

# CW Type of Inverse Limits and Function Spaces

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## Abstract

We continue the investigation of CW homotopy type of spaces of continuous functions between two CW complexes begun by J. Milnor in 1959 and P. Kahn in 1984. Viewing function spaces as particular cases of inverse limits we also study certain inverse systems of fibrations between CW homotopy type spaces.

If the limit space  $Z_\infty$  of an inverse sequence  $\{Z_i\}$  of fibrations between CW type spaces has CW type then a subsequence of  $\{\Omega Z_i\}$  splits into the product of a sequence of homotopy equivalences and one of nullhomotopic maps.

If for some  $N > 0$ , all spaces  $Z_i$  have  $\pi_k(Z_i) = 0$  for  $k > N$ , then the question of CW type of  $Z_\infty$  depends solely on the induced functions  $\pi_k(Z_j) \rightarrow \pi_k(Z_i)$ . This applies to  $Z_i = Y^{L_i}$  where  $\pi_k(Y) = 0$  for  $k > N$  and  $\{L_i\}$  is an ascending sequence of finite complexes. Here  $Z_\infty = Y^{\cup L_i}$ , the space of continuous functions  $(\cup L_i) \rightarrow Y$  with the compact open topology.

In general, if the path component of  $g \in Y^X$  has CW type then  $\Omega(Y^X, g) \rightarrow \Omega(Y^L, g|_L)$  is a homotopy equivalence for a countable subcomplex  $L$  of  $X$ . A suitable converse holds as well.

Function spaces of CW type lack phantom phenomena in a strong sense. This provides interesting examples. One is a space of pointed maps that is weakly contractible but not contractible. Next, for  $X$  the localization of a finite complex at a set of primes  $P$ , the question of CW type of  $Y^X$  is related, and sometimes equivalent, to that of eventual geometric H-space exponents of  $Y$ .

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If  $Y$  is a  $P$ -local and  $X$  a simply connected complex then localization  $X \rightarrow X_{(P)}$  induces a genuine homotopy equivalence  $Y^{X_{(P)}} \rightarrow Y^X$  regardless of whether  $Y^X$  has CW type or not.

For  $Y = K(G, n)$  we give necessary and sufficient conditions for  $Y^X$  to have CW type in terms of homology of  $X$ . If  $\bigoplus_n \pi_n(Y)$  is finitely generated and  $X$  is 1-connected then we give necessary and ‘almost’ sufficient conditions.

Some properties of CW complexes  $X$  are equivalent to  $Y^X$  having CW homotopy type for a certain family of complexes  $Y$ . For example,  $X$  is finitely dominated if and only if  $\pi_1(X)$  is finitely presented and  $Y^X$  has CW type for all complexes  $Y$ .

## Introduction

For topological spaces  $X$  and  $Y$  we denote by  $Y^X$  the space of continuous functions  $X \rightarrow Y$  with the compact open topology.

One of the first results on CW homotopy type of function spaces is a result of Kuratowski [30], saying that if  $Y$  is an absolute neighbourhood retract for the class of metric spaces and  $K$  is a compact metric space, then  $Y^K$  is also an absolute neighbourhood retract for the class of metric spaces. (Later it was shown that  $Y$  has the homotopy type of a countable CW complex if and only if it has the homotopy type of an absolute neighbourhood retract for the class of metric spaces; see Milnor [43] for details.)

In his fundamental study of spaces having the homotopy type of a CW complex, Milnor [43] strengthened Kuratowski's result to show that  $Y^X$  has CW homotopy type of a CW complex if  $Y$  has and  $X$  is a compact Hausdorff space. Some effort has already been made to extend Milnor's theorem. P.J.Kahn [29] proved that  $Y^X$  has the homotopy type of a CW complex when  $Y$  has the homotopy type of a CW complex with  $\pi_k(Y)$  trivial for  $k \geq n + 1$  and  $X$  has the homotopy type of a CW complex with finite  $n$ -skeleton.

We are interested in both sufficient and necessary conditions on CW complexes  $X$  and  $Y$  for the function space  $Y^X$  to have CW homotopy type. The restriction that  $X$  be a CW complex (or have homotopy type of one) is motivated by Kahn's result and is natural for homotopy theory.

Previous work of the author [56] on this problem shows that the conditions of Kahn are not necessary and gives other examples of function spaces of CW homotopy type.

To develop sufficient conditions for the space  $Y^X$  to have CW homotopy type, we view  $Y^X$  as the limit of inverse system  $\{Y^L \mid L \in \mathcal{L}\}$  where  $\mathcal{L}$  is a suitable set of subcomplexes of  $X$ , and the bond for  $L \leq L'$  is the restriction fibration  $Y^{L'} \rightarrow Y^L$ . Notably, choosing  $\mathcal{L}$  to be the set  $\mathcal{K}$  of finite subcomplexes of  $X$ , the associated inverse system is one of spaces of CW homotopy type, by Milnor's theorem.

This inverse system approach makes it natural to consider the problem in a slightly more general setting. Namely, if  $Z$  is the limit space of an

inverse system  $\{Z_\lambda | \lambda\}$  of fibrations between CW homotopy type spaces, when has  $Z$  the homotopy type of a CW complex?

Milnor has shown that CW homotopy type is intimately related to local contractibility notions. One of them is equilocal convexity. Milnor introduced it in [43] to show that if a space is paracompact and equilocally convex, then it has the homotopy type of a CW complex. He used this to prove his fundamental result concerning CW homotopy type of function spaces.

To investigate necessary conditions for a topological space to have the homotopy type of a CW complex it is natural to consider a much weaker form of local contractibility, namely semilocal contractibility, since it is homotopy invariant.

The lack of semilocal contractibility has been used a few times to demonstrate that spaces do not have the homotopy type of a CW complex; see for example Skljarenko [55] as well as Dydak and Geoghegan [12]. In presence of additional properties, semilocal contractibility can be a necessary and sufficient condition for a space to have CW homotopy type, see Allaud [1] and Schön [53].

In tune with the cited existing results on CW homotopy type of function spaces, we pursue the intuition that CW complex  $X$  must be ‘small’ with respect to CW complex  $Y$  for the space  $Y^X$  to have CW type. This intuition is justified in various ways in this paper.

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## Overview of results

In Chapter 1 we treat CW homotopy type of certain inverse systems of fibrations. Let  $X$  be a CW complex. The set  $\mathcal{X}$  of all subcomplexes of  $X$  is a complete lattice, and the subset  $\mathcal{K}$  of finite subcomplexes is a sublattice with the property that every  $L \in \mathcal{X}$  can be expressed as the supremum of a subset of  $\mathcal{K}$ .

In §1.1 and §1.3 we study CW homotopy type of limits of inverse systems indexed over a complete lattice that admits an appropriate sublattice over which the system is one of CW homotopy type spaces. We give necessary conditions for CW type which serve as our main obstruction throughout the paper, and some sufficient conditions. The developed results are mostly results of technical nature to be applied to function spaces in subsequent sections.

However, the results obtained for sequences in §1.2 are independent of function spaces. Consider an inverse sequence  $\cdots \rightarrow W_3 \rightarrow W_2 \rightarrow W_1$  of fibrations between spaces of CW homotopy type. We show the following. If the limit space  $W_\infty$  also has CW homotopy type, then a subsequence of the sequence of loop spaces  $\{\Omega W_i\}$  splits into the product of a sequence of nullhomotopic fibrations and a sequence of homotopy equivalences. Conversely, if  $\{P_i\}$  is an inverse sequence of fibrations that are nullhomotopic, and  $\{Q_i\}$  is an inverse sequence of fibrations that are homotopy equivalences, then the limit space of the sequence  $\{P_i \times Q_i\}$  has CW homotopy type and is homotopy equivalent to  $Q_1$ .

Thus the limit space of an inverse sequence of fibrations between spaces of CW type has itself CW type if and - up to a looping - only if it is trivial in a certain sense; see Theorem 1.2.6 below. In particular, this settles a problem arising from a misstatement in Theorem B of Dydak and Geoghegan [12].

In §1.4 we introduce the notion of ‘phantom path components’ of the limit space  $Z$  of an inverse system of fibrations  $\{Z_\lambda | \lambda\}$  between spaces of CW type. In the particular case when  $Z = Y^X$  is the limit of  $\{Y^L | L \text{ finite } \leq X\}$  this notion coincides with path components of phantom maps  $X \rightarrow Y$  with respect to the class of finite subcomplexes of  $X$ .

We show that if  $Z$  has CW type then the systems of groups  $\{\pi_k(Z_\lambda) | \lambda\}$  satisfy the Mittag-Leffler condition ‘uniformly with respect to  $k$ ’ (see Theorem 1.4.4). In particular,  $Z$  has no ‘phantom components’.

In §1.5 we consider sequences of fibrations  $\cdots \rightarrow Y_3 \rightarrow Y_2 \rightarrow Y_1$  of spaces of CW type with  $\pi_k(Y_i) = 0$  for  $k > N$  independently of  $i$ . We show that, roughly, (see Theorem 1.5.1) the limit space  $Y_\infty$  has CW homotopy type if and only if for each  $k \geq 1$  the sequence  $\{\pi_k(Y_i) | i\}$  satisfies the Mittag-Leffler condition and the canonical morphisms  $\lim_j \pi_k(Y_j) \rightarrow \pi_k(Y_i)$  are injective for all but finitely many  $i$ , and all  $k \geq 0$ .

In Chapter 2 we specialize to spaces of continuous functions between CW complexes.

In §2.1 we define a family  $\mathcal{F}(X, Y)$  of subspaces of  $Y^X$  that contains ‘all function spaces arising in practice’ and is suitable for application of machinery developed in §1.1. We give two simple-minded examples of function spaces not of CW type.

In §2.2 we apply the machinery of §1.1 to spaces of functions between CW complexes  $X$  and  $Y$ .

We show that  $Y^X$  has CW homotopy type if and only if it admits a numerable covering of open sets contractible within  $Y^X$  (see Theorem 2.2.1). Such a covering exists in every paracompact semilocally contractible space. For countable  $X$ , the space  $Y^X$  is always paracompact, but it may lack semilocal contractibility.



Next, if  $Y^X$  has CW homotopy type, then for every map  $g: X \rightarrow Y$  and every countable subcomplex  $L$  of  $X$  there exists a bigger countable subcomplex  $L'$  (depending on  $g$  and  $L$ ) such that the map

$$\Omega(Y^X, g) \rightarrow \Omega(Y^{L'}, g|_{L'})$$

is a homotopy equivalence (see Theorem 2.2.5). This is to say that if  $Y^X$  has CW type, then  $X$  is essentially *countable with respect to  $Y$* .

Conversely, if for each countable subcomplex  $L$  of  $X$  there exists a bigger countable subcomplex  $L'$  such that  $Y^X \rightarrow Y^{L'}$  is a weak equivalence onto image which has CW type, then  $Y^X$  itself has CW type.

Both Theorem 2.2.1 and Theorem 2.2.5 are appropriately stated for arbitrary elements of  $\mathcal{F}(X, Y)$ .

A particular case of Theorem 2.2.6 is the following. If for each countable subcomplex  $L$  there exists a larger countable subcomplex  $L'$  such that  $(Y, *)^{(L', *)}$  is contractible, then  $(Y, *)^{(X, *)}$  is contractible.

In §2.3 we study restrictions on the set of path components of  $Y^X$  of CW type, an issue which is delicate because this set does not in general admit a natural group structure. We show that (Proposition 2.3.4) if  $X$  has  $\aleph_\xi$  cells and all homotopy groups of  $Y$  have cardinality at most  $\aleph_\eta$  then  $Y^X$  has at most  $\aleph_{\max\{\xi, \eta\}}$  path components.

If  $X$  is countable and  $Y$  has finite homotopy groups then (Proposition 2.3.6)  $Y^X$  has actually finitely many path components.

In §2.4 we note that results of §1.4 imply that if  $Y^X$  has CW type then it does not contain nontrivial phantom maps. In fact there are no nontrivial phantom maps  $X \rightarrow \Omega^k Y$  for any  $k$ , in a ‘uniform way’ with respect to  $k$ .

We give an example of a function space with essential phantom maps (which consequently does not have CW type) and study its properties. Then we recall some examples from existing literature on phantom maps. In particular, it follows from a famous theorem of Zabrodsky [65] that for simply connected CW complexes  $X$  and  $Y$  where  $X$  has finitely many nontrivial homotopy groups and  $Y$  is a finite complex the space  $Y^X$  does not have CW type if  $(Y, *)^{(X, *)}$  is not weakly contractible.

In §2.5 we apply results of §2.2 and §1.5 to show that, roughly, for  $Y$  a CW complex with finitely many nontrivial homotopy groups the question of whether  $Y^X$  has CW type only depends on the induced morphisms  $\pi_k(Y^M) \rightarrow \pi_k(Y^L)$  for  $L \leq M \leq X$ . (See Theorem 2.5.1.) In particular, vanishing of phantom phenomena together with an additional condition suffice for CW homotopy type in this case.

For  $X$  a countable complex the statement becomes very simple. Let  $L_1 \leq L_2 \leq \dots$  be a filtration of finite subcomplexes for  $X$  and assume that  $Y$  has only finitely many nontrivial homotopy groups. Then the path

component  $C$  of  $g \in Y^X$  has CW homotopy type if and only if the sequences  $\{\pi_k(Y^{L_i}, g|_{L_i}) \mid i\}$ , for  $k \geq 1$ , satisfy the Mittag-Leffler condition, and the arrows  $\lim_j \pi_k(Y^{L_j}, g|_{L_j}) \rightarrow \pi_k(Y^{L_i}, g|_{L_i})$  are injective for all but finitely many  $i$ , and all  $k \geq 0$  (for  $k = 0$  in the usual sense). In particular, if the homotopy groups of  $Y$  are countable, then  $(Y, *)^{(X,*)}$  is contractible if and only if it is weakly contractible. The latter is ‘far from true’ for  $Y$  with infinitely many nontrivial homotopy groups, see Example 8 on page 56.

In §2.6 we present a few general sufficient conditions for a function space to have CW homotopy type; which we use in subsequent sections. In particular, in Proposition 2.6.1 we recover Theorem 1.1 of Kahn [29].

In Chapter 3 we investigate some effects of localization of both domain and target complexes.

In §3.1 we prove an auxiliary result used later on; a strengthened ‘Zabrodsky lemma’. The well known ‘Zabrodsky lemma’ (see Miller [40]) says that if  $E \rightarrow B$  is a fibration with fibre  $F$  and for some CW complex  $X$  the function space  $(X, *)^{(F,*)}$  is weakly contractible, then  $X^B \rightarrow X^E$  is a weak homotopy equivalence. Upon assuming that  $(X, *)^{(F,*)}$  is contractible, we observe that then  $X^B \rightarrow X^E$  is a genuine homotopy equivalence.

In §3.2 we consider localizations of the target complex  $Y$ . We prove (see Theorem 3.2.1) that if  $Y$  is a  $P$ -local CW complex with respect to a set of primes  $P$  then the natural map  $(Y, *)^{(X_{(P)},*)} \rightarrow (Y, *)^{(X,*)}$  is a homotopy equivalence for any simply connected CW complex  $X$ .

In §3.3 we study the space of functions from  $K_{(P)}$  where  $K$  is the suspension of a finite complex and  $P$  is a set of primes. Applying §1.4 we show that, given a CW complex  $Y$  of finite type over  $\mathbb{Z}_{(R)}$  where  $P \cap R$  is nonempty, the space  $(Y, *)^{(K_{(P \setminus P)},*)}$  has CW type only if  $(Y_{(P)}, *)^{(K_{(P \setminus P)},*)}$  is contractible and the H-space  $(\Omega Y_{(P)}, *)^{(K,*)}$  has an H-space exponent. We show that a partial converse holds in that  $(Y_{(P)}, *)^{(K_{(P \setminus P)},*)}$  is contractible if  $(Y_{(P)}, *)^{(K,*)}$  has an H-space exponent. (See Theorem 3.3.8.) Applying this to spheres in place of  $K$  (see Proposition 3.3.1) we relate the problem to the Moore conjecture on homotopy (and H-space) exponents. We provide a restatement of the (geometric part of the) conjecture.

Relying on results from H-space exponent theory we give non-trivial examples of both function spaces that have CW homotopy type, and of such that do not.

Relying on Serre’s theorem on torsion in homotopy groups of spheres we show that for any  $m, n \geq 2$ , the space  $Z = (S^n, *)^{(S^m_{(0)},*)}$  does not have CW homotopy type. In particular, if  $m > n$  then  $Z$  is a weakly contractible space which is not contractible (indeed this holds for all  $\Omega^k Z$ ,  $k \geq 0$ ).

In Chapter 4 we investigate spaces of maps into Eilenberg-MacLane spaces.

In §4.1 we note (see Theorem 4.1.1) that the homotopy type of  $K(G, n)^X$  (for abelian  $G$ ) depends only on homology groups of  $X$ , a result whose weak version was already established by Thom [61].

According to §4.1 it suffices to study spaces of functions  $M(A, m) \rightarrow K(G, n)$  where  $M(A, m)$  denotes a Moore space of type  $(A, m)$ . In §4.2 we give sharpened versions of results of §2.2 for this particular case.

Applying almost everything of the preceding sections, we give in §4.3 necessary and sufficient conditions on abelian groups  $A$  and  $G$  for  $K(G, n)^{M(A, m)}$  (see Theorem 4.3.1) respectively  $K(G, n)^{K(A, m)}$  (see Theorem 4.3.3) to have CW homotopy type. The conditions fit well into the intuition that ‘ $A$  must be small with respect to  $G$ ’ in order for the mentioned function spaces to have CW type.

In §4.4 we complement §4.1 and §4.3 to give necessary and sufficient conditions for a function  $K(G, 1)^X$  to have CW type, for arbitrary  $G$ .

The results of P.J.Kahn [29] as well as those of the author [56] exhibit function spaces  $Y^X$  of CW homotopy type where the restriction map  $Y^X \rightarrow Y^L$ , for a suitable finite subcomplex  $L$  of  $X$ , is a homotopy equivalence onto the union of some path components, reducing the problem to Milnor’s theorem. With this in mind one might conjecture that - at least for some CW decomposition of  $X$  - this is the only possible way for a function space  $Y^X$  to have CW type. This is not true, as we note in Example 10 on page 75 as a consequence of results of §4.3.

However, Milnor’s result is best possible in a global sense, as we discuss in §4.5 where we give necessary and sufficient conditions on  $X$  for  $Y^X$  to have CW homotopy type for all CW complexes  $Y$  belonging to certain classes. For example (see Theorem 4.5.3), a connected CW complex  $X$  is finitely dominated if and only if  $\pi_1(X)$  is finitely presented and  $Y^X$  has CW type for *all* CW complexes  $Y$ . For another example,  $X$  has homological finite type if and only if  $Y^X$  has CW type for all nilpotent CW complexes  $Y$  of finite type.

