Galois theory of graded fields

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Abstract

In this paper, we investigate Galois theory of a graded field extension S/R. In particular, we restate the Galois theory results given in [1]. On the other hand, the Galois notions defined in [2] do not cover many situations, for examples the group algebras. We show that these notions are particular cases of Galois theory of commutative ring extensions.

Introduction

Let $(\Gamma, *)$ be a multiplicative group and R a graded commutative ring with respect to Γ, i.e., $R = \oplus_{\sigma\in\Gamma} R_\sigma$ such that R_σ is an R_0-module and $R_\sigma R_\tau \subset R_{\sigma*\tau}$ for all $(\sigma, \tau) \in \Gamma^2$. Set $\Gamma_R = \{\sigma \in \Gamma \mid R_\sigma \neq 0\}$ and $R^h = \cup_{\sigma\in\Gamma} R_\sigma$ the set of homogeneous elements of R. For every non-zero homogeneous element $x \in R_\sigma$, we write $\deg(x) = \sigma$ and we call it the degree of x. The graded ring R is called a graded field if every non-zero homogeneous element in R is invertible. In this case, R_1 is a field and R_σ is an R_1-vector space of dimension 1 for every $\sigma \in \Gamma$ and Γ_R is a subgroup of Γ, called the grade group of R.

Throughout this paper, S/R is a graded field extension with grade groups $\Gamma_R = \Gamma$ and $\Gamma_S = \triangle$, i.e., $R_\sigma \subset S_\sigma$ for every $\sigma \in \Gamma$. For a subgroup Λ such that $\Gamma \subseteq \Lambda \subseteq \triangle$, define $S(\Lambda) := \oplus_{x\in\Lambda} S_x$. In particular $S(\Gamma)$ is a graded field with grade group Γ and we have $R \subseteq S(\Gamma) \subseteq S$. In the first section, we split the extension S/R into two graded ring extensions $R \subset S(\Gamma)$ and $S(\Gamma) \subset S$. The first one is made of graded fields which are graded over the same group Γ, while in the second one $S(\Gamma)$ and S have the same homogeneous components of degree 1, namely S_1. We split also the extension S/R into two graded field extensions $R \subseteq \mathcal{U}$ and $R \subseteq T$. The first one is a graded field extension with ramification index 1, while the second one is a totally ramified graded field extension. We characterize the separability of graded field extensions via the discriminant. In the second section, we investigate Galois theory of graded field extensions. In particular, we restate the Galois theory results given in [1]. Moreover the Galois theory notions defined in [2] do not cover many situations, for examples the group algebras. We show that these notions are particular cases of Galois theory of commutative ring extensions.

1 Preliminaries

In this section, S/R is a graded field extension with grade groups $\Gamma_R = \Gamma$ and $\Gamma_S = \triangle$, i.e., $R_\sigma \subset S_\sigma$, for every $\sigma \in \Gamma$. For a subgroup Λ such that $\Gamma \subseteq \Lambda \subseteq \triangle$, we define $S(\Lambda) := \oplus_{x\in\Lambda} S_x$. In particular $S(\Gamma)$ is a graded field with grade group Γ, and we have $R \subseteq S(\Gamma) \subseteq S$. In this way we split the extension S/R into two graded ring extensions $R \subset S(\Gamma)$ and $S(\Gamma) \subset S$.

We list here some useful results without proofs but with a exact references see [3].

Proposition 1 *1) S is a crossed product of $S(\Gamma)$ by the group $\triangle/\Gamma$. In particular, S is a free $S(\Gamma)$-module.*
2) and $S_1 \otimes_{R_1} R \simeq S(\Gamma)$. In particular, $S(\Gamma)$ is a free R-module.

Remark. If S is a finitely generated R-module, then S is a free R-module of finite rank and $[S : R] = [S_1 : R_1][\triangle : \Gamma]$. Indeed, S is a free $S(\Gamma)$-module of rank $[\triangle : \Gamma]$ and $S(\Gamma)$ is a free R-module of rank $[S_1 : R_1]$.

$[\triangle : \Gamma]$ is called the ramification index of the extension S/R and $[S_1 : R_1]$ is called its residual degree.

