ω as would be necessary for such a result.

Most of the set theoretic notation used in the paper is standard, but it may be useful to remind the reader of a couple of items.

If $X \subseteq \text{On}$ then $\text{ssup}(X)$, the *strong supremum of* X, is the least ordinal α such that $X \subseteq \alpha$, and if τ, ν are ordinals with $\tau < \nu$ then $[\tau, \nu)$ is the interval $\{\xi \in \text{On} \mid \tau \leq \xi < \nu\}$.

It is useful when one has a function defined on increasing pairs of ordinals to have notation to denote the function applied to a pair of ordinals which one knows is in its domain, but for which one does not know which ordinal is the greater. If f is a function on increasing pairs of ordinals and $\{(\nu, \tau), (\tau, \nu)\} \cap \text{dom}(f) \neq \emptyset$ let $f\{\nu, \tau\}$ denote $f(\nu, \tau)$ if $\nu < \tau$ and $f(\tau, \nu)$ if $\tau < \nu$.

Definitions, Theorems, Lemmas and so on are numbered separately in each section. Thus, for example, Fact (4) of §1 is referred to as Fact (4) within §1 and as Fact (1.4) elsewhere. The symbol '▲' indicates the end of the statement of a result (a theorem, proposition, lemma or fact) or of a proof.

§0. Background material on cofinal trees of intervals.

Definition 1. An *ordinal interval* is an interval of the form $[\alpha, \beta)$, where α, β are ordinals with $\alpha < \beta$. Given an ordinal interval I, write I^- for $\min(I)$ and I^+ for $\mathrm{ssup}(I)$.

Definition 2. Let η, ρ be ordinals with $\eta < \rho$. A *tree of intervals* on $[\eta, \rho)$ is a collection of ordinal intervals $\mathcal{I} = \bigcup\{\mathcal{I}_n \mid n < \omega\}$ such that

(1) $\quad \mathcal{I}_0 = \{[\eta, \rho)\}$

(2) $\quad \forall I,\, J \in \mathcal{I}\ \big(I \subseteq J \text{ or } J \subseteq I \text{ or } I \cap J = \emptyset\big)$

(3) $\quad \forall I,\, J \in \mathcal{I}\ \big(I \subsetneq J\ \ \&\ \ J^+ \text{a limit ordinal} \longrightarrow I^+ < J^+\big)$

(4) $\quad \forall n < \omega\ \ \mathcal{I}_n$ is a partition of $[\eta, \rho)$

(5) $\quad \forall n < \omega\ \ \mathcal{I}_{n+1}$ refines $\mathcal{I}_n$

(6) $\quad \forall \alpha \in [\eta, \rho)\ \ \{\alpha\} \in \mathcal{I}$

Notation 3. Suppose that $\mathcal{I} = \bigcup\{\mathcal{I}_n \mid n < \omega\}$ is a tree of intervals on $[\eta, \rho)$.

(1) For $n < \omega$ let $E_n = \{I^- \mid I \in \mathcal{I}_n\}$.

(2) For $\alpha \in [\eta, \rho)$ and $n < \omega$ let $I(\alpha, n)$ be the interval $I \in \mathcal{I}_n$ with $\alpha \in I$.

(3) For $\alpha \in [\eta, \rho)$, let $l(\alpha)$ be the least n such that there is an $I \in \mathcal{I}_n$ for which $I^- = \alpha$.

(4) For $n < \omega$ let i_n be the increasing enumeration of E_n.

(5) For $n < \omega$, $I \in \mathcal{I}_n$, let e_I be the increasing enumeration of $I \cap E_{n+1}$.

(6) Write e for the (partial) function defined by $e(\nu, j, \gamma) = e_{I(\nu,j)}(\gamma)$.

Definition 4. Let $\mathcal{I}$ be a tree of intervals on $[\eta, \rho)$. Let $\eta \le \alpha < \beta < \rho$ and $I \in \mathcal{I}$. Then α, β *separate at* I if there is some $n < \omega$ such that $I = I(\alpha, n) = I(\beta, n)$ but $I(\alpha, n+1) \neq I(\beta, n+1)$. In this case n is *the level where* α and β *separate*; set $j(\alpha, \beta) = n$. Note that any pair of distinct elements of ρ separate at some interval in $\mathcal{I}$ by property 6 of Definition (2).

The following elementary lemma about j is useful later and including it here may help fix the name of the function in the reader's mind.

Lemma 5. Let $\mathcal{I}$ be a tree of intervals. Suppose $\gamma < \alpha < \beta < \rho$. Then $j(\gamma, \beta) \le j(\gamma, \alpha),\ j(\alpha, \beta)$. If $j(\gamma, \alpha) = j(\alpha, \beta)$ then $j(\gamma, \alpha) = j(\gamma, \beta) = j(\alpha, \beta)$.

If $j(\gamma,\alpha) < j(\alpha,\beta)$ then $j(\gamma,\beta) = j(\gamma,\alpha)$. And if $j(\gamma,\alpha) > j(\alpha,\beta)$ then $j(\gamma,\beta) = j(\alpha,\beta)$. ▲

Proof. If γ, β are in an ordinal interval then α must also be in the same interval. Hence $j(\gamma,\beta) \leq j(\gamma,\alpha)$, $j(\alpha,\beta)$. If for all $I \in \mathcal{I}$ one has γ, $\alpha \in I$ if and only if α, $\beta \in I$, then for all $I \in \mathcal{I}$ one has γ, $\alpha \in I$ if and only if γ, $\beta \in I$. Thus if $j(\gamma,\alpha) = j(\alpha,\beta)$ then $j(\gamma,\alpha) = j(\gamma,\beta) = j(\alpha,\beta)$. If $j = j(\gamma,\alpha) < j(\alpha,\beta)$ then $I(\gamma,j+1) \neq I(\alpha,j+1) = I(\beta,j+1)$. Thus $j(\gamma,\beta) = j(\gamma,\alpha)$. The final proposition is similar. ▲

Definition 6. A tree of intervals $\mathcal{I} = \bigcup\{\mathcal{I}_n \mid n < \omega\}$ is *cofinal* if

(1) $\quad \forall I \in \mathcal{I}_n \,\big(I^+ \text{ is a limit ordinal} \longrightarrow \mathrm{otp}(E_{n+1} \cap I) = \mathrm{cf}\,(I^+)\big).$

(2) $\quad \forall I \in \mathcal{I}_n \,\big(I^+ \text{ is a successor ordinal} \longrightarrow \overline{\overline{E_{n+1} \cap I}} < \omega\big).$

Fact 7. ([Mart]) For every ordinal interval $[\eta,\rho)$ there is a cofinal tree of intervals on $[\eta,\rho)$. ▲

Proof. By induction on $\mathrm{otp}(\rho \setminus \eta)$. Let $\langle \rho_i \mid i < \mathrm{cf}\,(\rho)\rangle$ be a continuous cofinal sequence in ρ with $\rho_i \in [\eta,\rho)$ for all $i < \mathrm{cf}\,(\rho)$. Let $\mathcal{I}_1 = \{\,[\eta,\rho_0)\,\} \cup \{\,[\rho_i,\rho_{i+1}) \mid i < \mathrm{cf}\,(\rho)\,\}$ if ρ is a limit ordinal, and $\mathcal{I}_1 = \{\,[\eta,\rho_0)\,\} \cup \{\,\rho_0\,\}$ if ρ is a successor (when $\rho = \rho_0 + 1$). Finally apply the induction hypothesis to each of the intervals in $\mathcal{I}_1$. ▲

Definition 8. Let $\rho \in [\mu^{++},\mu^{+3})$ and let $\mathcal{I}$ be a cofinal tree of intervals on $[0,\rho)$. Suppose α, β are such that $\alpha < \beta < \rho$. Let $k = j(\alpha,\beta)$, $J = I(\alpha,k+1)$ and $K = I(\beta,k+1)$. For $l < \omega$, let $\beta_l = I(\beta,l)^-$. Define *the walk from α to β along $\mathcal{I}$*, in symbols $w(\alpha,\beta)$, as follows.

$$w(\alpha,\beta) = \begin{cases} \langle \alpha,\, \beta\,\rangle & \text{if } \alpha = J^- \text{ and } \beta = K^-, \\ \langle \alpha,\, J^+,\, \beta\,\rangle & \text{if } \alpha \neq J^- \text{ and } \beta = K^-, \\ \langle \alpha,\, \beta_{k+1},\, \beta_{k+2}, \ldots, \beta_{l(\beta)-1},\, \beta\,\rangle & \text{if } \alpha = J^- \text{ and } \beta \neq K^-, \text{ and} \\ \langle \alpha,\, J^+, \beta_{k+1},\, \beta_{k+2}, \ldots, \beta_{l(\beta)-1},\, \beta\,\rangle & \text{if } \alpha \neq J^- \text{ and } \beta \neq K^-. \end{cases}$$

This concludes the basic definitions about trees of ordinal intervals taken directly from [Mart*]. Since the main idea of the proof is to show that a forcing is μ^+-$\mathbb{M}$-proper rather than having the μ^+-chain condition, the constraining functions for forcing conditions will be given synthetically with the conditions rather than *a priori*. So instead of fixing a single stretched out auxiliary function what is needed is a recipe for taking (auxiliary) functions and stretching them out. This is the point of the next definition.

Definition 9. Let $\rho \in [\mu^{++}, \mu^{+3})$ and let $\mathcal{I}$ be a cofinal tree of intervals on $[0, \rho)$. Without loss of generality, by the proof of Fact (7), suppose that $[0, \mu^{++}) \in \mathcal{I}_1$ and $\{\alpha\} \in \mathcal{I}_2$ for all $\alpha < \mu^{++}$. Suppose that $m^* : [\mu^{++}]^2 \longrightarrow \mu^+ + 1$. Define a function $m : [\rho]^2 \longrightarrow \mu^+ + 1$ extending m^* as follows.

For each $n < \omega$, $I \in \mathcal{I}_n$ and each pair ν, τ with $\nu < \tau < \mathrm{otp}(E_{n+1} \cap I)$ let $m_I(e_I(\nu), e_I(\tau)) = m^*(\nu, \tau)$. Thus $m_I : [E_{n+1} \cap I]^2 \longrightarrow \mu^+ + 1$.

Now suppose $\nu < \tau < \rho$ and write I for $I(\nu, j(\nu, \tau))$. Define

$$m(\nu, \tau) = m_I(I(\nu, j(\nu, \tau) + 1)^-, I(\tau, j(\nu, \tau) + 1)^-).$$

(Asking for agreement between m^* and m on μ^{++} plays no essential part in the proof and is only for notational convenience, allowing one to use the same name for the two functions without causing confusion.)

Definition 10. As before, if $s = (\alpha, \gamma) \in \rho \times \mu$ let $\pi(s) = \alpha$ and $l(s) = l(\pi(s))$.

§1. Background material on morasses and μ^+-$\mathbb{M}$-properness.

Definition 1. ([V]) Let κ be a regular cardinal.

$$\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa\rangle\rangle$$

is a $(\kappa, 1)$*-simplified morass* if $\langle\theta_\alpha \mid \alpha < \kappa\rangle$ is an increasing sequence of ordinals less than κ, $\theta_\kappa = \kappa^+$, and each $\mathcal{F}_{\alpha\beta}$ is a collection of maps from θ_α to θ_β such that the following properties hold:

(i) $\forall\alpha \leq \kappa\ \ \mathcal{F}_{\alpha\alpha} = \{\mathrm{id}\}$

(ii) $\forall\alpha \leq \beta \leq \gamma \leq \kappa\ \ \mathcal{F}_{\alpha\gamma} = \{g \cdot f \mid f \in \mathcal{F}_{\alpha\beta}\ \ \&\ \ g \in \mathcal{F}_{\beta\gamma}\}$

(iii) $\forall\alpha < \kappa\ \big((\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}\}\ \ \&\ \ \theta_{\alpha+1} = \theta_\alpha + 1\big)$ or $\big(\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}, h\}\ \ \&$
$\exists\sigma < \theta_\alpha\ \ (h \restriction \sigma = \mathrm{id}\ \ \&\ \ \forall\tau\ (\sigma + \tau < \theta_\alpha \longrightarrow h(\sigma + \tau) = \theta_\alpha + \tau))\big)\big)$

(iv) $\forall\alpha \leq \kappa$ (α is a limit ordinal $\longrightarrow \forall\beta_0, \beta_1 < \alpha\ \forall f_0 \in \mathcal{F}_{\beta_0\alpha}\ \forall f_1 \in \mathcal{F}_{\beta_1\alpha}$
$\exists\gamma \in [\beta_0 \cup \beta_1, \alpha)\ \exists h \in \mathcal{F}_{\gamma\alpha}\ \exists g_0 \in \mathcal{F}_{\beta_0\gamma}\ \exists g_1 \in \mathcal{F}_{\beta_1\gamma}$
$(f_0 = h \cdot g_0\ \ \&\ \ f_1 = h \cdot g_1))$

(v) $\bigcup\{f``\theta_\alpha \mid \alpha < \kappa\ \ \&\ \ f \in \mathcal{F}_{\alpha\kappa}\} = \kappa^+$

Definition 2. Let $\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa\rangle\rangle$ be a $(\kappa, 1)$-simplified morass. Then $\mathcal{F} = \bigcup\{(\alpha, f) \mid \alpha < \kappa\ \ \&\ \ f \in \mathcal{F}_{\alpha\kappa}\}$.

Definition 3. A $(\kappa, 1)$-simplified morass, $\mathbb{M}$, with $\mathcal{F}$ as in Definition (2), is *stationary* if $\{\mathrm{rge}(f) \mid \exists\alpha < \kappa\ (\alpha, f) \in \mathcal{F}\}$ is a stationary subset of $[\kappa^+]^{<\kappa}$.

Stationary $(\kappa, 1)$-simplified morasses exist in L, K_{DJ} and so on, for all regular cardinals κ, and the usual forcing ([V]) for adding $(\kappa, 1)$-simplified morasses adds stationary ones. (Notice that, as the forcing is κ-directed closed, it is consistent that κ is also supercompact.) The following well-known, fundamental fact due to Velleman always comes into play when dealing with simplified morasses.

Fact 4. (Velleman, [V, Lemma 3.2]) Let $\alpha \leq \beta \leq \kappa$, and $f, g \in \mathcal{F}_{\alpha\beta}$. If $\nu \in \mathrm{rge}(f) \cap \mathrm{rge}(g)$ there is some $\overline{\nu} < \theta_\alpha$ such that $f(\overline{\nu}) = \nu = g(\overline{\nu})$ and $f \restriction \overline{\nu} + 1 = g \restriction \overline{\nu} + 1$. ▲

Proof. By induction on β for each α. ▲

Another two useful facts from [V] are the following.

Fact 5. (Stanley, [V, Theorem 3.9]) If $\alpha \leq \beta \leq \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $\nu < \theta_\alpha$ then there is some $g \in \mathcal{F}_{\alpha\beta}$ such that $g \restriction \nu = f \restriction \nu$ and $g(\nu + \xi) = \mathrm{ssup}(g``\nu) + \xi$ for $\nu + \xi < \theta_\alpha$. ▲

Proof. Again by induction on β for each α. ▲

Fact 6. (Velleman, [V, Corollary 3.5]) If $\langle f_i \mid i < \chi \rangle$ is a collection of maps with each $f_i \in \mathcal{F}_{\alpha_i \beta}$ for $i < \chi$ and $\chi < \mathrm{cf}(\beta)$ then there is some $\alpha \in [\sup(\{\alpha_i \mid i < \chi\}, \beta)$, some $f \in \mathcal{F}_{\alpha\beta}$ and maps $g_i \in \mathcal{F}_{\alpha_i \alpha}$ for all $i < \chi$ such that $f_i = f \cdot g_i$ for all $i < \chi$. In particular any collection of fewer than κ maps in $\mathcal{F}$ can be factored through some single common map $(\alpha, f) \in \mathcal{F}$. ▲

A strengthening of Fact (4) is also well known.

Fact 7. Let $\alpha \leq \beta \leq \kappa$, and $f,\ g \in \mathcal{F}_{\alpha\beta}$. If $\nu,\ \tau \leq \theta_\alpha$ and $\mathrm{ssup}(f``\nu) = \mathrm{ssup}(g``\tau)$ then $\nu = \tau$ and $f \restriction \nu = g \restriction \nu$. ▲

Proof. By induction on β for each α. (For $\nu,\ \tau < \theta_\alpha$ this is an immediate corollary of Facts (5) and (6), without the necessity of a separate inductive proof.) ▲

In contrast the following Δ-system type proposition does not seem to have been noted previously. The salient point is that no cardinal arithmetic assumption is necessary.

Proposition 8. (A Δ-system lemma for morass maps). Let $\langle (\alpha_\eta, F_\eta) \mid \eta < \kappa^+ \rangle$ be a collection of distinct maps in $\mathcal{F}$. Then there is a (stationary) set $S \subseteq \kappa^+$ such that $\langle \mathrm{rge}(F_\eta) \mid \eta \in S \rangle$ forms a Δ-system. ▲

Proof. Let $S_\kappa = \{\gamma < \kappa^+ \mid \mathrm{cf}(\gamma) = \kappa\}$. By Fodor's lemma one may as well assume that there is some $\alpha < \kappa$ and a stationary set $S \subseteq S_\kappa$ such that $\alpha_\eta = \alpha$ for all $\alpha \in S$.

Next set $G(\eta) = \mathrm{ssup}(\mathrm{rge}(F_\eta) \cap \eta)$ for $\eta \in S$. As $\mathrm{cf}(\eta) = \kappa$ and $\mathrm{otp}(\mathrm{rge}(F_\eta)) = \theta_\alpha < \kappa$ for all $\eta \in S$, one has that $G(\eta) < \eta$ for every $\eta < \kappa^+$. By Fodor's lemma again one has that there is some $\tau < \kappa^+$ and some stationary $T \subseteq S$ such that $G(\eta) = \tau$ for all $\eta \in T$.