In Chapter 5 we establish sufficient and ‘almost necessary’ conditions for  $Y^X$  to have the homotopy type of a CW complex when  $X$  is a simply connected CW complex, and  $Y$  is simply connected of finite type with only finitely many nontrivial homotopy groups. (See Theorem 5.0.1.) Thereby we enhance Kahn’s result in this case, and give a suitable partial converse.

In Appendix A we provide an explicit determination of a homotopy equivalence arising from a fibration with fibre that contracts in the total space; a result that involves exhibiting particular lifting functions for certain fibrations, and is (directly) applied once in the paper. Furthermore we

give the proof of a result (see Proposition A.0.2) which also involves lifting functions and is essentially applied once in the paper.

In Appendix B we introduce the notions of topological ‘quasi-groups’ and of ‘quasi-LEC’ spaces. We use this essentially to circumvent difficulties arising from the fact that the cartesian product of two CW complexes is not in general compactly generated. The results of this appendix are of technical character and are used on two occasions to extend properties that hold for countable CW complexes to the general case.

### Conventions and basic tools

Both domain and target spaces of our function spaces are assumed to be CW complexes (or have CW homotopy type). Let  $(X; A_1, \dots, A_n)$  be a CW  $n$ -ad and  $(Y; B_1, B_2, \dots, B_n)$  of the homotopy type of a CW  $n$ -ad. We may form the  $n$ -ad function space  $(Y; B_1, B_2, \dots, B_n)^{(X; A_1, \dots, A_n)}$  of maps  $f: X \rightarrow Y$  for which  $f(A_i) \subset B_i$  for all  $i$ . This  $n$ -ad function space is a subspace of  $Y^X$  and belongs to the class  $\mathcal{F}(X, Y)$  discussed above and defined in §2.1. Some results, in particular those of more topological nature in Chapters 2 and 2.6, are naturally stated for  $n$ -ad function spaces. See also Milnor [43], Theorem 3.

However, in later chapters where we treat particular families of function spaces, and determine explicit homological conditions, we consider spaces of free maps  $Y^X$  and spaces of pointed maps  $(Y, *)^{(X, *)}$ , for  $X$  a CW complex and  $Y$  a Hausdorff space of CW type. In this case the space  $Y^X$  has CW type if and only if the spaces  $(Y, y_0)^{(X, *)}$  have, for  $y_0$  ranging over different path components of  $Y$ , see Corollary 0.0.10 below. At the end of this section we note that for questions concerning  $Y^X$  and  $(Y, y_0)^{(X, x_0)}$  it suffices to consider path-connected spaces and assume that  $x_0$  is a 0-cell of  $X$  and  $y_0$  is a nondegenerate base point of  $Y$ .

When topological spaces are involved we use continuous function, function, and map synonymously.

We use throughout the compact open topology. While its compactly generated refinement (used for example by Kahn in [29]) is more appealing from a categorical point of view, it is not very useful when discussing properties concerning open sets, such as semilocal contractibility. Fortunately when  $X$  is a countable CW complex, and  $Y$  is any CW complex, the compact open topology on  $Y^X$  and its refinement are homotopy equivalent, see Lemma 3.4 of [56]. In general only our ‘positive’ results can be applied to the compactly generated refinement; if  $Y^X$  has CW type then so has  $\mathfrak{R}(Y^X)$  (see [56], Remark 2.5).

We distinguish strictly between homotopy equivalent and *weakly* homotopy equivalent, shortly just equivalent and weakly equivalent, respectively.

Indeed, in this paper we are concerned exactly with the difference between homotopy equivalences and weak homotopy equivalences, since every topological space is *weakly* equivalent to a CW complex. In the course of our investigation we prove some results which in their weak version are well known or obvious, but not so in the genuine homotopy equivalence setting.

We use ‘type’ for homotopy type.

The term fibration is used for a Hurewicz fibration, that is a not necessarily surjective map with the homotopy lifting property with respect to all spaces.

We refer to Chapter 6 of Maunder’s book [35] for a concise treatment of basic facts regarding the compact open topology that circumvents in a very nice way the problem of local compactness of the domain. (Specific CW decompositions of our infinite domain complexes will almost never be locally compact.)

We use the following notation. By  $Y^X$  we denote the space of continuous functions  $X \rightarrow Y$ . Function spaces are endowed with the compact open topology, and we use  $G(K, V)$  for the standard subbasic open set,  $G(K, V) = \{f \in Y^X \mid f(K) \subset V\}$ , with  $K$  compact and  $V$  open.

The symbol  $SX$  denotes the reduced suspension of the complex  $X$  (with the standard CW structure).

We denote the  $n$ -skeleton of  $X$  by  $X^{(n)}$ . When there is no danger of confusion, we denote by  $Y_i$  the  $i$ -th stage of the Postnikov tower of  $Y$ .

The space of paths in a space  $Z$  starting with a (sometimes tacitly understood) fixed base point is denoted by  $PZ$ . The corresponding evaluation at endpoint is denoted by  $\varepsilon_1: PZ \rightarrow Z$ .

We adopt the convention that all spaces considered are Hausdorff. For a brief explanation, we want the domain space of a function space to be Hausdorff since every compact subspace is normal. On the other hand, compact open topology is Hausdorff if the target space is. Moreover, if two Hausdorff spaces are homotopy equivalent, then so also are their compactly generated refinements. Occasionally we add the Hausdorff assumption to the statement of a proposition, for emphasis.

By  $\mathbb{Z}_p$  we will denote the group of integers mod  $p$ , by  $\mathbb{Z}_{p^\infty}$  the quasicyclic  $p$ -group (divisible  $p$ -torsion group of rank 1), and by  $\hat{\mathbb{Z}}_p$  the group of  $p$ -adic integers. For a short exact sequence of groups  $0 \rightarrow B \rightarrow A \rightarrow Q \rightarrow 0$  we will say that  $A$  is an extension of  $B$  by  $Q$  (in accord with Fuchs [20]).

For a set of primes  $P$  we will denote by  $A_{(P)}$  the localization of  $A$  at  $P$  (for  $A$  an abelian group or a suitable CW complex).

We record here a handful of basic results on (CW) homotopy type of function spaces, which will be extensively used in the paper.

LEMMA 0.0.1. *Let  $X$  be a Hausdorff space.*

- (i) If  $Y$  is homeomorphic to  $\prod_{\lambda} Y_{\lambda}$ , then  $Y^X$  is homeomorphic to  $\prod_{\lambda} Y_{\lambda}^X$  and consequently  $(Y, *)^{(X, *)}$  is homeomorphic to  $\prod_{\lambda} (Y_{\lambda}, *_{\lambda})^{(X, *)}$ .
- (ii) If  $X$  is homeomorphic to  $\vee_{\lambda} X_{\lambda}$  then  $(Y, *)^{(X, *)}$  is homeomorphic to the product  $\prod_{\lambda} (Y, *)^{(X_{\lambda}, *_{\lambda})}$ .  $\square$

PROPOSITION 0.0.2. Let  $\varphi: A \rightarrow X$  and  $\psi: Y \rightarrow B$  be continuous maps, and define  $\Phi(\varphi, \psi): Y^X \rightarrow B^X$  by  $f \mapsto \psi \circ f \circ \varphi$ . If  $\varphi \simeq \varphi'$  and  $\psi \simeq \psi'$  then  $\Phi(\varphi, \psi) \simeq \Phi(\varphi', \psi')$ . (If  $\varphi \simeq \varphi'$  and  $\psi \simeq \psi'$  are homotopic as pointed maps, then  $\Phi(\varphi, \psi)$  and  $\Phi(\varphi', \psi')$  are homotopic in the pointed category, taking constant maps to be base points of function spaces.)

Consequently if  $\varphi$  and  $\psi$  are homotopy equivalences, so is  $\Phi(\varphi, \psi)$ .  $\square$

PROOF. See Maunder [35], Theorem 6.2.25.  $\square$

The following is Theorem 3 of Milnor [43]. In what follows we will simply call it ‘Milnor’s theorem.’

THEOREM 0.0.3 (Milnor’s theorem). Let  $(A; A_1, A_2, \dots, A_n)$  be a compact  $n$ -ad ( $A$  is a compactum and the  $A_i$  are closed in  $A$ ) and let  $(Y; Y_1, Y_2, \dots, Y_n)$  have the homotopy type of a CW  $n$ -ad. Then  $(Y; Y_1, Y_2, \dots, Y_n)^{(A; A_1, A_2, \dots, A_n)}$  has the homotopy type of a CW complex.  $\square$

A topological space is called *finitely dominated* if it is dominated by a finite CW complex (see J. Whitehead [64]), that is there exist a finite CW complex  $K$  and continuous maps  $i: X \rightarrow K$  and  $p: K \rightarrow X$  such that  $p \circ i$  is homotopic to  $\text{id}_X$ .

COROLLARY 0.0.4. Let  $X$  be finitely dominated. Then  $Y^X$  has CW homotopy type for every space  $Y$  of CW homotopy type.

PROOF. If  $X$  is dominated by finite complex  $K$ , then  $Y^X$  is dominated by  $Y^K$  by Proposition 0.0.2. The space  $Y^K$  has CW homotopy type by Theorem 0.0.3. Then  $Y^X$  is dominated by a CW complex and hence has CW homotopy type by Theorem 23 of Whitehead [64].  $\square$

LEMMA 0.0.5. If  $X$  is homotopy equivalent to  $\vee_{\lambda} X_{\lambda}$  in the pointed sense, then  $(Y, *)^{(X, *)}$  is homotopy equivalent to  $\prod_{\lambda} (Y, *)^{(X_{\lambda}, *_{\lambda})}$ .

Consequently if  $(Y, *)^{(X, *)}$  has CW type, so has  $(Y, *)^{(X_{\lambda}, *_{\lambda})}$ , for each  $\lambda$ . If  $\Lambda$  is finite, then the converse also holds.

PROOF. The first statement follows from (ii) of Lemma 0.0.1 and Proposition 0.0.2, the second statement follows from Proposition 3 of Milnor [43].  $\square$

LEMMA 0.0.6. *Consider the following topological pushout diagram.*

$$(*) \quad \begin{array}{ccc} A & \xrightarrow{i} & B \\ \varphi \downarrow & & \downarrow \Phi \\ C & \xrightarrow{j} & D = B \sqcup_A C \end{array}$$

*Given a space  $Y$ , we get an induced diagram*

$$(**) \quad \begin{array}{ccc} Y^D & \xrightarrow{\Phi^\#} & Y^B \\ j^\# \downarrow & & \downarrow i^\# \\ Y^C & \xrightarrow{\varphi^\#} & Y^A \end{array}$$

*Then  $(**)$  is a pullback diagram if either*

- $i$  is a closed embedding, and  $j$  is a proper closed map, or
- $(B, A)$  is a CW pair, and  $C$  admits a CW structure such that  $\varphi: A \rightarrow C$  is a cellular map.

*If  $a_0 \in A$ ,  $b_0 \in B$ ,  $c_0 \in C$ ,  $d_0 \in D$  are base points coherent with respect to  $(*)$ , and  $y_0$  is a base point in  $Y$ , then  $(**)$  is a pullback also if  $Y^A, Y^B, Y^C, Y^D$  are replaced by, respectively,  $(Y, y_0)^{(A, a_0)}$ ,  $(Y, y_0)^{(B, b_0)}$ ,  $(Y, y_0)^{(C, c_0)}$ ,  $(Y, y_0)^{(D, d_0)}$ .  $\square$*

Taking the diagram  $* \leftarrow L \hookrightarrow X$  for a CW pair  $(X, L)$  we obtain the following corollary (which is not true for general pairs  $(X, L)$ ).

COROLLARY 0.0.7. *Let  $(X, L)$  be a CW pair and  $(Y, y_0)$  a pointed space. Then the space  $(Y, y_0)^{(X, L)}$  is homeomorphic with  $(Y, y_0)^{(X/L, *)}$ .  $\square$*

COROLLARY 0.0.8. *Let  $\varphi: A \rightarrow L$  be a cellular map of CW complexes (with respect to some decomposition of  $L$ ) and let  $C_\varphi = CA \sqcup_\varphi L$  denote the (either reduced or unreduced) mapping cone of  $\varphi$ . Choose  $a_0 \in A$  and set  $x_0 = \varphi(a_0)$ . Then the following is a pullback diagram.*

$$(\dagger) \quad \begin{array}{ccc} (Y, y_0)^{(C_\varphi, x_0)} & \longrightarrow & (Y, y_0)^{(CA, a_0)} \\ \downarrow & & \downarrow \\ (Y, y_0)^{(L, x_0)} & \longrightarrow & (Y, y_0)^{(A, a_0)} \end{array}$$

*The vertical arrows are fibrations and  $(Y, y_0)^{(CA, a_0)}$  is contractible. This exhibits*

$$(Y, y_0)^{(C_\varphi, x_0)} \rightarrow (Y, y_0)^{(L, x_0)}$$

as a principal fibration with all fibres either empty or homotopy equivalent to the space  $\Omega((Y, y_0)^{(A, a_0)}, \text{const}_{y_0}) \approx (Y, y_0)^{(SA, *)}$ .

In addition, if  $(X, L)$  is a CW pair such that  $\varphi$  induces a homotopy equivalence  $C_\varphi \rightarrow X$  then

$$(\ddagger) \quad (Y, y_0)^{(C_\varphi, x_0)} \rightarrow (Y, y_0)^{(X, x_0)}$$

(respectively  $Y^{C_\varphi} \rightarrow Y^X$ ) is a fibre homotopy equivalence over  $(Y, y_0)^{(L, x_0)}$  (respectively  $Y^L$ ).

PROOF. The space  $(Y, y_0)^{(CA, a_0)}$  is contractible by Proposition 0.0.2, hence  $(\ddagger)$  is the diagram of a principal fibration. (It is equivalent to the standard one when we are pulling back the path fibration  $P(Y, y_0)^{(A, a_0)} \rightarrow (Y, y_0)^{(A, a_0)}$  by the coglueing theorem of Brown and Heath [3].)

By Proposition 0.0.2 the map  $(\ddagger)$  is a homotopy equivalence hence it is a fibre homotopy equivalence by [3], Corollary 3.7.  $\square$

The following is a result of Stasheff [59] (see also Schön [53]). We will simply call it ‘Stasheff’s theorem’ in what follows.

**THEOREM 0.0.9** (Stasheff’s theorem). *Let  $p: E \rightarrow B$  be a fibration where  $B$  has the homotopy type of a CW complex.*

*If  $E$  has the homotopy type of a CW complex then so have all fibres of  $p$ .*

*Conversely if for every path component  $C$  of  $B$  and some  $b_0 \in C$  the fibre  $p^{-1}(b_0)$  has CW type, then the total space  $E$  has CW homotopy type.*  $\square$

The quoted theorems of Milnor and of Stasheff, as well as the above cited coglueing theorem of Brown and Heath (see [3]), and Theorem 3.4.1 of Edwards and Hastings [14] together with its simplified proof by Geoghegan [22] form the basis for our work.

**COROLLARY 0.0.10.** *Let  $X$  be a Hausdorff space with nondegenerate base point  $x_0$  and let  $Y$  be a space of CW homotopy type. For each path component  $D_\lambda$  of  $Y$  choose a point  $y_\lambda \in D_\lambda$ . Then  $Y^X$  has CW homotopy type if and only if all spaces  $(Y, y_\lambda)^{(X, x_0)}$  have CW homotopy type.*

*Fix  $y_0 \in Y$ . The path component of  $\text{const}_{y_0}$  in  $Y^X$  has CW type if and only if the path component of  $\text{const}_{y_0}$  in  $(Y, y_0)^{(X, x_0)}$  has CW type. (If  $\pi_1(Y, y_0)$  is trivial then this holds for any map  $g: (X, x_0) \rightarrow (Y, y_0)$  in place of  $\text{const}_{y_0}$ .)*

PROOF. Since  $x_0$  is nondegenerate, the evaluation map

$$Y^X \rightarrow Y, \quad f \mapsto f(x_0),$$

is a fibration. (See also Lemma 2.1.1 and the remark after it.)



Let  $D$  be the path component of  $y_0 \in Y$ . Pick a pointed map  $g: (X, x_0) \rightarrow (Y, y_0)$  and let  $C$  be the path component of  $g$  in  $Y^X$ . Then  $C \cap (Y, y_0)^{(X, x_0)}$  is a set of path components of  $(Y, y_0)^{(X, x_0)}$ . In case  $C$  is the constant component,  $C \subset D^X$  and  $C \cap (Y, y_0)^{(X, x_0)} = C \cap (D, y_0)^{(X, x_0)}$  is path-connected. (And if  $\pi_1(Y, y_0)$  is trivial this holds for any  $g$ .) The restricted map  $C \rightarrow Y$  is still a fibration.

Now apply Stasheff's theorem. □

**COROLLARY 0.0.11.** *If  $Z$  has CW type and  $z_0$  is any point in  $Z$  then  $\Omega(Z, z_0)$  has CW type.*

**PROOF.** The space  $Z^{S^1}$  has CW type by Milnor's theorem, hence so has the space  $\Omega(Z, z_0) = (Z, z_0)^{(S^1, *)}$ , by Corollary 0.0.10. □

Let  $X$  be a CW complex and  $Y$  a space of CW type. Write  $X = \cup_{\lambda \in \Lambda} X_\lambda$  where  $\{X_\lambda \mid \lambda\}$  is the set of path component of  $X$ . Pick  $x_0 \in X_{\lambda_0} \subset X$ . Then  $Y^X$  is homeomorphic to  $\prod_{\lambda} Y^{X_\lambda}$ , and  $(Y, y_0)^{(X, x_0)}$  is homeomorphic to  $(Y, y_0)^{(X_{\lambda_0}, x_0)} \times \prod_{\lambda \neq \lambda_0} Y^X$ . In light of Example 1 on page 4 it suffices to consider  $X$  path-connected.

Let  $X$  be a path-connected CW complex and let  $x_0 \in X$ . Then any given CW decomposition may be refined in such a way that  $x_0$  becomes a 0-cell. In particular,  $x_0$  is a nondegenerate base point.

Let  $Y$  be a space of CW type and let  $y_0 \in Y$ . Then  $(Y, y_0)^{(X, x_0)} = (D, y_0)^{(X, x_0)}$  where  $D$  is the path component of  $y_0$ , and by Corollary 0.0.10 the space  $Y^X$  has CW type if and only if  $(Y, y_0)^{(X, x_0)}$  has for  $y_0$  ranging over all different path components of  $Y$ . In this way it suffices to consider  $Y$  path-connected.

Let  $X$  be a path-connected CW complex with a 0-cell  $x_0$  and let  $T$  be a maximal tree with vertex  $x_0$  in  $X$ . Then  $(X, x_0) \rightarrow (X/T, *)$  is a homotopy equivalence of pairs and in light of Proposition 0.0.2 it suffices to consider CW complexes  $X$  with a single 0-cell.