Recall that, for a free R-algebra S of finite rank, every $x \in S$ induces an R-homomorphism l_x of S defined by $l_x(s) = xs$ for every $s \in S$. Define $T_{S/R}(x) = \mathrm{tr}(l_x)$ the trace of l_x. Then $T_{S/R}$ is a linear form of S which induces a bilinear form of S defined by $T_{S/R}(x, y) = T_{S/R}(xy)$ for every $(x, y) \in S^2$. The determinant of the bilinear form $T_{S/R}$ with respect to an R-basis $(e_1, \dots, e_n)$ of S is denoted by $D(e_1, \dots, e_n)$, and called the discriminant of $(e_1, \dots, e_n)$. The discriminant ideal of the R-algebra S is the principal ideal generated by $D(e_1, \dots, e_n)$, where $(e_1, \dots, e_n)$ is an R-basis of S. From [[4], Théorème III 4.7, page 89], if S is a commutative free R-algebra of finite rank, then S/R is separable if and only if the trace map $T_{S/R}$ induces a non-singular bilinear form, i.e., $D_R(S) = R$ (see [5]).

Lemma 1 *Let S/R be a graded field extension. If S/R is separable, then S is a free R-module of finite rank.*

Proof. By the previous Remark S is a free R-module. Then by [[4], Proposition III.3.2], S is finitely generated as an R-module.

In the sequel S/R is a graded field extension such that S is a free R-module of finite rank. We split the extension S/R into two graded field extensions $R \subseteq \mathcal{U}$ and $R \subseteq T$. The first one is a graded field extension with ramification index 1, while the second one is a totally ramified graded field extension.

Proposition 2 *S may be decomposed as $S \simeq \mathcal{U} \otimes_R T$, where T/R is totally ramified and $\mathcal{U}/R$ is a graded field extension with ramification index 1.*

Proof. Since S is a free R-module of finite rank, then $\triangle/\Gamma$ is a finite abelian group. Set $\triangle/\Gamma =< \bar{\sigma_1} > \times \cdots \times < \bar{\sigma_r} >$. For each σ_i fix an homogeneous element u_{σ_i} of S with degree σ_i and for every $\sigma \in \Gamma$, fix $u_\sigma \in R_\sigma - \{0\}$ and define $u_{\sigma * \sigma_1^{s_1} * \cdots * \sigma_r^{s_r}} = u_\sigma (u_{\sigma_1})^{s_1} \dots (u_{\sigma_r})^{s_r}$. Somehow $u_\sigma u_{\sigma'} = \alpha_{\sigma,\sigma'} u_{\sigma * \sigma'}$, where $\alpha_{\sigma,\sigma'} \in R_1$ for each $(\sigma, \sigma') \in \triangle^2$. Set $T = \displaystyle\sum_{i=1}^{n} R u_{\tau_i}$, where $\triangle/\Gamma = \{\bar{\tau_1}, \dots, \bar{\tau_n}\}$ and $\mathcal{U} = \displaystyle\sum_{\sigma \in \Gamma} S_0 u_\sigma$. Then $\mathcal{U}/R$ is a graded field extension with ramification index 1 and T/R is totally ramified graded field extension. The multiplication of S induces an isomorphism of R-algebras

$$\begin{array}{cccc} \mu : & \mathcal{U} \otimes T & \longrightarrow & S \\ & e_i \otimes u_{\tau_j} & \longmapsto & e_i u_{\tau_j}. \end{array}$$

The following Theorem give us a characterization of the separability of graded field extensions. This Theorem appears in [3], but for the convince of the reader, we give a new simple proof.

Proposition 3 *S/R is separable if and only if S_1/R_1 is separable and $[\triangle : \Gamma]$ is invertible in R_1.*