Hence, for each $\eta \in T$ one has that there is some $\tau_\eta \leq \theta_\alpha$ such that $\tau = \mathrm{ssup}(F_\eta``\tau_\eta)$. By Fact (7) one has that $\tau_\eta = \tau_\xi, = \overline{\tau}$, say, and that $F_\eta \restriction \overline{\tau} = F_\xi \restriction \overline{\tau}$ for all $\eta,\ \xi \in T$. Consequently $\mathrm{rge}(F_\eta) \cap \eta = \mathrm{rge}(F_\xi) \cap \xi$ $(\subseteq \tau)$ for all $\eta,\ \xi \in T$. (It is by the use of Fact (7) here that the necessity for cardinal arithemetic assumptions is avoided.)

Note also that $\tau < \theta_\alpha$ as otherwise all of the maps F_η for $\eta \in T$ are identical. Finally, choose by induction, $\{\eta_i \mid i < \kappa^+\} \subseteq T$ by letting η_ζ be the least $\eta \in T$ such that $\bigcup\{\mathrm{rge}(F_{\eta_i}) \mid i < \zeta\} \subseteq \eta$. ▲

The following notation is now more or less standard for simplified morasses (see, for example, [M]).

Definition 9. Let $\alpha \leq \beta \leq \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $f(\overline{\nu}) = \nu$. Then $\psi_{\langle \alpha, \overline{\nu} \rangle, \langle \beta, \nu \rangle}$ is the function $f \restriction \overline{\nu} + 1$. By Fact (3) this is a good definition. If $\beta = \kappa$ write ψ^{ν}_{α} for $\psi_{\langle \alpha, \overline{\nu} \rangle, \langle \beta, \nu \rangle}$. Write ν_α for the $\overline{\nu}$ such that $\psi^{\nu}_{\alpha}(\overline{\nu}) = \nu$.

Definition 10. If $\beta \leq \kappa$ and $\nu < \tau < \theta_\beta$ then $c_\beta(\nu, \tau) =$ the least $\alpha \leq \beta$ such that there is some $f \in \mathcal{F}_{\alpha\beta}$ with ν, $\tau \in \mathrm{rge}(f)$. Write c for c_κ and note that $c(\nu, \tau) < \kappa$ for all ν, $\tau < \kappa^+$ by properties (iii) and (v) of gap-one simplified morasses. c is the *coupling function* for $\mathbb{M}$.

As motivation for some of the properties of the functions defined in Definition (12) below I also recall the following fact.

Fact 11. ([M]) c has the following properties:

(a) If $\alpha < \kappa$ and $\tau < \kappa^+$ then $\overline{\overline{\{\xi \mid c(\xi, \tau) \leq \alpha\}}} < \kappa$.

(b) c is *subadditive*: if $\nu < \tau < \xi < \kappa^+$ then

(i) $$c(\nu, \tau) \leq \max(\{c(\nu, \xi), c(\tau, \xi)\})$$
(ii) $$c(\nu, \xi) \leq \max(\{c(\nu, \tau), c(\tau, \xi)\})$$

(c) If $\xi < \lambda = \bigcup \lambda < \varepsilon < \kappa^+$ there then is some $\zeta \in [\xi, \lambda)$ such that for all $\nu \in [\zeta, \lambda)$ one has $c(\nu, \lambda) \leq c(\nu, \varepsilon)$. ▲

Now I turn to definitions concerned with the local connectedness functions discussed in [M*1].

Definition 12. ([M*1, Definition 1.2 and Lemma 1.9]) Let $A \subseteq \mathcal{F}$. Define $c_A(\nu, \tau) =$ the least $\alpha < \kappa$ such that there are $(\alpha_0, f_0), \ldots (\alpha_{k-1}, f_{k-1}) \in A$, with $\alpha_0 < \ldots < \alpha_{k-1} = \alpha$, and $\xi_k = \nu \leq \xi_{k-1} \leq \ldots \leq \xi_0 = \tau$ with $\forall i < k$ ($\alpha_i \leq \alpha$ & $f_i \in \mathcal{F}_{\alpha_i \kappa}$ & $\xi_i, \xi_{i+1} \in \mathrm{rge}(f_i)$), if there is such an α, and $c_A(\nu, \tau) = \kappa$ otherwise.

A handful of elementary facts are very useful.

Fact 13. ([M*1, Lemma 1.3]) Suppose that $A \subseteq B \subseteq \mathcal{F}$ and $\nu < \tau < \kappa^+$. Then $(A \subseteq B \longrightarrow c_B(\nu, \tau) \leq c_A(\nu, \tau))$. ▲

Fact 14. ([M*1, Lemma 1.4]) Suppose that $A \subseteq \mathcal{F}$ and $\nu < \tau < \kappa^+$. If $f \in \mathcal{F}_{\beta\kappa}$, $\tau \in \mathrm{rge}(f)$ and $c_A(\nu, \tau) \leq \beta$ then $\nu \in \mathrm{rge}(f)$. ▲

Fact 15. ([M*1, Lemma 1.6]) Suppose $A \subseteq \mathcal{F}$, $A = B \cup \{(\beta, g)\}$, and $\alpha \leq \beta$ for all $(\alpha, f) \in A$. Suppose $\nu < \tau < \zeta < \kappa^+$ and $\nu \in \operatorname{rge}(g)$ if $\tau \in \operatorname{rge}(g)$. Then $c_A(\nu, \tau) \leq c_A(\tau, \zeta)$ implies that $c_B(\nu, \tau) \leq c_B(\tau, \zeta)$. ▲

Fact 16. ([M*1, Lemma 1.11]) $\forall A, B \subseteq \mathcal{F}\ \forall \tau < \zeta < \nu < \kappa^+ \big(A \subseteq B \longrightarrow (c_B(\tau, \zeta) \leq c_B(\zeta, \nu) \longrightarrow c_A(\tau, \zeta) \leq c_A(\zeta, \nu)\big)$. ▲

Fact 17. ([M*1, Lemma 1.14]) Suppose $\alpha < \kappa$, $D \subseteq \bigcup\{\mathcal{F}_{\gamma\alpha} \mid \gamma < \alpha\}$ and (α, F), $(\alpha, G) \in \mathcal{F}$. Set $A = \{(\gamma, F \cdot g) \mid \exists \gamma < \alpha\ g \in D \cap \mathcal{F}_{\gamma\alpha}\}$, $B = \{(\gamma, G \cdot g) \mid \exists \gamma < \alpha\ g \in D \cap \mathcal{F}_{\gamma\alpha}\}$ and $E = A \cup B$. Suppose $\xi < \zeta < \kappa^+$ and $c_E(\xi, \zeta) \leq \alpha$. If $\zeta \in \operatorname{rge}(F)$, then $c_E(\xi, \zeta) = c_A(\xi, \zeta)$, and if $\zeta \in \operatorname{rge}(G)$ then $c_E(\xi, \zeta) = c_B(\xi, \zeta)$. ▲

Finally, one of the most important basic facts about the functions c_A is that they are subadditive.

Fact 18. ([M*1, Theorem 1.12] If $A \subseteq \mathcal{F}$ and $\nu < \tau < \zeta < \kappa^+$ then

(i) $$c_A(\nu, \tau) \leq \max(\{c_A(\nu, \tau), c_A(\tau, \zeta)\}),$$
(ii) $$c_A(\nu, \zeta) \leq \max(\{c_A(\nu, \tau), c_A(\tau, \zeta)\}).$$ ▲

Now suppose that $\mathbb{M}$ is stationary and let $\mu = \kappa^-$ be the cardinal predecessor of κ. (Up until now everything mentioned was true for any regular κ, but the following definition as stated only makes sense for successor κ.)

Definition 19. Let λ be a regular cardinal greater than ω_2. Then $(\mathcal{N}, \epsilon) \prec (H_\lambda, \in)$ is a *good model* if $\overline{\overline{\mathcal{N}}} = \mu$, $\{\mathbb{M}, \mathcal{F}, c, \kappa, \kappa^+\} \cup \mu \subseteq \mathcal{N}$, $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$, there is some $F \in \mathcal{F}_{\delta\kappa}$ such that $\operatorname{rge}(F) = \mathcal{N} \cap \kappa^+$, and for each $(\alpha, f) \in \mathcal{F}$ with $\alpha < \delta$ if there is some $f' \in \mathcal{F}_{\alpha\delta}$ such that $f = F \cdot f'$ then (α, f), $\operatorname{rge}(f) \in \mathcal{N}$.

The following two observations about good models are useful in the next section.

Lemma 20. If $\mathcal{N}$ is good, $\alpha < \delta$, $g \in \mathcal{F}_{\alpha\kappa}$, $g' \in \mathcal{F}_{\alpha\delta}$, $g'' \in \mathcal{F}_{\delta\kappa}$ and $g = g'' \cdot g'$ then $h(g) = F \cdot g' \in \mathcal{F}_{\alpha\kappa} \cap \mathcal{N}$ (and so $(\alpha, h(g)) \in \mathcal{F} \cap \mathcal{N}$). ▲

Proof. As $\mathcal{N}$ is good, if $\alpha < \delta$ and $g' \in \mathcal{F}_{\alpha\delta}$ then $F \cdot g' \in \mathcal{N}$. ▲

Lemma 21. If $\mathcal{N}$ is good and $(\alpha, f) \in \mathcal{N}$ then $\exists f' \in \mathcal{F}_{\alpha\delta}\ (f = F \cdot f')$. ▲

Proof. If $(\alpha, f) \in \mathcal{N}$ then, firstly, $f \in \mathcal{N}$, and, secondly, $\alpha < \delta$, since $\delta = \mathcal{N} \cap \kappa$. Since $\mathbb{M} \in \mathcal{N}$ and $\alpha \in \mathcal{N}$ one has that $\theta_\alpha < \delta$, and hence that $\theta_\alpha \subseteq \mathcal{N}$. So if $\xi < \theta_\alpha$ then $f(\xi) \in \mathcal{N}$. Hence $\operatorname{rge}(f) \subseteq \kappa^+ \cap \mathcal{N} = \operatorname{rge}(F)$.

Now factor f as $k \cdot h$, where $(\delta, k) \in \mathcal{F}$ and $h \in \mathcal{F}_{\alpha\delta}$. Let $\overline{\theta} = \mathrm{ssup}(h``\theta_\alpha)$. Then there is some $\zeta \leq \theta_\delta$ such that $\mathrm{ssup}(F``\zeta) = \mathrm{ssup}(k``\overline{\theta})$. Consequently, by Fact (7), one has that $\zeta = \overline{\theta}$ and $F \restriction \overline{\theta} = k \restriction \overline{\theta}$. Thus one also has that $f = F \cdot h$. ▲

However the main point of introducing the notion of a good model is in order to formulate the following definition and fact.

Definition 22. Let $\mathbb{P}$ be a forcing notion with $\mathbb{P} \in H_\lambda$ for some regular cardinal λ. $\mathbb{P}$ is κ-$\mathbb{M}$-*proper* if there is some $x \in [H_\lambda]^{\leq\mu}$ such that the following holds. Suppose $p \in \mathbb{P}$ and $\mathcal{N}$ is a good model with $\{\mathbb{P}, p\} \cup x \cup p \subseteq \mathcal{N}$. Then there is some $p^* \leq p$ which is $(\mathbb{P}, \mathcal{N})$-generic.

Fact 23. If $\mathbb{P}$ is κ-$\mathbb{M}$-proper then $\Vdash_{\mathbb{P}} \mu^+ = \kappa$. ▲

§2. A general framework for proofs of κ-$\mathbb{M}$-properness.

Now I give a general framework of proofs of κ-$\mathbb{M}$-properness. In order to complete the proof of κ-$\mathbb{M}$-properness in any specific case one will have to complete the steps labeled 'Step i' for i from A to D.

Suppose that $\mathbb{P}$ is a forcing and that conditions in $\mathbb{P}$ are of the form $p = (a_p, A_p, \ldots)$, where $a_p \in [\kappa^+]^{<\mu}$ is the *realm* of p and $A_p \in [\mathcal{F}]^{<\mu}$ is the *side condition part* of p.

Fix $p \in \mathbb{P}$. Suppose that $\mathcal{N} \prec H_{\kappa^{++}}$ has size κ, $\mathbb{P}$, p, $\mathbb{M}$, $\mathcal{F}$, κ, $\kappa^+ \in \mathcal{N}$, p, $\mu \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$ and that there is some $F \in \mathcal{F}_{\delta\omega_1}$ such that $F``\theta_\delta = \mathcal{N} \cap \kappa$.

The aim of the proof will be to show that $(a_p, A_p \cup \{(\delta, F)\})$ is $(P, \mathcal{N})$-generic. The first step that needs to be carried out in a particular case is

Step A. Writing p^* for $(a_p, A_p \cup \{(\delta, F)\}, \ldots)$, show that $p^* \in \mathbb{P}$ and $p^* \leq p$. ▲

If this can be done, suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$ with $\mathcal{D} \in \mathcal{N}$. Suppose also that $q \in \mathcal{D}$ and that $q \leq (a_p, A_p \cup \{(\delta, F)\}, \ldots)$. For technical reasons one also needs to be able to do the following

Step B. Show that one may, without loss of generality, assume that $c_{A_q}(\nu,\tau) < \kappa$ for all ν, $\tau \in a_q$. ▲

Next let $q \restriction \mathcal{N} = q \cap \mathcal{N} = (a_q \cap \mathcal{N}, A_q \cap \mathcal{N}, \ldots)$.

Step C. Prove that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and that $q \leq q \restriction \mathcal{N}$. ▲

If Step (C) can be carried out then next make the following long notational definition.

Notation.

Let $\langle (\beta_i, f_i) \mid i < \chi \rangle$ enumerate $\langle (\alpha, f) \in A_q \mid \alpha < \delta \rangle$ for some $\chi < \mu$.
Let $Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(h(f_i))$ for $i < \chi$.
Note that $Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(F)$ for all $i < \chi$ as well.
Let $\rho_i = \mathrm{ssup}(\{\rho < \theta_{\alpha_i} \mid f_i(\rho) = h(f_i)(\rho)\})$.
Then $Y_i = \mathrm{rge}(f_i \restriction \rho_i) = \mathrm{rge}(\psi_{(\alpha_i,\rho_i),(\kappa,f_i(\rho_i))} \restriction \rho_i)$.
Let $\beta^* = \mathrm{ssup}(\{\beta_i \mid i < \chi\})$.
Let $C_q = \{c_{A_q}(\zeta,\xi) \mid \zeta, \xi \in a_q\} = c_{A_q}``[a_q]^2$.
Let $e_q \in [\kappa]^{<\mu}$ be arbitrary, $e_{q\restriction\mathcal{N}} = e_q \cap \mathcal{N}$, and
$\beta^\dagger = \max(\{\beta^*, \mathrm{ssup}(e_{q\restriction\mathcal{N}})\})$.

Write $c^q(\zeta,\xi)$ in place of $c_{A_q}(\zeta,\xi)$, and similarly for other conditions in $\mathbb{P}$, in order to save on subscripts!

Note that as Y_i is an initial segment of $\mathrm{rge}(h(f_i)) \subseteq \mathrm{rge}(F) = \mathcal{N}$ for $i < \chi$ (by Fact (1.4)), one has that $Y_i \subseteq \mathcal{N}$ for each $i < \chi$. Note also that $Y_i \in \mathcal{N}$ and as $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$ one has that $\langle Y_i \mid i < \chi \rangle \in \mathcal{N}$.

Let $\phi(x)$ be the conjunction of the following:

(i) $x \in \mathcal{D}$, (ii) $x \leq q \restriction \mathcal{N}$, and

(iii) $\exists (\alpha^*, h^*) \in A_x$ with $\beta^\dagger < \alpha^*$ such that

(1) $\forall (\beta, f) \in A_x \ \big(\beta < \beta^* \longrightarrow \exists i < \chi\ (\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i)\big)$,

(2) $\forall i < \chi\ \exists (\beta_i, f) \in A_x\ \big(\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i\ \ \&\ \ \exists f'' \in \mathcal{F}_{\alpha^*\kappa}$
$\exists f' \in \mathcal{F}_{\beta\alpha^*}\, f = f'' \cdot f' \longrightarrow \mathrm{rge}(h^* \cdot f') \cap \mathrm{rge}(f) = Y_i\big)$,

(3) $a_{q\restriction\mathcal{N}} \subseteq \mathrm{rge}(h^*)$,

(4) $(a_x \setminus a_{q\restriction\mathcal{N}}) \cap \mathrm{rge}(h^*) = \emptyset$,

(5) $e_{q\restriction\mathcal{N}} \subseteq e_x\ \ \&\ \ (e_x \setminus \beta^\dagger) \cap \alpha^* = \emptyset$,

(6) $\forall \gamma\ (\gamma \in C_x \longrightarrow \gamma < \kappa)$.

Then $H_{\kappa^{++}} \models$ "$\phi(q)$". So by the elementarity of $\mathcal{N}$ in $H_{\kappa^{++}}$ there is some $s \in \mathcal{N}$ with $\mathcal{N} \models \phi(s)$. Let $(\alpha^*, h^*) \in \mathcal{N}$ be the witness to (iii) in the definition of ϕ for s in $\mathcal{N}$. Set $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$.

Step D. Show that there is a condition r with realm containing a_r and side condition part A_r such that $r \in \mathbb{P}$ and $r \leq q$, s. ▲

The pay-off of this elaborate abstract set-up is that it is proved in [M*1] that one now has the following properties which one can use in order to complete Step (D).