Let  $y_0$  be an arbitrary base point of a path-connected space  $Y$  of CW type. If  $y_0$  is degenerate, attach a whisker at  $y_0$  to obtain  $Y'$  with nondegenerate base point  $y'_0$ . (That is  $Y'$  is the mapping cylinder of  $\{y_0\} \rightarrow Y$ .) The retraction  $Y' \rightarrow Y$  is a homotopy equivalence (sending  $y'_0$  to  $y_0$ ), hence so is the induced function  $Y'^X \rightarrow Y^X$ . The diagram

$$\begin{array}{ccc}
 Y'^X & \longrightarrow & Y^X \\
 \downarrow & & \downarrow \\
 Y' & \xrightarrow{\text{retraction}} & Y
 \end{array}$$

where the vertical arrows are evaluations at  $x_0$ , is a map of fibrations. Since the horizontal arrows are homotopy equivalences, so is the induced map

$$(\dagger) \quad (Y', y'_0)^{(X, x_0)} \rightarrow (Y, y_0)^{(X, x_0)}$$

on the fibres, by Brown and Heath [3], Corollary 1.5.

## CHAPTER 1

### CW homotopy type of certain inverse limits

A topological space  $Z$  is *semilocally contractible* (see for example Dydak and Geoghegan [12] or Fritsch and Piccinini [19]) if every point  $z \in Z$  has a neighbourhood that deforms in  $Z$  to a point. The deformation is not required to be  $z$ -preserving, but we may clearly assume that the final point is  $z$ .

Semilocal contractibility is also referred to as *weak local contractibility*, see for example [16], [11].

DEFINITION. A space  $Z$  is a Dold space if it admits a numerable covering of open sets deformable in  $Z$  to a point.

Such spaces are called *locally contractible in the large* in Allaud [1].

Obviously a Dold space is semilocally contractible.

A space dominated by a semilocally contractible (respectively Dold) space is itself semilocally contractible (respectively Dold); see Fadell [16] and Dold [8]. In particular, the two properties are homotopy invariant, and we note (see [8])

LEMMA 1.0.1. *Spaces of CW homotopy type are Dold spaces.*  $\square$

Path components of a semilocally contractible space are open, and therefore a space is semilocally contractible (respectively Dold) if and only if its path components are open and semilocally contractible (respectively Dold). Similarly a space has the homotopy type of a CW complex if and only if its path components are open and each has the type of a CW complex.

It is easy to see (see [8])

LEMMA 1.0.2. *A paracompact and semilocally contractible space is a Dold space.*  $\square$

#### 1.1. General necessary conditions for CW type of certain inverse limits

Let  $(\mathcal{X}, \leq, \vee, \wedge)$  be a complete lattice with a subset  $\mathcal{K}$ .

DEFINITION. The subset  $\mathcal{K}$  is an *ideal* in  $\mathcal{X}$  if for each  $\kappa, \kappa' \in \mathcal{K}$  and  $\xi \in \mathcal{X}$  also  $\kappa \vee \kappa'$  and  $\kappa \wedge \xi$  belong to  $\mathcal{K}$ . (In particular,  $\mathcal{K}$  is a sublattice.)

The subset  $\mathcal{K}$  of  $\mathcal{X}$  is *generating* if for each  $\xi \in \mathcal{X}$  there exists a subset  $K$  of  $\mathcal{K}$  with  $\xi = \sup K$ .

The subset  $\mathcal{K}$  of  $\mathcal{X}$  is *cofinite* if for each  $\kappa \in \mathcal{K}$  the set

$$(*) \quad \{\lambda \in \mathcal{X} \mid \lambda < \kappa\}$$

is finite. Note that if  $\mathcal{K}$  is an ideal,  $(*)$  is contained in  $\mathcal{K}$ .

If  $\mathcal{K}$  is a cofinite generating ideal we will say that it is a *set of finite elements* of  $\mathcal{X}$ .

Let  $\mathcal{K}$  be a generating subset of  $\mathcal{X}$  and  $\xi \in \mathcal{X}$ . We define the cardinality of  $\xi$  with respect to  $\mathcal{K}$  as

$$\text{card}_{\mathcal{K}} \xi = \min \{\text{card } K \mid K \subset \mathcal{K} \text{ such that } \xi = \sup K\}.$$

As every set of cardinal numbers is well-ordered, the above definition makes sense. Note that  $\text{card}_{\mathcal{K}} \kappa = 1$  for each  $\kappa \in \mathcal{K}$ . We will suppress the index  $\mathcal{K}$  whenever understood from the context.

We say that  $\mathcal{X}$  is  $\mathcal{K}$ -finite if  $\text{card}(\max \mathcal{X})$  is finite. Note that if  $\mathcal{K}$  is a set of finite elements and  $\mathcal{X}$  is  $\mathcal{K}$ -finite, then  $\mathcal{X}$  is actually a finite set.

Let  $\mathbf{Z}$  be an inverse system of topological spaces indexed by  $\mathcal{X}$ , that is a cofunctor  $\mathcal{X} \rightarrow \mathcal{T}op$ . As customary we write  $Z_\lambda$  for  $\mathbf{Z}(\lambda)$  and employ an additional symbol  $p_{\lambda, \mu}$  for the arrow, or bonding morphism,  $\mathbf{Z}(\mu) \rightarrow \mathbf{Z}(\lambda)$  whenever  $\lambda < \mu$ .

Let  $\mathcal{L}$  be a directed subset of  $\mathcal{X}$  and let  $\mu = \sup \mathcal{L}$ . The morphisms  $Z_\mu \rightarrow Z_\lambda$ , for  $\lambda \in \mathcal{L}$ , induce a map

$$(**) \quad Z_\mu \rightarrow \lim_{\lambda \in \mathcal{L}} Z_\lambda.$$

DEFINITION. The inverse system  $\mathbf{Z}$  is restricted if the map  $(**)$  is a homeomorphism, for each directed subset  $\mathcal{L}$  of  $\mathcal{X}$ .

REMARK. The only type of lattice that we are going to consider (apart from the lattice of natural numbers together with  $\infty$ ) is the lattice  $\mathcal{X}$  of all subcomplexes of a given CW complex  $X$ . An obvious set of finite elements is the subset of all finite subcomplexes. Given a topological space  $Y$  we may form the induced inverse system of function spaces  $\{Y^L \mid L \in \mathcal{X}\}$  together with restriction fibrations as bonding maps. This system is certainly restricted in the sense defined above, and systems of this type will be considered below.

Therefore we restrict our attention to (restricted) inverse systems indexed by complete lattices. Some results of general nature below, for instance Proposition 1.1.1, could be formulated in a more general setting. However, we insist that all morphisms in the system be fibrations. This is

a strong requirement which is difficult to check in ‘practice.’ In a complete lattice we can choose a maximal totally ordered subset which is well-ordered because of completeness. For a restricted inverse system indexed by a well-ordered set all bonding maps are fibrations if and only if all consecutive bonding maps are fibrations. (See Corollary 1.3.2.)

With ‘target lattice’ being subcomplexes of a CW complex, it may appear artificial to work with abstract complete lattices. We have two justifications. We exhibit some interesting results for inverse sequences which do not involve function spaces. On the other hand, for function spaces with domain an uncountable complex our results follow from easy transfinite induction arguments that rely mainly upon properties of the lattice of subcomplexes of domain and not particularly on the function space structure. Stripped of the ‘unnecessary extra’ structure, proofs involving transfinite arguments appear more natural. Thus we have taken an intermediate approach.

**PROPOSITION 1.1.1.** *Let  $\mathbf{Z}$  be a restricted system of fibrations indexed by a complete lattice  $\mathcal{X}$  with set of finite elements  $\mathcal{K}$ . Set  $\alpha = \sup \mathcal{X} = \max \mathcal{X}$ . Assume that for each  $\kappa \in \mathcal{K}$  the space  $Z_\kappa$  has the homotopy type of a CW complex. Assume also that  $\alpha \notin \mathcal{K}$ , for nontriviality.*

*Denote the limit space  $Z_\alpha = \lim_{\lambda \in \mathcal{X}} Z_\lambda = \lim_{\kappa \in \mathcal{K}} Z_\kappa$  simply by  $Z$ . Let  $C$  be an arbitrary path component of  $Z$ , and let  $C_\kappa$  denote the image of  $C$  under  $Z \rightarrow Z_\kappa$ .*

- (i)  *$C$  is semilocally contractible if and only if for each  $\zeta \in C$  there exists  $\kappa \in \mathcal{K}$  such that the fibre of  $C \rightarrow C_\kappa$  over  $\zeta|_\kappa$  contracts in  $C$ .*
- (ii)  *$C$  is semilocally contractible and open if and only if for each  $\zeta \in C$  there exists  $\kappa \in \mathcal{K}$  such that the fibre of  $Z \rightarrow Z_\kappa$  over  $\zeta|_\kappa$  contracts in  $Z$ .*
- (iii) *If  $C$  is semilocally contractible then the loop space  $\Omega(Z, \zeta)$  has CW homotopy type for each  $\zeta \in C$ .*
- (iv)  *$Z$  has the homotopy type of a CW complex if and only if it is a Dold space.*

**PROOF.** Assume that  $C$  is semilocally contractible and let  $\zeta \in C$ . There exists a neighbourhood, with no loss of generality basic, of  $\zeta \in C$  that contracts in  $C$ . Suppose this neighbourhood has the form

$$(\dagger) \quad \langle U_{\kappa_1}, \dots, U_{\kappa_r} \rangle \cap C$$

where  $\kappa_1, \dots, \kappa_r \in \mathcal{K}$ . Since  $\mathcal{K}$  is a sublattice,  $\kappa = \sup \{\kappa_1, \dots, \kappa_r\}$  belongs to  $\mathcal{K}$ . Thus the neighbourhood  $(\dagger)$  contains the neighbourhood  $\langle U_\kappa \rangle \cap C$

where

$$U_\kappa = \bigcap_{i=1}^r p_{\kappa_i, \kappa}^{-1}(U_{\kappa_i}).$$

The latter contains the fibre  $F_\kappa$  of  $C \rightarrow C_\kappa$  over  $\zeta_\kappa = \zeta|_\kappa$ . Thus  $F_\kappa$  contracts in the total space  $C$  and therefore the loop space  $\Omega(C_\kappa, \zeta_\kappa)$  has the same homotopy type as the product  $F_\kappa \times \Omega(C, \zeta)$ . (See Spanier [58], Corollary 2.8.15, as well as Proposition A.0.6.) In particular,  $\Omega(Z, \zeta) = \Omega(C, \zeta)$  is dominated by  $\Omega(C_\kappa, \zeta_\kappa)$  which has CW homotopy type by assumption and Milnor's theorem. This shows (iii) and the necessity part of (i). To prove necessity of (ii), note that if  $C$  is open in  $Z$ , then (†) may be assumed equal to  $\langle U_{\kappa_1}, \dots, U_{\kappa_r} \rangle \cap Z$ .

For the sufficiency part of (i), let  $\zeta \in C$  and assume that the fibre  $F_\kappa$  over  $\zeta|_\kappa$  of  $C \rightarrow C_\kappa$  contracts in  $C$ . Since  $C_\kappa$  has CW type, there exists a neighbourhood  $U_\kappa$  of  $\zeta|_\kappa$  that contracts in  $C_\kappa$  to  $\zeta|_\kappa$ . Let  $U$  be the preimage of  $U_\kappa$  in  $C$ . By homotopy lifting property the deformation of  $U_\kappa$  lifts to a homotopy that deforms  $U$  into the fibre  $F_\kappa$ . Since the latter deforms to a point, we may concatenate to obtain a deformation of  $U$  into a point. This concludes (i).

Note that  $C_\kappa$  is open in  $Z_\kappa$ , hence  $U_\kappa$  is open in  $Z_\kappa$ , and the preimage  $\tilde{U}$  of  $U_\kappa$  in  $Z$  is an open neighbourhood of  $\zeta$ . If the fibre of  $Z \rightarrow Z_\kappa$  contracts in  $Z$ , then so does  $\tilde{U}$ , by the above. The image of the contracting homotopy is a path-connected subset of  $Z$ , hence  $\tilde{U}$  is contained in  $C$ . This shows that  $C$  is open, and proves the sufficiency part of (ii).

If  $Z$  is semilocally contractible it follows by (iii) that for each  $\zeta \in Z$ , the loop space  $\Omega(Z, \zeta)$  has CW homotopy type. If, in addition,  $Z$  is a Dold space, it has CW homotopy type by the 'delooping' theorem of Allaud [1].  $\square$

**EXAMPLE 1.** *Let  $I$  be an infinite set and let  $\{W_i \mid i \in I\}$  be a family of nonempty spaces of CW type. Let  $\mathcal{K}$  denote the set of all finite subsets of  $I$ , and  $\mathcal{X}$  the set of all subsets of  $I$ . For  $\lambda \in \mathcal{X}$  let  $Z_\lambda$  denote the cartesian product of  $W_i$  for  $i \in \lambda$ , and for  $\lambda \subset \mu$  let  $Z_\mu \rightarrow Z_\lambda$  denote the projection. The limit space of the system  $\{Z_\lambda \mid \lambda \in \mathcal{X}\}$  is the cartesian product  $Z = \prod_i W_i$ .*

*Proposition 1.1.1 implies that  $Z$  is semilocally contractible if and only if there exists  $\kappa \in \mathcal{K}$  such that  $Z' = \prod_{i \notin \kappa} W_i$  contracts in  $Z$ . By virtue of the projection  $Z \rightarrow Z'$  this implies that  $Z'$  is contractible and hence so are the spaces  $W_i$  for  $i \notin \kappa$ . In this case  $Z$  is homotopy equivalent to the finite product  $\prod_{i \in \kappa} W_i$  and has CW homotopy type.  $\square$*

PROPOSITION 1.1.2. *Given hypotheses of Proposition 1.1.1, assume that  $C$  has the homotopy type of a CW complex. Then, given any  $\kappa \in \mathcal{K}$  there exist  $\lambda, \lambda' \in \mathcal{K}$  with  $\kappa \leq \lambda \leq \lambda'$  so that the following is true.*

- (i) *For every  $\mu \in \mathcal{X}$  such that  $\mu \geq \lambda$  the fibre  $F_\mu$  of  $C \rightarrow C_\mu$  over  $\zeta_\mu$  contracts in  $C$  and the fibration  $\Omega(C, \zeta) \rightarrow \Omega(C_\mu, \zeta_\mu)$  is a domination map. Moreover, there exists a commutative diagram*

$$\begin{array}{ccc} \Omega(Z_\mu, \zeta_\mu) & \xrightarrow{\Omega(p_{\lambda, \mu, \zeta_\mu})} & \Omega(Z_\lambda, \zeta_\lambda) \\ \simeq \uparrow & & \uparrow \simeq \\ F_\mu \times \Omega(Z, \zeta) & \xrightarrow{\text{inclusion}} & F_\lambda \times \Omega(Z, \zeta) \end{array}$$

where the vertical arrows are homotopy equivalences of the form

$$F_\lambda \times \Omega(Z, \zeta) \rightarrow \Omega(Z_\lambda, \zeta_\lambda), \quad (z, \gamma) \mapsto p_{\lambda, \alpha \#}(\gamma) * \theta_\lambda(z)$$

where  $\theta_\lambda: F_\lambda \rightarrow \Omega(Z_\lambda, \zeta_\lambda)$  is a map, and  $*$  denotes concatenation.

- (ii) *For every  $\mu \in \mathcal{X}$  with  $\mu \geq \lambda'$  the inclusion  $F_\mu \rightarrow F_\lambda$  (and hence also  $F_\mu \rightarrow F_\kappa$ ) is homotopic to a constant map. Consequently the inclusion  $F_\mu \times \Omega(Z, \zeta) \hookrightarrow F_\lambda \times \Omega(Z, \zeta)$  is homotopic to the projection onto  $\{\zeta\} \times \Omega(Z, \zeta)$ , and the fibration  $\Omega(Z_\mu, \zeta_\mu) \rightarrow \Omega(Z_\lambda, \zeta_\lambda)$  factors as*

$$\Omega(Z_\mu, \zeta_\mu) \xrightarrow{j} \Omega(Z, \zeta) \xrightarrow{\Omega(p_{\lambda, \alpha, \zeta})} \Omega(Z_\lambda, \zeta_\lambda)$$

where  $j$  is a right homotopy inverse of the fibration  $\Omega(Z, \zeta) \rightarrow \Omega(Z_\mu, \zeta_\mu)$ .

PROOF. If  $\mu \geq \lambda$  then  $F_\mu \subset F_\lambda$ , so by (i) of Proposition 1.1.1 and directedness of  $\mathcal{K}$  there exists  $\lambda \geq \kappa$  so that  $F_\lambda$  contracts in  $C$  and consequently so does  $F_\mu$  for any  $\mu \geq \lambda$ .

View  $\Omega(Z, \zeta)$  and  $\Omega(Z_\lambda, \zeta_\lambda)$  as subspaces of  $C^I$  and  $C_\lambda^I$ , respectively. Let  $k: F_\lambda \rightarrow C^I$  denote the adjoint of a contracting homotopy. By Proposition A.0.6 a homotopy equivalence  $F_\lambda \times \Omega(Z, \zeta) \rightarrow \Omega(Z_\lambda, \zeta_\lambda)$  is given by

$$(\ddagger) \quad (z, \gamma) \mapsto p_{\lambda, \alpha} \circ [\gamma * k^{-1}(z)] = [p_{\lambda, \alpha} \circ \gamma] * [p_{\lambda, \alpha} \circ k^{-1}(z)].$$

For fixed  $z$ , the map  $(\ddagger)$  amounts to the induced fibration  $\Omega(Z, \zeta) \rightarrow \Omega(Z_\lambda, \zeta_\lambda)$  followed by a multiplication homotopy equivalence; which concludes the proof of (i).

Since both  $C$  and  $C_\lambda$  have CW type, so has  $F_\lambda$  by Stasheff's theorem. Hence also  $F_\lambda$  is semilocally contractible and there exists a basic neighbourhood of  $\zeta$  in  $F_\lambda$  that contracts in  $F_\lambda$ . Using directedness of  $\mathcal{K}$  as in the proof of Proposition 1.1.1 we may assume that the neighbourhood has the form  $\langle U_{\lambda'} \rangle \cap F_\lambda$  for some  $\lambda' \in \mathcal{K}$ ,  $\lambda' \geq \lambda$ , and open  $U_{\lambda'} \subset Z_{\lambda'}$ . But this

neighbourhood contains the set  $\{z \mid z_{\lambda'} = \zeta_{\lambda'}\} \cap F_{\lambda}$  which equals  $F_{\lambda'}$ . The commutativity of the diagram of (ii) follows from (‡) upon having chosen  $k|_{F_{\mu}}$  for the ‘contraction’  $F_{\mu} \rightarrow C^I$ .  $\square$

## 1.2. Results on sequences

In this section we study inverse sequences of fibrations between spaces of CW type, and show that the inverse limit space has CW type only if the sequence of loop spaces splits into the product of a sequence of nullhomotopic maps and one of homotopy equivalences. Moreover, a converse holds as well (see Theorem 1.2.6 below). Thus the limit space has CW type if and only if the sequence is trivial in some sense, in the spirit of Theorem B of Dydak and Geoghegan [12] (see also [13]).