Proof. Under the same notations as in the proof of the Proposition 2, $\{e_i u_{\tau_j} \,|\, 1 \le i \le m\,,\, 1 \le j \le n\}$ is an R-basis of S. For every $1 \le i \le m\,,\, 1 \le j \le n$, define $e_i^* * u_{\tau_j}^*$ a linear form of S by $e_i^* * u_{\tau_j}^*(x \otimes y) = e_i^*(x) u_{\tau_j}^*(y)$ for all $(x, y) \in \mathcal{U} \times T$. Then $\{e_i \otimes u_{\tau_j}\,,\, 1 \le i \le m\,,\, 1 \le j \le n\}$ is an R-basis of S, with dual basis $\{e_i^* * u_{\tau_j}^*\,,\, 1 \le i \le m\,,\, 1 \le j \le n\}$. Hence $T_{\mathcal{U} \otimes T/R}(x \otimes y) = \sum_{i,j} e_i^* * u_{\tau_j}^*((x \otimes y)(e_i \otimes u_{\tau_j})) = \sum_{i,j} e_i^*(x e_i) u_{\tau_j}^*(y u_{\tau_j}) = \sum_{i=1}^m e_i^*(x e_i) \sum_{j=1}^n u_{\tau_j}^*(y u_{\tau_j}) = T_{\mathcal{U}/R}(x) T_{T/R}(y)$. Since $T_{T/R}(u_{\tau_i}) = \delta_{1i} u_{\tau_i} n$, where $\tau_1 = 1$, then the determinant of the bilinear form $T_{T/R}$ with respect to the basis $\{u_{\tau_j}\ ,\ 1 \leq j \leq n\}$ is $D(u_{\tau_1}, \ldots, u_{\tau_n}) = sn^n$, where s is an invertible element of R_0. Set $T_{\mathcal{U}/R}(e_i e_j) = t_{ij}$, $T_{T/R}(\tau_k \tau_l) = s_{kl}$, $M = (t_{ij})_{i,j}$ and $N = (s_{kl})_{k,l}$. Then the determinant of $T_{S/R}$ with respect to the basis $\{e_i u_{\tau_j}\ ,\ 1 \leq i \leq m\ ,\ 1 \leq j \leq n\}$ is $\det((x_{uv})_{u,v})$, where $x_{u,v} = t_{ij} s_{kl},\ u = (i,k)$ and $v = (j,l)$. From [[6], p 101], $(x_{uv})_{u,v} = M \otimes N$ and $\det((x_{uv})_{u,v}) = (\det(M))^r (\det(N))^n$. Consequently, $D_R(S) = (D_{R_1}(S_1))^n (n)^{nm} R$. Hence S/R is separable if and only if S_1/R_1 is separable and n is invertible in R_1.

Proposition 4 *Let S/R be a graded field extension. Then S/R is separable if and only if for every graded subfield extension S'/R of S/R, S/S' and S'/R are separable. In particular, S/R is separable if and only if $S/S(\Gamma)$ and $S(\Gamma)/R$ are separable.*

Proof. Let S'/R be a graded subfield extension of S/R. Then its grade group Λ is a subgroup of $\triangle$ and its homogeneous component of degree 1, S'_1 is a subfield of S_1. Set n, m and r the cardinal orders of $\triangle/\Gamma$, $\triangle/\Lambda$ and Λ/Γ respectively. Then $n = mr$. If S/R is separable, then S_1/R_1 is separable and n is invertible in R_1. Hence m and r are invertible in R_1, S_1/S'_1 and S'_1/R_1 are separable. Consequently, S/S' and S'/R are separable. Conversely, if S/S' and S'/R are separable, then m and r are invertible in R_1, S_1/S'_1 and S'_1/R_1 are separable. Therefore S_1/R_1 is separable and n is invertible in R_1, i.e., S/R is separable.

2 Galois extension of graded fields

In the this section, we investigate Galois theory of graded field extensions. First we characterize simple Galois extensions by the minimal polynomial of a primitive element of a such extension and then we characterize Galois extension of graded fields by its ramification index and its homogeneous components of degree 1. In particular, we restate the Galois theory results given in [1].

For two commutative free R-algebras A and B, let G_1 (resp. G_2) be a finite subgroup of $\mathrm{Aut}_R(A)$ (resp. of $\mathrm{Aut}_R(B)$) such that $A^{G_1} = R$ and $B^{G_2} = R$. Then $G_1 \times G_2$ is a finite subgroup of $\mathrm{Aut}_R(A \otimes_R B)$ such that $\mathrm{Aut}_R(A \otimes_R B)^{G_1 \times G_2} = R$.