[M*1, 4.12] $\forall \zeta_0 < \zeta_1 < \kappa^+ \; \big(\zeta_0,\ \zeta_1 \in a_s \longrightarrow c^s(\zeta_0, \zeta_1) = c^r(\zeta_0, \zeta_1) < \delta\big)$.

[M*1, 4.15] $\forall \sigma < \xi < \kappa^+ \; \big(\xi \in a_q \;\;\&\;\; \beta < \alpha^* \longrightarrow \big(c^r(\sigma, \xi) = \beta \longleftrightarrow c^q(\sigma, \xi) = \beta\big)\big)$.

[M*1, 4.16] $\forall \xi_0 < \xi_1 < \kappa^+,\ \forall \beta \in [\delta, \kappa) \; \big(c^q(\xi_0, \xi_1) \leq \beta \longleftrightarrow c^r(\xi_0, \xi_1) \leq \beta\big)$.

[M*1, 4.17] $\forall \xi < \zeta < \kappa^+ \; \big(\xi \in a_q \setminus a_s \;\;\&\;\; \zeta \in a_s \longrightarrow \delta \leq c^r(\xi, \zeta) = c^q(\xi, \zeta)\big)$.

[M*1, 4.19] $\forall \zeta < \xi < \kappa^+ \; \big(\zeta \in a_s \setminus a_q \;\;\&\;\; \xi \in a_q \longrightarrow \alpha^* \leq c^r(\zeta, \xi)\big)$.

[M*1, 4.21] $\forall \tau < \xi < \kappa^+ \; \big(\tau \in a_s \cap a_q \;\;\&\;\; \xi \in a_q \setminus a_s \longrightarrow \big(c^r(\tau, \xi) \leq \alpha^* \text{ or } c^r(\tau, \xi) \geq \delta\big)\big)$.

[M*1, 4.24] $\forall \zeta,\ \tau,\ \xi < \kappa^+ \; \big(\zeta \in a_s \setminus a_q \;\;\&\;\; \tau \in a_s \cap a_q \;\;\&\;\; \xi \in a_q \setminus a_s \longrightarrow c^r\{\zeta, \tau\} \leq c^r\{\zeta, \xi\}\big)$.

(Recall that $c^r\{\nu, \upsilon\} = c^r(\min\{\nu, \upsilon\}, \max\{\nu, \upsilon\})$)

[M*1, 4.27] $\forall \xi_0 < \xi_1 < \kappa^+ \; \big(\xi_0,\ \xi_1 \in a_q \setminus a_s \longrightarrow c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)\big)$.

[M*1, 4.28] $\forall \tau_0 < \tau_1 < \kappa^+ \; \big(\tau_0,\ \tau_1 \in a_q \cap a_s \longrightarrow c^r(\tau_0, \tau_1) \leq \alpha^*\big)$.

One further fact, which is immediate from (iii.3) of the definition of ϕ for q and s, (iii.5) of the definition of ϕ for q, [M*1, 4.12], [M*1, 4.15] and [M*1, 4.28], is perhaps also worth isolating.

$$\begin{aligned} &\forall \tau_0 < \tau_1 < \kappa^+ \; \big(\tau_0,\ \tau_1 \in a_q \cap a_s \longrightarrow \\ &\qquad \big(c^r(\tau_0, \tau_1) = c^q(\tau_0, \tau_1) = c^s(\tau_0, \tau_1) < \alpha^* \text{ or} \\ &\qquad c^r(\tau_0, \tau_1) = c^s(\tau_0, \tau_1) = \alpha^* < \delta = c^q(\tau_0, \tau_1)\big)\big) \end{aligned} \; .$$

§3. Proof of Theorem (4) of the Introduction.

Definition 1. Let $p = (x_p, \leq_p, A_p) \in \mathbb{P}$ if $x_p \in [\rho \times \mu]^{<\mu}$, $\leq_p$ is a partial order on x_p, $A_p \in [\mathcal{F}]^{<\mu}$ and, writing c^p for the function obtained as in Definition (0.9) from c_{A_p},

$$\forall s, t \in x_p \ \big(s \leq_p t \longrightarrow \pi(s) \leq \pi(t)\big)$$

$$(\bullet) \quad \forall u, s, t \in x_p \ \big(\pi(s) < \pi(t) \ \& \ s \not\leq_p t \ \& \ u \leq_p s, t \ \& \ \neg\exists v \ (u <_p v \leq_p s, t) \ \longrightarrow \big(\pi(u) < I(\pi(s), j(\pi(s), \pi(t)) + 1)^- \ \& \ c^p(\pi(u), \pi(t)) \leq c^p(\pi(s), \pi(t)) \text{ or } \pi(u) = I(\pi(s), j(\pi(s), \pi(t)))^-\big)\big)$$

$$(\star) \quad \forall s, t \in x_p \ \big(\pi(s) = \pi(t) \longrightarrow \neg\exists u \in x_p \ (u \leq_p s, t)\big).$$

$$(\circ) \quad \forall s, t \in x_p \ \big(s <_p t \longrightarrow \exists u_1, \ldots, u_n \in x_p \ \big(s <_p u_1 \leq_p \ldots \leq_p u_n \leq_p t \ \& \ w(\pi(s), \pi(t)) = \langle \pi(s), \pi(u_1), \ldots, \pi(u_n), \pi(t)\rangle\big)\big).$$

If $p \in \mathbb{P}$, $s, t \in x_p$, $s \not\leq_p t$ and $\pi(s) \leq \pi(t)$ set

$$i_p\{s, t\} = \{u \in x_p \mid u \leq_p s, t \ \& \ \neg\exists v \ (u <_p v \leq_p s, t)\}.$$

The ordering of $\mathbb{P}$ is that $q \leq_{\mathbb{P}} p$ if $x_p \subseteq x_q$, $A_p \subseteq A_q$, $\leq_q \restriction x_p \times x_p = \leq_p$ and $i_q \restriction [x_p]^2 = i_p$.

Definition 2. (Similarly to [Mart].) If $p \in \mathbb{P}$, $s, t \in a_p$ and $u_1, \ldots, u_n \in x_p$ are witnesses to $(\circ)$ for s and t then $\langle u_1, \ldots, u_n\rangle$ is a *walk from s to t in p*. Note that walks (from s to t) may include repeated elements. If a walk does include a repeated element then any tuple which is a contraction of it in which repeats are elimated will also be called a walk (from s to t).

$\mathbb{P}$ is similar to the forcing of [M*2,§3] since that forcing is the natural κ-$\mathbb{M}$-proper analogue of the Baumgartner-Shelah forcing, modulo the isomorphism of sending $\mu^{++} \times \mu$ to μ^{++} by $(\mu\nu, \xi) \mapsto \mu\nu + \xi$ for $\nu < \mu^{++}$ and $\xi < \mu$. However $(\star)$ and $(\circ)$ are modifications, inherited from Martinez's forcing.

Remark. If $u \in i_p\{s, t\}$, where s and $t \in x_p$ are incomparable in $\leq_p$, and the second alternative of the conclusion of $(\bullet)$ fails for u, s and t, so that $\pi(u) \neq I(\pi(s), j(\pi(s), \pi(t)))^-$, then, as will be shown in Corollary (5) below, it doesn't matter whether I use $c^p(\pi(u), \pi(t)) \leq c^p(\pi(s), \pi(t))$ or $c^p(\pi(u), \pi(s)) \leq c^p(\pi(s), \pi(t))$ as the constraint in the first alternative since all three instances of c^p are calculated using the same, subadditive, function. The second alternative of the conclusion of $(\bullet)$ says that elements of $i\{s, t\}$

can be at the level of the left hand end point of the interval where $\pi(s)$ and $\pi(t)$ separate. ▲

Since it is convenient to fit $\mathbb{P}$ into the framework for μ^+-$\mathbb{M}$-proper forcing set out in §2, it is incumbent on me to define a notion of *realm* for conditions $p \in \mathbb{P}$. A first idea is to take $\pi``x_p$ as the realm of p. One problem with this is that it gives a subset of ρ rather than of μ^{++} as required in the framework. A deeper problem, however, is that this putative definition ignores the fact that ordinals other than $\pi(s)$ and $\pi(t)$ enter into the computations of $c^p(\pi(s), \pi(t))$ for pairs of elements $s, t \in x_p$. As the analysis of [M*1, §4] will be heavily used in order to ensure that a suitable amalgamation of two conditions proves the μ^+-$\mathbb{M}$-properness of $\mathbb{P}$, a more elaborate definition of realm, which includes these extra ordinals, is necessary.

Notation 3. Let $p \in \mathbb{P}$. Define a_p, the *realm of* p, to be

$$\{ e^{-1}_{I(\nu,j)}(I(\nu, j+1)^-) \mid j < \omega \ \ \& \ \ \exists v \in x_p \ (\pi(v) = \nu) \}.$$

Note that each $v \in x_p$ only contributes finitely many ordinals to a_p since if $\nu = \pi(v)$ there is some $j < \omega$ such that $I(\nu, j)^- = \nu$ and hence $I(\nu, k)^- = \nu$ for all $k > j$. Thus if $\mu = \omega$ one has that a_p is finite.

Note also that, as with i_p, one can please oneself as to whether a_p should be added as a constituent of conditions in the official definition of the forcing notion. For more on this see the remark after the introduction of $\mathcal{N}$ at the start of the proof of Proposition (6) below.

Before really getting down to business it is worthwhile spending a moment thinking about what the possible levels of elements in $i_p\{s, t\}$ can be for non-comparable elements s, t in a condition p.

Lemma 4. Suppose $p \in \mathbb{P}$. Let $u, s, t \in x_p$, with $u \in i_p\{s, t\}$, $\pi(s) < \pi(t)$ and $s \not\leq_p t$. Write γ for $\pi(u)$, α for $\pi(s)$, and β for $\pi(t)$.

(a) $k = j(\gamma, \beta) < j(\gamma, \alpha) \longrightarrow \gamma = I(\alpha, k)^- = I(\alpha, k+1)^-$.
(One also has that $c^p(\gamma, \beta)$ and $c^p(\alpha, \beta)$ are both calculated using $c_{I(\gamma,k)}$, although this is not particularly useful in this case.)

(b) $j = j(\gamma, \beta) = j(\gamma, \alpha) \longrightarrow j(\alpha, \beta) = j \ \ \& \ \ \gamma = I(\gamma, j+1)^-$.
Note that this implies that $c^p(\gamma, \alpha)$, $c^p(\gamma, \beta)$ and $c^p(\alpha, \beta)$ are all calculated using $c_{I(\gamma,j)}$.

▲

Proof. (a). Suppose that $k = j(\gamma,\beta) < j(\gamma,\alpha)$. Then $I(\gamma,k+1) = I(\alpha,k+1)$ and so $\gamma \in I(\alpha,k+1)$. By Lemma (0.5) one has that $j(\alpha,\beta) = k$. As $\gamma \not< I(\alpha,k+1)^-$ it must be the case that $\gamma = I(\alpha,k)^- = I(\alpha,k+1)^-$, by ($\bullet$). ▲

(b). Suppose $j(\gamma,\alpha) = j(\gamma,\beta)$, $= j$, say. If $\gamma \neq I(\gamma,j+1)^-$, then, by ($\circ$), there are s', t' with $u <_p s' <_p s$ and $u \leq_p t' \leq_p t$ and $\pi(s') = \pi(t') = I(\gamma,j+1)^+$. This is a contradiction to ($\star$) unless $s' = t'$. But that would contradict u being maximal below both s and t. Hence one must have that $\gamma = I(\gamma,j+1)^-$.

If $j < j(\alpha,\beta)$ then $I(\alpha,j+1) = I(\beta,j+1)$, so, by ($\circ$), there are s' and t' with $u \leq_p s' \leq_p s$ and $u \leq_p t' \leq_p t$ and $\pi(s') = \pi(t') = \beta_j$. As before this contradicts ($\star$) if $s' \neq t'$ and the maximality of u if $s' = t'$. Hence $j = j(\alpha,\beta)$. ▲

Note. The proof of (b) requires only that p satisfies ($\star$) and ($\circ$). ($\bullet$) is not used in the proof. This allows one to eliminate some of the subcases in Case (C) in the proof of ($\bullet$) of Claim (16) below.

Corollary 5. Suppose $p \in \mathbb{P}$. Let u, s, $t \in x_p$, with $u \in i_p\{s,t\}$, $\pi(s) < \pi(t)$ and $s \not\leq_p t$. Write γ for $\pi(u)$, α for $\pi(s)$, and β for $\pi(t)$. Let $j = j(\alpha,\beta)$. Then $j = j(\gamma,\beta)$ and $\gamma = I(\gamma,j+1)^-$. Moreover, if $\gamma \neq I(\alpha,j)^-$ then $j = j(\gamma,\beta) = j(\gamma,\alpha)$ and $c^p(\gamma,\alpha)$, $c^p(\gamma,\beta)$ and $c^p(\alpha,\beta)$ are all calculated using $c_{I(\gamma,j)}$. ▲

Proof. $j(\gamma,\beta) \leq j(\gamma,\alpha)$, by the first part of the conclusion of Lemma (0.5), so the hypothesis of either (a) or (b) of Lemma (4) is satisfied. Hence, if the conclusion of Lemma (4.a) fails then that of Lemma (4.b) must hold. ▲

Now I launch off into the μ^+-$\mathbb{M}$-properness proof.

Proposition 6. $\mathbb{P}$ is μ^+-$\mathbb{M}$-proper. ▲

Proof. Suppose that $p \in \mathbb{P}$ and that $\mathcal{N}$ is as in the general framework from [M*2] but for this specific $\mathbb{P}$ and that $\{\rho,\mathcal{I}\} \cup \{i_k \mid k < \omega\} \subseteq \mathcal{N}$. ($\{\rho,\mathcal{I}\} \cup \{i_k \mid k < \omega\}$ plays, here, the rôle of the "x" of Definition (1.22).)

As $\mathcal{I}_k$ is a partition of ρ one has that $I(\nu,k)^+ \in E_k$. Let $i_k(\gamma) = I(\gamma,k)^-$. If $\nu \in \mathcal{N} \cap \rho$ then $i_k(\gamma) \in \mathcal{N}$ and hence $\gamma \in \mathcal{N}$. Thus $i_k(\gamma+1) = I(\gamma,k)^+ \in \mathcal{N}$. And if ν, $\tau \in \mathcal{N} \cap \rho$ then $\mathcal{N}$ calculates $j(\nu,\tau)$ correctly, by the elementarity of $\mathcal{N}$ and since ρ, $i \in \mathcal{N}$. Note, also, that $e \in \mathcal{N}$ and if ν, $\tau \in \mathcal{N} \cap \rho$ then $e^{-1}_{I(\nu,j\{\nu,\tau\})}(I(\nu,j\{\nu,\tau\}+1)^-) \in \mathcal{N}$. Hence if $s \in \mathbb{P} \cap \mathcal{N}$ one has that $a_s \subseteq \mathcal{N} \cap \mu^{++} = \mathrm{rge}(F)$, and that $a_s \in \mathcal{N}$ since $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$.

Step A. Checking that $p' = (x_p, \leq_p, i_p, A_p \cup \{(\delta, F)\})$ is a condition and is stronger than p. ▲

Proof. The only thing that needs to be checked is that p' satisfies ($\bullet$). Suppose that s, t, $u \in x_p$, $s \not\leq_p t$, $u \in i_p\{s, t\}$, $\pi(u) < \pi(s) < \pi(t)$. There are two cases to consider. First of all, suppose that $\pi(u) = I^-$, where $I = I(\pi(s), j(\pi(s), \pi(t)))$. This is just a statement about the tree of ordinal intervals $\mathcal{I}$ and so remains true for p'. Otherwise, if $c^p(\pi(u), \pi(s)) \leq c^p(\pi(s), \pi(t))$ then $c^p = c^p_I$ in both instances by Lemma (4). Likewise $c^{p'} = c^{p'}_I$ when applied to any pair from $\pi(u)$, $\pi(s)$ and $\pi(t)$. Observe that $(\alpha, f) \in \mathcal{N}$ for each $(\alpha, f) \in A_p$, since $p \subseteq \mathcal{N}$. So for each $(\alpha, f) \in A_p$ one has, by (i) of the properties of $\mathcal{N}$ given in §2, that there is some $f' \in \mathcal{F}_{\alpha\beta}$ such that $f = F \cdot f'$. Thus, as $a_p \subseteq \mathcal{N} \cap \mu^{++} = \mathrm{rge}(F)$, one has that $c^{p'}(\pi(u), \pi(s)) \leq c^{p'}(\pi(s), \pi(t))$ by Fact (1.15). Hence p' satisfies ($\bullet$), and thus $p' \in \mathbb{P}$ and $p' \leq p$. ▲

Suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$, $\mathcal{D} \in \mathcal{N}$ and $q \in \mathcal{D}$ where $q \leq p'$.

Step B. Show that without loss of generality one may assume that $c^q(\nu, \tau) < \kappa$ for all ν, $\tau \in \pi``x_q$. ▲

Proof. First of all, note that if $\nu < \tau < \rho$ then one has, in the notation of Definition (0.9), that

$$c^q(\nu, \tau) = c^q(e^{-1}_{I(\nu, j(\nu,\tau))}(I(\nu, j(\nu, \tau) + 1)^-), e^{-1}_{I(\nu, j(\nu,\tau))}(I(\tau, j(\nu, \tau) + 1)^-)).$$

Thus if it is not the case that $c^q(\nu, \tau) < \kappa$ for all ν, $\tau \in \pi``x_q$, it suffices simply to add to A_q, by applying Fact (1.6), a map (β, f) such that $\alpha \leq \beta$ for all $(\alpha, g) \in A_q$ and

$$e^{-1}_{I(\nu,j)}(I(\nu, j + 1)^-) \in \mathrm{rge}(f) \text{ for all } \nu \in x_q \text{ and } j < \omega$$
$$\text{such that } I(\nu, j)^- \neq \nu.$$

By Fact (1.15), this augmentation, $(x_q, \leq_q, A_q \cup \{(\beta, f)\})$, of q is still a condition since, as in the proof of Step (A), it satisfies ($\bullet$). It is stronger than q by the definition of $\leq_{\mathbb{P}}$, and is in $\mathcal{D}$ because $\mathcal{D}$ is open. ▲

Let $q \restriction \mathcal{N} = q \cap \mathcal{N}$.