We first recall a well known result due to Edwards and Hastings (see Geoghegan [22] for details).

PROPOSITION 1.2.1. *Assume given a commutative diagram of topological spaces and continuous functions*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & W_3 & \xrightarrow{p_3} & W_2 & \xrightarrow{p_2} & W_1 \\ & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 \\ \cdots & \longrightarrow & Z_3 & \xrightarrow{r_3} & Z_2 & \xrightarrow{r_2} & Z_1 \end{array}$$

where the maps  $p_i$  as well as  $r_i$  are fibrations and the maps  $f_i$  are homotopy equivalences. Then the induced inverse limit map  $f_{\infty}: W_{\infty} \rightarrow Z_{\infty}$  is also a homotopy equivalence.  $\square$

COROLLARY 1.2.2. *Suppose  $\cdots \rightarrow Z_3 \rightarrow Z_2 \rightarrow Z_1$  is an inverse sequence of fibrations which are homotopy equivalences. Then any canonical projection from the inverse limit  $Z_{\infty} \rightarrow Z_i$  is a homotopy equivalence.  $\square$*

Apply homotopy lifting property to inductively construct commutative squares in order to infer

COROLLARY 1.2.3. *In order to conclude that  $W_{\infty}$  and  $Z_{\infty}$  are homotopy equivalent it is enough to assume commutativity only up to homotopy in Proposition 1.2.1.  $\square$*

The following proposition generalizes Lemma 2.8 of Kahn [29], and is crucial for understanding the problem of CW homotopy type of inverse limits.

PROPOSITION 1.2.4. *Let  $\cdots \rightarrow Z_3 \xrightarrow{r_3} Z_2 \xrightarrow{r_2} Z_1$  be an inverse sequence of fibrations and let  $Z_{\infty}$  denote the inverse limit space. If each fibration*



$r_{i+1}: Z_{i+1} \rightarrow Z_i$  is homotopic to a constant map, then the limit space  $Z_\infty$  is contractible.

PROOF. Define  $W_1 = Z_1$  and let  $f_1: W_1 \rightarrow Z_1$  be the identity. Assume having constructed a suitable homotopy equivalence  $f_i: W_i \rightarrow Z_i$ . Let  $r_{i+1}: Z_{i+1} \rightarrow Z_i$  be homotopic to the constant  $\zeta_i \in Z_i$ . Let  $w_i$  be a point in  $W_i$  that is mapped by  $f_i$  into the path component of  $\zeta_i$ . Let  $PW_i$  denote the space of paths in  $W_i$  that start in  $w_i$  and set  $W_{i+1} = Z_{i+1} \times PW_i$ . Let  $p_{i+1}: W_{i+1} \rightarrow W_i$  be the composite  $Z_{i+1} \times PW_i \xrightarrow{\text{pr}} PW_i \xrightarrow{\varepsilon} W_i$  where  $\text{pr}$  is the obvious projection and  $\varepsilon$  denotes the evaluation at end point. Let  $f_{i+1}: W_{i+1} \rightarrow Z_{i+1}$  be the projection  $Z_{i+1} \times PW_i \rightarrow Z_{i+1}$ . Clearly  $f_{i+1}$  is a homotopy equivalence. The composite  $r_{i+1} \circ f_{i+1}$  is homotopic to  $\text{const}_{\zeta_i}$  while the composite  $f_i \circ p_{i+1}$  is homotopic to  $\text{const}_{f_i(w_i)}$ . The two constant maps are homotopic by choice of  $w_i$ . We proceed inductively, and appeal to Corollary 1.2.3 to conclude that  $Z_\infty$  is homotopy equivalent to  $W_\infty = \lim W_i$ . But the latter is homeomorphic to the limit space of the sequence  $\cdots \rightarrow PW_{i+1} \rightarrow PW_i \rightarrow \cdots \rightarrow PW_1$  where the map  $PW_{i+1} \rightarrow PW_i$  is a fibration. By Corollary 1.2.2 the limit space is contractible.  $\square$

DEFINITION. Let  $\mathbf{Z}$  be an inverse system indexed by  $\mathcal{X}$  and let  $\zeta = \{\zeta_\lambda \mid \lambda \in \mathcal{X}\}$  be a coherent choice of base points, that is  $\zeta \in \lim \mathbf{Z}$ . We denote by  $\Omega(\mathbf{Z}, \zeta)$  the inverse system of spaces  $\Omega(Z_\lambda, \zeta_\lambda)$  with the induced bonds

$$\Omega(Z_\mu, \zeta_\mu) \xrightarrow{\Omega(Z_\mu \rightarrow Z_\lambda)} \Omega(Z_\lambda, \zeta_\lambda).$$

Let  $\mathbf{P}$  and  $\mathbf{Q}$  be inverse systems indexed by the same set  $\mathcal{X}$ . We define the product system  $\mathbf{P} \times \mathbf{Q}$  as the inverse system of products with product bonding maps.

LEMMA 1.2.5. *Let  $\mathbf{Z}, \mathbf{P}, \mathbf{Q}$  be restricted inverse systems of fibrations indexed by complete lattice  $\mathcal{X}$ , and let  $\zeta \in \lim \mathbf{Z}$ . Then  $\Omega(\mathbf{Z}, \zeta)$  and  $\mathbf{P} \times \mathbf{Q}$  are restricted inverse systems of fibrations.*  $\square$

DEFINITION. We say that the inverse sequence  $\mathbf{W}$  splits as a product of inverse sequences  $\mathbf{P}$  and  $\mathbf{Q}$  if there exists a level preserving morphism (up to homotopy) of sequences  $\mathbf{f}: \mathbf{P} \times \mathbf{Q} \rightarrow \mathbf{W}$  which is a homotopy equivalence on each level.

(Care should be taken to define splitting up to homotopy of a general inverse system. We will not need this.)

THEOREM 1.2.6. (i) *Let  $(\mathbf{W}, \mathbf{w})$  be an inverse sequence of fibrations that splits into the product of  $(\mathbf{P}, \mathbf{p})$  and  $(\mathbf{Q}, \mathbf{q})$  where  $\mathbf{P}$  is a sequence of nullhomotopic maps and  $\mathbf{Q}$  is a sequence of homotopy*

equivalences. Then the limit space  $W_\infty$  is homotopy equivalent to  $Q_1$ .

In particular, if the space  $Q_1$  has CW type, so has  $W_\infty$ .

- (ii) Let  $(\mathbf{Z}, \mathbf{r})$  be an inverse sequence of fibrations between spaces of CW homotopy type. If the limit space  $Z_\infty$  also has CW type then, for any  $\zeta \in Z_\infty$ , a subsequence of the induced inverse sequence  $\Omega(\mathbf{Z}, \zeta)$  splits into the product of a sequence of nullhomotopic maps and the identity sequence  $\{\dots = \Omega(Z_\infty, \zeta_\infty) = \Omega(Z_\infty, \zeta_\infty)\}$ .

PROOF. We change  $(\mathbf{P}, \mathbf{p})$  and  $(\mathbf{Q}, \mathbf{q})$  inductively to obtain sequences of fibrations  $(\mathbf{P}', \mathbf{p}')$  and  $(\mathbf{Q}', \mathbf{q}')$  together with (strict) level preserving morphisms  $\mathbf{P} \rightarrow \mathbf{P}'$  and  $\mathbf{Q} \rightarrow \mathbf{Q}'$  consisting of homotopy equivalences. Thus there exists a level preserving morphism (up to homotopy)  $\mathbf{P}' \times \mathbf{Q}' \rightarrow \mathbf{W}$  consisting of homotopy equivalences. The limit space of  $\mathbf{P}' \times \mathbf{Q}'$  is  $P'_\infty \times Q'_\infty$  where  $P'_\infty = \lim \mathbf{P}'$  is contractible by Proposition 1.2.4, and  $Q'_\infty = \lim \mathbf{Q}'$  is homotopy equivalent to  $Q'_1 = Q_1$  by Corollary 1.2.2. By Corollary 1.2.3 the space  $W_\infty = \lim \mathbf{W}$  is homotopy equivalent to  $P'_\infty \times Q'_\infty$  which in turn is homotopy equivalent to  $Q'_\infty$ . This proves (i).

Statement (ii) follows immediately from Proposition 1.1.2.  $\square$

COROLLARY 1.2.7. *Given the assumptions of (ii) of Theorem 1.2.6, if  $Z_\infty$  has CW type and  $\Omega(Z_\infty, \zeta_\infty)$  is contractible, then a subsequence of  $\Omega(\mathbf{Z}, \zeta)$  is a sequence of nullhomotopic maps.*  $\square$

Note that while Proposition 1.2.4 yields a number of counterexamples to Theorem B of Dydak and Geoghegan [12], Theorem 1.2.6 says these represent essentially the only kind of counterexample. In a sense therefore the cited theorem is ‘not far’ from being correct.

### 1.3. Some sufficient and some necessary conditions

The purpose of this section is proving Theorem 1.3.9 and Theorem 1.3.12, to be applied in later sections. We begin with some results on inverse systems indexed by well-ordered sets; then we show that given a complete lattice with a set of finite elements, suitable well-ordered subsets can be chosen, for better grip on the inverse limit space.

DEFINITION. For any ordinal number  $\alpha$  we will denote by  $W(\alpha)$  the ordinal number segment consisting of the set of ordinal numbers  $\lambda$  such that  $\lambda < \alpha$ . Recall that every well-ordered set  $W$  can be indexed uniquely in an order preserving manner by an ordinal number segment  $W(\alpha)$  for  $\alpha = \text{ord } W$ .

Let  $\alpha$  be a limit ordinal and let  $(\mathbf{Z}, \mathbf{p})$  be an inverse system of fibrations indexed by  $W(\alpha)$ . Note that  $W(\alpha) \cup \{\alpha\} = W(\alpha + 1)$  is a complete lattice.

We let  $Z_\alpha$  denote the limit of  $\mathbf{Z}$ , and may view  $\mathbf{Z}$  as an inverse system indexed by a complete lattice. Our notion of restrictedness of the system  $(\mathbf{Z}, \mathbf{p})$  now coincides with terminology used by J. Cohen [6].

For emphasis, the system  $(\mathbf{Z}, \mathbf{p})$  is restricted if for each limit ordinal  $\mu \in W(\alpha)$  the canonical map

$$\vartheta_\mu: Z_\mu \rightarrow \bar{Z}_\mu = \lim \mathbf{Z}|_{W(\mu)}$$

is a homeomorphism. (In fact for our applications it would suffice for  $\vartheta_\mu$  to be a fibration and a homotopy equivalence.)

The projections  $p_{\lambda, \lambda+1}$  are called *consecutive* bonding maps (or bonds).

REMARK. If  $(\mathbf{Z}, \mathbf{p})$  is indexed by  $W(\alpha)$  where  $\alpha$  has a predecessor  $\alpha - 1$  then  $Z_{\alpha-1}$ , together with  $p_{\lambda, \alpha-1}$ , is the limit of the system.  $\square$

We note the well known

LEMMA 1.3.1. *Let  $\eta$  be a limit ordinal (bigger than  $\omega$ ) and let  $\mu$  be the largest limit ordinal smaller than  $\eta$ . Then  $\eta = \mu + \omega$  where as usual  $\omega$  denotes the smallest infinite ordinal.*  $\square$

Using Lemma 1.3.1 and transfinite induction it is straightforward to prove

COROLLARY 1.3.2. *Let  $(\mathbf{Z}, \mathbf{p})$  be a restricted inverse system indexed by  $W(\alpha)$  where the consecutive bonds are fibrations. Then all bonds are fibrations.*  $\square$

The following transfinite analogue of Theorem 1.2.1 follows immediately upon an application of Lemma 1.3.1 which reduces the problem to sequences.

COROLLARY 1.3.3. *Let  $\alpha$  be a limit ordinal number and let  $(\mathbf{W}, \mathbf{p})$  and  $(\mathbf{Z}, \mathbf{q})$  be inverse systems of fibrations indexed by the well-ordered set  $W(\alpha) = \{\lambda \mid \lambda < \alpha\}$ . If  $\mathbf{f}: \mathbf{W} \rightarrow \mathbf{Z}$  is a level-preserving morphism consisting of homotopy equivalences, then the limit map  $f_\alpha: W_\alpha \rightarrow Z_\alpha$  is a homotopy equivalence.*  $\square$

We view the following result as the nontrivial generalization of Proposition 1.2.1 to systems indexed by well-ordered sets.

THEOREM 1.3.4. *Let  $\mathbf{f}: \mathbf{W} \rightarrow \mathbf{Z}$  be a level-preserving morphism of restricted inverse systems of fibrations as above. Let  $f_0: W_0 \rightarrow Z_0$  be a homotopy equivalence. If upon assuming that  $f_\lambda$  is a homotopy equivalence it follows that also  $f_{\lambda+1}$  is a homotopy equivalence for each  $\lambda < \alpha$ , then the limit map  $f_\alpha: W_\alpha \rightarrow Z_\alpha$  is a homotopy equivalence.*

PROOF. Let

$$S = \{\mu \in W(\alpha) \mid f_\lambda \text{ is a homotopy equivalence for all } \lambda < \mu\}.$$

Then  $S$  is nonempty since it contains 1. Suppose  $W(\eta) \subset S$ . If  $\eta$  is limiting, then for each  $\lambda \in W(\eta)$ , also  $\lambda + 1 \in W(\eta) \subset S$ , hence trivially  $\eta \in S$ . Assume that  $\eta$  has a predecessor  $\eta - 1$ . If  $\eta - 1$  is limiting, then Proposition 1.3.3 shows that the limit map  $\bar{f} = \lim_{\lambda < \eta-1} f_\lambda: \lim_{\lambda} W_\lambda \rightarrow \lim_{\lambda} Z_\lambda$  is a homotopy equivalence. The commutative diagram

$$\begin{array}{ccc} Z_{\eta-1} & \longrightarrow & \lim_{\lambda < \eta-1} Z_\lambda \\ \uparrow f_{\eta-1} & & \uparrow \simeq \bar{f} \\ W_{\eta-1} & \longrightarrow & \lim_{\lambda < \eta-1} W_\lambda \end{array}$$

together with the restrictedness hypothesis on  $\mathbf{W}$  and  $\mathbf{Z}$  implies that  $f_{\eta-1}$  is a homotopy equivalence. Hence also  $\eta \in S$ .

If  $\eta - 1$  has a predecessor  $\eta - 2$  then  $f_{\eta-1}$  is a homotopy equivalence by assumption in the statement of the proposition. Transfinite induction guarantees  $S = W(\alpha)$  and a reapplication of Corollary 1.3.3 finishes the proof.  $\square$

**COROLLARY 1.3.5.** *Let  $(\mathbf{W}, \mathbf{p})$  be a restricted inverse system of fibrations indexed by  $W(\alpha)$  for some limit ordinal  $\alpha$ . If each consecutive bonding map  $p_{\lambda, \lambda+1}$  is a homotopy equivalence then the limit projection  $W_\alpha \rightarrow W_0$  is a homotopy equivalence.*  $\square$

For an abelian group  $G$ , a well-ordered ascending chain of subgroups

$$N_0 \leq N_1 \leq \dots \leq N_\lambda \leq \dots \quad (\lambda < \alpha)$$

is called a *smooth chain* if for each limit ordinal  $\mu$  the subgroup  $N_\mu$  equals the union (‘supremum’) of all  $N_\lambda$  with  $\lambda < \mu$ . (See for example Fuchs [21] or Loth [33].)

Thus we may think of the assignment  $\lambda \rightarrow N_\lambda$  as ‘limit preserving,’ and could call it ‘continuous.’ In accord with the group-theoretic terminology and to avoid confusion we agree to the following

**DEFINITION.** Let  $f: W(\alpha) \rightarrow \mathcal{X}$  be a monotone function from the segment  $W(\alpha)$  to a complete lattice  $\mathcal{X}$ . The function  $f$  is *smooth* if for each limit ordinal  $\mu$  in  $W(\alpha)$ , the following equality holds

$$f(\mu) = \sup \{f(\lambda) \mid \lambda < \mu\}.$$

If a smooth chain  $\{N_\lambda \mid \lambda < \alpha\}$  in abelian group  $G$  has all successive quotients  $N_{\lambda+1}/N_\lambda$  cyclic groups (of infinite or prime order), and  $N_0 = 0$ ,  $\cup_{\lambda < \alpha} N_\lambda = G$ , then  $\{N_\lambda\}$  is called a *composition series* for  $G$ . In the

lemma below, we view the notion of *good filtration* as analogous to that of composition series for abelian groups.

LEMMA 1.3.6. *Let  $\mathcal{X}$  be a complete lattice with set of finite elements  $\mathcal{K}$ . Set  $\xi = \sup \mathcal{X}$ . Let  $\alpha$  denote the initial ordinal of cardinality  $\text{card}_{\mathcal{K}}(\xi)$ .*

*There exists an order preserving smooth injection  $U: W(\alpha) \rightarrow \mathcal{X}$  whose image  $\mathcal{U}$  is well-ordered, and exhaustive in the sense that  $\sup \mathcal{U} = \xi$ . Moreover, for each subset  $\Lambda \subset \mathcal{U}$  either  $\sup \Lambda = \xi$  or  $\sup \Lambda \in \mathcal{U}$  (in  $\mathcal{X}$ ). In addition, the following two properties hold.*

- *For each  $\mu \in W(\alpha)$ ,  $U(\mu + 1) = U(\mu) \vee \kappa = \sup \{U(\mu), \kappa\}$  for some  $\kappa = \kappa(\mu) \in \mathcal{K}$ , and*
- *for each  $\mu < \alpha + 1$ ,  $\text{card}_{\mathcal{K}} U(\mu) \leq \text{card } W(\mu)$ .*

*We name  $\mathcal{U}$  a good filtration for  $\mathcal{X}$  with respect to  $\mathcal{K}$ .*

PROOF. Let  $E$  be a subset of  $\mathcal{K}$  with  $\sup E = \xi$  and  $\text{card } E = \text{card}_{\mathcal{K}} \xi$ . Abusing notation we assume that  $E$  is an initial ordinal corresponding to the cardinal of  $E$ . To avoid ambiguity we denote the (new) relation in  $E$  by  $\prec$ . We define  $u: E \rightarrow \mathcal{X}$  by setting

$$u(f) = \sup \{e \mid e \in E, e \prec f\}.$$

Obviously  $u$  is monotone. Since it is initial,  $E$  is limiting, and hence  $f \leq u(f+1)$  (in  $\mathcal{X}$ ) for each  $f \in E$ . In particular,  $\sup \{u(f) \mid f \in E\} = \sup E = \xi$ . Moreover, note that if  $g$  is limiting in  $E$ , then

$$u(g) = \sup \{u(f) \mid f \prec g\}.$$

This says that  $u$  is smooth.

