Invariant Factors of an Endomorphism of a projective Module

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Abstract

Let R be a commutative integrally closed domain. For any endomorphism u of a projective R-module M of constant rank, we define its characteristic polynomial and we give a finite sequence of monic polynomials $\{s_i(u,X)\}_{1\le i\le t}$ such that their product is the characteristic polynomial of u and $s_i(u,X)$ divides $s_{i+1}(u,X)$. Like the invariant factors in the case of vector spaces, these polynomials are the monic generators of the ideals $[F_{t-i}(M_u) : F_{t-i+1}(M_u)]$, where $F_i(M_u)$ is the i^{th} Fitting ideal of the $R[X]$-module M_u defined on M via u. We also give the tie between the $s_i(u,X)$ and the invariant factors of M_u, as defined in [3].

0 Introduction

Let R be a commutative ring with unit, and M a finitely generated R-module. In this paper we are concerned with the classical problem of classification of R-endomorphisms of M, when M is a projective R-module of constant rank. We shall assign to each endomorphism u of M a finite set of monic polynomials such that when $R{=}K$ is a field and $M{=}E$ is a finite dimensional vector space these polynomials are the classical invariant factors associated to u.

Let M be an R-module and $u\colon M \longrightarrow M$ an R-endomorphism. The endomorphism u converts M into an $R[X]$-module by: $X.x = u(x)$, for each $x \in M$. We will denote M as $R[X]$-module via u by M_u.

In the classical case, i.e., when $R = K$ is a field and $M = E$ is a finite dimensional vector space, the invariant factors Theorem implies that $M_u = \oplus_{i=1}^t K[X]/(s_i(u,X))$, where $\{s_i(u,X)\}_{1\leq i\leq t}$ are the invariant factors of u, i.e., they are monic generators of the ideals $[F_{t-i}(E_u) : F_{t-i+1}(E_u)]_{1\leq i\leq t}$, where $F_i(E_u)$ is the i^{th} Fitting ideal of E_u. Two endomorphisms u and v are similar if and only if $s_i(u,X) = s_i(v,X)$ for each i, or equivalently, if and only if $F_i(E_u){=}F_i(E_v)$ for each i.

In Section 1, we recall the notion of the Fitting ideals of a finitely presented module over a commutative ring and describe some of their properties. We also give the characteristic exact sequence which allows us to compute the Fitting ideals of M_u when M is a finite free R-module and u is an endomorphism of M. In Section 2, we prove Lemma 1 and Proposition 1 which are a crucial tool in the proof of our main result: Theorem 1, in which we give the construction of invariant factors $\{s_i(u,X)\}_{1\leq i\leq t}$ associated to an endomorphism u of a projective R-module of constant rank. We show that the invariant factors of M_u, according to the definition of [2], are the smallest principal ideals of $R[X]$ generated by monic polynomials that contain the $s_i(u,X)$. In the third section, we give the construction of invariant factors $\{s_i(u,X)\}_{1\leq i\leq t}$ associated to an endomorphism u of a projective module of constant rank.

In [7] J. A. Hermida and M. Pisoniro give the same construction in the context of finite free resolution module over reduced rings. Our method is different, and the explicit obtaining of invariant factors following their construction is more difficult.

1 Preliminaries and notation

Let R be a commutative ring with unit, and let M be a finitely presented module. We denote by $F_i(M)$ the i^{th} Fitting ideal of M (see [7]), i.e, if

$$R^m \xrightarrow{u} R^n \longrightarrow M \longrightarrow 0$$

is a presentation of M, and A is the matrix of u relatively to two R-bases of R^m and R^n, then

$$F_k(M) = \begin{cases} \mathcal{U}_{n-k}(A) & \text{if} k = 0, 1, 2, \ldots, n-1. \\ R & \text{if} k \geq n. \end{cases}$$

where $\mathcal{U}_{n-k}(A)$ is the ideal generated by the $(n-k)\times(n-k)$ minors of A. The Fitting ideals satisfy the following properties

(I) $0 \subseteq F_0(M) \subseteq F_1(M) \subseteq \ldots.$

(II) $(\mathrm{Ann}_R(M))^n \subseteq F_0(M) \subseteq \mathrm{Ann}_R(M)$.