Step C. Show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and $q \leq q \restriction \mathcal{N}$. ▲

Lemma 7. If $s, t \in x_q \cap \mathcal{N}$ then $i_q\{s, t\} \subseteq \mathcal{N}$. And if $u \in i\{s, t\}$, s and t are not comparable in $\leq_q$ and $\pi(s) < \pi(t)$, and $c^q(\pi(u), \pi(s)) \leq c^q(\pi(s), \pi(t))$ then $c^{q \restriction \mathcal{N}}(\pi(u), \pi(s)) \leq c^{q \restriction \mathcal{N}}(\pi(s), \pi(t))$. ▲

Proof. If $s \leq_q t$ then $i_q(s,t) = s \in \mathcal{N}$. So suppose that s and t are incomparable in $\leq_q$ and, without loss of generality, suppose also that $\pi(s) = \sigma < \tau = \pi(t)$. Suppose that $u \in i\{s,t\}$ with $\pi(u) = \gamma$. First of all, suppose that $\pi(u) = I(\sigma, j(\sigma,\tau))^-$. As σ, $\tau \in \mathcal{N}$ one has that $j(\sigma,\tau)$ is calculable in $\mathcal{N}$ and so $I(\sigma, j(\sigma,\tau))^- \in \mathcal{N}$. As $\mu \subseteq \mathcal{N}$ and $\mathcal{N}$ is closed under pairing, $u \in \mathcal{N}$.

Now suppose instead that $u \in i_q(\sigma,\tau)$ and $\pi(u) = \gamma \neq I(\sigma, j(\sigma,\tau))^-$. Write j for $j(\sigma,\tau)$ and recall from Lemma (4.b) that $j = j(\gamma,\tau) = j(\gamma,\sigma)$. Write I for $I(\nu,j) = I(\tau,j) = I(\gamma,j)$, and write J_γ for $I(\gamma,j+1)$, J_σ for $I(\sigma,j+1)$, and J_τ for $I(\tau,j+1)$. Lemma (4.b) says that these latter three intervals are distinct.

As σ, $\tau \in \mathcal{N}$, one has that $e_I^{-1}(J_\sigma^-)$, $e_I^{-1}(J_\tau^-) \in \mathcal{N} \cap \mu^{++} = \mathrm{rge}(F)$, and hence

$$c^q(\sigma,\tau) = c^q(e_I^{-1}(J_\sigma^-), e_I^{-1}(J_\tau^-)) \leq \delta.$$

Then $c^q(\gamma,\sigma) \leq \delta$ by the constraint ($\bullet$) on i_q in the definition of what it is to be a condition in $\mathbb{P}$. Recall from Lemma (4.b) that c^q is calculated using the same map c_I for each pair from the triple γ, σ, τ. Thus

$$c^q(\gamma,\sigma) = c^q(e_I^{-1}(J_\gamma^-), e_I^{-1}(J_\sigma^-)).$$

Hence $e_I^{-1}(J_\gamma^-) \in \mathrm{rge}(F)$ by Fact (1.14). As e, σ, and $\tau \in \mathcal{N}$, one has that $J_\gamma^- \in \mathcal{N}$. But $\gamma = J_\gamma^-$, also by Lemma (4.b), hence $\gamma \in \mathcal{N}$. As in the previous case, since $\mu \subseteq \mathcal{N}$ and $\mathcal{N}$ is closed under taking pairs, this immediately gives that $u \in \mathcal{N}$.

Finally, if $u \in i_q(\sigma,\tau)$ and $\pi(u) = \gamma \neq I(\sigma, j(\sigma,\tau))^-$, then, with the same notation as before, by Fact (1.16), one has that

$$c^{q\restriction\mathcal{N}}(\gamma,\sigma) = c^{q\restriction\mathcal{N}}(e_I^{-1}(J_\gamma^-), e_I^{-1}(J_\sigma^-)) \leq$$
$$c^{q\restriction\mathcal{N}}(e_I^{-1}(J_\sigma^-), e_I^{-1}(J_\tau^-)) = c^{q\restriction\mathcal{N}}(\sigma,\tau).$$ ▲

Corollary 8. $q \leq q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$. ▲

Proof. If s, $t \in x_q \cap \mathcal{N}$ then, by the first half of Lemma (7), $i_{q\restriction\mathcal{N}}\{s,t\} = i_q\{s,t\}$. Also, $i_{q\restriction\mathcal{N}}\{s,t\}$ satisfies ($\bullet$) by the second half of Lemma (7). Hence as far as the pair s, t are concerned, $q \restriction \mathcal{N}$ satisfies the requirement ($\bullet$) for being a condition, and q and $q \restriction \mathcal{N}$ satisfy the requirement that $i_q \restriction x_{q\restriction\mathcal{N}} = i_{q\restriction\mathcal{N}}$ for q to be a stronger condition than $q \restriction \mathcal{N}$.

Lastly, suppose that s, $t \in x_{q\restriction\mathcal{N}}$ and $s \leq_q t$. Let $s <_q u_1 \leq_q \ldots \leq_q u_n \leq_q t$ witness ($\circ$) for q. Thus $w(\pi(s),\pi(t)) = \langle \pi(s), \pi(u_1), \ldots, \pi(u_n), \pi(t) \rangle$. Write

j for $j(\pi(s), \pi(t)) + 1$. As $s, t \in \mathcal{N}$ one has, as remarked immediately after the introduction of $\mathcal{N}$ at the start of the proof of Proposition (6), that $I(\pi(s), j)^+ \in \mathcal{N}$ and $I(\pi(t), k)^- \in \mathcal{N}$ for each $k \in [j, l(\pi(t)) - 1)$. Hence $u_1, \ldots, u_n \in \mathcal{N}$ (again because $\mu \subseteq \mathcal{N}$ and $\mathcal{N}$ is closed under pairing), and provide a sequence witnessing ($\circ$) for s and t in $q \restriction \mathcal{N}$. ▲

Now obtain s as in §2 using b_q as the arbitrary set e_q in the Notation in §2.

By elementarity one may as well also assume that s is chosen so that one has an order preserving map from x_s to x_q which fixes $x_{q\restriction\mathcal{N}}$ and such that $\leq_s$ is mapped to $\leq_q$ and is fixed on $x_{q\restriction\mathcal{N}} \times x_{q\restriction\mathcal{N}}$, and hence i_s is mapped to i_q as well. One may also assume that this order preserving map induces an order preserving map of $\pi``x_s$ onto $\pi``x_q$ which fixes (exactly) $\pi``x_s \cap x_q$. One may also assume that one has an order preserving map from a_q to a_s which fixes $a_q \cap a_s$ and that if z, $z' \in x_s$ are mapped to u, $u' \in x_q$ then $j(\pi(z), \pi(z')) = j(\pi(u), \pi(u'))$.

Step D. Define some r with realm $a_s \cup a_q$ and side condition part $A_s \cup A_q$. Show that $r \in \mathbb{P}$ and that $r \leq s$, q. ▲

Proof. I define r as follows. Let $x_r = x_s \cup x_q$ and $A_r = A_s \cup A_q$. Let $t \leq_r u$ if $t \leq_q u$ for $t \leq_s u$ or $\exists v\ (t \leq_s v \leq_q u)$ or $\exists v\ (t \leq_q v \leq_s u)$. The remainder of Step D occupies several pages and is broken into a sequence of intermediary results. The relatively simple arguments that r satisfies all of the requirments for being a condition apart from ($\circ$) and ($\bullet$), and to show that it is stronger than q and s are dealt with first in Lemmas (9)-(12). Then comes the proof that ($\circ$) holds, and finally the lengthy argument which breaks into three distinct cases which shows that ($\bullet$) also holds.

Lemma 9. If $\{t, v\} \in [x_q]^2$, $u \in x_s$ and w, $z \in x_s \cap x_q$ are such that $t \leq_q w \leq_s u \leq_s z \leq_q v$ then $t \leq_q v$. The same conclusion also holds with s and q exchanged. ▲

Proof. One has that $t \leq_q w \leq_s z \leq_q v$ by the transitivity of $\leq_s$. Moreover, $w \leq_q z$ since $\leq_q \restriction x_{q\restriction\mathcal{N}} = \leq_s \restriction x_{q\restriction\mathcal{N}}$, so $t \leq_q v$ by the transitivity of $\leq_q$. The proof with q and s exchanged is identical. ▲

Corollary 10. $\leq_r \restriction x_q \times x_q = \leq_q$ and $\leq_r \restriction x_s \times x_s = \leq_s$. ▲

Lemma 11. $i_r \restriction [x_q]^2 = i_q$ and $i_r \restriction [x_s]^2 = i_s$. ▲

Proof. Suppose u_0, $u_1 \in x_q$ and $v \leq_r u_0, u_1$. If $v \in x_q$ then $v \leq_q u_0, u_1$, by Corollary (9), so there is some $u \in i_q(u_0, u_1)$ such that $v \leq_q u$. If $v \in x_s \setminus x_q$ then there are t_0, $t_1 \in x_s \cap x_q$ such that $v \leq_s t_0 \leq_q u_0$ and $v \leq_s t_1 \leq_q u_1$.

Let $t \in i_s(t_0, t_1)$ be such that $v \leq_s t$. Now, as q, $s \leq q \restriction \mathcal{N}$ one has that $i_q(t_0, t_1) = i_{q \restriction \mathcal{N}}(t_0, t_1) = i_s(t_0, t_1)$. So $t \in i_q(t_0, t_1)$. Let $u \in i_q(u_0, u_1)$ be such that $t \leq_q u$. Then $v \leq_r t \leq_r u \leq_r u_0$, u_1, so $v \leq_r u \in i_q(u_0, u_1)$. This shows that $i_r \restriction [x_q]^2 = i_q$. The argument that $i_r \restriction [x_s]^2 = i_s$ is identical with the rôles of s and q swapped. ▲

Lemma 12. $\leq_r$ is a partial order on $x_r \times x_r$. ▲

Proof. $\leq_r$ is clearly reflexive and anti-symmetric, so only transitiviy needs to be shown. Suppose $t \leq_r u \leq_r v$. If $\{t, u, v\} \in [x_q]^3 \cup [x_s]^3$ then $t \leq_r v$. If $\{t, v\} \in [x_q]^2$ and $u \in x_s$ then $t \leq_q v$ by Lemma (8). If $\{t, u\} \in [x_q]^2$ and $v \in x_s$ let $z \in x_s \cap x_q$ be such that $t \leq_q u \leq_q z \leq_s v$. Then $t \leq_q z \leq_s v$ by the transitivity of $\leq_q$. The argument in each of the other cases is identical to one of these latter two. ▲

The arguments for Lemmas (9), (11) and (12) are essentially the same as analgous arguments used in [BS, §8] in the proof that the forcing there has the ccc. Similarly to that proof, as a consequence of Lemmas (9), (11) and (12), it suffices to show that $r \in \mathbb{P}$ in order to show that q and s are compatible in $\mathbb{P}$. Here this will show that $\mathbb{P}$ is $\mathbb{M}$-proper. What needs to be done in order to complete this is to show that if t, $u \in x_r$ are incomparable in $\leq_r$ with $\pi(t) < \pi(u)$ and $v \in i_r(t, u)$ then $\pi(v) = I(t, j(\pi(t), \pi(u)))^-$ or $c^r(\pi(v), \pi(u)) \leq c^r(\pi(t), \pi(u))$, showing (•) for r (and to check that (∘) holds for r). But the proof of this is markedly different from (and considerably harder than) the analgous proof in [BS, §8]. In conformity with the usage in [M*1, §4], I try to use t, possibly adorned with subscripts, for variables in $x_s \cap x_q$, z for variables in x_s and u for variables in x_q. I shall also try to write $\tau_{(i)}$ for $\pi(t_{(i)})$, $\zeta_{(i)}$ for $\pi(z_{(i)})$, and $\xi_{(i)}$ for $\pi(u_{(i)})$, and so on. I remind the reader that the facts [M*1, 4.xx] from [M*1] cited below are listed for their convenience at the end of §2 of this paper.

Proposition 13. $r \in \mathbb{P}$. ▲

Proof. It is clear by the definition of r that r satisfies ($\star$).

Lemma 14. r satisfies (∘). ▲

Proof. This is just as in the proof of the analagous claim in [Mart*].

If one has two elements of either x_s or x_q then the witnesses to (∘) for them for s or q, respectively, provide witnesses to (∘) for the pair for r.

So suppose, aligning notation with the convention above, that $u \in x_q \setminus x_s$ and $z \in x_s \setminus x_q$. Write ξ for $\pi(u)$ and ζ for $\pi(z)$, and let $j = j(\xi, \zeta)$,

$J_\xi = I(\xi, j+1)$ and $J_\zeta = I(\zeta, j+1)$.

First of all, suppose $u <_r z$. Let $t \in x_s \cap x_q$ be such that $u <_r t <_r z$ and write τ for $\pi(t)$. There are three cases to consider according to the relationship between $j(\xi,\tau)$ and $j(\tau,\zeta)$. In each I apply the relevant instance of Lemma (0.5).

Case A. $j = j(\xi,\tau) < j(\tau,\zeta)$. Note that $\tau \in J_\xi$, so $\tau \neq J_\xi^-$. Let $\langle t, v_1, v_2, \ldots, v_m, z\rangle$ be a walk from t to z in s. If $\xi \neq J_\xi^-$ then a walk from u to z in r is given by $\langle u, v_1, v_2, \ldots, v_m, z\rangle$. On the other hand, if $\xi = J_\xi^-$ then $\langle u, v_2, \ldots, v_m, z\rangle$ is a walk from u to z in r.

Case B. $j = j(\tau,\zeta) < j(\xi,\tau)$. Let $\langle u, w_1, w_2 \ldots, w_m, t\rangle$ be a walk from u to t in q and let $\langle t, v_1, v_2 \ldots, v_n, z\rangle$ be a walk from t to z in s. Then $\langle u, w_1, w_2 \ldots, w_{j(\xi,\tau)}, v_{j(\xi,\tau)+1}, \ldots, v_n, z\rangle$ is a walk from u to z in r.

Case C. $j = j(\xi,\tau) = j(\tau,\zeta)$. $\langle t, t', v_1, v_2, \ldots, v_m, z\rangle$ be a walk from t to z in s with $\pi(t') = J_\tau^+$ if $\tau \neq J_\tau^-$ and $t' = t$ otherwise. If $u \neq J_\xi^-$ let $u' \in x_q$ be such that $u <_q u' \leq_q s$ and $\pi(u') = J_\xi^+$ and otherwise let $u' = u$. Then $\langle u, u', v_1, v_2, \ldots, v_m, z\rangle$ is a walk from t to z in r.

The proof that ($\circ$) holds when $z <_r u$ is identical. ▲

Lemma 15. r satisfies ($\bullet$). ▲

Case A. $u_0, u_1 \in x_q$ and $\pi(u_0) = \xi_0 < \pi(u_1) = \xi_1$. ▲

Proof. Lemma (11) shows that $i_r(u_0,u_1) = i_q(u_0,u_1)$. Suppose $u \in i_q(u_0,u_1)$ and let $\xi = \pi(u)$. Clearly, if $\xi = I(\xi_0, j(\xi_0,\xi_1))^-$ then this fact witnesses ($\bullet$) for r as well as for q. So what remains to be shown is that otherwise $c^r(\xi,\xi_0) \leq c^r(\xi_0,\xi_1)$. By Corollary (5) applied to q let $j = j(\xi,\xi_0) = j(\xi_0,\xi_1) = j(\xi,\xi_1)$, $I = I(\xi,j) = I(\xi_0,j) = I(\xi_1,j)$, $J = I(\xi,j+1)$, $J_0 = I(\xi_0,j+1)$ and $J_1 = I(\xi_1,j+1)$. Recall that $\xi = J^-$. Also, let $\overline{\xi} = e_I^{-1}(J^-)$, $\overline{\xi}_0 = e_I^{-1}(J_0^-)$, $\overline{\xi}_1 = e_I^{-1}(J_1^-)$. Thus $c^q(\xi,\xi_0) = c^q(\overline{\xi},\overline{\xi}_0)$, $c^q(\xi,\xi_1) = c^q(\overline{\xi},\overline{\xi}_1)$, and $c^q(\xi_0,\xi_1) = c^q(\overline{\xi}_0,\overline{\xi}_1)$.

Now I finish with Case A by dealing with four subcases depending on whether $\overline{\xi_i} \in \mathcal{N}$ (that is, $\overline{\xi_i} \in a_q \cap a_s$) or not for $i < 2$. I use the facts from [M*1, §4] listed at the end of §2.