Since  $u$  is monotone, its preimages are of the form  $W(f_2) \setminus W(f_1)$ . Thus the quotient set  $\mathcal{E}$  of preimages of  $u$  is well-ordered, and the induced function  $U: \mathcal{E} \rightarrow \mathcal{X}$  is a monotone injection.

Let  $E_f = \{g \in E \mid u(g) = u(f)\}$  be a preimage of  $u$ ;  $E_f = W(f_2) \setminus W(f_1)$ . Assume that  $f_2$  is limiting. By smoothness,  $u(f_2) = \sup \{u(g) \mid g \prec f_2\}$ , which we rewrite as

$$u(f_2) = u(f_1) \vee \sup \{u(g) \mid g \in E_f\}.$$

But by definition of  $E_f$  this shows  $u(f_2) = u(f_1)$ , and hence  $f_2 \in E_f$ , contradiction. So  $f_2$  has a predecessor  $f_2 - 1 \in E_f$ , and  $E_f$  is a ‘closed segment’  $[f_1, f_2 - 1]$  of  $E$ . In particular,

$$u(f_2) = u(f_2 - 1) \vee (f_2 - 1) = u(f_1) \vee (f_2 - 1).$$

If the class  $[f_1, f_2 - 1]$  is limiting in  $\mathcal{E}$  then  $f_1$  is limiting in  $E$ . Therefore  $u(f_1) = \sup \{u(f) \mid f \prec f_1\}$  by smoothness of  $u$ . Thus also the induced function  $U: \mathcal{E} \rightarrow \mathcal{X}$  is smooth.

Define  $j: \mathcal{E} \rightarrow E$  by setting  $j(\lambda) = \max_E \lambda$ . By preceding discussion this is well defined. Evidently it is an order preserving injection, and hence  $\text{ord } \mathcal{E} \leq \text{ord } E$ . Note that  $U = u \circ j$ , and an easy transfinite argument shows that

$$U(\mu) = \sup \{j(\lambda) \mid \lambda < \mu\}.$$

This yields  $\text{card}_{\mathcal{K}} U(\mu) \leq \text{card } W(\mu)$ . Moreover,  $\xi = \sup E = \sup \{j(\lambda) \mid \lambda \in \mathcal{E}\}$  from which it follows that  $\text{card } \mathcal{E} = \text{card } E$ . Since  $E$  is initial it follows that  $\text{ord } \mathcal{E} = \text{ord } E$ , as claimed.  $\square$

We note the evident

LEMMA 1.3.7. *Let  $\alpha$  be a limit ordinal and let  $(\mathbf{W}, \mathbf{p})$  be an inverse system indexed by  $W(\alpha)$  with  $\mathbf{W}|_{W(\mu)}$  restricted for all  $\mu \in W(\alpha)$ . Then  $(\mathbf{W}, \mathbf{p})$  is restricted.*  $\square$

LEMMA 1.3.8. (i) *Let  $\{Z_i \mid i \in \mathbb{N}\}$  be an inverse sequence of fibrations and let  $Z$  denote the limit space of the sequence. Let  $C$  be a set of path components of  $Z$  and let  $C_i$  denote the image of  $C$  under the projection  $Z \rightarrow Z_i$ . Suppose that for all  $i$  the induced map  $C_i \rightarrow C_{i-1}$  is a homotopy equivalence. Then the inverse limit of  $\{C_i \mid i\}$  is homotopy equivalent to  $C$  (and  $C \rightarrow C_i$  is a homotopy equivalence for each  $i$ ).*

(ii) *Let  $\alpha$  be a limit ordinal and let  $\mathbf{Z}$  be a restricted inverse system of fibrations indexed by  $W(\alpha)$ . Let  $Z$  denote the limit space and let  $C$  be a set of path components of  $Z$ . For each  $\lambda < \alpha$  denote by  $C_\lambda$  the image of  $C$  under  $Z \rightarrow Z_\lambda$ . If for each  $\lambda$  the induced fibration  $C_{\lambda+1} \rightarrow C_\lambda$  is a homotopy equivalence, then the system  $\{C_\lambda \mid \lambda \in W(\alpha)\}$  is restricted and hence a system of homotopy equivalences. In particular, its limit space is  $C$  and is homotopy equivalent to  $C_0$ .*

PROOF. Let  $\bar{C}$  denote the inverse limit of  $\{C_i \mid i \in \mathbb{N}\}$ . Then  $\bar{C}$  is naturally a subspace of  $Z$ . Evidently  $\bar{C}$  contains  $C$  while by definition, the fibration  $C \rightarrow C_1$  is surjective. Since  $\bar{C} \rightarrow C_1$  is a homotopy equivalence by Corollary 1.2.2,  $\bar{C}$  cannot intersect path components of  $Z$  others than  $C$ . Consequently  $C = \bar{C}$ , and (i) follows.

For (ii), let  $\eta$  be a limit ordinal in  $W(\alpha)$ . Let  $\mu$  be the largest limit ordinal smaller than  $\eta$  (or 0 if  $\eta = \omega$ ). Then  $\eta = \mu + \omega$ . By restrictedness of  $\mathbf{Z}$ , the natural map

$$Z_\eta \rightarrow \lim_{\lambda < \eta} Z_\lambda = \lim_{i \in \mathbb{N}} Z_{\mu+i}$$

is a homeomorphism. By (i), also  $C_\eta \rightarrow \lim_{i \in \mathbb{N}} C_{\mu+i} = \lim_{\lambda < \eta} C_\lambda$  is a homeomorphism. Hence  $\{C_\lambda\}$  is a restricted system. The rest follows from Corollary 1.3.5.  $\square$

We come to the two main results of the section.

**THEOREM 1.3.9.** *Let  $\mathbf{Z}$  be a restricted inverse system of fibrations indexed by a complete lattice  $\mathcal{X}$  with a set of finite elements  $\mathcal{K}$ . Assume that  $\mathcal{X}$  is uncountable with respect to  $\mathcal{K}$ . Let  $Z$  denote the inverse limit space of  $\mathbf{Z}$  and let  $C$  be a set of path components of  $Z$ . For  $L \in \mathcal{X}$  let  $C_L$  denote the image of  $C$  under  $Z \rightarrow Z_L$ . Let  $\Gamma$  be a topological space which maps to the system  $\{C_L \mid L\}$ .*

*Let  $\mathcal{L}$  be the set of those  $L \in \mathcal{X}$  for which  $C_L$  has CW type and the map  $\Gamma \rightarrow C_L$  is a weak equivalence.*

*Let  $\mathcal{X}(\eta)$  (respectively  $\mathcal{L}(\eta)$ ) denote the subset of  $\mathcal{X}$  (respectively  $\mathcal{L}$ ) of elements of  $\mathcal{K}$ -cardinality at most  $\aleph_\eta$ .*

*If  $\mathcal{L}(0)$  is cofinal in  $\mathcal{X}(0)$ , then  $\mathcal{L}(\eta)$  is cofinal in  $\mathcal{X}(\eta)$ , for any ordinal  $\eta$ . In particular,  $C$  has CW homotopy type, and  $\Gamma \rightarrow C$  is a weak equivalence.*

**REMARK.** The role of  $\Gamma$  mapping into  $\{C_L \mid L\}$  is the implication

(\*)  $L', L'' \in \mathcal{L}, L' \leq L'' \implies C_{L''} \rightarrow C_{L'}$  is a homotopy equivalence,

by Whitehead's theorem.

We will make use of the following observation.

(\*\*)

Let  $\eta$  be a limit ordinal and let  $f: W(\eta) \rightarrow \mathcal{L}$  be an order preserving function. Then  $L = \sup \{f(\lambda) \mid \lambda < \eta\} \in \mathcal{L}$ .

**PROOF OF (\*\*).** Write  $\eta = \mu + \omega$  where  $\mu$  is the largest limit ordinal smaller than  $\eta$ . For  $i < \omega$  set  $L_i = f(\mu + i)$ . Then  $L_0 \leq L_1 \leq \dots$  is an increasing sequence of elements of  $\mathcal{X}$  with supremum  $L$ . It follows from (\*) that for each  $i$ , the map  $C_{L_i} \rightarrow C_{L_{i-1}}$  is a homotopy equivalence. By (i) of Lemma 1.3.8,  $C_L$  is the limit of  $\{C_{L_i} \mid i\}$ , and  $C_L \rightarrow C_{L_0}$  is an equivalence. Thus  $L \in \mathcal{L}$ .  $\square$

**PROOF OF THE THEOREM.** Assume that the  $\mathcal{K}$ -cardinality of  $X = \sup \mathcal{X}$  is strictly smaller than  $\aleph_\xi$  where  $\xi$  is an ordinal. Let  $S$  be the set of those  $\lambda \in W(\xi)$  for which  $\mathcal{L}(\lambda)$  is cofinal in  $\mathcal{X}(\lambda)$ . By hypothesis  $0 \in S$ .

Assume that  $W(\eta) \subset S$  and let  $L$  have  $\mathcal{K}$ -cardinality  $\aleph_\eta$ . Apply Lemma 1.3.6 to the complete lattice  $\{M \in \mathcal{X} \mid M \leq L\}$  with set of finite elements  $\mathcal{K} \cap L$  to obtain a good filtration  $\{L_\lambda \mid \lambda \in W(\alpha)\}$  for  $L$ . In particular,  $\alpha$  is an initial ordinal of cardinality  $\aleph_\eta$ , and for each  $\lambda < \alpha$  the element  $L_\lambda$  has  $\mathcal{K}$ -cardinality at most  $\text{card } W(\lambda)$ .

Note that  $\mathcal{L}(\eta)$  is not empty and that it is partially ordered by set inclusion.

Let  $T$  be the set of those pairs  $(\tau, f)$  where  $\tau < \alpha$  and  $f: W(\tau) \rightarrow \mathcal{L}(\eta)$  is an order-preserving smooth function with  $f(\lambda) \geq L_\lambda$  for all  $\lambda$ . Note that  $T$  is not empty. Order  $T$  by setting

$$(\tau, f) \leq (\tau', f') \iff \tau \leq \tau' \text{ and } f'|_{W(\tau)} = f.$$

Let  $A$  be a maximal totally ordered subset of  $T$ . The natural projection  $A \rightarrow W(\alpha)$  is injective, and hence  $A$  is well-ordered. Denote the image of  $A$  in  $W(\alpha)$  by  $A'$ . Maximality of  $A$  implies that  $A'$  is an initial segment of  $W(\alpha)$ .

Assume  $A' \neq W(\alpha)$ . Then  $A' = W(\mu)$  for some  $\mu \in W(\alpha)$ . In particular,  $\text{card } W(\mu) < \aleph_\eta$  and the following implication holds

$$(\dagger) \quad (\forall \lambda < \mu: \text{card}_{\mathcal{K}} M_\lambda < \aleph_\eta) \implies \text{card}_{\mathcal{K}} (\sup \{M_\lambda \mid \lambda < \mu\}) < \aleph_\eta.$$

Assume first that  $\mu$  has a predecessor  $\mu - 1$ . Then  $(\mu - 1, f) \in T$  for some  $f: W(\mu - 1) \rightarrow \mathcal{L}(\eta)$ . If  $\mu - 1$  is limiting then set  $L_{\mu-1} = \sup \{L_\lambda \mid \lambda < \mu - 1\}$ . Define  $F: W(\mu) \rightarrow \mathcal{L}(\eta)$  by letting  $F|_{W(\mu-1)} = f$  and  $F(\mu - 1) = \sup \{f(\lambda) \mid \lambda < \mu - 1\}$ . Trivially  $F(\mu - 1) \geq L_{\mu-1}$  while  $F(\mu - 1) \in \mathcal{L}(\eta)$  by  $(\dagger)$  and  $(**)$ .

If  $\mu - 1$  has a predecessor  $\mu - 2$  then the element  $f(\mu - 2) \cup L_{\mu-1}$  has  $\mathcal{K}$ -cardinality strictly less than  $\aleph_\eta$ , thereby yielding a bigger element  $F(\mu - 1) \in \mathcal{L}(\eta)$ , by ‘inductive hypothesis’. Adjoining  $(\mu, F)$  to  $A$  contradicts maximality.

Assume then that  $\mu$  is itself limiting and index  $A = \{(\tau, f_\tau)\}$ . Since  $\cup_{\tau < \mu} W(\tau) = W(\mu)$ , we may define  $(\mu, F) \in T$  by  $F|_{W(\tau)} = f_\tau$ . Adjoining  $(\mu, F)$  to  $A$  again contradicts maximality hence  $A' = W(\alpha)$ .

The same procedure as above furnishes a smooth function  $F: W(\alpha) \rightarrow \mathcal{L}$ . Set  $L' = \sup \{F(\lambda) \mid \lambda < \alpha\}$ . The  $\mathcal{K}$ -cardinality of  $L'$  is at most  $\text{card } W(\alpha) = \aleph_\eta$ . Moreover,  $L'$  is bigger than  $L$  since  $\sup \{L_\lambda \mid \lambda < \alpha\} = L$ . By  $(**)$  the element  $L'$  belongs to  $\mathcal{L}$ , and, consequently,  $\eta \in S$ .

Transfinite induction guarantees  $S = W(\xi)$  and applying the definition of  $S$  to  $X$  concludes the proof.  $\square$

**LEMMA 1.3.10.** *Let  $p: E \rightarrow B$  be a fibration where  $E$  is a Dold space. Suppose  $p(e_0) = b_0$  and let  $F$  be the fibre of  $p$  over  $b_0$ . Then the induced map  $\Omega p: \Omega(E, e_0) \rightarrow \Omega(B, b_0)$  is a homotopy equivalence if and only if  $F$  is contractible.*

**PROOF.** The proof is contained in the proof of the ‘delooping theorem’ of Allaud [1].  $\square$



DEFINITION. Let  $\mathcal{X}$  be a complete lattice with set of finite elements  $\mathcal{K}$ . A property  $P = P(\alpha, \beta)$  of pairs  $(\alpha, \beta) \in \mathcal{K} \times \mathcal{K}$  is an *order preserving property* if for any  $\alpha', \alpha, \beta, \beta' \in \mathcal{K}$  the implication

$$\alpha' \leq \alpha, \beta \leq \beta', P(\alpha, \beta) \implies P(\alpha', \beta').$$

(For example  $\leq$  is an order preserving property.)

If, in addition, for each  $\alpha \in \mathcal{K}$  there exists  $\beta \in \mathcal{K}$  with  $\beta \geq \alpha$  and  $P(\alpha, \beta)$  then  $P$  is a *directed order preserving property*.

We say that an ascending sequence

$$\lambda_1 \leq \lambda_2 \leq \dots$$

of elements of  $\mathcal{K}$  is a  $P$ -sequence if  $P(\lambda_i, \lambda_{i+1})$  holds for all  $i \geq 1$ .

The proof of the following lemma is trivial.

LEMMA 1.3.11. *Let  $\mathcal{X}$  be a complete lattice with set of finite elements  $\mathcal{K}$ . Suppose  $P$  is a directed order preserving property of ordered pairs. Then for any ascending sequence*

$$\kappa_1 \leq \kappa_2 \leq \dots$$

*of elements of  $\mathcal{K}$  there exists a  $P$ -sequence*

$$\lambda_1 \leq \lambda_2 \leq \dots$$

*of elements of  $\mathcal{K}$  which dominates  $\{\kappa_i\}$  in the sense that  $\lambda_i \geq \kappa_i$ .  $\square$*

THEOREM 1.3.12. *Let  $\mathbf{Z}$  be a restricted inverse system of fibrations indexed by a complete lattice  $\mathcal{X}$  with set of finite elements  $\mathcal{K}$ , and let  $Z$  denote the limit space of  $\mathbf{Z}$ . Let  $\zeta \in Z$ .*

- (i) *If the path component of  $\zeta$  has CW homotopy type then for every  $\mathcal{K}$ -countable  $\kappa_\infty$  there exists a bigger  $\mathcal{K}$ -countable  $\lambda_\infty$  such that*

$$(\star) \quad \Omega(Z, \zeta) \rightarrow \Omega(Z_{\lambda_\infty}, \zeta|_{\lambda_\infty})$$

*is a homotopy equivalence. Consequently the fibre of  $C \rightarrow C_{\lambda_\infty}$  is contractible.*

*In addition, if  $P$  is a directed order preserving property of ordered pairs in  $\mathcal{K}$ , then  $\lambda_\infty$  may be chosen to be the supremum of a  $P$ -sequence.*

- (ii) *Conversely, if for each  $\mathcal{K}$ -countable  $\kappa_\infty$  there exists a bigger  $\mathcal{K}$ -countable element  $\lambda_\infty$  such that  $\Omega(Z_{\lambda_\infty}, \zeta|_{\lambda_\infty})$  has CW type and the map  $(\star)$  is a weak homotopy equivalence, then  $\Omega(Z, \zeta)$  has CW homotopy type.*

PROOF. Assume the path component  $C$  containing  $\zeta$  has CW homotopy type. As usual, let  $C_\mu$  denote the image of  $C$  under  $Z \rightarrow Z_\mu$ .

Assume  $\kappa_\infty = \sup\{\kappa_1, \kappa_2, \dots\}$  where  $\kappa_i \in \mathcal{K}$ . By Proposition 1.1.2 there exists  $\kappa_0 \in \mathcal{K}$  such that for each  $\mu \in \mathcal{X}$  with  $\mu \geq \kappa_0$  the fibre  $F_\mu$  of  $C \rightarrow C_\mu$  over  $\zeta_\mu$  contracts in  $C$ .

For  $\alpha, \beta \in \mathcal{K}$  we define  $Q(\alpha, \beta)$  if  $\alpha \leq \beta$ ,  $P(\alpha, \beta)$ , and the inclusion  $F_\beta \rightarrow F_\alpha$  is nullhomotopic. Clearly  $Q$  is an order preserving property. By assumption on  $P$  and by Proposition 1.1.2 the property  $Q$  is also directed.

We apply Lemma 1.3.11 to obtain a  $Q$ -sequence  $\{\lambda_i\}$  dominating the sequence  $\{\kappa_i \vee \kappa_0\}$ . In particular  $P(\lambda_i, \lambda_{i+1})$  for all  $i \geq 1$  and  $\lambda_i \geq \kappa_0$  for all  $i$ . Let  $\lambda_\infty = \sup\{\lambda_i\}$ . By domination  $\lambda_\infty \geq \kappa_\infty$ .

To continue we write  $i$  instead of  $\lambda_i$  for  $1 \leq i \leq \infty$  to ease the notation. Proposition 1.1.2 further guarantees commutative diagrams

$$\begin{array}{ccc} \Omega(Z_i, \zeta_i) & \xrightarrow{p_{i-1, i\#}} & \Omega(Z_{i-1}, \zeta_{i-1}) \\ \simeq \uparrow & & \uparrow \simeq \\ F_i \times \Omega(Z, \zeta) & \xrightarrow{\text{inclusion}} & F_{i-1} \times \Omega(Z, \zeta) \end{array}$$

where the vertical homotopy equivalences are given by

$$F_i \times \Omega(Z, \zeta) \rightarrow \Omega(Z_i, \zeta_i), \quad (z, \gamma) \mapsto p_{i, \alpha\#}(\gamma) * \theta_i(z),$$

and the maps  $\theta_i$  satisfy  $p_{i-1, i\#} \circ \theta_i = \theta_{i-1} \circ \text{inclusion}$ .