(III) Let $u\colon R \longrightarrow R^{'}$ be an homomorphism of commutative rings. Then

$$F_k(R^{'} \otimes_R M) = F_k(M).R^{'}, \text{ for } k \geq 0.$$

In particular, if S is a multiplicatively closed subset of R, and I is an ideal of R, then $F_k(S^{-1}M) = S^{-1}F_k(M)$, and $F_k(M/IM) = F_k(M).R/I$ for any positive integer k.

Let M be a finitely generated R-module and $u\colon M \longrightarrow M$ an R-endomorphism. We have the characteristic exact sequence

$$0 \longrightarrow R[X] \otimes_R M \xrightarrow{\psi_u} R[X] \otimes_R M \xrightarrow{\varphi} M_u \longrightarrow 0$$

where

$$\psi_u(f(X) \otimes m) = X.f(X) \otimes m - f(X) \otimes u(m)$$

and

$$\varphi(f(X) \otimes m) = f(X).m = f(u)(m)$$

for each $(m, f(X)) \in M \times R[X]$ (see [7]). Therefore if M is a finite free R-module and A a matrix of u, then $XI_n - A$ is a matrix of ψ_u, where I_n is the $(n \times n)$-unit matrix. So, $F_k(M_u) = \mathcal{U}_{n-k}(XI_n - A)$ for any positive integer k. In consequence, the ideal $F_0(M_u)$ is a principal ideal generated by the characteristic polynomial $\chi_u(X)$ of u, while the ideals $F_k(M_u)$ for $k \geq 1$ are not principal in general. So, in this case we can talk about a characteristic polynomial but not about invariant factors.

In the second section, we show that for an endomorphism of a finite free module over an integrally closed domain, we can construct a sequence of monic polynomials analogous to the invariant factors in the case of vector spaces.

2 Invariant factors of an endomorphism

In this section, we suppose that R is an integrally closed domain and u an R-endomorphism of a finite free R-module M of rank n. Let K be the quotient field of R. Set $E = K \otimes_R M$ and $u(K) = 1_K \otimes u$. Then $E_{u(K)} \cong K[X] \otimes_{R[X]} M_u$ and $F_k(E_{u(K)}) = F_k(M_u).K[X]$ for any positive integer k (see [[7], Proposition 1.2]).

Lemma 1 *Let J be an ideal of $R[X]$ which contains a monic polynomial of $R[X]$. Set K.J=f(X)K[X], where $f(X) \in K[X]$ is monic. Then*

(i) $f(X) \in R[X]$.

(ii) $J \subseteq f(X)R[X]$.

(iii) $f(X)R[X]$ *is the smallest principal ideal of* $R[X]$ *generated by a monic polynomial that contains* J.

Proof. (i) Let $g(X)$ be a monic polynomial in J. Then there exists $h(X) \in K[X]$ such that $g(X) = f(X)h(X)$. Hence every root of $f(X)$ is a root of $g(X)$ and then every root of $f(X)$ is integral over R. Since $f(X)$ is monic, then its coefficients are integral over R.Since R is integrally closed, then $f(X) \in R[X]$.

(ii) Let $g(X) \in J$. Then $g(X) = f(X)h(X)$, with $h(X) \in K[X]$. On the other hand, since $g(X)$ and $f(X)$ are monic, there exist polynomials $q(X)$ and $r(X)$ in $R[X]$ such that $g(X) = f(X)q(X) + r(X)$, where the degree of $r(X)$ is smaller than the degree of $f(X)$. Hence $r(X) = 0$ and $h(X) = q(X) \in R[X]$. So, $g(X) \in f(X)R[X]$ and then $J \subseteq f(X)R[X]$.