Subcase A.i. $\overline{\xi_0}$, $\overline{\xi_1} \in \mathcal{N}$. As in the proof of Corollary (10) one has that $\overline{\xi} \in \mathcal{N}$. By the fact given at the bottom of the list at the end of §2, if $c^r(\overline{\xi},\overline{\xi}_0) \neq c^q(\overline{\xi},\overline{\xi}_0)$ then $c^r(\overline{\xi},\overline{\xi}_0) = \alpha^*$ and $c^q(\overline{\xi},\overline{\xi}_0) = \delta$. Similarly, if $c^r(\overline{\xi}_0,\overline{\xi}_1) \neq c^q(\overline{\xi}_0,\overline{\xi}_1)$ then $c^r(\overline{\xi}_0,\overline{\xi}_1) = \alpha^*$ and $c^q(\overline{\xi}_0,\overline{\xi}_1) = \delta$. As $c^q(\overline{\xi},\overline{\xi}_0) \leq c^q(\overline{\xi}_0,\overline{\xi}_1)$, this implies that $c^r(\overline{\xi},\overline{\xi}_0) \leq c^r(\overline{\xi}_0,\overline{\xi}_1)$.

Subcase A.ii. $\overline{\xi_0}$, $\overline{\xi_1} \notin \mathcal{N}$. Thus u_0, $u_1 \in x_q \setminus x_s$. By Fact (1.13) one has that $c^r(\overline{\xi},\overline{\xi}_0) \leq c^q(\overline{\xi},\overline{\xi}_0)$, and, by ($\bullet$) for q and the hypothesis that $\xi \neq I(\xi_0, j(\xi_0,\xi_1))^-$, one also has that $c^q(\xi,\xi_0) \leq c^q(\xi_0,\xi_1)$. Moreover, $c^q(\overline{\xi}_0,\overline{\xi}_1) = c^r(\overline{\xi}_0,\overline{\xi}_1)$ by [M*1, 4.27]. Hence $c^r(\xi,\xi_0) \leq c^r(\xi_0,\xi_1)$.

Subcase A.iii. $\overline{\xi_0} \notin \mathcal{N}$, $\overline{\xi_1} \in \mathcal{N}$. One has $c^r(\overline{\xi},\overline{\xi}_0) \leq c^q(\overline{\xi},\overline{\xi}_0)$ by Fact (1.13), again, and $c^q(\xi,\xi_0) \leq c^q(\xi_0,\xi_1)$ by ($\bullet$) for q and the hypothesis that $\xi \neq I(\xi_0, j(\xi_0,\xi_1))^-$. One also has $\delta \leq c^r(\overline{\xi}_0,\overline{\xi}_1) = c^q(\overline{\xi}_0,\overline{\xi}_1)$ by [M*1, 4.17]. Hence $c^r(\overline{\xi},\overline{\xi}_0) \leq c^r(\overline{\xi}_0,\overline{\xi}_1)$. Thus $c^r(\xi,\xi_0) \leq c^r(\xi_0,\xi_1)$.

Subcase A.iv. $\overline{\xi_0} \notin \mathcal{N}$, $\overline{\xi_1} \in \mathcal{N}$. If $\delta \leq c^r(\overline{\xi}_0,\overline{\xi}_1)$ then $c^r(\overline{\xi}_0,\overline{\xi}_1) = c^q(\overline{\xi}_0,\overline{\xi}_1)$ by [M*1, 4.16], when $c^r(\overline{\xi},\overline{\xi}_0) \leq c^r(\overline{\xi}_0,\overline{\xi}_1)$ as in Cases (A.ii) and (A.iii) again.

Otherwise, $c^r(\overline{\xi}_0,\overline{\xi}_1) \leq \alpha^*$ by [M*1, 4.21]. If $c^r(\overline{\xi}_0,\overline{\xi}_1) < \alpha^*$ then one has $c^r(\overline{\xi}_0,\overline{\xi}_1) = c^q(\overline{\xi}_0,\overline{\xi}_1)$ by [M*1, 4.15], and $c^r(\overline{\xi},\overline{\xi}_0) \leq c^r(\overline{\xi}_0,\overline{\xi}_1)$ as in Cases (A.ii) and (A.iii) once more.

The remaining possibility is that $c^r(\overline{\xi}_0,\overline{\xi}_1) = \alpha^*$ and $c^q(\overline{\xi}_0,\overline{\xi}_1) = \delta$. As $\overline{\xi} \in a_q$ one has that $c^q(\overline{\xi},\overline{\xi}_0) \leq c^q(\overline{\xi},\overline{\xi}_1) = \delta$. By Fact (1.14) one has that $\overline{\xi} \in \mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, so that $\overline{\xi} \in a_q \cap a_s$. Thus, by clause (iii.3) of the definition of ϕ for s, one then has that $\overline{\xi}$, $\overline{\xi}_0 \in \mathrm{rge}(h^*)$, so that $c^r(\overline{\xi},\overline{\xi}_0) \leq \alpha^*$, as required. ▲ (Case A)

Case B. z_0, $z_1 \in x_s$ and $\pi(z_0) = \zeta_0 < \zeta_1 = \pi(z_1)$. ▲

Proof. Lemma (11) shows that $i_r\{z_0,z_1\} = i_s\{z_0,z_1\}$. Suppose that $z \in i_s(z_0,z_1)$ and let $\zeta = \pi(z)$. Clearly, if $\zeta = I(z_0, j(\zeta_0,\zeta_1))^-$ then this fact witnesses ($\bullet$) for r as well as for q. So what needs to be shown is that otherwise $c^r(\zeta,\zeta_0) \leq c^r(\zeta_0,\zeta_1)$.

By Corollary (5) applied to s let $j = j(\zeta,\zeta_0) = j(\zeta_0,\zeta_1) = j(\zeta,\zeta_1)$, $I = I(\zeta,j) = I(\zeta_0,j) = I(\zeta_1,j)$, $J = I(\zeta,j+1)$, $J_0 = I(\zeta_0,j+1)$ and $J_1 = I(\zeta_1,j+1)$. Recall that $\zeta = J^-$. Also, let $\overline{\zeta} = e_I^{-1}(J^-)$, $\overline{\zeta}_0 = e_I^{-1}(J_0^-)$, $\overline{\zeta}_1 = e_I^{-1}(J_1^-)$. Thus $c^s(\zeta,\zeta_0) = c^s(\overline{\zeta},\overline{\zeta}_0)$, $c^s(\zeta,\zeta_1) = c^s(\overline{\zeta},\overline{\zeta}_1)$ and $c^s(\zeta_0,\zeta_1) = c^s(\overline{\zeta}_0,\overline{\zeta}_1)$. Note that $z \in \mathcal{N}$ since $s \in \mathcal{N}$ and $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, so $\zeta \in \mathcal{N}$ and $\overline{\zeta} \in \mathcal{N} \cap \mu^{++}$.

As previously, one has that $c^r(\overline{\zeta},\overline{\zeta}_0) \leq c^s(\overline{\zeta},\overline{\zeta}_0)$ by Fact (1.13), and that $c^s(\zeta,\zeta_0) \leq c^s(\zeta_0,\zeta_1)$ by ($\bullet$) and the hypothesis that $\zeta \neq I(z_0, j(\zeta_0,\zeta_1))^-$. But [M*1, 4.12] shows that $c^r(\overline{\zeta}_0,\overline{\zeta}_1) = c^s(\overline{\zeta}_0,\overline{\zeta}_1)$. So $c^r(\overline{\zeta},\overline{\zeta}_0) \leq c^r(\overline{\zeta}_0,\overline{\zeta}_1)$, and hence $c^r(\zeta,\zeta_0) \leq c^r(\zeta_0,\zeta_1)$, as required. ▲ (Case B)

Case C. $z \in x_s \setminus x_q$ and $u \in x_q \setminus x_s$. Let $\zeta = \pi(z)$ and $\xi = \pi(u)$.

Proof. Observe that $\zeta \in \mathcal{N}$ and $\xi \notin \mathcal{N}$, since $\mu \subseteq \mathcal{N}$ and $\mathcal{N}$ is closed under taking co-ordinates of ordered pairs and (ordered) pairing.

If $z \leq_r u$ then $i_r(z,u) = \{z\}$ and, clearly, $c^r(\zeta,\zeta) \leq c^r(\zeta,\xi)$. Similarly if $u \leq_r z$ then $i_r(u,z) = \{u\}$ and $c^r(\xi,\xi) \leq c^r(\xi,\zeta)$. So suppose that $z \not\leq_r u$ and $u \not\leq_r z$. Suppose also that $v \in i_r\{z,u\}$ and let $\pi(v) = \nu$. If $\nu = I(\zeta, j(\zeta,\xi))^-$ there is nothing more to check, so suppose otherwise.

Claim 16. Suppose $z \in x_s \setminus x_q$ and $u \in x_q \setminus x_s$ are incomparable in $\leq_r$ and $v \in i_r\{z,u\}$. Let $\zeta = \pi(z)$, $\xi = \pi(u)$ and $\nu = \pi(v)$, and suppose $\nu \neq I(\zeta, j\{\zeta,\xi\})^-$. Then $j(\nu,\xi) = j(\nu,\zeta) = j\{\zeta,\xi\}, = j$, say, and letting $J_\nu = I(\nu, j{+}1)$, $J_\zeta = I(\zeta, j{+}1)$, $J_\xi = I(\xi, j{+}1)$, $\overline{\nu} = e_I^{-1}(J_\nu^-)$, $\overline{\zeta} = e_I^{-1}(J_\zeta^-)$, $\overline{\xi} = e_I^{-1}(J_\xi^-)$, and $I = I(\nu,j)$, then $\nu = J_\nu^-$ and $c^r(\nu,\xi) = c^r(\overline{\nu},\overline{\xi})$, $c^r(\nu,\zeta) = c^r(\overline{\nu},\overline{\zeta})$, and $c^r\{\zeta,\xi\} = c^r\{\overline{\zeta},\overline{\xi}\}$. ▲

Proof. By Lemma (0.5) and Lemma (4.b) and the note subsequent to the proof of the latter, since it has already been shown that r satisfies ($\circ$) and ($\star$), either the conclusion of the claim is true, or $\nu < \xi < \zeta$ and $k = j(\nu,\zeta) = j(\xi,\zeta) < j(\nu,\xi)$, or $\nu < \zeta < \xi$ and $k = j(\nu,\xi) = j(\zeta,\xi) < j(\nu,\zeta)$.

In either of the latter two cases note that one has that $\nu \in I(\xi,k)$ and $\nu \in I(\xi \cap \zeta, k+1)$. Suppose that one is in one of the latter two cases. I show that these cases both lead to contradictions (by identical arguments with the rôles of s and q exchanged) and thus cannot occur.

Suppose, first of all, that $v \in x_q \cap x_s$ and that $\nu \in I(\xi \cap \zeta, k+1)$. Let z' and u' be the images of z and u under the order preserving, $x_q \cap x_s$-fixing map from x_s to x_q and its inverse, respectively. Let $\pi(u') = \xi'$ and $\pi(z') = \zeta'$. Recall that s was chosen, by elementarity, so that $j(\nu,\zeta) = j(\nu,\zeta')$ and $j(\nu,\xi) = j(\nu,\xi')$.

Suppose that $\xi < \zeta$ and that $k = j(\nu,\zeta) = j(\xi,\zeta) < j(\nu,\xi)$. Then $u' \in I(\xi, k+1)$ and $z' \notin I(\xi,k+1)$. This then gives that $\xi < \zeta'$ and $\xi' < \zeta$ as well, since the map taking x_s to x_q is order preserving.

If u and z' are not comparable in $\leq_q$ there is some $w \in i_q\{u,z'\}$ such that $v \leq_q w <_q u$, z'. Now apply ($\bullet$) for q. One has that $I(\xi,k)^- < \nu \leq \pi(w)$, so $\pi(w) \neq I(\xi,k)^-$. Hence $\pi(w) < I(\xi,k+1)^-$. But this contradicts the fact that $\nu \in I(\xi,k+1)$.

On the other hand, if u and z' are comparable in $\leq_q$ then $u <_q z'$ and there is some $w \in x_q$ with $u <_q w \leq_q z'$ and $\pi(w) = I(\xi,k+1)^+$, by ($\circ$) for q.

Likewise, u' and z are not incomparable in $\leq_s$. If u' and z are comparable in $\leq_s$ there is some $w' \in x_s$ such that $u' <_q w' \leq_q z$ and $\pi(w') = I(\xi,k+1)^+$, by ($\circ$) for s. So $I(\xi,k+1)^+ \in \pi``(x_q \cap x_s)$ and hence $w' \in x_q \cap x_s$ (since if

$w' \in x_q$ and $\pi(w') \in \mathcal{N}$ then $w \in \mathcal{N}$ since $\mu \subseteq \mathcal{N}$ and $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, again). Thus $w \in x_q \cap x_s$ since $\pi(w) = \pi(w')$. Hence $v <_q w$, w'. By $(\star)$ for q this gives that $w = w'$. Hence $u <_r z$. This contradicts the assumption that u and z are incomparable. Hence the assumption that $\xi < \zeta$ yields a contradiction. By an identical argument, the assumption that $\zeta < \xi$ also gives a contradiction.

Now suppose that $v \in x_q \setminus x_s$. Let $w \in x_q \cap x_s$ be such that $v <_q w <_s z$. As u and z are incomparable in $\leq_r$ it must be the case that $u \not\leq_q w$ if $\xi < \zeta$. It must also be the case that $u \not\leq_q w$ if $(\pi(w) <)\ \zeta < \xi$. And $w \not\leq_q u$ since otherwise v would not be maximal below both u and z (*i.e.* a member of $i_r\{u, z\}$). So u and w are incomparable in $\leq_q$, and, again by the maximality of v below u and t, one has that $v \in i_q\{u, w\}$. Set $\tau = \pi(w)$.

Suppose that $\xi < \zeta$. I claim that $\tau \notin I(\zeta, k{+}1)$. Otherwise $j(\xi,\tau) = j(\xi,\zeta) < j(\tau,\zeta)$. Now $\nu < I(\xi, j(\xi,\tau) + 1)^-$, by $(\bullet)$ for q and the assumption that $\nu \neq I(\xi, j(\xi,\zeta))^-$. Thus $\nu < I(\xi, j(\xi,\zeta) + 1)^-$. But this contradicts the case assumption that $j(\xi,\zeta) < j(\nu,\xi)$.

Let z' be the image of z in x_q under the order preserving map between x_s and x_q. Let $\zeta' = \pi(z')$. Then $j(\tau,\zeta) = j(\tau,\zeta')$, $= j(\xi,\zeta)$, and so $l(\tau) = j(\xi,\zeta) + 1$ by $(\circ)$ and $(\star)$ for r. Again by $(\bullet)$ for q applied for u and w, this implies that $\nu < I(\xi, j(\xi,\zeta) + 1)^-$. This again contradicts the assumption that $j(\xi,\zeta) < j(\nu,\xi)$.

Hence $\nu < I(\xi, j(\xi,\zeta) + 1)^-$ and $j(\xi,\zeta) = j(\nu,\xi)$.

Next suppose that $v \in x_q \setminus x_s$ and that $\zeta < \xi$. I claim that $\tau \notin I(\zeta, k + 1)$. Otherwise $j(\xi,\tau) = j(\xi,\zeta) < j(\tau,\zeta)$. Now $\nu < I(\tau, j(\tau,\xi) + 1)^-$, by $(\bullet)$ for q and the assumption that $\nu \neq I(\xi, j(\xi,\zeta))^-$. Thus $\nu < I(\zeta, j(\xi,\zeta) + 1)^-$. But this contradicts the case assumption that $j(\xi,\zeta) < j(\nu,\zeta)$.

The assumption that $v \in x_s \setminus x_q$ leads to a contradiction by identical arguments with the rôles of s and q exchanged. ▲(Claim (16))

If $J^-_\xi \in \mathcal{N}$ then $\xi \neq J^-_\xi$ and, by $(\circ)$ for r there is some $u' \in x_q$ such that $\pi(u') = J^-_\xi$ and $v <_r u' <_q u$. But $u' \in \mathcal{N}$ since $\pi(u') \in \mathcal{N}$, so $u' \in x_s \cap x_q$. Now $u' \not\leq_s z$ since otherwise u' would contradict the maximality of v in being below both u and z. Hence, by the maximality of v again, $v \in i_r\{u', z\} = i_s\{u', z\}$. Hence $v \in x_s$. By $(\bullet)$ for s and the assumption that $v \neq I^-$, this implies that $c^s(\nu, \pi(u')) \leq c^s(\pi(u'), \zeta)$. Hence, by [M*1, 4.12], $c^r(\nu,\xi) = c^r(\nu, \pi(u')) \leq c^r(\pi(u'), \zeta) = c^r(\xi,\zeta)$. Thus in this instance $(\bullet)$ holds.

Now suppose that $J_\xi^- \notin \mathcal{N}$ (and hence $\bar{\xi} = e_I^{-1}(J_\xi^-) \notin \mathcal{N}$). The remainder of the proof of CASE C breaks into two cases depending on whether $v \in x_s$ (SUBCASE C.I) or not (SUBCASE C.II).

SUBCASE C.I. $v \in x_s$. ▲

Proof. By the definition of $\leq_r$ there is some point between v and u in $x_s \cap x_q$. So let $t \in x_s \cap x_q$ be such that $v \leq_s t \leq_q u$ with $\pi(t)$ least possible. (Thus $t = v$ if $v \in x_s \cap x_q$.) Although it is not needed here, note that t is unique by ($\star$) for s. Set $\tau = \pi(t)$.

I next show that even if $\nu < \tau$ one has that $j\{\tau, \zeta\} = j(\nu, \tau) = j$.

Now $v \in i_r\{u, z\}$, so $v \in i_r\{t, z\}$. But, by Lemma (11), $i_r\{t, z\} = i_s\{t, z\}$, so $v \in i_s\{t, z\}$.

If $\nu \neq I(\min\{\tau, \zeta\}, j\{\tau, \zeta\})^-$ then by Corollary (5) one has that $j(\nu, \tau) = j(\nu, \zeta) = j\{\tau, \zeta\} = j$.