Set  $\Phi_1 = F_1$  and let  $\dots \rightarrow \Phi_3 \rightarrow \Phi_2 \rightarrow \Phi_1$  be the inverse sequence obtained by changing  $\dots \rightarrow F_2 \rightarrow F_1$  inductively into a sequence of fibrations. Thus we obtain (strictly) commutative diagrams

$$\begin{array}{ccc} F_i & \longrightarrow & F_{i-1} \\ f_i \downarrow \simeq & & \downarrow \simeq f_{i-1} \\ \Phi_i & \xrightarrow{q_{i-1, i}} & \Phi_{i-1} \end{array}$$

where the  $f_i : F_i \rightarrow \Phi_i$  are homotopy equivalences. Pick homotopy inverses  $g_i$  for  $f_i$ . Since the maps  $p_{i-1, i\#}$  are fibrations, we may homotope the composites  $\theta_i \circ g_i$  inductively to maps  $\varphi_i : \Phi_i \rightarrow \Omega(Z_i, \zeta_i)$  for which

$$p_{i-1, i\#} \circ \varphi_i = \varphi_{i-1} \circ q_{i-1, i}.$$

By defining  $\psi_i: \Phi_i \times \Omega(Z, \zeta) \rightarrow \Omega(Z_i, \zeta_i)$  as  $\psi_i(x, \gamma) = p_{i, \alpha\#}(\gamma) * \varphi_i(x)$ , we get a (strict) map between towers of fibrations

$$\begin{array}{ccc} \Omega(Z_i, \zeta_i) & \xrightarrow{p_{i-1, i\#}} & \Omega(Z_{i-1}, \zeta_{i-1}) \\ \psi_i \uparrow \simeq & & \simeq \uparrow \psi_{i-1} \\ \Phi_i \times \Omega(Z, \zeta) & \xrightarrow{\text{inclusion}} & \Phi_{i-1} \times \Omega(Z, \zeta) \end{array}$$

consisting of homotopy equivalences. By Proposition 1.2.1 the limit map  $\psi_\infty$  is also a homotopy equivalence.

The limit space of  $\{\Phi_i \times \Omega(Z, \zeta)\}$  is  $\Phi_\infty \times \Omega(Z, \zeta)$  where  $\Phi_\infty$  is the limit space of  $\{\Phi_i\}$ . If we let  $\varphi_\infty: \Phi_\infty \rightarrow \Omega(Z_\infty, \zeta_\infty)$  denote the limit map of the sequence  $\{\varphi_i\}$ , then by universality,

$$(\star\star) \quad \psi_\infty(x, \gamma) = p_{\infty, \alpha\#}(\gamma) * \varphi_\infty(x).$$

Let  $x \in \Phi_\infty$ . By Proposition 1.2.4 the space  $\Phi_\infty$  is contractible, hence the inclusion

$$\iota: \Omega(Z, \zeta) \equiv \{x\} \times \Omega(Z, \zeta) \hookrightarrow \Phi_\infty \times \Omega(Z, \zeta)$$

is a homotopy equivalence. Therefore so is the map  $\psi_\infty \circ \iota$ . But by  $(\star\star)$ , the map  $\psi_\infty$  for a fixed  $x$  is the restriction  $\Omega(Z, \zeta) \rightarrow \Omega(Z_\infty, \zeta_\infty)$  followed by a multiplication homotopy equivalence. Hence the map  $(\star)$  is a homotopy equivalence, and an application of Lemma 1.3.10 concludes the proof of (i).

Statement (ii) follows immediately from Theorem 1.3.9, by setting  $\Gamma = \Omega(Z, \zeta)$ , and using Lemma 1.2.5.  $\square$

#### 1.4. Phantom components and the uniform Mittag-Leffler property

We introduce the notion of phantom path components of the limit space of an inverse system of fibrations between CW type spaces. If  $X$  is a CW complex then  $Y^X$  is the limit of  $\{Y^K\}$  with  $K$  ranging over the set  $\mathcal{K}$  of finite subcomplexes of  $X$ . Phantom path components of  $Y^X$  then coincide with path components of phantom maps  $X \rightarrow Y$  with respect to  $\mathcal{K}$  (see also Section 2.4 below).

**DEFINITION.** Let  $\Lambda$  be a directed set and let  $\{G_\lambda\}$  be an inverse system of groups indexed by  $\Lambda$ . For  $\lambda_1 \leq \lambda_2$  we denote the corresponding bonding morphism  $p_{\lambda_1 \lambda_2}: G_{\lambda_2} \rightarrow G_{\lambda_1}$ . We say that  $\{G_\lambda\}$  satisfies the Mittag-Leffler condition if for each  $\lambda \in \Lambda$  there exists  $\mu \in \Lambda$  with  $\mu \geq \lambda$  so that for each  $\nu \geq \mu$  the images of  $p_{\lambda \nu}$  and  $p_{\lambda \mu}$  coincide.

Recall the following well known result (see for example Mardešić and Segal [34] II, Theorem 7.1.1, and J. Cohen [6] for generalizations)

PROPOSITION 1.4.1. *Let  $(Z, \{P^i\})$  be the limit of inverse sequence of fibrations  $\cdots \rightarrow Z_3 \xrightarrow{p_3} Z_2 \xrightarrow{p_2} Z_1$ . For each number  $k \geq 0$  there exists a natural exact sequence (of pointed sets)*

$$* \rightarrow \lim^1 \pi_{k+1}(Z_i, *_i) \xrightarrow{\phi} \pi_k(Z, *) \xrightarrow{\lim \pi_k(P^i)} \lim \pi_k(Z_i, *_i) \rightarrow *.$$

In particular,  $\phi$  is an injection.  $\square$

We refer to [34], II, Theorems 6.2.10 and 6.2.11 for a proof of

PROPOSITION 1.4.2. *Let  $\{G_j, p_j\}$  be an inverse sequence of groups. Then  $\lim^1 G_j$  is trivial if the sequence satisfies the Mittag-Leffler condition. If the groups  $G_j$  are countable, the converse also holds.*  $\square$

DEFINITION. Let  $\mathbf{Z}$  be a restricted system of fibrations indexed by a complete lattice  $\mathcal{X}$  with set of finite elements  $\mathcal{K}$ . Let  $Z$  denote the inverse limit of  $\mathbf{Z}$ .

Path components  $C, D$  of  $Z$  form a phantom pair with respect to  $\mathcal{K}$  if the images of  $C$  and  $D$  under  $Z \rightarrow Z_\kappa$  coincide for each  $\kappa \in \mathcal{K}$ . They form a nontrivial phantom pair if  $C \neq D$ .

Clearly the relation ‘phantom pair’ is an equivalence relation, and we denote by  $\text{Ph}(C)$  the equivalence class of  $C$ , i.e. the set of all path components  $D$  such that  $C, D$  form a phantom pair.

LEMMA 1.4.3. *Assume that  $\mathcal{X}$  is  $\mathcal{K}$ -countable, and let  $\sup \mathcal{X}$  equal  $\sup \{\kappa_i\}$  where  $\kappa_1 \leq \kappa_2 \leq \dots$  is a countable ascending chain in  $\mathcal{K}$ . Clearly  $\{\kappa_i\}$  is cofinal in  $\mathcal{K}$  and by Proposition 1.4.1 we may identify*

$$\text{Ph}(C) = \lim^1 \pi_{k+1}(Z_{\kappa_i}, \zeta_i)$$

where  $\zeta$  belongs to  $C$  and  $\zeta_i = \zeta|_{\kappa_i}$ .  $\square$

THEOREM 1.4.4. *Given hypotheses of Proposition 1.1.1, assume that  $C$  has the homotopy type of a CW complex. Then, given any  $\kappa \in \mathcal{K}$  there exists  $\lambda' \in \mathcal{K}$  such that for all  $k \geq 1$  and any  $\mu \in \mathcal{X}$ , the image of*

$$\pi_k(C_\mu, \zeta_\mu) \rightarrow \pi_k(C_\kappa, \zeta_\kappa)$$

equals that of

$$(*) \quad \pi_k(C, \zeta) \rightarrow \pi_k(C_\kappa, \zeta_\kappa).$$

Moreover, for all large enough  $\kappa$ , the morphism  $(*)$  is injective for  $k \geq 1$ .

In particular, the  $\mathcal{K}$ -indexed induced inverse system of groups

$$(**) \quad \{\pi_k(C_\lambda, \zeta_\lambda)\}$$

satisfies the Mittag-Leffler condition uniformly with respect to  $k \geq 1$ .

If  $C$  is open then for all large enough  $\kappa \in \mathcal{K}$  the preimage of  $C_\kappa$  under  $Z \rightarrow Z_\kappa$  equals  $C$ . Consequently  $\text{Ph}(C) = \{C\}$ .

If  $\mathcal{X}$  is  $\mathcal{K}$ -countable then  $\text{Ph}(C) = \{C\}$  regardless of whether  $C$  is open or not.

PROOF. The first statement follows from (ii) of Proposition 1.1.2 by applying homotopy groups to the diagram of (i) of the same proposition.

We may first choose  $\kappa' \in \mathcal{K}$  such that the fibre of  $C \rightarrow C_{\kappa'}$  over  $\zeta_{\kappa'}$  contracts in  $C$ . Then if  $\kappa \geq \kappa'$  the morphism  $(*)$  is an injection.

If  $\mathcal{X}$  is  $\mathcal{K}$ -countable then by Lemma 1.4.3 we may identify  $\text{Ph}(C)$  with  $\lim^1$  of a subsequence of  $(**)$  which is trivial by Proposition 1.4.2.

If  $C$  is open then by Proposition 1.1.1 we may assume that the fibre of  $Z \rightarrow Z_{\kappa'}$  over  $\zeta_{\kappa'}$  contracts in the total space. By applying  $\pi_0$  this shows that the preimage of  $C_{\kappa'}$  under  $Z \rightarrow Z_{\kappa'}$  is exactly  $C$ .  $\square$

### 1.5. Sequences of Postnikov sections

Let  $\cdots \rightarrow Y_i \xrightarrow{p_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1$  be an inverse sequence of fibrations between CW type spaces. We do not know whether in general the question of CW type of the limit space  $Y_\infty$  can be reduced to studying the morphisms  $\pi_k(Y_i) \rightarrow \pi_k(Y_{i-1})$  induced on homotopy groups. However, this is the case when all the  $Y_i$  have trivial homotopy groups above a certain level (independent of  $i$ ).

While Proposition 1.5.7, the key ingredient of Theorem 1.5.1, is more general in nature, we need the vanishing of homotopy groups to ‘reach an end’ of an inductive if and only if argument.

DEFINITION. Let  $\cdots \rightarrow G_3 \rightarrow G_2 \rightarrow G_1$  be an inverse sequence of groups with inverse limit  $G_\infty$ . We say that  $\{G_i\}$  is *injectively Mittag-Leffler* if it satisfies the Mittag-Leffler condition and the canonical projections  $G_\infty \rightarrow G_i$  are injective for all but finitely many  $i$ .

Let  $\mathfrak{K}$  denote the ‘ $k$ -ification functor’, that is the functor which to every topological space assigns the space with the same underlying set whose topology is the compactly generated refinement of the original one. (See Steenrod [60].)

THEOREM 1.5.1. *Let  $\cdots \rightarrow Y_i \xrightarrow{p_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1$  be an inverse sequence of fibrations with a coherent set of nondegenerate base points  $\{\eta_i\}$ . Let  $Y_\infty$  be the limit space and let  $C$  be the path component of  $\eta_\infty = \{\eta_i\}$  in  $Y_\infty$ . Further let  $C_i$  be the image of  $C$  under  $Y_\infty \rightarrow Y_i$ . Assume that all spaces  $Y_i$  have CW homotopy type and that there exists a number  $N$  so that  $\pi_k(Y_i, \eta_i) = 0$  for  $k \geq N + 1$  and all  $i$ .*

- (i) *If  $C$  has the homotopy type of a CW complex then for each  $k \geq 1$  the sequence  $\{\pi_k(Y_i, \eta_i) \mid i\}$  is injectively Mittag-Leffler.*

If, in addition,  $C$  is open, then the preimage of  $C_i$  under  $Y_\infty \rightarrow Y_i$  equals  $C$  for all but finitely many  $i$ .

- (ii) Conversely, if for each  $k \geq 1$  the sequence  $\{\pi_k(Y_i, \eta_i) \mid i\}$  is injectively Mittag-Leffler, then  $\mathfrak{K}(C)$  has CW homotopy type. If, in addition,  $\pi_k(Y_i, \eta_i)$  is countable for  $k \geq 2$  and all  $i$  then  $C$  has CW type.

If, in addition, the preimage of  $C_i$  under  $Y_\infty \rightarrow Y_i$  equals  $C$  then  $C$  is open in  $Y_\infty$ .

**COROLLARY 1.5.2.** Let  $\cdots \rightarrow Y_i \xrightarrow{p_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1$  be an inverse sequence of fibrations with a coherent set of nondegenerate base points  $\{\eta_i\}$ . Assume that all spaces  $Y_i$  have homotopy types of countable CW complexes and that there exists a number  $N$  so that  $\pi_k(Y_i, \eta_i) = 0$  for  $k \geq N + 1$  and all  $i$ .

Then the inverse limit of  $\{Y_i\}$  is contractible if and only if it is weakly contractible.

**PROOF OF COROLLARY.** Weak contractibility implies the vanishing of  $\lim^1$  terms by Proposition 1.4.1. By Proposition 1.4.2 the sequences  $\{\pi_k(Y_i, *) \mid i\}$  satisfy the Mittag-Leffler property.  $\square$

**EXAMPLE 2.** Consider the inverse sequence of circles

$$\cdots \rightarrow S^1 \xrightarrow{p_2} S^1 \xrightarrow{p_1} S^1.$$

where  $p_i$  denotes the  $n_i$ -sheeted cover  $\zeta \mapsto \zeta^{n_i}$  for a number  $n_i$ , and we understand  $S^1$  as the set of complex numbers of absolute value 1.

The induced sequence on  $\pi_1$  is  $\cdots \rightarrow \mathbb{Z} \xrightarrow{n_2} \mathbb{Z} \xrightarrow{n_1} \mathbb{Z}$ . This has the Mittag-Leffler property if and only if either

- $n_i = \pm 1$  for all but finitely many  $i$  in which case the limit is  $S^1$ ,  
or
- $n_i = 0$  for all but finitely many  $i$  in which case the limit is a point.

In particular, the  $p$ -adic solenoid  $T_p$  for a prime  $p$  obtained by setting  $n_i = p$  for all  $i$ , has  $\hat{\mathbb{Z}}_p/\mathbb{Z}$  path components all of which form a single phantom class, and each is weakly contractible but not contractible.  $\square$

The rest of this section is devoted to the proof of Theorem 1.5.1.

**PROPOSITION 1.5.3.** Let  $k$  be a natural number and let

$$\cdots \rightarrow K_i \xrightarrow{p_i} K_{i-1} \rightarrow \cdots \rightarrow K_1$$

be an inverse sequence of fibrations where the  $K_i$  are Eilenberg-MacLane spaces with the single nonvanishing homotopy group in dimension  $k$ .

Let  $K_\infty$  denote the inverse limit of the sequence, together with natural projections  $P^i: K_\infty \rightarrow K_i$ . Pick a base point  $\zeta = \{\zeta_i\} \in K_\infty$  and let

$G_i = \pi_k(K_i, \zeta_i)$ . Denote the induced morphism  $G_i \rightarrow G_{i-1}$  simply by  $p_i$  and let  $G_\infty$  denote the inverse limit of  $\{G_i, p_i\}$ .

Assume that the sequence  $\{G_i, p_i\}$  is injectively Mittag-Leffler, and, in case  $k = 1$ , that the point  $\zeta_i$  is nondegenerate in  $K_i$ , for each  $i$ .

Then the limit space  $K_\infty$  has the homotopy type of  $K(G_\infty, k)$ . In particular, it has the homotopy type of a CW complex.

PROOF. Note that since  $\pi_{k+1}(K_i, \zeta_i) = 0$ ,  $\pi_k(K_\infty, \zeta) \cong G_\infty$ , by Proposition 1.4.1. By the same proposition, the space  $K_\infty$  has a single nonvanishing homotopy group, since the sequence  $\{G_i, p_i\}$  has the Mittag-Leffler property.

More precisely, the Mittag-Leffler property implies that the images of  $G_j \rightarrow G_i$  stabilize for large enough  $j$ . By replacing the sequence  $\{G_i\}$  with an appropriate subsequence we may assume that for each  $i$  the image  $S_{i-1}$  of  $G_j \rightarrow G_{i-1}$  equals that of  $G_j \rightarrow G_{i-1}$  for all  $j > i$  (including  $j = \infty$ ).

In addition, we may assume that for each  $i$ , the morphism  $G_\infty \rightarrow G_i$  is injective. This implies that the morphisms  $G_\infty \rightarrow S_i$  are bijective and consequently so are  $S_i \rightarrow S_{i-1}$ , for all  $i$ .

Now we treat cases  $k \geq 2$  and  $k = 1$  separately.

Assume first that  $k \geq 2$ . The composite  $G_\infty \rightarrow G_2 \rightarrow G_1$  is then an isomorphism onto  $S_1$ . Since the groups are abelian, there is a short exact sequence  $0 \rightarrow G_\infty \rightarrow G_2 \rightarrow S_1 \rightarrow 0$  which splits. By possibly neglecting the first term we may therefore assume that all injections  $G_\infty \rightarrow G_i$  are split. Let  $K_0$  be a CW complex of type  $K(G_\infty, k)$  and let  $p_1: K_1 \rightarrow K_0$  be a map inducing a ‘projector’  $G_1 \rightarrow G_\infty$  on the homotopy group. Split  $p_1$  canonically as  $K_1 \xrightarrow{f_1} L_1 \xrightarrow{q_1} K_0$  where  $f_1$  is a homotopy equivalence and  $q_1$  is a fibration. Given  $f_i: K_i \rightarrow L_i$  split  $f_i p_{i+1}: K_{i+1} \rightarrow L_i$  as  $K_{i+1} \xrightarrow{f_{i+1}} L_{i+1} \xrightarrow{q_{i+1}} L_i$  with  $f_{i+1}$  a homotopy equivalence and  $q_{i+1}$  a fibration. Let  $f_\infty: K_\infty \rightarrow L_\infty$  denote the induced homotopy equivalence on the limit spaces. We note that the projection  $L_\infty \rightarrow L_1$  is a fibration and consequently so is the composite map  $L_\infty \rightarrow K_0$ . Clearly the fibre of  $L_\infty \rightarrow K_0$  over  $b_0 \in K_0$  is the inverse limit of the fibres  $F_i$  of  $L_i \rightarrow K_0$  over  $b_0$ . But the induced maps  $F_i \rightarrow F_{i-1}$  are nullhomotopic, and hence the limit  $F_\infty$  is contractible. Therefore  $L_\infty \simeq K_0$ , as claimed.