(iii) Suppose that there exist another monic polynomial $g(X)$ in $R[X]$ such that $J \subseteq g(X)R[X]$. Then $K.J \subseteq g(X)K[X]$. So $f(X)K[X] \subseteq g(X)K[X]$. Hence $f(X) = g(X)h(X)$, where $h(X) \in K[X]$. Then, by Gauss Lemma, $h(X) \in R[X]$ ($f(X)$ and $g(X)$ are monic). Hence $f(X)R[X] \subseteq g(X)R[X]$.□

For two ideals I and J of R. We denote by $[I : J]$ the quotient ideal of I and J, i.e., $(I : J) = \{a \in R \mid aJ \subseteq I\}$.

Proposition 1 *For any positive integer* k*, there exists a monic plynomial* $q_k(X)$ *(resp.* $q_k^{'}(X)$*) in* $R[X]$ *such that* $q_k(X)R[X]$ *(resp.* $q_k^{'}(X)R[X]$*) is the smallest principal ideal of* $R[X]$ *generated by a monic polynomial that contains* $F_k(M_u)$ *(resp.* $(F_k(M_u) : F_{k+1}(M_u))$*).*

Proof. Let k be a positive integer. Let $J{=}F_k(M_u)$ and $J^{'}{=}(F_k(M_u) : F_{k+1}(M_u))$. We know that $F_r(E_{u(K)}) = F_r(M_u)K[X]$ for any positive integer r. Then $F_k(E_{u(K)}) = J.K[X]$. So, by [[2], corollary 3.15, page 43] $(F_k(E_{u(K)}) : F_{k+1}(E_{u(K)})) = J^{'}.K[X]$. Thereby we have two monic polynomials $q_k(X)$ and $q_k^{'}(X)$ in $K[X]$ such that $J.K[X] = q_k(X)K[X]$ and $J^{'} = q_k^{'}(X)K[X]$. On the other hand $J = F_k(M_u) = \mathcal{U}_{n-k}(XI_n - A)$, where n is the rank of M, and A is a matrix of u. Then J and $J^{'}$ contain monic polynomials. So the result holds by Lemma 1 (iii). □

The following result give the construction of the "invariant factors" of an endomorphism of a finite free module.

Theorem 1 *Suppose* R *is an integrally closed domain, and* u *an* R*-endomorphism of a finite free* R*-module* M*. There exist monic polynomials* $s_1(u, X)$*,* $\ldots, s_t(u, X)$ *of positive degree which are unique verifying:*

(i) $\chi_u(X) = s_1(u,X)\dots s_t(u,X)$, *where* $\chi_u(X)$ *is the characteristic polynomial of* u.

(ii) For $i = 1, \dots, t-1$, $s_i(u,X)$ *divides* $s_{i+1}(u,X)$.

(iii) For each positive integer $k \leq t-1$, *the ideal* $(\prod_{i=1}^{t-k} s_i(u,X))$ *is the smallest principal ideal of* $R[X]$ *generated by a monic polynomial that contains* $F_k(M_u)$.

(iv) For each positive integer $k \geq t$, *the smallest principal ideal of* $R[X]$ *generated by a monic polynomial that contains* $F_k(M_u)$ *is* $R[X]$.

Proof. Let $t = min\{k \geq 0 \mid F_k(M_u) = R[X]\}$. For any positive integer k, there exists a monic polynomial $q_k^{'}(X)$ such that $q_k^{'}(X)R[X]$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $[F_k(M_u) : F_{k+1}(M_u)]$ (by Proposition 1). Then we put $s_i(u,X) = q_{t-i}^{'}(X)$ for $i = 1, \dots, t$. So, we have a sequence of monic polynomials of $R[X]$. The uniqueness of these polynomials is insured by definition of the $q_i^{'}(X)$. Since the polynomials $s_1(u,X), \dots, s_t(u,X)$ are the invariant factors of $u(K)$, then (i) and (ii) holds. (iii) and (iv) are consequences of the Proposition 1.