On the other hand, if $\nu = I(\min\{\tau, \zeta\}, j\{\tau, \zeta\})^-, = I(\zeta, j\{\tau, \zeta\})^-$, by the definition of $j\{\tau, \zeta\}$, then $j\{\tau, \zeta\} \leq j$ since $\nu \notin J_\zeta$ $(= I(\zeta, j)$, while $j\{\tau, \zeta\} \geq j$ since $\nu < \tau$, $\zeta < \xi$. So $j\{\tau, \zeta\} = j(\nu, \tau) = j$ once more.

Thus the calculations of c^s and c^r applied to pairs from $\{\tau,\ \nu,\ \zeta,\ \xi\}$ all use (the same) m_I and j (*cf.* Definition (0.9)) whether $\nu = \tau$ or not. In view of this, set $\bar{\tau} = e_I^{-1}(I(\tau, j+1)^-)$. Recall that $c^w\{\mu, \lambda\} = c^w\{\bar{\mu}, \bar{\lambda}\}$ for all pairs $\{\mu, \lambda\}$ from $\{\tau,\ \nu,\ \zeta,\ \xi\}$ and $w \in \{r, s\}$. This is used without further comment repeatedly below.

If $\nu = I(\min \tau, \zeta, j)^-$ then $\nu = I(\min \xi, \zeta, j)^-$ and this shows (•) for v, z and u. So suppose that this is not the case, that is, that $\nu \neq I(\min \tau, \zeta, j)^-$.

By CASE B above, if $\nu \neq \tau$ one has $c^r(\nu, \tau)$, $c^r(\nu, \zeta) \leq c^r\{\tau, \zeta\}$. By [M*1, 4.24] one has that $c^r\{\bar{\tau}, \bar{\zeta}\} \leq c^r\{\bar{\zeta}, \bar{\xi}\}$. So $c^r(\bar{\nu}, \bar{\zeta}) \leq c^r\{\bar{\zeta}, \bar{\xi}\}$, and, by the sub-additivity of c^r, one has that $c^r(\nu, \xi)$, $c^r(\nu, \zeta) \leq c^r\{\xi, \zeta\}$. ▲

SUBCASE C.II $v \in x_q \setminus x_s$. ▲

Proof. The whole proof is very similar to that of Subcase (C.i). Let $t \in x_s \cap x_q$ be such that $v \leq_q t \leq_s z$ and $\pi(t)$ is minimal such that this is the case. Set $\tau = \pi(t)$. Note that $\nu < \tau < \zeta$. Note also that $I(\min \tau, \xi, j\{\tau, \xi\}) = I(\xi, j\{\tau, \xi\})$ by the definition of $j\{\tau, \xi\}$.

As $v \in i_r\{z, u\}$ one has that $v \in i_r\{t, u\}, = i_q\{t, u\}$, the latter by Lemma (11).

If $\nu \neq I(\xi, j\{\xi, \tau\})^-$ then, by Corollary (5), $j\{\tau, \xi\} = j(\nu, \xi) = j, = j(\nu, \tau)$.

On the other hand, if $\nu = I(\xi, j\{\xi,\tau\})^-$ then $j\{\xi,\tau\} \leq j$ since $\nu \notin J_\xi$ $(= I(\xi, j+1))$, while $j\{\xi,\tau\} \geq j$ since either $\xi < \tau$ and $\nu \notin I(\tau, j\{\tau,\xi\}+1)$, and hence $\nu \notin I(\zeta, j\{\tau,\xi\}+1)$; or $\tau < \xi$ and $\nu \notin I(\xi, j\{\tau,\xi\}+1)$.

Thus in either case one has that $I(\xi,j)^- = I(\zeta,j)^- = I(\tau,j)^- = I(\nu,j)^-$. Moreover, similarly to Subcase C.i, the calculations of c^q and c^r applied to pairs from $\{\tau,\ \nu,\ \zeta,\ \xi\}$ all use (the same) m_I and j (*cf.* Definition (0.9)). In view of this, set $\bar{\tau} = e_I^{-1}(I(\tau, j+1)^-)$.

If $\nu = I(\xi,j)^- = I(\zeta,j)^-$ then this shows ($\bullet$) for v, t and u and there is nothing more to prove.

So suppose instead that $\nu \neq I(\xi,j)^- = I(\zeta,j)^- = I(\tau,j)$.

One has that $c^r(\nu,\tau), c^r(\nu,\xi) \leq c^r\{\tau,\xi\}$ by Case (A) and the subadditivity of c^r. By [M*1, 4.24] one has that $c^r(\bar{\tau},\bar{\zeta}) \leq c^r(\bar{\zeta},\bar{\xi})$. And, since $\tau < \zeta$ and regardless of whether $\tau < \xi$ or $\xi < \tau$, one has $c^r\{\bar{\tau},\bar{\xi}\} \leq c^r\{\bar{\zeta},\bar{\xi}\}$, by the subadditivity of c^r. Hence $c^r(\bar{\nu},\bar{\xi}) \leq c^r\{\bar{\tau},\bar{\xi}\} \leq c^r\{\bar{\zeta},\bar{\xi}\}$. Applying the subadditivity of c^r one more time one gets $c^r(\bar{\nu}) \leq c^r\{\bar{\zeta},\bar{\xi}\}$ and so $c^r(\nu,\zeta)$, $c^r(\nu,\xi) \leq c^r\{\zeta,\xi\}$, as required, concluding the proof of this subcase. ▲ (Case C)

Cases (A), (B) and (C) taken together show that ($\bullet$) holds for r and so that $r \in \mathbb{P}$, thus concluding the proof of Proposition (13).
▲ (Lemma (15), (Proposition (13))

This shows that q and s are compatible and hence $\mathbb{P}$ is μ^+-$\mathbb{M}$-proper and forcing with $\mathbb{P}$ preserves μ^+. ▲ (**Step D**, Proposition (6))

One great advantage of μ^+-$\mathbb{M}$-proper forcing over (μ^+-)proper forcing is that one has a reasonable chance of preserving μ^{++} (and all greater cardinals) as well as μ^+!

Proposition 17. If $2^{<\mu} \leq \mu^+$ then $\mathbb{P}$ has the μ^{++}-cc. ▲

Proof. Unsurprisingly, the proof essentially consists of taking a collection of μ^{++} many conditions, thinning the collection using cardinality and Δ-system arguments and then showing that any two conditions from the thinned collection are compatible. However, it is helpful to work with a collection of conditions p which have a single map in their A_p through which all of the other maps in A_p factor, and which also has each element of a_p in its range. So I start by showing how working with such a collection suffices for the proof that $\mathbb{P}$ has the μ^{++}-chain condition.

Let $\{p'_\eta \mid \eta < \mu^{++}\}$ be a subset of $\mathbb{P}$ of size μ^{++}. Extend each p'_η to a stronger condition p_η by adding, if necessary, a single map (α_η, F_η) to $A_{p'_\eta}$ to form A_{p_η}, in order to ensure that there is a map $(\alpha_\eta, F_\eta) \in A_{p_\eta}$ such that for all $(\beta, f) \in A_{p_\eta}$ one has $\beta \leq \alpha_\eta$ and there is some $g \in \mathcal{F}_{\beta\alpha}$ such that $f = F_\eta \cdot g$, and such that $a_{p_\eta} \subseteq \mathrm{rge}(F_\eta)$. This is possible by Fact (1.6), similarly to the argument in Step (B) of the proof of Proposition (6). If η, $\xi < \mu^{++}$ are such that p_η and p_ξ are compatible then, clearly, one will have that p'_η and p'_ξ are compatible as well.

Now thin by applying the Δ-system lemma for morass maps, Proposition (1.8), to get $\alpha < \mu^+$ such that $\alpha_\eta = \alpha$ for all $\eta < \mu^{++}$ and to get that $\langle \mathrm{rge}(F_\eta) \mid \eta < \mu^{++} \rangle$ forms a Δ-system with $\mathrm{ssup}(\mathrm{rge}(F_\eta) \cap \mathrm{rge}(F_\xi)) \leq \min(\mathrm{rge}(F_\eta) \setminus \mathrm{rge}(F_\xi))$ and $\mathrm{ssup}(\mathrm{rge}(F_\eta) \setminus \mathrm{rge}(F_\xi)) \leq \min(\mathrm{rge}(F_\xi) \setminus \mathrm{rge}(F_\eta))$ for $\eta < \xi < \mu^{++}$. Let X be the root of the Δ-system $\langle \mathrm{rge}(F_\eta) \mid \eta < \mu^{++} \rangle$.

As $\{ \{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha \}$ has size at most μ one has that $[\{ \{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha \}]^{<\mu}$ has size at most $2^{<\mu}$. By the hypothesis that $2^{<\mu} \leq \mu^+$, by thinning again if necessary, one may as well assume that there is some $A \in [\{ \{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha \}]^{<\mu}$ such that $A_\eta = \{ (\beta, F_\eta \cdot f) \mid (\beta, f) \in A \}$ for all $\eta < \mu^{++}$.

Having thinned with respect to the side-condition part of conditions, turn next to their working parts. In order to simplify notation, write x_η for x_{p_η}, $<_\eta$ for $<_{p_\eta}$, i_η for i_{p_η}, A_η for A_{p_η} and a_η for a_{p_η} for each $\eta < \mu^{++}$. Let $d_\eta = \pi``x_\eta$ for $\eta < \mu^{++}$.

One may as well assume that $\langle d_\eta \mid \eta < \mu^{++} \rangle$ forms a Δ-system with root d and that there are order preserving bijections $h_{\eta\xi} : d_\eta \longrightarrow d_\xi$ for $\eta, \xi < \mu^{++}$. Let δ be the common order type of the d_η for $\eta < \mu^{++}$. By a cardinality argument one may assume that the $h_{\eta\xi}$ extend to maps from $(x_\eta, \leq_\eta)$ to $(x_\xi, \leq_\xi)$ such that $h_{\eta\xi}(\nu, \gamma) = (h_{\eta\xi}(\nu), \gamma)$.

In order to see this, first of all, note that as $\overline{\overline{x_\eta}} < \mu$ for each $\mu < \mu^{++}$ there is some $\gamma_0 < \mu$ such that $x_\mu \in [\rho \times \gamma_0]^{<\mu}$. The characteristic function of x_η can thus be read off from a function from $\delta \times \gamma_0$ to 2 via the enumeration of d_η. Now δ, $\gamma_0 < \mu$, so there are at most $2^{<\mu}$ such functions. By the cardinal arithmetic hypothesis, $2^{<\mu} < \mu^{++}$, of the proposition one can thin again to obtain a collection of μ^{++} conditions such that the characteristic function of each x_η can be read off from the same function from $\delta \times \gamma_0$ to 2 via the enumeration of d_η. For this collection the $h_{\eta\xi}$ do extend to maps from x_η to x_ξ as claimed.

As X, the root of the Δ-system $\langle \mathrm{rge}(F_\eta) \mid \eta < \mu^{++} \rangle$, has size μ there are only $\mu^{<\mu} \leq \mu^+$ many possibilities for $a_\eta \cap X$, so, thinning again if necessary, one may as well assume that $a_\eta \cap X$ is fixed for $\eta < \mu^{++}$, and equals a, say. Since $a_\eta \subseteq \mathrm{rge}(F_\eta)$ for $\eta < \mu^{++}$ one has that $\langle a_\eta \mid \eta < \mu^{++} \rangle$ forms a Δ-system with root a and such that $\mathrm{ssup}(a) \le \min(a_\eta \setminus a)$ and $\mathrm{ssup}(a_\eta) \le \min(a_\xi \setminus a)$ for $\eta < \xi < \mu^{++}$.

For each $\eta < \mu^{++}$ let

$$b_\eta = \{ \langle |\nu|, j, e_I^{-1}(\nu, j)(I(\nu, j+1)^-) \rangle \mid \nu \in d_\eta, \\ j < \omega,\ \nu \neq I(\nu, j)^- \text{ and } |\nu| = \mathrm{otp}(d_\eta \cap \nu) \}.$$

Note that $a_\eta = \bigcup \{ \sigma \mid \langle \gamma, j, \sigma \rangle \in b_\eta \}$ for all $\eta < \mu^{++}$. However at the moment it is possible to have $\langle \gamma, j, \sigma \rangle \in b_\eta$ and $\langle \gamma', j', \sigma \rangle \in b_\xi$ for some distinct η, $\xi < \mu^{++}$, and hence that $\sigma \in a$, even though there is no γ^*, j^* such that $\langle \gamma^*, j^*, \sigma \rangle \in b_\eta \cap b_\xi$. In order to avoid this phenomenon further thinning of the collection of conditions is needed.

Firstly, by thinning, if necessary, one may as well assume that $\{ \langle \gamma, j \rangle \in \delta \times \omega \mid \exists \sigma\, \langle \gamma, j, \sigma \rangle \in b_\eta \}$ is fixed, *i.e.*, the same for each $\eta < \mu^{++}$, since there are at most $2^{<\mu}$, $\le \mu^+$ by hypothesis, many characteristic functions with domain $\delta \times \omega$. Note that if $\langle \gamma, j, \sigma \rangle \in b_\eta$ then σ is uniquely determined by γ, j and η. For each $\gamma < \delta$ let j_γ be minimal such that there is no σ such that $\langle \gamma, j, \sigma \rangle \in b_\eta$. (The thinning just done ensures that the definitions of the j_γ are independent of η.)

Thinning again, if necessary, one may as well assume, since $2^{<\mu} \le \mu^+$, that the $\langle b_\eta \mid \eta < \mu^{++} \rangle$ form a Δ-system with root b.

For each indexing pair $\gamma < \delta$ and $j < j_\delta$ consider $Y_{\langle \gamma, j \rangle} = \{ \sigma \in a \mid \exists \eta < \mu^{++}\ \langle \gamma, j, \sigma \rangle \in b_\eta \setminus b \}$. Note that if $\sigma \in Y_{\langle \gamma, j \rangle}$ the η witnessing this is unique since the b_η form a Δ-system and $\langle \gamma, j, \sigma \rangle \notin b$. Thus one can discard the (at most μ many) conditions p_η such that there are $\gamma < \delta$, $j < j_\gamma$, and $\sigma \in a$ such that $\langle \gamma, j, \sigma \rangle \in b_\eta \setminus b$ and still have μ^{++} many conditions. For these conditions one has that $Y_{\langle \gamma, j \rangle} = \emptyset$ for all $\gamma < \delta$ and $j < j_\gamma$, and that $a = \bigcup \{ \sigma \mid \langle \gamma, j, \sigma \rangle \in b \}$.

Lastly one may also 'thin to fix j"$[d_\eta]$.' Since $\delta < \mu$ there are at most $2^{<\mu}$ functions from $[\delta]^2$ to ω and so one may as well assume that for all η, $\xi < \mu^{++}$ and s, $t \in x_\eta$ one has $j(\pi(s), \pi(t)) = j(h_{\eta\xi}(\pi(s)), h_{\eta\xi}(\pi(t)))$.

After this exhaustive homogenizing of the conditions one can now show that any two which have survived so far are compatible. I break this proof into a series of Lemmas and Propositions.

Lemma 18. If s, $t \in x$ and $\eta < \mu^{++}$ then $i_\eta\{s, t\} \subseteq x$. ▲

Proof. Let $u \in i_\eta\{s, t\} \setminus x$. Then, by the Δ-system property of the $\{ x_\xi \mid \xi < \mu^{++} \}$, one has that $u_\xi = h_{\eta\xi}(u) \in i_\xi\{s, t\} \setminus x$ for each $\xi < \mu^{++}$ and that the $\{u_\xi \mid \xi < \mu^{++}\}$ are all distinct. For each $\xi < \mu^{++}$ one has either $\pi(u_\xi) = I(\pi(s), j(\pi(s), \pi(t)))^-$ or $c^\xi(\pi(s), \pi(t)) \leq c^\xi(\pi(s), \pi(t)),\ = \gamma_\xi$, say. Note that in the latter case $\gamma_\xi \leq \alpha(< \mu^+)$ since $a \subseteq \mathrm{rge}(F_\xi)$.

So $\pi(u_\xi) \in \{I(\pi(s), j(\pi(s), \pi(t)))^-\} \cup \bigcup_{j<\omega}\{\beta \mid j(\beta, \pi(t)) = j,\ c^\xi(\beta, \pi(\tau))$ is calculated using $I(\pi(t), j)\ \&\ c^\xi(\beta, \pi(\tau)) \leq \alpha\}$, where the union is a countable union of sets of size at most μ. So the set of possibilities for $\pi(u_\xi)$ as ξ runs through μ^{++} has size at most μ. As one has that $u_\xi \in \{\pi(u)\} \times \mu$ for all $\xi < \mu^{++}$ one has that there can only be μ many distinct u_ξ, a contradiction. Hence $u \in x$ as required. ▲

Now let $\eta,\ \nu < \mu^{++}$. Define $r = (x_r, \leq_r, A_r)$ by $x_r = x_\eta \cup x_\xi$, $A_r = A_\eta \cup A_\xi$ and $s \leq_r t$ if $s \leq_\eta t$, $s \leq_\xi t$ or there is some $u \in x$ such that either $s \leq_\eta u \leq_\xi t$ or $s \leq_\xi u \leq_\eta t$.