Proposition 5 *Under the above hypotheses, $A \otimes_R B$ is a $G_1 \times G_2$-Galois extension over R if and only if A/R is a G_1-Galois extension and B/R is a G_2-Galois extension.*

Proof. Denote by $A^{(G_1)}$ the A-algebra with basis consisting of idempotents $\{e_\sigma | \sigma \in G_1\}$ and consider the following homomorphisms,

$$\begin{array}{rccc} h_A: & A\otimes A & \longrightarrow & A^{(G_1)} \\ & x\otimes y & \longmapsto & \sum_{\sigma\in G_1} x\sigma(y)e_\sigma. \end{array}$$

$$\begin{array}{rccc} h_B: & B\otimes B & \longrightarrow & B^{(G_2)} \\ & x\otimes y & \longmapsto & \sum_{\sigma\in G_2} x\sigma(y)e_\sigma. \end{array}$$

$$\begin{array}{rccc} h: & A\otimes_R B\otimes_{A\otimes B} A\otimes_R B & \longrightarrow & A\otimes_R B^{(G_1\times G_2)} \\ & (a\otimes b)\otimes(a'\otimes b') & \longmapsto & \sum_{(\sigma,\tau)\in G_1\times G_2} a\sigma(a')\otimes b\tau(b').e_{\sigma,\tau}. \end{array}$$

Then h is an isomorphism if and only if h_A and h_B are isomorphisms. Indeed, set $\mathcal{A} = (a_1,\dots,a_n)$ and $\mathcal{B} = (b_1,\dots,b_m)$ two R-bases of A and B respectively. Then $\mathcal{F} = (a_i \otimes b_j)_{i,j}$ is an R-basis of $A\otimes B$. Let $M = (\alpha_{ij})_{1\le i,j\le n}$ (resp. $N = (\beta_{ij})_{1\le i,j\le m}$) be the matrix of h_A with respect to bases $\mathcal{A}$ and G_1 (resp. the matrix of h_B with respect to the bases $\mathcal{B}$ and G_2). Then $H = ((\alpha_{ik}\beta_{jl})_{(i,j)(k,l)})$ is the matrix of h with respect to bases $\mathcal{F}$ and G. Hence $H = M\otimes N$ and $\det(M\otimes N) = (\det(M))^r(\det(N))^n$.

Corollary 1 *Let $S = \mathcal{U}\otimes_R T$ as defined in the Proposition 2 and let G_1 (resp. G_2) be a finite subgroup of $\mathrm{Aut}_R(\mathcal{U})$ (resp. of $\mathrm{Aut}_R(T)$) such that $\mathcal{U}^{G_1} = R$ and $T^{G_2} = R$. Then S/R is a $G_1\times G_2$-Galois extension over R if and only if $\mathcal{U}/R$ is a G_1-Galois extension and T/R is a G_2-Galois extension.*

In the sequel Γ is considered as an additive group. Recall that for a polynomial $P = \sum_{i=0}^n a_i X^i \in R[X]$ such that $a_n \neq 0$, P is called an homogeneous polynomial if every a_i is homogeneous and $r = \frac{\deg(a_i)-\deg(a_j)}{i-j} \in \mathbb{Q}\Gamma$ does not depend of the choice of (i,j) for every $i\neq j$ and $a_ia_j \neq 0$. Set $\lambda = \deg(a_n) + nr$. λ is called the degree of the homogeneous polynomial P, i.e., P is an homogeneous element of degree λ in the graded ring $R[X]$, when we put $\deg(X) = r$. We show that if S/R is a graded field extension, then every homogeneous element x of S, which is integral over R has its minimal polynomial over R, which is homogeneous.

The following Theorem appears in [3], but for the convince of the reader, we give the same proof.

Proposition 6 *Let S/R be a graded field extension and $\alpha \in S_x$ be an homogeneous element of degree $x \in \triangle$. If α is integral over R, then α has its minimal polynomial over R, which is homogeneous, i.e., the ideal $I(\alpha) = \{P \in R[X] \,|\, P(\alpha) = 0\}$ of $R[X]$ is an homogeneous principal ideal of the graded ring $R[X]$.*

Proof. Since α is integral over R, then there exists a non-zero polynomial of smallest degree in $R[X]$, which annihilate α. Let $P \in R[X]$ be a such polynomial and set $P(X) = a_nX^n + \cdots + a_0$, where $a_n \neq 0$ and $a_n = s_{1n}+\cdots+s_{rn}$ its decomposition of homogeneous elements of R. Consider the $(\deg(s_{1n})+n\deg(\alpha))$-homogeneous component of $a_n\alpha^n+\cdots+a_0$, then there exist homogeneous elements of S $s_n, \dots, s_0$ such that $s_n\alpha^n + \cdots + s_0 = 0$ and $\deg(s_n) + n\deg(\alpha) = \deg(s_{n-1}) + (n-1)\deg(\alpha) = \cdots = \deg(s_0)$. Set $\mu_\alpha(X) = X^n + r_{n-1}X^{n-1} + \cdots + r_0$, where $r_i = \frac{s_i}{s_n}$. Then $\mu_\alpha \in R[X]$ is an homogeneous polynomial of minimal degree, which annihilate α. Since μ_α is a monic polynomial then $I(\alpha) = \mu_\alpha R[X]$.