Definition 1 *Let the assumptions and notation be as in Theorem 1. We define the invariant factors of* u *as the polynomials* $\{s_i(u,X)\}_{1 \leq i \leq t}$ *given in Theorem 1.*

The following corollary makes the tie between the invariant factors of u and the invariant factors $\{\delta_k(M_u)\}_{k \geq 1}$ of M_u defined in [3] by:

$$\delta_k(M_u) = \operatorname*{Ann}_{R[X]}(\bigwedge^k (M_u)),$$

where $\bigwedge^k (M_u)$ is the k^{th} exterior power of M_u.

Corollary 1 *Let* k *be a positive integer* $\leq t-1$. *Then* $s_{t-k}(u,X)R[X]$ *is the smallest principal ideal of* $R[X]$ *generated by a monic polynomial that contains* $\delta_{k+1}(M_u)$.

Proof. Let k be a positive integer $\leq t-1$. Then

$$\delta_k(E_{u(K)}) = \operatorname*{Ann}_{K[X]}(\bigwedge^k (E_{u(K)})).$$

On the other hand,

$$\bigwedge^{k+1}(E_{u(K)}) = \bigwedge^{k+1}(K[X]\otimes_{R[X]} M_u) \cong K[X]\otimes_{R[X]}\bigwedge^{k+1}(M_u)$$

(see [[11], (4) in page 204]). Then

$$\begin{aligned}\operatorname*{Ann}_{K[X]}(\bigwedge^{k+1}(E_{u(K)})) &= \operatorname*{Ann}_{K[X]}(K[X]\otimes_{R[X]}\bigwedge^{k+1}(M_u)) \\ &= S^{-1}(\operatorname*{Ann}_{R[X]}(R[X]\otimes_{R[X]}\bigwedge^{k+1}(M_u))),\end{aligned}$$

where $S = R$-$\{0\}$. So, $\mathrm{Ann}_{K[X]}(\bigwedge^{k+1}(E_{u(K)})) = \mathrm{Ann}_{R[X]}(\bigwedge^{k+1}(M_u))K[X]$. Hence $\delta_{k+1}(E_{u(K)}) = \delta_{k+1}(M_u)K[X]$. Since

$$\begin{aligned}\delta_{k+1}(E_{u(K)}) &= [F_k(E_{u(K)}) : F_{k+1}(E_{u(K)})] \\ &= [F_k(M_u) : F_{k+1}(M_u)]K[X] = s_{t-k}(u,X)K[X],\end{aligned}$$

then $\delta_{k+1}(M_u)K[X] = s_{t-k}(u,X)K[X]$. By [[5], corollary 1.4], $\chi_u(X)R[X] = F_0(M_u) \subseteq \delta_{k+1}(M_u)$, then $\delta_{k+1}(M_u)$ contains a monic polynomial of $R[X]$. Also the result holds by Lemma 1 (iii).

3 Invariant factors of an endomorphism of a projective module of constant rank

In [1], M.A. Knus and M. Ojanguren have given a generalization of the characteristic polynomial of an endomorphism u of a projective R-module M as follows: For a Zariski recouvrement $S = \oplus_{i=1}^{n} R_i$ of R and $S\otimes M = \oplus_{i=1}^{n} M_i$. The characteristic polynomial of u is the tuple $\chi_u(X) = (\chi_{u_i}(X))_{1\leq i\leq n} \in S[X]$, where $\chi_{u_i}(X)$ is the characteristic polynomial of u_i, the R_i endomorphism of M_i induced by u. In particular, for $S = \oplus_{\mathcal{P}\in max(R)} R_{\mathcal{P}}$, the characteristic polynomial of u is $\chi_u(X) = (\chi_{\mathcal{P},u}(X))_{\mathcal{P}\in max(R)}$. A natural question is to ask what's link between the components of $\chi_u(X)$? In this section, if R is a domain and M is a projective R-module of constant rank n, then we show that the components of $\chi_u(X)$, as defined in [1], are equals. Let $\mathcal{P}$ be a maximal ideal of R, then $M_{\mathcal{P}}$ is a free $R_{\mathcal{P}}$-module of rank n. Set $I_{\mathcal{P}}$ the identity of $M_{\mathcal{P}}$. The characteristic polynomial of $u_{\mathcal{P}}$, denoted by $\chi_{\mathcal{P},u}(X)$, is the determinant $det(XI_{\mathcal{P}} - u_{\mathcal{P}})$.