Lemma 19. $\leq_r$ is a partial order on x_r. ▲

Proof. There are many cases, and I leave the proofs of most of them to the reader. Perhaps among those with a trace of interest are: if $s \leq_\eta u \leq_\xi v \leq_\xi z \leq_\eta t$, with $u,\ z \in x$, when one has that $s \leq_\eta u \leq_\xi z \leq_\eta x$, and so $s \leq_\eta u \leq_\eta z \leq_\eta x$, and so $s \leq_\eta t$; or if $s \leq_\eta u \leq_\xi v \leq_\xi t$ with $u \in x$, when $s \leq_\eta u \leq_\xi t$. ▲

Corollary 20. $(\star)$ holds for r. ▲

Proof. If $u \leq_\eta s$ and $u \leq_\xi t$ and $\pi(s) = \pi(t)$ then $s,\ t \in x$ (by the thinning of the collection of conditions). This gives a contradiction to $(\star)$ for η (and ξ) unless $s = t$. ▲

Lemma 21. If $s,\ t \in x_\eta$ then $i_r\{s, t\} = i_\eta\{s, t\}$, and if $s,\ t \in x_\xi$ then $i_r\{s, t\} = i_\xi\{s, t\}$. ▲

Proof. Suppose $s,\ t \in x_\eta$. The only interesting case is when $s,\ t$ are incomparable in $\leq_\eta$, and hence, by Lemma (19), in $\leq_r$. If $u \leq_r s,\ t$ and $u \in x_\xi \setminus x$ then there are $v,\ w \in x$ such that $u <_\xi v \leq_\eta s$ and $u <_\xi w \leq_\eta t$. By Lemma (18) one has that $i_\eta\{v, w\} = i_\xi\{v, w\} = i_r\{s, t\} \subseteq x$. So there is some $z \in x \subseteq x_\eta$ such that $u <_r z \leq_r s,\ t$. The case $s,\ t \in x_\xi$ is similar. ▲

Corollary 22. If one can show that $r \in \mathbb{P}$ then one has that $r \leq p_\eta, p_\xi$. ▲

Lemma 23. $(\circ)$ holds for r. ▲

Proof. Immediate from the definition of r and $(\circ)$ for p_η, p_ξ alone. ▲

Lemma 24. If s, $t \in x_\eta$ or s, $t \in x_\xi$ then $(\bullet)$ holds for s, t, and if s, $t \in x_\xi$ then $(\bullet)$ holds for s, t. ▲

Proof. Let s, $t \in x_\eta$ and $u \in i_r\{s,t\}$. Then $u \in i_\eta\{s,t\}$ by Lemma (21). Applying $(\bullet)$ for p_η, if $\pi(u) = I(\pi(s), j(\pi(s), \pi(t)))^-$ there is nothing more to prove. Otherwise one must have that $\pi(u) < I(\pi(s), j(\pi(s), \pi(t)) + 1)^-$ and $c^\eta(\pi(u), \pi(t)) \leq c^\eta(\pi(s), \pi(t)) \leq \alpha (< \mu^+)$ (the latter since $a_\eta \subseteq \text{rge}(F_\eta)$). Fact (1.17) now gives both that $c^r(\pi(s), \pi(t)) = c^\eta(\pi(s), \pi(t))$ and that $c^r(\pi(u), \pi(t)) = c^\eta(\pi(u), \pi(t))$, since $\pi(t) \in a_\eta \subseteq \text{rge}(F_\eta)$, and hence $c^r(\pi(u), \pi(t)) \leq c^r(\pi(s), \pi(t))$. ▲

Now comes the main part of the proof.

Claim 25. If $s \in x_\eta \setminus x$ and $t \in x_\xi \setminus x$ then $(\bullet)$ holds for s, t. ▲

Proof. Suppose, without loss of generality that $\pi(s) < \pi(t)$. (Note that nothing has been assumed about which of η and ξ is the larger and which the smaller, so there really is no loss of generality.) If $s \leq_r t$ there is nothing to prove, so suppose on the contrary that s, t are incomparable in r. Let $u \in i_r\{s,t\}$. If $\pi(u) = I(\pi(s), j(\pi(s).\pi(t)))^-$ then there is again nothing more to be done, so suppose also that this is not the case. If $c^r(\pi(s), \pi(t)) = \mu^+$ then once more nothing more need be proven, so suppose that $c^r(\pi(s), \pi(t)) < \mu^+$

Set $j = j(\pi(s), \pi(t))$ and $I = I(\pi(s), j)$ $(= I(\pi(t), j))$. Summarizing the case to which the proposition has been reduced one has: $c^r(\pi(s), \pi(t)) \leq \alpha$, $\pi(u) < \pi(s) < \pi(t)$, $u \in i_r\{s,t\}$ and $\pi(u) \neq I^-$.

It is also useful to have some notation for the translation of elements of x_η to x_ξ and vice versa. So let $v' = h_{\eta\xi}(v)$ if $v \in x_\eta$ and let $v' = h_{\xi\eta}(v)$ if $v \in x_\xi$. Thus $v = v'$ if $v \in x$.

Lemma 26. $\pi(u) > I^-$. ▲

Proof. Suppose that $\pi(u) < I^-$. By Lemma (23), that $(\circ)$ holds for r, there are v, w such that $\pi(v) = \pi(w) = I^-$, $u <_r v \leq_r s$ and $u <_r w <_r t$. By Corollary (20), that $(\star)$ holds for r one has that $v = w$. But then $u <_r v \leq s$, t, contradicting the fact that $u \in i_r\{s,t\}$. As $\pi(u) \neq I^-$ one has $\pi(u) > I^-$ as required. ▲ (Lemma (26))

Corollary 27. $j(\pi(u), \pi(s)) \geq j$ and $j(\pi(u), \pi(t)) = j$. ▲

Lemma 28. $I(\pi(s'), j) = I = I(\pi(t'), j)$. ▲

Proof. If $u \in x$ then $j(\pi(u), \pi(t')) = j$ and $j(\pi(u), \pi(s')) \geq j$ by the thinning to 'fix j' and the fact that $u = u'$. Thus $I(\pi(s'), j)$, $I(\pi(t', j) = I(\pi(u), j) = I(\pi(s), j) = I$.

If $u \in x_\eta \setminus x$ then there is some $w \in x$ with $u <_\eta w <_\xi t$. As $j(\pi(u), \pi(t)) = j$ one has $j(\pi(u), \pi(w))$, $j(\pi(w), \pi(t)) = j$ (by Lemma (0.5)). By the fixedness of j one thus has $j(\pi(w), \pi(t')) \geq j$ and hence (by the other part of Lemma (0.5)) that $j(\pi(u), \pi(t')) \geq j$. Using Lemma (0.5) for a third time one has, regardless of whether $\pi(s) \leq \pi(t')$ or vice versa, that $j(\pi(s), \pi(t')) \geq j$. Hence, by the fixedness of j again, $j(\pi(s'), \pi(t)) \geq j$. Thus $I(\pi(s'), j) = I(\pi(t), j) = I = I(\pi(s), j) = I(\pi(t'), j)$ as required.

The case $u \in x_\xi \setminus x$ is similar. ▲ (Lemma (28))

Lemma 29. $j(\pi(s), \pi(s')) \geq j + 1$ and $j(\pi(u), \pi(u')) \geq j + 1$. ▲

Proof. Set $\sigma = e_I^{-1}(I(\pi(s), j + 1)^-)$, $\tau = e_I^{-1}(I(\pi(t), j + 1)^-)$ and $\nu = e_I^{-1}(I(\pi(u), j + 1)^-)$. As $a_\xi \subseteq \mathrm{rge}(F_\xi)$ one has that $c^r(\pi(s), \pi(t)) = c^r(\sigma, \tau) = c^\xi(\sigma, \tau)$. If $\sigma \in a_\eta \setminus a$ then $\sigma \notin \mathrm{rge}(F_\xi)$ and $c^r(\sigma, \tau) = \mu^+$, a contradiction. Hence $\sigma \in a$.

As $u <_r s$ and $\pi(u)$, $\pi(s) \in I$, one has $\nu < \sigma$. Since $\sigma \in a$, $\nu \in a_\eta$ and a is an initial segment of a_η, one has that $\nu \in a$ as well.

By the thinning of the collection of conditions the fact that $\sigma \in a$ gives that $\langle |\pi(s)|, j, \sigma \rangle \in b$. Hence $\sigma = e^{-1}_{I(\pi(s'), j)}(I(\pi(s'), j + 1)^-)$, since $b \subseteq b_\xi$. But Lemma (28) shows that $I(\pi(s'), j) = I$, so $I(\pi(s'), j + 1)^- = e_I(\sigma) = I(\pi(s), j + 1)^-$ and $j(\pi(s), \pi(s)) \geq j + 1$.

The remainder of the argument for u is exactly as in the previous paragraph replacing σ by ν and s by u. ▲ (Lemma (29))

Lemma 30. $\pi(u) = I(\pi(u), j + 1)^-$. ▲

Proof. If not then there are v and w such that $\pi(v) = \pi(w) = I(\pi(u), j + 1)^+$, $u <_r v \leq_r s$ and $u <_r w \leq_r t$, by ($\circ$) for r. But then by ($\star$) for r one has that $v = w$, a contradiction to the fact that $u \in i_r\{s, t\}$. ▲ (Lemma (30))

Corollary 31. $u = u' \in x$. ▲

Proof. By a proof identical to that for Lemma (30) one has that $\pi(u') = I(\pi(u'), j+1)^-$. But Lemma (30) shows that $I(\pi(u), j+1) = I(\pi(u'), j+1)$, so $\pi(u) = \pi(u')$. But then $\pi(u) \in d$ and for all $w \in x_r$ with $\pi(w) \in d$ one has that $w \in x$. Hence $u \in x$ and $u = u'$. ▲(Corollary (31))

Lemma 32. $s \not<_\eta t'$ and $s' \not<_\xi t$. ▲

Proof. As $h_{\eta\xi}$ sends $<_\eta$ to $<_\xi$, either the Lemma holds or $s <_\eta t'$ and $s' <_\xi t$, so suppose the latter. Then by ($\circ$) for r there are v, w such that $u <_\eta v \leq_\eta s$ and $u <_\xi w \leq_\xi s'$ such that $\pi(v) = \pi(w) = I(\pi(s), j+1)^-$. Note that $\pi(s)$, $\pi(s') \neq I(\pi(s), j+1)^-$ since s, $s' \notin x$ and hence $\pi(s)$, $\pi(s') \notin \pi``x = d$. Thus $v <_\eta s$ and $w <_\xi s'$. By ($\star$) for r one has that $v = w$ and hence $v <_r s, s'$. But then $u <_r v <_r s' <_r t$ and $u <_r s$, contradicting the fact that $u \in i_r\{s, t\}$. ▲ (Lemma (32))

Lemma 33. $u \in i_\eta\{s, t'\} \cap i_\xi\{s', t\}$. ▲

Proof. As $s \not<_\eta t'$ let $v \in i_\eta\{s, t'\}$ be such that $u \leq_\eta v$. Recall that Lemma (26) shows that $\pi(u) > I^-$ so $\pi(v) = I^-$ would imply $\pi(v) < \pi(u) \leq \pi(v)$, a contradiction. So, by ($\bullet$) for p_η, one has $\pi(v) < I(\pi(s), j+1)^-$. By ($\circ$) for p_η let z be such that $u \leq_\eta v <_\eta z <_\eta s$ and $\pi(z) = I(\pi(s), j+1)^-$, (Note that $\pi(s), \pi(s') \neq I(\pi(s), j+1)^-$ since s, $s' \notin x$ and hence $\pi(s)$, $\pi(s') \notin \pi``x = d$.) Clearly $v' \in i_\xi\{s', t\}$, $u \leq_\xi v'$ and $\pi(v') < I(\pi(s'), j+1) = I(\pi(s), j+1)$.

By ($\circ$) for p_ξ let z^* be such that $\pi(z^*) = I(\pi(s), j+1)^-$ and $u \leq_\xi v' <_\xi z^* < s'$. Then $z = z^*$ by ($\star$) for r (which holds by Corollary (19)). But then one has $v' <_r z <_r s$ and $v' \in i_\xi\{s', t\}$ and so $v' <_r t$. Hence $u <_r v' <_r s, t$, a contradiction to the fact that $u \in i_r\{s, t\}$.

Consequently $v = u = v'$ and $u \in i_\eta\{s, t'\}$. As $h_{\eta\xi}$ sends $<_\eta$ to $<_\xi$, this also gives that $u \in i_\xi\{s', t\}$. ▲ (Lemma (33)

Lemma 34. $j(\pi(s), \pi(t')) = j$ and $j(\pi(s'), \pi(t)) = j$. $\pi(s) < \pi(t')$ and $\pi(s') < \pi(t)$.

Proof. Lemma (28) gives that $j(\pi(s), \pi(t')) \geq j$. If $j(\pi(s), \pi(t')) > j$, then $j(\pi(s'), \pi(t')) > j$ by Lemma (0.5) and Lemma (29). But then the fixedness of j gives $j(\pi(s), \pi(t)) > j$, a contradiction to the definition of j as being $j(\pi(s), \pi(t))$, This gives $j(\pi(s'), \pi(t)) = j$ by the fixedness of j. Finally, $\pi(s) < \pi(t')$ and $\pi(s') < \pi(t)$ since $j(\pi(s), \pi(s')) > j+1$ while $j(\pi(s), \pi(t')) = j$ and $j(\pi(s'), \pi(t)) = j$. ▲ (Lemma (34))

Conclusion of Proof of Claim (25). As $u \in i_\xi\{s', t\}$ one has $c^\xi(\pi(u), \pi(t)) \leq c^\xi(\pi(s'), \pi(t))$. But $j(\pi(s), \pi(s') > j$, by Lemma (27), so $I(\pi(s), j+1)^- = I(\pi(s'), j+1)^-$, while $j(\pi(s'), \pi(t)) = j$ by Lemma (32). Consequently $c^\xi(\pi(s), \pi(t)) = c^\xi(e_I^{-1}(I(\pi(s), j+1)^-), e_I^{-1}(I(\pi(t), j+1)^-)) = c^\xi(e_I^{-1}(I(\pi(s'), j+1)^-), e_I^{-1}(I(\pi(s), j+1)^-)) = c^\xi(\pi(s'), \pi(t))$. Thus $c^\xi(\pi(u), \pi(t)) \leq c^\xi(\pi(s), \pi(t))$. Hence, by Fact (1.17), $c^r(\pi(u), \pi(t)) \leq c^r(\pi(s), \pi(t))$.

This completes the proof of Claim (25) and with it the proof (of Proposition (17)) that $\mathbb{P}$ has the μ^{++}-chain condition. ▲ (Claim (25), Proposition (17))

Proposition 35. $\mathbb{P}$ is μ-closed. ▲

Proof. Let $\langle p_\eta \mid \eta < \lambda \rangle$ be a descending sequence of conditions for some limit ordinal λ with $p_\eta = (x_\eta, \leq_\eta, A_\eta)$ for each $\eta < \lambda$. Set $p = (x_p, \leq_p, A_p)$ where $x_p = \bigcup\{ x_\eta \mid \eta < \lambda \}$, $s \leq_p t$ if there is some (equivalentely, for all sufficiently large) $\eta < \lambda$ such that $s \leq_\eta t$ for $s, t \in x_p$, and $A_p = \bigcup\{ A_\eta \mid \eta < \lambda \}$. The definition of $c^p(s,t)$ is finitary (see Definition (1.12)), that is uses only finitely many members of A_p for each $s, t \in x_p$. So $c^p(s,t) = c^\eta(s,t)$ for all sufficiently large $\eta < \lambda$. Using this it is easy to see that p is a condition and that $p \leq p_\eta$ for all $\eta < \lambda$. ▲

Corollary 36. Forcing with $\mathbb{P}$ over any model in which $2^{<\mu} \leq \mu^+$ preserves cardinals. ▲

Proof. By Propositions (6), (17) and (35), μ^+, cardinals above μ^+ and cardinals below μ^+, respectively, are preserved. ▲

Lemma 37. For each $\xi < \rho$ and $\gamma < \mu$ the collection $\{ p \in \mathbb{P} \mid (\xi, \gamma) \in x_p \}$ is a dense (and open) subset of $\mathbb{P}$. ▲

Proof. Let $\xi < \rho$, $\gamma < \mu$ and $p \in \mathbb{P}$. If $(\xi, \gamma) \in a_p$ there is nothing to prove. Otherwise define q by $x_q = x_p \cup \{ (\xi, \gamma) \}$, $\leq_q = \leq_p$, $A_q = A_p$. Then $q \in \mathbb{P}$ and $(\xi, \gamma) \in x_q$. ▲

Proposition 38. Let G be $\mathbb{P}$-generic over V and define a partial order on $\rho \times \mu$ by $s \leq t$ if there is some $p \in G$ such that $s, t \in x_p$ and $s \leq_p t$. Then $\leq$ is a (μ, ρ)-admissible partial order. ▲

Proof. One has that $\leq$ is indeed a partial order on $\rho \times \mu$ by Lemma (37). It is clear that $\leq$ respects the partial ordering by first co-ordinates, that every pair of elements has a set of maximal common predecessors of size less than μ, and that $(\star)$ holds: every pair of elements with the same first co-ordinate have no common predecessors, since these are true for conditions in $\mathbb{P}$. The one remaining thing to show is that each element has μ-many predecessors at every level below its own.

So suppose that $\alpha < \beta < \rho$ and ε, $\gamma < \mu$. I show that $\mathcal{D}_\varepsilon = \{ q \in \mathbb{P} \mid \exists \delta \in (\varepsilon, \mu) (\alpha, \delta) \leq_q (\beta, \gamma) \}$ is dense in $\mathbb{P}$. ($\mathcal{D}$ is clearly open.)

Let $p \in \mathbb{P} \setminus \mathcal{D}_\varepsilon$. Without loss of generality suppose $t = (\beta, \gamma) \in x_p$. Let

$\delta \in (\varepsilon, \mu)$ be such that for all $(\beta', \gamma') \in x_p$ one has that $\gamma' < \delta$. I shall define a condition q such that $s = (\alpha, \delta) \in q$. This entails adding additional points between in order to witness $(\star)$ for s, t, and, merely for convenience, I take these to each have second co-ordinate δ as well.