Proposition 7 *Let T/R be a totally ramified graded field extension and $s \in T_x$ an homogeneous element of degree x. Let G be a finite subgroup of $\mathrm{Aut}_R(R[s])$. Then $R[s]/R$ is a G-Galois extension if and only if the minimal polynomial $\mu_s(X)$, of s, splits in $R[s]$ and its roots are simple. In this case $G = \{g_1, \dots, g_n\}$, where $g_i(s) = \zeta_n^i s$, $n = [R[s] : R]$ and ζ_n is a primitive root of $X^n - 1$.*

Proof. Let n be the cardinal order of the group $\Gamma < x > /\Gamma$. Set $a = s^n$. Then $\deg(a) = nx \in \Gamma$ and then there exists $v \in R[s]_0$ such that $a = vu_{nx}$. Since T/R is a totally ramified graded field extension, then $R[s]_0 \subseteq T_0 = R_0$ and then $v \in R$. Consequently, $a \in R$. Let $f(X) = X^m + \cdots + r_0$ be an homogeneous polynomial in $R[X]$, which annihilate s. Then $m\deg(s) = \deg(r_0) \in \Gamma$. Hence n divides m and then $\mu_s(X) = X^n - a$ is the minimal polynomial of s over R. Since the cardinal order of G is equal to n, then $\mu_s(X)$ splits in $R[s]$ with simple roots $s_1, \dots, s_n$, where $\frac{s_1}{s}, \dots, \frac{s_n}{s}$ are the roots of $X^n - 1$. Conversely, assume that $\mu_s(X)$ splits in $R[s]$ with simple roots $s_1, \dots, s_n$. Then $\frac{s_1}{s}, \dots, \frac{s_n}{s}$ are the roots of $X^n - 1$ in $R[s]$. Consequently, $s_i = \zeta_n^i s$ for each i. Let $y = \sum_{i=0}^{n-1} r_i s^i \in R[s]$. Then $g_k(y) = r_0 + \sum_{i=1}^{n-1} r_i \zeta_n^{ik} s^i$ for each k. Hence if $y \in R[s]^G$, then $y = r_0 \in R$. Now we compute $\det(h)$, the determinant of the homomorphism h of $R[s] \otimes R[s]$ into $R[s]^{(G)}$, with respect to $R[s]-$bases $\mathcal{B} = (1, s, \dots, s^{n-1})$ and G respectively. We obtain

$$\det(h) = \begin{vmatrix} 1 & s & . & . & . & s^{n-1} \\ 1 & \zeta_n s & . & . & . & (\zeta_n s)^{n-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & \zeta_n^{n-1} s & . & . & . & \zeta_n^{(n-1)^2} s^{n-1} \end{vmatrix} =$$

$$= s^{n(n+1)/2} \zeta_n^{n(n-1)/2} \prod_{1 \leq i \leq j \leq n, i \neq j} (\zeta_n^i - \zeta_n^j)$$

Since s is an homogeneous element and $\zeta_n^i - \zeta_n^j$ is a non-zero homogeneous element of degree 0 for each $i \neq j$, then $\det(h)$ is invertible in R. Finally, $R[s]/R$ is a $G-$Galois extension.