Lemma 2 *If R is a domain, then $\chi_{\mathcal{P},u}(X)$ is a monic polynomial in $R[X]$ of degree n, defined independently of the choice of the maximal ideal $\mathcal{P}$ of R.*

Proof. Set K the quotient field of R and let $\mathcal{P}$ be a maximal ideal of R and $(e_1, \ldots, e_n)$ an $R_{\mathcal{P}}$-basis of $M_{\mathcal{P}}$. Then $(e_1, \ldots, e_n)$ is a K-basis of $K \otimes_R M$. Since the $R_{\mathcal{P}}$-endomorphism $u_{\mathcal{P}}$ and the K-endomorphism $u(K)$, induced by u, operate with the same manner over $(e_1, \ldots, e_n)$, then they have the same characteristic polynomial. Consequently, $\chi_{u(K)}(X) =\in \cap_{\mathcal{P} \in max(R)} R_{\mathcal{P}}[X]$ and then $\chi_{\mathcal{P},u}(X)$ is a monic polynomial in $R[X]$ of degree n which defined independently of the choice of the maximal ideal $\mathcal{P}$ of R.

Set $I(u) = \{f \in R[X] \mid f(u) = 0\}$ the annihilator ideal of M_u. Then we have the natural extension of Mc Coy's Theorem

Proposition 2 *Let R be a domain, M be a projective R-module of constant rank and u an R-endomorphism M. Then*

$$I(u) = [(\chi_u(X)) : F_1(M_u)].$$

Proof. Since $\oplus_{\mathcal{P} \in max(R)} R_{\mathcal{P}}$ is a faithfully-flat R-algebra, it suffices to check that for localization. Since $(F_1(M_u))_{\mathcal{P}} = F_1(M_{u_{\mathcal{P}}})$, $\chi_{u_{\mathcal{P}}}(X) = \chi_u(X)$ and $[I : J]_{\mathcal{P}} = [I_{\mathcal{P}} : J_{\mathcal{P}}]$ for every ideals I and J of R, then we can use the classical Mc Coy's Theorem.

Corollary 2 *Let R be a domain, M be a projective R-module of constant rank and u an R-endomorphism M. Then $\chi_u(u) = 0$.*
Consequently, u is integral over R.

Remark If R is an integrally closed domain, then u has its minimal polynomial over R, i.e., $I(u)$ is a principal ideal which is generated by a monic polynomial of $R[X]$.

For every maximal ideal $\mathcal{P}$ of R, $M_{\mathcal{P}}$ is a free $R_{\mathcal{P}}$-module. Let

$$s_1(\mathcal{P}, u, X), \ldots, s_t(\mathcal{P}, u, X)$$

be the invariant factors of $u_{\mathcal{P}}$, then they don't depend of the choice of the maximal ideal $\mathcal{P}$ of R. Define the invariant factors of u as the invariant factors of $u_{\mathcal{P}}$ for some maximal ideal $\mathcal{P}$ of R. Since $R_{\mathcal{P}}$ is an integrally closed domain, then we have

Theorem 2 *(i) $\chi_u(X) = s_1(u, X) \ldots s_t(u, X)$, where $\chi_u(X)$ is the characteristic polynomial of u.*

(ii) $s_i(u, X)$ divides $s_{i+1}(u, X)$ for $i = 1, \ldots, t-1$.

(iii) For each positive integer $k \leq t-1$ the ideal $(\prod_{i=1}^{t-k} s_i(u, X))$ is the

smallest principal ideal of R[X] generated by a monic polynomial that contains $F_k(M_u)$.

(iv) For each positive integer $k \geq t$, the smallest principal ideal of R[X] generated by a monic polynomial that contains $F_k(M_u)$ is $R[X]$.

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