So let $w(\alpha, \beta) = \langle \alpha, \nu_1, \ldots, \nu_n, \beta \rangle$ and let $W = \{ \alpha, \nu_1, \ldots, \nu_n \} \setminus \{ \beta \}$. Set $x_q = x_p \cup \{ (\tau, \delta) \mid \tau \in W \}$, $u \leq_q v$ if $u \leq_p v$ or $u, v \in \{ (\tau, \delta) \mid \tau \in W \} \cup \{ z \in x_p \mid t \leq_p z \}$, and $\pi(u) \leq \pi(v)$, β.

It is now elementary using Definition (0.8), the definition of a walk, to check that q satisfies $(\circ)$. (The proof is exactly as in [Mart, Lemma (1)].) It is then clear that $q \in \mathbb{P}$ and that $q \leq p$.

G, being generic, meets $\mathcal{D}_\varepsilon$ for every $\varepsilon < \mu$. Since μ is regular this ensures that t has μ-many predecessors with first co-ordinate equal to α. Thus $\leq$ is (μ, ρ)-admissible. ▲

Proof of Theorem (4) of the Introduction. Proposition (38) and Corollary (36) give exactly what is required. ▲

§4. Proof of Theorem (5) of the Introduction.

Definition 1. A topological space X is *μ-Lindelöf* if every open cover has a subcover of size *less than* μ. Thus ω-Lindeöf means compact and ω_1-Lindelöf means Lindelöf. A space is *locally μ-Lindelöf* if every point has an open neighbourhood whose closure is μ-Lindelof. A space is *0-dimensional* if its topology has a basis of sets which are both closed and open. A space X is *scattered* or *right-separated* if there is a well ordering $<_X$ of it such that for every $x \in X$ there is an open neighbourhood $\mathcal{O}$ of x such that $\mathcal{O} \cap \{ y \in X \mid x <_X y \} = \emptyset$. A space X is *μ-tight* if for every $Y \subseteq X$ and x is in the closure of Y there is some $Z \in [Y]^{\leq\mu}$ such that x is in the closure of Z. The *height* of a scattered space is its Cantor-Bendixson rank.

I now show that in the forcing extension of the previous section Theorem (5) of the Introduction holds.

Let $\mathbb{P}$ be the forcing defined in §3, and let G be $\mathbb{P}$-generic over V. For each $t \in \rho \times \mu$ set $B_t = \{ s \in \rho \times \mu \mid \exists p \in G \; s \leq_p t \}$. Let $\mathcal{B} = \{B_t \setminus \bigcup\{ B_s \mid s \in b\} \mid t \in \rho \times \mu \;\; \& \;\; b \in [\pi(t) \times \mu]^{<\mu}\}$. Let τ be the topology on κ^+ generated by taking $\mathcal{B}$ as a sub-basis. Let $<_{\rm lex}$ be the lexicographic well-ordering of $\rho \times \mu$.

For each pair $s,\, t \in \rho \times \mu$ let $i\{ s,t \} = \{ u \mid \exists p \in G \; u \in i_p\{ s,t \} \}$. Note that $\overline{\overline{i\{ s,t \}}} < \mu$ for all $s,\, t \in \rho \times \mu$.

Lemma 2. $\mathcal{B}$ is a basis for a topology on $\rho \times \mu$, not just a sub-basis. ▲

Proof. Let $\mathcal{O}_0 = B_t \setminus \bigcup\{ B_w \mid w \in a \}$ and $\mathcal{O}_1 = B_s \setminus \bigcup\{ B_v \mid v \in b \}$ be sets in $\mathcal{B}$. Then $\mathcal{O}_0 \cap \mathcal{O}_1 = B_s \cap B_t \cap (\rho \times \mu \setminus \bigcup\{ B_u \mid v \in a \cup b \})$.

Now $B_s \cap B_t = \bigcup\{ B_u \mid u \in i\{ s,t \} \}$, so

$$\mathcal{O}_0 \cap \mathcal{O}_1 = \bigcup\{B_u \setminus \bigcup\{B_v \mid v \in a \cup b\} \mid u \in i\{s,t\}\}.$$

So in order to show that $\mathcal{O}_0 \cap \mathcal{O}_1$ is a union of sets in $\mathcal{B}$ it suffices to show that each $B_u \setminus \bigcup\{ B_v \mid v \in a \cup b \}$ is in $\mathcal{B}$.

But $B_u \setminus \bigcup\{ B_v \mid v \in a \cup b \} = B_u \setminus \bigcup\{ B_u \cap B_v \mid v \in a \cup b \}$, and each $B_u \cap B_v = \bigcup\{ B_z \mid z \in i\{ u,v \} \}$. Since each $i\{ u,v \}$ has size less than μ, one has that $\{ z \mid \exists v \in a \cup b \; z \in i\{ u,v \} \}$ is a union of fewer than μ sets each of size less than μ and so, as μ is regular, itself has size less than μ. Thus $B_u \setminus \bigcup\{ B_v \mid v \in a \cup b \} \in \mathcal{B}$ as required and so $\mathcal{O}_1 \cap \mathcal{O}_2$ is a union of sets in $\mathcal{B}$. ▲

Lemma 3. $\mathcal{B}$ is a clopen basis and so τ is 0-dimensional. ▲

Proof. If $\mathcal{O} = B_t \setminus \bigcup\{B_w \mid w \in a\}$ then $\rho \setminus \mathcal{O} = (\rho \times \mu \setminus B_t) \cup \{B_w \mid w \in a\}$. Clearly $B_w \in \mathcal{B}$ for $w \in a$. Also $\rho \times \mu \setminus B_t = \bigcup\{B_u \setminus B_u \mid u \in \rho \times \mu\}$. But $B_u \setminus B_t = B_u \setminus (B_u \cap B_t) = B_u \setminus \bigcup\{B_s \mid s \in i\{t,u\}\}$, as in the proof of Lemma (2). Thus each $B_u \setminus B_t$ is a union of sets in $\mathcal{B}$, and hence $\rho \times \mu \setminus \mathcal{O}$ is also a union of sets in $\mathcal{B}$. ▲

Lemma 4. τ is right-separated. ▲

Proof. If $t \in \rho \times \mu$ then $t \in B_t \in \mathcal{B}$, and $\pi``(B_t \setminus \{t\}) \subseteq \pi(t)$. So $u \notin B_t$ for any $u \in \rho \times \mu$ with $t <_{\text{lex}} u$. ▲

Lemma 5. $(\rho \times \mu, \tau)$ is locally μ-Lindelöf. ▲

Proof. Let $t \in \rho \times \mu$. $t \in B_t$, and B_t is closed (by Lemma (19)). So it suffices to show that each B_t is μ-Lindelöf. This is done by induction on ρ.

Suppose that B_u is μ-Lindelöf for each $u \in \rho \times \mu$ with $\pi(u) < \pi(t)$. Let $\langle \mathcal{O}_i \mid i \in I \rangle$ be an open cover of B_t. So there is some $i \in I$ and some $B_w \setminus \bigcup\{B_{u_j} \mid u_j \in b\} \subseteq \mathcal{O}_i$ such that $t \in B_w \setminus \bigcup\{B_{u_j} \mid u_j \in b\} \in \mathcal{B}$. So B_t has a subcover of size less than μ if and only if $B_t \cap B_{u_j}$ has a subcover of size less than μ for each $j \in b$ (using the facts that $\overline{\overline{b}} < \mu$ and that μ is regular). But $B_t \cap B_{u_j} = \bigcup\{B_s \mid s \in i\{t, u_j\}\}$ for each $j \in b$, and thus is the union of fewer than μ many sets of the form B_v with $\pi(v) < \pi(t)$ since each $i\{t, u_j\}$ has size less than μ.

By the inductive hypothesis each $B_v, \subseteq B_t \subseteq \bigcup\{\mathcal{O}_i \mid i \in I\}$, has a subcover of size less than μ, and hence $B_t \cap B_{u_j}$ has the union of these subcovers as a subcover of size less than μ. Hence B_t is μ-Lindelöf in τ. ▲

Lemma 6. Suppose $\mathcal{O} \in \mathcal{B}$. If $\mathcal{O} \neq \emptyset$ then $\mathcal{O} \cap \{0\} \times \mu \neq \emptyset$. ▲

Proof. Let $p \in \mathbb{P}$ be such that $p \Vdash$ "$\mathcal{O} = B_t \setminus \bigcup\{B_w \mid w \in a\} \in \mathcal{B}$" and suppose, without loss of generality, using the μ-closure of $\mathbb{P}$, that $\{t\} \cup a \subseteq x_p$. Let $\beta \in \mu \setminus \{\gamma \mid \exists s \in x_p \to s = (\pi(s), \gamma)\}$. Define q by $x_q = x_p \cup \{(0,\beta)\}$, $A_q = A_p$ and $\leq_q \restriction x_p \times x_p = \leq_p$ and for each $v \in x_p$ set $(0,\beta) \leq_q v$ if and only if $t \leq_p v$. It is clear that $q \in \mathbb{P}$ and $q \leq p$. And it is also clear that $q \Vdash (0,\beta) \in \mathcal{O} \cap \{0\} \times \mu$. ▲

Corollary 7. The separability degree of τ is μ. ▲

Proof. Lemma (6) gives μ as an upper bound, but it is clear from the definition of $\mathbb{P}$ that given any smaller sized set and some condition there is a stronger condition that forces some set in $\mathcal{B}$ to miss it. ▲

Propostion 8. τ has tightness μ. ▲

Proof. Fix $Z \subseteq \rho \times \mu$. Write $cl(Z)$ for the closure of Z in the topology generated by $\mathcal{B}$. I prove by induction on ρ that if $t \in \mathrm{cl}(Z)$ for some $Z \subseteq \rho \times \mu$ then there is some $Y \in [Z]^{\mu}$ such that $t \in \mathrm{cl}(Y)$.

So let $t \in \rho \times \mu$ and assume that for all $s \in \rho \times \mu$ such that $\pi(s) < \pi(t)$ there is some $Y' \in [Z]^{\mu}$ such that $s \in \mathrm{cl}(Y')$. Assume also that $t \notin Z$ as otherwise one can merely take $Y = \{z\}$.

First of all $Z \setminus B_t \subseteq \rho \times \mu \setminus B_t$, so $\mathrm{cl}(Z \setminus B_t) \subseteq \mathrm{cl}(\rho \times \mu \setminus B_t) = \rho \times \mu \setminus B_t$, since B_t is open in τ. Thus $\mathrm{cl}(Z \setminus B_t) \cap B_t = \emptyset$.

This gives us that $t \in \mathrm{cl}(Z \cap B_t)$, because $\mathrm{cl}(Z) = \mathrm{cl}(Z \cap B_t) \cup \mathrm{cl}(Z \setminus B_t)$ since $\mathrm{cl}(.)$ is a closure operator.

Now B_t is closed in τ by Lemma (3), so $\mathrm{cl}(Z \cap B_t) \subseteq B_t$. I now proceed by a series of claims.

Claim 9. There is some $Y \subseteq Z \cap B_t$ such that $t \in \mathrm{cl}(Y)$ and the set $\{s \in \mathrm{cl}(Y)) \mid s <_{\mathrm{lex}} t\}$ has no $<_{\mathrm{lex}}$-maximal element. ▲

Proof. Let $X_0 = Z \cap B_t$ and inductively set v_i to be $<_{\mathrm{lex}}$-maximal in $\{s \in \mathrm{cl}(X_i) \mid s <_{\mathrm{lex}} t\}$ and $X_{i+1} = X_i \setminus B_{v_i}$ for as long as possible. By the above observations applied to X_i in place of Z, $\mathrm{cl}(X_{i+1}) \subseteq (\rho \times \mu \setminus B_{v_i})$ and $\mathrm{cl}(X_i \cap B_{v_i}) \subseteq B_{v_i}$, one has that $t \in \mathrm{cl}(X_{i+1}) \setminus \mathrm{cl}(X_i \cap B_{v_i})$ and $v_i \in \mathrm{cl}(X_i \cap B_{v_i}) \setminus \mathrm{cl}(X_{i+1})$ for each i for which v_i is defined. As $X_{i+1} \subseteq X_i$ one has that $v_{i+1} \leq v_i$ for all i for which v_{i+1} is defined.

But $v_{i+1} \in \rho \times \mu \setminus B_{v_i}$, so $v_{i+1} \neq v_i$ and, hence, $v_{i+1} < v_i$ for each i for which v_{i+1} is defined. So let $k < \omega$ be least such that $\mathrm{cl}(X_k) \cap \gamma$ has no $<_{\mathrm{lex}}$-maximal element. Set $Y = X_k$. Then $Y \subseteq X_0 = Z \cap B_t$, $\gamma \in \mathrm{cl}(Y)$ and $\{s \in \mathrm{cl}(Y) \mid s <_{\mathrm{lex}} t\}$ has no $<_{\mathrm{lex}}$-maximal element. ▲ (Claim (9))

Write W for the set $\{s \in \mathrm{cl}(Y) \mid s <_{\mathrm{lex}} t\}$.

Claim 10. There is some $Y' \in [W]^{\mu}$ such that $t \in \mathrm{cl}(Y')$. ▲

Proof. Set $W^* = (\{0\} \times \mu) \cap \bigcup\{B_u \mid u \in W\}$. Note that $Y \subseteq W$ since $\mathrm{cl}(Y) \subseteq \mathrm{cl}(B_t) = B_t$ and $t \notin Y$, and that $\mathrm{cl}(Y) = \mathrm{cl}(W)$. Choose $Y' \in [W]^{\mu}$ such that $W^* \subseteq \bigcup\{B_u \mid u \in Y'\}$.

Subclaim 11. $t \in \mathrm{cl}(W^*)$. ▲

Proof. Suppose that $\mathcal{O}$ is an open set (in τ) and that $\mathcal{O} \cap W^* = \emptyset$. Then for all $w \in W$ one has that $\mathcal{O} \cap B_w \cap (\{0\} \times \mu) = \emptyset$. By Lemma (6) this implies that $\mathcal{O} \cap B_w = \emptyset$ for all $w \in W$. Hence $\mathcal{O} \cap \bigcup\{B_w \mid w \in W\} = \emptyset$. But $t \in \mathrm{cl}(Y) \subseteq \mathrm{cl}(W) \subseteq \mathrm{cl}(\bigcup\{B_w \mid w \in W\})$, so one must have $t \notin \mathcal{O}$.

▲ (Subclaim (11))

Since $\mathrm{cl}(Y') \subseteq \mathrm{cl}(Y) \subseteq B_t$, one has that $\mathrm{cl}(Y')$ is μ-Lindelöf in τ. Suppose that $t \notin \mathrm{cl}(Y')$.

Then $\{B_w \mid w \in W\}$ is a cover of $\mathrm{cl}(Y')$, so there is some $\chi < \mu$ and some $\{u_i \mid i < \chi\}$ such that $u_i \in W$ for each $i < \chi$ and $\{B_{u_i} \mid i < \chi\}$ is also a cover of $\mathrm{cl}(Y')$. This means that if $w \in \mathrm{cl}(Y')$ there is some $i < \chi$ such that $w \in B_{u_i}$, *i.e.*, such that $w \leq u_i$, and hence such that $B_w \subseteq B_{u_i}$.

Thus $\bigcup\{B_w \mid w \in W\} \subseteq \{B_{u_i} \mid i < \chi\}$, and consequently $W^* \subseteq \bigcup\{B_{u_i} \mid i < \chi\}$. *A fortiori*, $\mathrm{cl}(W^*) \subseteq \mathrm{cl}(\bigcup\{B_{u_i} \mid i < \chi\})$.

However, $B_t \setminus \bigcup\{B_{u_i} \mid i < \chi\}$ is an open set that contains t and has empty intersection with $\{B_{u_i} \mid i < \chi\}$, showing that $t \notin \mathrm{cl}(\{B_{u_i} \mid i < \chi\})$ and hence $t \notin \mathrm{cl}(W^*)$, contradicting Subclaim (11). Thus $t \in \mathrm{cl}(Y')$ as required.

▲ (Claim (10))

Conclusion of proof of Proposition 8. Suppose $\pi(s) = \pi(t)$ and $s <_{\mathrm{lex}} t$. Then $B_s \cap B_t = \emptyset$ (since the admissible partial order satisfies $(\star)$). So $B_s \cap Y' = \emptyset$.

Hence if $v \in Y'$ then $v \in \mathrm{cl}(Z) \cap \{u \in \rho \times \mu \mid u <_{\mathrm{lex}} t \,\&\, \pi(u) < \pi(t)\}$ and so by the inductive hypothesis there is some $W_v \in [Z]^{\mu}$ such that $v \in \mathrm{cl}(W_v)$. Set $Y'' = \bigcup\{W_v \mid v \in Y'\}$. Then $\overline{\overline{Y''}} \leq \mu$, and $Y' \subseteq \mathrm{cl}(Y'')$, so one has that $t \in \mathrm{cl}(Y') \subseteq \mathrm{cl}(\mathrm{cl}(Y'')) = \mathrm{cl}(Y'')$. ▲ (Proposition (8))

Proof of Theorem (5) of the Introduction. By Lemmas (2) to (7) and Proposition (8), $(\rho \times \mu, \tau)$ is a scattered, 0-dimensional, μ-tight, locally μ-Lindelöf space of height ρ. Its one-point μ-Lindelöfization is a 0-dimensional, μ-tight, μ-Lindelöf scattered space of height ρ. Both clearly have width μ.

▲

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