Corollary 2 *Let T/R be a totally graded field extension and $s \in T_x$ be an homogeneous element of degree x. Then $R[s]/R$ is a Galois extension if and only if n is invetible in R_0 and R_0 contains a primitive root ζ_n of $X^n - 1$, where n is the cardinal order of the group $\Gamma < x > /\Gamma$.*

Proof. We have already $\mu_s(X) = X^n - a$ is the minimal polynomial of s over R. Since $R[s]/R$ is a Galois extension, then it is separable, i.e., the polynomial $\mu_s(X) = X^n - a$ is separable over R. Hence its discriminant $n^n a^{n-1}$ is invertible in R and then n is invertible in R_0. Moreover $\mu_s(X)$ splits in $R[s]$ with simple roots $s, \zeta_n s, \ldots, \zeta_n^{n-1} s$. Consequently, $\zeta_n \in R_0$. Conversely, assume that n is invertible in R_0 and $\zeta_n \in R_0$. Set $G = \{g_1, \ldots, g_n\}$, where $g_i(s) = \zeta_n^i s$. Then the same proof of the previous Proposition implies that $R[s]/R$ is a $G-$Galois extension.

Now, we are ready to generalize the Proposition 3.3, given in [1].

Theorem 1 *Let S/R be a graded field extension. Let l be the exponent of $\triangle/\Gamma$. Then S/R is a Galois extension if an d only if S_0/R_0 is a Galois extension, l is invertible in R_0, which contains ζ_l.*

Proof. Set $\triangle/\Gamma = < \bar{\sigma_1} > \cdots < \bar{\sigma_r} >$ and let $s_1, \ldots, s_r$ be homogeneous elements of T such that $\deg(s_i) = \sigma_i$ for each i. Then the multiplication of T induces an isomorphism of R-algebras $\mu : R[s_1] \otimes \cdots \otimes R[s_r] \longrightarrow T$. Consequently, T/R is a Galois extension if an d only if $R[s_i]/R$ is a Galois extension for every $1 \leq i \leq r$. Set $S \simeq \mathcal{U} \otimes_R T$ as defined in the Proposition 2. Then S/R is a Galois extension if an d only if T/R and $\mathcal{U}/R$ are Galois extensions. Since $\mathcal{U} \simeq S_0 \otimes_{R_0} R$ and R_0 is a field, then $\mathcal{U}/R$ is a Galois extension if an d only if S_0/R_0 is a Galois extension. Then by Corollary 2, S/R is a Galois extension if an d only if S_0/R_0 is a Galois extension and l is invertible in R_0, which contains ζ_l.

Proposition 8 *Let S/R be a graded field extension and l the exponent of $\triangle/\Gamma$. Then If S/R is a Galois extension, then $S/S(\Gamma)$ and $S(\Gamma)/R$ are Galois extensions. The converse is true if R_0 contains ζ_l.*

Proof. Set n, m and r the cardinal orders of $\triangle/\Gamma$, $\triangle/\Lambda$ and Λ/Γ respectively. Then $n = mr$. If S/R is a Galois extension, then S_0/R_0 is a Galois extension and n is invertible in R_0. Hence m and r are invertible in R_0, R_0 contains ζ_l and S_0/R_0 is a Galois extension. Consequently, $S/S(\Gamma)$ and $S(\Gamma)/R$ are Galois extensions. Conversely, assume that $S/S(\Gamma)$ and $S(\Gamma)/R$

are Galois extensions and R_0 contains ζ_l. Then m and r are invertible in R_0, R_0 contains ζ_l and S_0/R_0 is a Galois extension. Consequently, S/R is a Galois extensions.

Theorem 2 *Let T/R be a totally ramified graded field extension. Then T/R is a Galois extension if and only if for every homogeneous element of $s \in T$, $R[s]/R$ is a Galois extension.*

Proof. Let $s \in T_x$ be homogeneous element. Then $R \subseteq R[s]$ Γ and $\Gamma < x >$ respectively. Let l be the cardinal order of $\Gamma < x > /\Gamma$. Then l divides the cardinal order of $\triangle/\Gamma$. Consequently, R_0 contains ζ_l and l is invertible in R_0, i.e.,$R[s]/R$ is a Galois extension. Conversely, assume that for every homogeneous element $s \in T$, $R[s]/R$ is a Galois extension. Set $\triangle/\Gamma =< \bar{\sigma_1} > \cdots < \bar{\sigma_r} >$ and let $s_1, \ldots, s_r$ be homogeneous elements of T such that $\deg(s_i) = \sigma_i$ for each i. Then $R[s_1] \otimes \cdots \otimes R[s_r] \simeq T$ and then T/R is a Galois extension.