Now assume that  $k = 1$ . For each  $i$  choose a CW complex  $C_i$  having the homotopy type of  $K_i$ . Next choose a base point  $\gamma_i \in C_i$ . Since the base points  $\gamma_i \in C_i$  and  $\zeta_i \in K_i$  are nondegenerate,  $(K_i, \zeta_i)$  and  $(C_i, \gamma_i)$  are equivalent as pairs; that is there exist pointed maps  $g_i: K_i \rightarrow C_i$ ,  $f_i: C_i \rightarrow K_i$  such that  $g_i \circ f_i$  and  $f_i \circ g_i$  are homotopic to the respective identity maps via base point preserving homotopies.

Set  $\bar{S}_i = g_{i\#}(S_i)$  and let  $\bar{C}_i \rightarrow C_i$  be a covering associated to the subgroup  $\bar{S}_i \leq \pi_1(C_i, \gamma_i)$ . Pulling back this covering over  $g_i: K_i \rightarrow C_i$  yields a fibration  $q_i: W_i \rightarrow K_i$  with discrete fibres, which maps  $\pi_1(W_i, w_i)$  isomorphically onto  $S_i$ .

By construction the composite  $C_{i+1} \xrightarrow{f_{i+1}} K_{i+1} \rightarrow K_i$  lifts to a map into  $W_i$  (sending  $\gamma_{i+1}$  to  $w_i$ ). Precomposition with  $g_{i+1}$  defines a map  $K_{i+1} \rightarrow W_i$  which lifts  $K_{i+1} \rightarrow K_i$  up to homotopy. Since  $W_i \rightarrow K_i$  is a fibration, a strict lifting  $r_{i+1}$  can be obtained. Since  $\zeta_{i+1}$  is nondegenerate in  $K_{i+1}$ , we may arrange that  $r_{i+1}(\zeta_{i+1}) = w_i$ .

Note that since the fibres of  $q_i$  are discrete, the map  $r_{i+1}$  is a fibration. The limit space is thus homeomorphic to the limit space of

$$\cdots \rightarrow W_3 \rightarrow W_2 \rightarrow W_1$$

which is a sequence of fibrations that are homotopy equivalences.  $\square$

We recall a definition due to J. Cohen [7] in a slightly modified version that is suitable for our purposes.

**DEFINITION.** Let  $\Phi: G \times E \rightarrow E$  denote a free left action of a topological group  $G$  on a space  $E$ . The action  $\Phi$  is *open* if for every space  $Z$  and every pair of maps  $f_1, f_2: Z \rightarrow E$  such that  $f_1(z)$  and  $f_2(z)$  lie in the same orbit for each  $z$  there exists a map  $f: Z \rightarrow G$  so that

$$\Phi(f(z), f_1(z)) = f_2(z), \quad \forall z.$$

A fibration  $p: E \rightarrow B$  is called a *principal open fibration* if  $E$  is a free left  $G$ -space with an open action such that  $p(e_1) = p(e_2)$  if and only if  $e_1$  and  $e_2$  lie in the same orbit. (This is to say that  $p$  induces a continuous bijection  $E/G \rightarrow B$  which we do not require to be a homeomorphism.)

For example, if  $G$  is a closed subgroup of a topological group  $H$ , then the action of  $G$  on  $H$  by multiplication on the left is an open action.

The proofs of the following two lemmas are straightforward.

**LEMMA 1.5.4.** (i) *Let  $p: E \rightarrow B$  be a morphism of topological groups and let  $G := \ker p$ . Let  $G$  act on the left on  $E$ . If  $p$  is a fibration then it is a principal open fibration.*

(ii) *Let  $p: E \rightarrow B$  be a principal open fibration and let  $f: X \rightarrow B$  be a map. Then the pullback fibration  $f^*E \rightarrow X$  is a principal open fibration.*  $\square$



LEMMA 1.5.5. *Let  $E'$  be a free  $G'$ -space with orbit space  $B'$  and let  $E$  be a free space with orbit space  $B$ . Consider a commutative diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\eta} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{\beta} & B \end{array}$$

where  $p'$  and  $p$  are principal open fibrations and  $\eta$  is an equivariant map in the sense that  $\eta(g \cdot x) = \omega(g) \cdot \eta(x)$  for some (continuous) morphism  $\omega: G' \rightarrow G$ . If  $\omega$  and  $\beta$  are fibrations, so is  $\eta$ . (Actually it is enough to assume that only the action on  $E$  is open.)  $\square$

The following proposition is due to J. Cohen; see [7], Theorem 1.1.

PROPOSITION 1.5.6 (J. Cohen). *Let  $\{p_i\}: \{E_i\} \rightarrow \{B_i\}$  be a level preserving morphism of inverse sequences where for each  $i$ , the space  $E_i$  is a free left  $G_i$ -space and  $p_i: E_i \rightarrow B_i$  is a principal open fibration. Assume that the maps  $E_i \rightarrow E_{i-1}$  are equivariant given by homomorphisms  $\omega_i: G_i \rightarrow G_{i-1}$ .*

*Let  $p_\infty: E_\infty \rightarrow B_\infty$  denote the induced inverse limit map, and set  $G_\infty = \lim G_i$ .*

*If the  $\omega_i$  are fibrations then  $p_\infty$  is a principal open fibration.*  $\square$

PROPOSITION 1.5.7. *Let  $\cdots \rightarrow Z_i \xrightarrow{p_i} Z_{i-1} \rightarrow \cdots \rightarrow Z_1$  be an inverse sequence of fibrations with a coherent set of base points  $\{\zeta_i\}$ . Assume that the  $Z_i$  are  $(k-1)$ -connected spaces of CW type where  $k \geq 1$  and denote  $G_i = \pi_k(Z_i, \zeta_i)$ . Further let  $G_\infty$  denote the limit group of the induced sequence  $\cdots \rightarrow G_i \rightarrow G_{i-1} \rightarrow \cdots \rightarrow G_1$ .*

*If  $k = 1$ , assume also that  $\zeta_i$  is a nondegenerate base point of  $Z_i$  for each  $i$ .*

*There exists a morphism of towers*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bar{Z}_3 & \longrightarrow & \bar{Z}_2 & \longrightarrow & \bar{Z}_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & Z_3 & \longrightarrow & Z_2 & \longrightarrow & Z_1 \end{array}$$

where all arrows are fibrations and the arrows  $\bar{Z}_i \rightarrow Z_i$  are  $k$ -connected covers. The limit map  $\bar{Z}_\infty \rightarrow Z_\infty$  fits into a pullback square

$$\begin{array}{ccc} \bar{Z}_\infty & \longrightarrow & P_\infty \\ \downarrow & & \downarrow \\ Z_\infty & \longrightarrow & K_\infty \end{array}$$

where  $P_\infty$  is contractible and the vertical arrows are always fibrations for the class of compactly generated spaces.

If the sequence  $\{G_i\}$  satisfies the Mittag-Leffler condition and the canonical projections  $G_\infty \rightarrow G_i$  are injective, then the space  $K_\infty$  has CW homotopy type. Hence  $\mathfrak{K}(Z_\infty)$  has CW homotopy type if and only if  $\mathfrak{K}(\bar{Z}_\infty)$  has.

If  $k = 1$  or the groups  $G_i$  are countable, then the arrows  $\bar{Z}_\infty \rightarrow Z_\infty$  and  $P_\infty \rightarrow K_\infty$  are fibrations and the space  $Z_\infty$  has CW homotopy type if and only if  $\bar{Z}_\infty$  has.

PROOF. Assume that  $k \geq 2$ . First inductively construct an inverse sequence

$$\cdots \rightarrow K(G_i, k) \xrightarrow{\alpha_i} K(G_{i-1}, k) \rightarrow \cdots \rightarrow K(G_1, k)$$

of fibration homomorphisms between topological abelian groups which on the nontrivial homotopy group agree with the induced maps  $\pi_k(Z_i, \zeta_i) \rightarrow \pi_k(Z_{i-1}, \zeta_{i-1})$ . This can be done as follows. Using infinite symmetric products (see Dold and Thom [9]) or geometric realization of simplicial groups (see Milnor [42]) it is possible to construct an inverse sequence  $\cdots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1$  with  $A_i \rightarrow A_{i-1}$  a morphism of topological abelian groups for all  $i$ . Since the space of free paths on a topological group is canonically equipped with a compatible topological group structure, the factorization of  $A_2 \rightarrow A_1$  as a composite of homotopy equivalence  $A_2 \rightarrow A'_2$  followed by fibration  $A'_2 \rightarrow A_1$  is one of morphisms of topological abelian groups. Hence  $A_3 \rightarrow (A_2 \rightarrow A'_2)$  is a morphism of topological groups and we may proceed inductively.

In case the groups  $G_i$  are not countable, the above construction yields an inverse sequence of morphism-fibrations of topological quasi-groups (see Appendix B).

By applying the functorial contractible path-space ‘covering’ construct a commutative ladder

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & K(G_3, k) & \longrightarrow & K(G_2, k) & \longrightarrow & K(G_1, k)
 \end{array}$$

where all arrows are both fibrations and morphisms of topological abelian groups.

Pick maps  $Z_i \rightarrow K(G_i, k)$  corresponding to an isomorphism on the homotopy group  $\pi_k$ . Using homotopy lifting property to change  $Z_i \rightarrow K(G_i, k)$  for homotopic maps inductively construct a commutative ladder

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & Z_3 & \longrightarrow & Z_2 & \longrightarrow & Z_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & K(G_3, k) & \longrightarrow & K(G_2, k) & \longrightarrow & K(G_1, k)
 \end{array}$$

Finally form an inverse sequence of pullback squares

$$\begin{array}{ccccccc}
 & & P_\infty & \longrightarrow & \cdots & \longrightarrow & P_i & \longrightarrow & P_{i-1} \\
 \bar{Z}_\infty & \nearrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & K_\infty & \longrightarrow & \cdots & \longrightarrow & K(G_i, k) & \longrightarrow & K(G_{i-1}, k) \\
 \bar{Z}_\infty & \longrightarrow & \cdots & \longrightarrow & \bar{Z}_i & \longrightarrow & \bar{Z}_{i-1} & \longrightarrow & \bar{Z}_{i-2} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Z_\infty & \longrightarrow & \cdots & \longrightarrow & Z_i & \longrightarrow & Z_{i-1} & \longrightarrow & Z_{i-2}
 \end{array}$$

Since  $P_i \rightarrow K(G_i, k)$  is a fibration morphism, it is a principal open fibration. Hence so is the pullback map  $\bar{Z}_i \rightarrow Z_i$  by Lemma 1.5.4. Further note that the restriction of  $P_i \rightarrow P_{i-1}$  to the fibre is trivially a fibration morphism. Hence the induced equivariant map  $\bar{Z}_i \rightarrow \bar{Z}_{i-1}$  is also a fibration by Lemma 1.5.5. The morphism  $P_\infty \rightarrow K_\infty$  is the limit of principal open fibrations, and is itself such by Proposition 1.5.6. Moreover, the square

$$\begin{array}{ccc}
 \bar{Z}_\infty & \longrightarrow & P_\infty \\
 \downarrow & & \downarrow \\
 Z_\infty & \longrightarrow & K_\infty
 \end{array}$$

is a pullback by construction. Hence  $\bar{Z}_\infty \rightarrow Z_\infty$  is a principal open fibration by Lemma 1.5.4.

In case the groups  $G_i$  are not countable, the actions are only compactly open (see Proposition B.1.6), and the limit maps under consideration have the homotopy lifting property with respect to compactly generated spaces (see Corollary B.1.7).

If the inverse sequence of groups  $\{G_i\}$  satisfies the hypotheses of the proposition,  $K_\infty$  has the homotopy type of a  $K(G_\infty, k)$  by Proposition 1.5.3. In particular, it has CW homotopy type.

Hence  $Z_\infty$  has CW homotopy type if and only if  $\bar{Z}_\infty$  has.

If the groups  $G_i$  are not countable then after  $\mathfrak{K}$ -ification, the square remains a pullback, and the vertical arrows are fibrations in the category of compactly generated Hausdorff spaces. Stasheff's theorem remains valid in that category (see for example P. Kahn [29]), hence in this case  $\mathfrak{K}(Z_\infty)$  has CW type if and only if  $\mathfrak{K}(\bar{Z}_\infty)$  has.

In case  $k = 1$ , first inductively construct a based inverse sequence of fibrations

$$\cdots \rightarrow (K_i, \kappa_i) \xrightarrow{\alpha_i} (K_{i-1}, \kappa_{i-1}) \rightarrow \cdots \rightarrow (K_1, \kappa_1)$$

where  $\kappa_i$  is nondegenerate in  $K_i$ , the pair  $(K_i, \kappa_i)$  has the type of a  $(K(G_i, 1), *)$ , and  $\alpha_{i\#}: \pi_1(K_i, \kappa_i) \rightarrow \pi_1(K_{i-1}, \kappa_{i-1})$  realizes the morphism  $G_i \rightarrow G_{i-1}$ .

For each  $i$ , let  $q_i: P_i \rightarrow K_i$  denote the fibration with discrete fibre, obtained by pulling back a universal covering  $E_i \rightarrow K(G_i, 1)$  over the homotopy equivalence  $K_i \rightarrow K(G_i, 1)$ . Since the  $P_i$  are contractible, any map  $P_i \rightarrow P_{i-1}$  is a lifting of  $K_i \rightarrow K_{i-1}$  up to homotopy, and since  $P_i \rightarrow K_i$  is a fibration, a strict lifting may be obtained. The composite  $P_i \rightarrow P_{i-1} \rightarrow K_{i-1}$  is a fibration since it equals  $P_i \rightarrow K_i \rightarrow K_{i-1}$  and since  $P_{i-1} \rightarrow K_{i-1}$  has discrete fibres, also  $P_i \rightarrow P_{i-1}$  is a fibration.

We use well-pointedness of the  $Z_i$  to inductively construct a level-preserving map of inverse sequences  $\{Z_i\}$  to  $\{K_i\}$ . Form  $\bar{Z}_i$  as the pullback of  $Z_i \rightarrow K_i \leftarrow P_i$  as before. Since  $P_i \rightarrow K_i$  and  $\bar{Z}_i \rightarrow Z_i$  are fibrations with discrete fibres, the limit maps  $P_\infty \rightarrow K_\infty$  and  $\bar{Z}_\infty \rightarrow Z_\infty$  are always fibrations.  $\square$

PROOF OF THEOREM 1.5.1. Statement (i) is clear by Theorem 1.4.4.

As for (ii), let  $C_\infty$  be the limit space of  $\{C_i\}$ . Since  $\{\pi_1(C_i, \eta_i) = \pi_1(Y_i, \eta_i)\}$  satisfies the Mittag-Leffler property,  $C_\infty$  is path-connected by Proposition 1.4.1. Evidently  $C \subset C_\infty$  and thus  $C = C_\infty$  since  $C$  is a path component. By inductive application of Proposition 1.5.7 it follows that

$\mathfrak{K}(C)$  has CW type if and only if  $\mathfrak{K}(Z)$  has where  $Z$  is the limit space of  $N$ -connected covers of  $C_i$ . By Proposition 1.2.4 the space  $Z$  and consequently  $\mathfrak{K}(Z)$  are contractible. Hence  $\mathfrak{K}(C)$  has CW type.  $\square$

## CHAPTER 2

### CW homotopy type of function spaces

In this chapter we discuss various properties of function spaces of CW homotopy type. We give some necessary conditions, and some sufficient conditions.

First we show that certain spaces of continuous functions between CW complexes fit into the framework of Chapter 1. In particular,  $n$ -ad function spaces  $(Y; Y_1, Y_2, \dots, Y_n)^{(X; A_1, A_2, \dots, A_n)}$  are suitable for study.

Then we give our main ‘topological’ characterizations of CW type function spaces. For CW complexes  $X$  and  $Y$ , the function space  $Y^X$  has CW type if and only if  $Y^X$  admits a numerable covering of open sets contractible within  $Y^X$ . In particular, if  $X$  is countable then  $Y^X$  has CW type if and only if it is semilocally contractible. Hence function spaces between CW complexes carry so much structure that local contractibility properties are sufficient.

Next,  $Y^X$  has CW type if and only if  $X$  is essentially countable with respect to  $Y$ . More precisely, if  $Y^X$  has CW type then for every  $g: X \rightarrow Y$  and every countable subcomplex  $L$  of  $Y$  there exists a bigger countable subcomplex  $L'$  such that the fibre over  $g|_{L'}$  of the fibration  $Y^X \rightarrow Y^{L'}$  is contractible and consequently  $\Omega(Y^X, g) \rightarrow \Omega(Y^{L'}, g|_{L'})$  is a homotopy equivalence.

Conversely, if for each countable subcomplex  $L$  there exists a bigger countable subcomplex  $L'$  such that  $Y^X \rightarrow Y^{L'}$  is a weak equivalence onto image of CW type, then  $Y^X$  has CW type.

The results of Section 1.1 (see also Section 1.4) impose severe restrictions on homotopy groups  $\pi_k(Y^X, g)$ , where  $k \geq 1$ , for  $Y^X$  of CW type. Below we show that there are also restrictions on the set of path components. In particular, path components must be small in number, and there can be no phantom path components.

For  $Y$  a CW complex with finitely many nontrivial homotopy groups, the question of CW type of  $Y^X$  depends only on the induced morphisms  $\pi_k(Y^M) \rightarrow \pi_k(Y^L)$  for  $L \leq M \leq X$ , and  $k \geq 0$ , and we give necessary and sufficient conditions.

At the end of this chapter we present three types of ‘constructions’ of CW type function spaces, which we apply in later chapters.

### 2.1. Function spaces as inverse limits

Let  $(X; A_1, \dots, A_n)$  be a CW  $n$ -ad and let  $(Y; B_1, \dots, B_n)$  have the homotopy type of a CW  $n$ -ad. Our general aim is to study the homotopy type of  $n$ -ad function spaces  $(Y; B_1, \dots, B_n)^{(X; A_1, \dots, A_n)}$ .

Let  $\mathcal{S}_X$  be the set of all subcomplexes of  $X$ , and let  $\mathcal{A}_X$  be the direct sum  $\bigoplus_{i=1}^{\infty} \mathcal{S}_X$  whose elements are sequences  $\mathbb{A} = (A_1, A_2, A_3, \dots)$  of subcomplexes of  $X$  with  $A_j = \emptyset$  for all but finitely many  $j$ . Let  $\mathcal{B}_Y$  denote the subset of  $\bigoplus_{i=1}^{\infty} 2^Y$  consisting of those sequences  $\mathbb{B} = (B_1, B_2, \dots)$  of subsets of  $Y$  where  $(Y; B_1, \dots, B_j)$  has the homotopy type of a CW  $j$ -ad for each  $j$ .