Corollary 3 *Let T/R be a graded field extension. Then T/R is a Galois extension if an d only if every homogeneous element $s \in T$ is a simple root of its minimal polynomial over R, which splits in $R[s]$.*

3 Concluding remarks

Let S/R be a graded field extension such that R is a domain with quotient field K.

1) Since $D_R(S)$ is generated by an homogeneous element of R, with degree 0, then S/R is separable if and only if $D_R(S) \neq 0$ that is equivalently to $D_K(KS) \neq 0$. Hence S/R is separable if and only if KS/K is separable. In particular, if $\triangle$ is abelian torsion free then R and S are domains with quotient fields K and KS respectively. We find also the separability results, given in [1].

2) If $\triangle$ is abelian torsion free then R and S are domains. Since S is a connected ring, then S/R is a Galois extension if and only if S/R is separable and $S^G = R$, where $G = \mathrm{Aut}_R(S)$ (see [9] Theorem 2.1, page 7). From [1], S is an integrally closed domain. Then $S^G = R$ if and only if $(KS)^G = K$. Consequently, S/R is a Galois extension if and only if KS/K is a Galois extension. This state the Galois theory results given in [1].

The following Theorem allows us to show that the notions defined in [2], are simple cases of the Galois extensions of commutative rings.

Theorem 3 *Let S/R be a graded field extension such that S is a connected ring. Then S/R is a Galois extension if and only if for every homogeneous element $s \in S$, s is a simple root of its minimal polynomial which splits in S.*

Proof. Since s is homogeneous element, $R[s]/R$ is a graded field extension. Since S/R is a Galois extension, then it is separable and then $R[s]/R$ is separable. Hence it is a simple root of its minimal polynomial. Since S is a connected ring, then the Galois group of S/R is $\mathrm{Aut}_R(S)$ and then $\sigma(s) \in S$ for every $\sigma \in \mathrm{Aut}_R(S)$. Consequently, $\mu_s(X)$ splits in S.

3) If $\triangle$ is abelian torsion free, then S is a connected ring. Hence S/R is a Galois extension if and only if for every homogeneous element $s \in S$, s is a simple root of its minimal polynomial which splits in S. That shows that the Galois theory notions defined in [2] are simple cases of Galois extensions of commutative rings.

Examples. Let K be a field with characteristic p.

1) Set $R = K[X_1, \dots, X_r, X_1^{-1}, \dots, X_r^{-1}]$ and $S = K[X_1{}^{\frac{1}{n_1}}, \dots, X_r{}^{\frac{1}{n_r}}, X_1^{\frac{-1}{n_1}}, \dots, X_r^{\frac{-1}{n_r}}]$. Then S/R is a Galois extension if and only if p does not divide $n_1 \dots n_r$ and $\zeta_n \in K$. In this case the Galois group of the extension S/R is $G =< \sigma_1 > \times \cdots \times < \sigma_r >$, where $\sigma_i(X_i^{\frac{1}{n_i}}) = \zeta_{n_i} X^{\frac{1}{n_i}}$ for each i.

2) Let L/K be a finite field extension. Set $S = L[X^{\frac{1}{n}}, X^{\frac{1}{n}}]$. Then S/R is a Galois extension if and only if p does not divide n, $\zeta_n \in L$ and L/K is a Galois extension. In this case the Galois group of the extension S/R is $G\times < \sigma >$, where $\sigma(X^{\frac{1}{n}}) = \zeta_n X^{\frac{1}{n}}$ and G is the Galois group of the extension L/K.

3) Let R be a graded field and $P(X) = X^n - 1 \in R[X]$. Set $S = R[X]/(P)$. Since $P(X) \in R[X]_0$ is an homogeneous polynomial, then S/R is a graded field extension. Hence S/R is a Galois extension extension if and only if n is invertible in R and $\zeta_n \in R$. In this case $S \simeq R^n$ as algebras. In particular, let G be a cyclic group with cardinal order n. Then $L[G]/K$ is a Galois extension if an d only if L/K is a Galois extension, n is invertible in R and $\zeta_n \in R$.

4) Generally, let G be a finite abelian group with exponent m. Then $L[G]/K$ is a Galois extension if and only if L/K is a Galois extension, p does not divide m and K contains ζ_m.

5) Let H be a subgroup of G. $L[G]/K[H]$ is a Galois extension if and only if L/K is a Galois extension, p does not divide l the exponent of G/H and K contains ζ_l.

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