Let  $K$  be a compact CW complex and let  $\mathcal{S}_K$  be the set of subcomplexes of  $K$ . Then  $X \times K$  is a CW complex and the cartesian product defines a function  $\mathcal{S}_X \times \mathcal{S}_K \rightarrow \mathcal{S}_{X \times K}$ .

We define obvious operations  $\mathcal{A}_X \times \mathcal{S}_X \xrightarrow{\cap} \mathcal{A}_X$ ,  $\mathcal{A}_X \times \mathcal{S}_K \xrightarrow{\times} \mathcal{A}_{X \times K}$ , and  $\mathcal{A}_X \times \mathcal{A}_X \xrightarrow{\cup} \mathcal{A}_X$ .

Let  $T$  be a subcomplex of  $X$  and let  $\gamma: T \rightarrow Y$  be a map. For  $\mathbb{A} \in \mathcal{A}_X$  and  $\mathbb{B} \in \mathcal{B}_Y$  let  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  denote the ‘ $n$ -ad’ function space, and  $(Y; \mathbb{B})^{(X; \mathbb{A})}[T, \gamma]$  its subspace

$$\{f: X \rightarrow Y \mid f(A_i) \subset B_i \ \forall i, \ f|_T = \gamma\}.$$

LEMMA 2.1.1. *Let  $L$  be a subcomplex of  $X$ . The restriction map*

$$(Y; \mathbb{B})^{(X; \mathbb{A})}[T, \gamma] \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}[T \cap L, \gamma|_{T \cap L}]$$

*is a fibration.*

PROOF. We may assume  $T \leq L$ , since  $(Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}[T \cap L, \gamma|_{T \cap L}]$  is homeomorphic to  $(Y; \mathbb{B})^{(L'; \mathbb{A} \cap L')}[T, \gamma]$  where  $L' = L \cup T$ . Then  $(Y; \mathbb{B})^{(X; \mathbb{A})}[T, \gamma]$  is the preimage of  $(Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}[T, \gamma]$  under the restriction map  $(Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$ . Thus it suffices to show that the latter is a fibration.

For a compactly generated space  $M$  and a compactum  $C$  the product  $M \times C$  is compactly generated and  $(Y^M)^C$  is homeomorphic to  $Y^{M \times C}$ . In particular, this holds for a CW complex  $M$ . Thus  $[(Y; \mathbb{B})^{(X; \mathbb{A})}]^I$  is homeomorphic to  $(Y; \mathbb{B})^{(X \times I, \mathbb{A} \times I)}$  and the pullback of  $(Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)} \leftarrow [(Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}]^I$  is homeomorphic to  $(Y; \mathbb{B})^{(X \times 0 \cup L \times I, \mathbb{A} \times 0 \cup (\mathbb{A} \cap L) \times I)}$ .

Let  $\rho: X \times I \rightarrow X \times 0 \cup L \times I$  be a retraction such that  $\rho(A_i \times I) \subset A_i \times 0 \cup (A_i \cap L) \times I$  for all  $i$ . Composition with  $\rho$  defines a lifting function

$$\lambda: (Y; \mathbb{B})^{(X \times 0 \cup L \times I, \mathbb{A} \times 0 \cup (\mathbb{A} \cap L) \times I)} \rightarrow (Y; \mathbb{B})^{(X \times I, \mathbb{A} \times I)}$$

for  $(Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$ .  $\square$

REMARK. Note that Lemma 2.1.1 remains valid if we assume that the entries  $A_i$  of  $\mathbb{A}$  are subsets of a space  $X$  such that all possible intersections  $\cap_i A_i$  are closed cofibered subsets of  $X$ .

If  $(Y; \mathbb{B})^{(X; \mathbb{A})}[T, \gamma]$  is non-empty then it contains a map  $\Gamma: X \rightarrow Y$ . Therefore it makes sense to define

$$\mathcal{F}(X, Y) = \{(Y; \mathbb{B})^{(X; \mathbb{A})}[T, \Gamma|_T] \mid \mathbb{A} \in \mathcal{A}_X, \mathbb{B} \in \mathcal{B}_Y, T \in \mathcal{S}_X, \\ \Gamma \in (Y; \mathbb{B})^{(X; \mathbb{A})}\}.$$

For  $Z = (Y; \mathbb{B})^{(X; \mathbb{A})}[T, \Gamma|_T] \in \mathcal{F}(X, Y)$  define

$$Z(L) = (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}[T \cap L, \Gamma|_{T \cap L}].$$

LEMMA 2.1.2. *Let  $Z \in \mathcal{F}(X, Y)$ .*

- (i) *For arbitrary subcomplexes  $L \leq L' \leq X$ , the restriction map  $Z(L') \rightarrow Z(L)$  is a fibration. The fibre of  $Z \rightarrow Z(L)$  is an element of  $\mathcal{F}(X, Y)$ .*
- (ii) *Let  $M \leq X$  and let  $\mathcal{L}$  be an exhaustive directed system of proper subcomplexes of  $M$ . Then  $Z(M)$ , together with restriction maps  $Z(M) \rightarrow Z(L)$ , is the limit of the inverse system  $\{Z(L) \mid L \in \mathcal{L}\}$ .*
- (iii) *For  $K$  a finite subcomplex of  $X$ , the space  $Z(K)$  has CW homotopy type.*

PROOF. Statement (i) is contained in Lemma 2.1.1.

To prove (ii), note that directedness of  $\mathcal{L}$  guarantees that every compact subset of  $M$  is already contained in a member of  $\mathcal{L}$ . Thus (ii) is an easy consequence of the fact that CW complexes have the weak topology with respect to the family of closed cells.

Let  $Z = (Y; \mathbb{B})^{(X; \mathbb{A})}[T, \Gamma|_T]$ . Then  $Z(K) = (Y; \mathbb{B})^{(K; \mathbb{A} \cap K)}[T \cap K, \Gamma|_{T \cap K}]$  is the fibre of  $(Y; \mathbb{B})^{(K; \mathbb{A} \cap K)} \rightarrow (Y; \mathbb{B})^{(T \cap K; \mathbb{A} \cap T \cap K)}$  over  $\Gamma|_{T \cap K}$ . The total space and the base space have CW homotopy types by Theorem 3 of [43] hence so has the fibre  $Z(K)$  by Stasheff's theorem.  $\square$

We have established

PROPOSITION 2.1.3. *Let  $Z \in \mathcal{F}(X, Y)$ , and let  $\mathcal{X}$  denote the set of all subcomplexes of  $X$ . Then the functor  $\mathcal{X} \rightarrow \text{Top}$ ,*

$$L \mapsto Z(L), \quad (L \leq M) \mapsto (Z(M) \xrightarrow{\text{restriction}} Z(L))$$

*is a restricted inverse system of fibrations indexed by the complete lattice  $\mathcal{X}$ . The sublattice  $\mathcal{K}$  of  $\mathcal{X}$  consisting of finite subcomplexes is a set of finite elements and  $Z(K)$  has CW homotopy type for each  $K \in \mathcal{K}$ .  $\square$*



One of the nice features of  $\mathcal{F}(X, Y)$  is that it is ‘closed under forming loop spaces.’ More precisely, let  $g \in Z = (Y; \mathcal{B})^{(X; \mathcal{A})}[T, \Gamma|_T]$ . Then  $\Omega(Z, g)$  is homeomorphic to

$$(Y; \mathcal{B})^{(X \times I; \mathcal{A} \times I)}[T \times I \cup X \times \partial I, \Gamma|_T \circ \text{pr}_T \sqcup g \circ \text{pr}_X].$$

In this way we may view  $\Omega(Z, g) \in \mathcal{F}(X \times I, Y)$ , and the restriction  $\Omega(Z, g) \rightarrow \Omega(Z(L), g|_L)$  can be identified with  $\Omega(Z, g) \rightarrow \Omega(Z, g)(L \times I)$ .

Also note that in this way  $\Omega(Z, g)$  may naturally be viewed as the limit of a restricted inverse system of fibrations indexed by  $\mathcal{X}$ .

CONVENTION. If  $X$  is a CW complex then under cardinality of  $X$  we understand the cardinality of the set of cells  $X$ .

Lemma 1.3.6 implies

LEMMA 2.1.4. *Let  $X$  be an infinite CW complex, and let  $\mathcal{X}$  be the set of all subcomplexes of  $X$ . For an initial ordinal  $\alpha$  of cardinality  $\text{card } X$  there exists an order-preserving injection  $W(\alpha) \rightarrow \mathcal{X}$ ,  $\lambda \mapsto X_\lambda$ , whose image is well-ordered and exhaustive in the sense that  $\cup_{\lambda < \alpha} X_\lambda = X$ . For each  $\mu \in W(\alpha)$  the quotient complex  $X_{\mu+1}/X_\mu$  is finite and, if  $\mu$  is limiting,  $X_\mu = \cup_{\lambda < \mu} X_\lambda$ . Moreover, for each  $\mu < \alpha + 1$ , the subcomplex  $X_\mu$  has  $\text{card } W(\mu)$  cells.  $\square$*

REMARK. If necessary, we may replace each ‘successive pair’  $(X_{\mu+1}, X_\mu)$  by a finite refinement  $X_\mu = X_0 < X_1 < \dots < X_r = X_{\mu+1}$  where for each  $i$ , the pair  $(X_i, X_{i-1})$  is the adjunction of a single cell.  $\square$

DEFINITION. Let  $X$  be a CW complex and let  $X_\lambda$ ,  $\lambda \in W(\alpha)$ , be a filtration of subcomplexes of  $X$ . We say that  $\{X_\lambda | \lambda\}$  is a good filtration if it is exhaustive in the sense that  $\cup_\lambda X_\lambda = X$ , the assignment  $\lambda \rightarrow X_\lambda$  is order-preserving, and for each limiting  $\mu < \alpha$  the subcomplex  $X_\mu$  is given by  $X_\mu = \cup_{\lambda < \mu} X_\lambda$ .  $\square$

Lemma 2.1.4 asserts that every complex  $X$  admits a good filtration indexed by  $W(\alpha)$  where  $\alpha$  is a limit ordinal such that consecutive steps are adjunctions of finitely many cells (or a single one, if desired) and for each  $\mu < \alpha + 1$ , the subcomplex  $X_\mu$  has exactly  $\text{card } W(\mu)$  cells.

The ‘goodness’ of a good filtration is apparent from

LEMMA 2.1.5. *Let  $\alpha$  be an ordinal number and let  $\{X_\lambda | \lambda < \alpha\}$  be a good filtration for  $X$ . Let  $Y$  be any CW complex. We may form an inverse system indexed by  $W(\alpha)$  consisting of spaces  $Y^{X_\lambda}$  where for each  $\lambda < \mu$  the bonding map  $Y^{X_\mu} \rightarrow Y^{X_\lambda}$  is the restriction fibration. Since  $\cup_\lambda X_\lambda = X$ , the limit of this system is  $Y^X$ , together with restriction projections.*

Moreover, if  $\mu$  is a limit ordinal,  $\mu < \alpha$ , then

$$Y^{X_\mu} = \lim_{\lambda < \mu} Y^{X_\lambda},$$

which is to say that  $\{Y^{X_\lambda} \mid \lambda \in W(\alpha)\}$  is a restricted inverse system.  $\square$

We close this section with two simple-minded examples of function spaces that are not of CW type.

LEMMA 2.1.6 (Splittings of domain). *Assume  $X = \bigvee_{\lambda \in \Lambda} X_\lambda$ . Let  $C$  (respectively  $C_\lambda$ ) denote the path component of the constant map in  $(Y, *)^{(X, *)}$  (respectively  $(Y, *)^{(X_\lambda, *)}$ ). If  $C$  has CW type, then  $C_\lambda$  is contractible for all but finitely many  $\lambda$ .*

PROOF. Since  $(Y, *)^{(X, *)}$  (respectively  $C$ ) is homeomorphic to  $\prod_{\lambda} (Y, *)^{(X_\lambda, *)}$  (respectively  $\prod_{\lambda} C_\lambda$ ), this follows immediately from Example 1 on page 4.  $\square$

EXAMPLE 3 (Infinitely many “free” cells). *Assume that  $Y^T$  has CW type and let  $(X, T)$  be an adjunction of cells  $e_\lambda^{m_\lambda}$  where  $m_\lambda$  denotes the dimension, and  $\lambda$  ranges over an infinite index set. If for each  $\lambda$  there exists a number  $n_\lambda \geq m_\lambda$  with  $\pi_{n_\lambda}(Y)$  nontrivial, then  $Y^X$  cannot have CW type.*

If  $Y^X$  has CW type then so has the space  $(Y, *)^{(X/T, \{T\})}$ , by Stasheff’s theorem. But  $X/T = \bigvee_{\lambda} S^{m_\lambda}$ , and  $\pi_k((Y, *)^{(S^{m_\lambda, *}), *}) \cong \pi_{k+m_\lambda}(Y, *)$ , by adjunction. The assumptions on  $Y$  contradict Lemma 2.1.6.

For an explicit example we may take  $X = M(\mathbb{F}, m)$  and  $Y = K(\mathbb{Z}_2, n)$  where  $m \leq n$ . Here  $M(\mathbb{F}, n)$  denotes a Moore space of type  $(\mathbb{F}, n)$ , and  $\mathbb{F}$  is the free abelian group on countably many generators.

The case  $m = n$  generalizes Milnor’s example  $X = \mathbb{N}$ ,  $Y = \{0, 1\}$  (see [43]) if we understand  $\mathbb{N}$  as a  $M(\mathbb{F}, 0)$  and  $\{0, 1\}$  as a  $K(\mathbb{Z}_2, 0)$ . It was pointed out by Milnor that  $(\{0, 1\}, *)^{(\mathbb{N}, *)}$  is homeomorphic to the Cantor set. We note that  $(K(\mathbb{Z}_2, n), *)^{(M(\mathbb{F}, n), *)}$  is homeomorphic to  $(\Omega^n K(\mathbb{Z}_2, n), *)^{(\mathbb{N}, *)}$  which in turn is homotopy equivalent to the Cantor set.  $\square$

EXAMPLE 4 (A stable splitting). *Let  $Y$  be a CW complex with infinitely many nontrivial homotopy groups. (By a theorem of Serre [52], see also Neisendorfer [46],  $Y$  may be a finite CW complex which is simply connected but not contractible.) Then the path component  $C$  of the constant map in  $(Y, *)^{(\Omega S^{p+1}, *)}$  does not have CW type for any  $p \geq 1$ .*

Since  $\Omega S^{p+1}$  is homotopy equivalent to the James construction  $J(S^p)$  (see James [28], and also Puppe [49]), we may take  $X = J(S^p)$ . If  $C$  has CW type, so has the loop space  $\Omega(C, *) \approx (Y, *)^{(SX, *)}$ . It is well known that  $SX = SJ(S^p)$  splits as  $SX \simeq \bigvee_{j=1}^{\infty} S^{pj+1}$ . By adjunction the group

$\pi_k((Y, *)^{(S^{pj+1}, *)}, *)$  is isomorphic to  $\pi_{k+pj+1}(Y, *)$ . The latter is nontrivial for infinitely many  $k$ , contradicting Lemma 2.1.6.  $\square$

## 2.2. Properties of function spaces of CW type

Proposition 1.1.1 says

**THEOREM 2.2.1.** *Let  $X$  and  $Y$  be CW complexes and  $Z \in \mathcal{F}(X, Y)$ . Let  $C$  be a set of path components of  $Z$ , and for each subcomplex  $L$  of  $X$  let  $C_L$  denote the image of  $C$  under  $Z \rightarrow Z(L)$ .*

- (i)  *$C$  is semilocally contractible if and only if for each map  $g \in C$  there exists a finite subcomplex  $K$  of  $X$  such that the fibre  $F_K$  over  $g|_K$  of  $C \rightarrow C_K$  contracts in the total space.*
- (ii)  *$C$  is semilocally contractible and open if and only if for each map  $g \in C$  there exists a finite subcomplex  $K$  such that the fibre of  $Z \rightarrow Z(K)$  over  $g|_K$  contracts in the total space.*
- (iii) *If  $C$  is semilocally contractible then for each map  $g \in C$  the loop space  $\Omega(Z; g)$  has CW homotopy type.*
- (iv)  *$Z$  has CW homotopy type if and only if it is a Dold space.*  $\square$

**LEMMA 2.2.2.** *If  $Z \in \mathcal{F}(X, Y)$  where  $X$  is a countable complex, then  $Z$  is hereditarily paracompact (in fact stratifiable).*

**PROOF.** By Cauty [4], the function space  $Y^X$  is stratifiable, and hence hereditarily paracompact.  $\square$

**COROLLARY 2.2.3.** *Let  $Z \in \mathcal{F}(X, Y)$  where  $X$  is countable, and let  $C$  be a set of path components of  $Z$ . Then  $C$  has CW homotopy type if and only if it is semilocally contractible.*

*In particular, if  $C$  is globally well-pointed, it has CW homotopy type. Compare Theorem 3.1 of [56].*  $\square$

**LEMMA 2.2.4.** (i) *Let  $X$  be a connected CW complex and  $K$  a finite subcomplex of  $X$ . Then there exists a finite subcomplex  $L$  of  $X$  such that the inclusion induced morphism  $\tilde{H}_*(L) \rightarrow \tilde{H}_*(X)$  is injective on the image of  $\tilde{H}_*(K) \rightarrow \tilde{H}_*(L)$ .*

- (ii) *Let  $L_0 \leq L_1 \leq L_2 \leq \dots$  be an ascending sequence of finite subcomplexes of  $X$  with union  $L_\infty$ . If for each  $i \geq 1$  the morphism  $\tilde{H}_*(L_i) \rightarrow \tilde{H}_*(X)$  is injective on the image of  $\tilde{H}_*(L_{i-1}) \rightarrow \tilde{H}_*(L_i)$  then the morphism*

$$\tilde{H}_*(L_\infty) \rightarrow \tilde{H}_*(X)$$

*is injective.*

**PROOF.** For (i) see the proof of Lemma 4.1.3 of Gray and McGibbon [24], and (ii) is evident.  $\square$

Theorem 1.3.12 says

**THEOREM 2.2.5.** *Let  $Z \in \mathcal{F}(X, Y)$ , let  $C$  be a set of path components of  $Z$ , and let  $g \in C$  be a map. Let  $C_L$  denote the image of  $C$  under  $Z \rightarrow Z_L$ .*

- (i) *If  $C$  has CW homotopy type then for every countable complex  $L$  of  $X$  there exists a bigger countable complex  $L'$  such that  $\Omega(C_{L'}, g|_{L'})$  has CW type and*

$$(*) \quad \Omega(C; g) \rightarrow \Omega(C_{L'}, g|_{L'})$$

*is a homotopy equivalence. Consequently the fibre of  $C \rightarrow C_{L'}$  over  $g|_{L'}$  is contractible.*

*In addition, if  $P$  is a directed order preserving property defined on pairs  $(K, K')$  of finite subcomplexes with  $K \leq K'$  then  $L'$  may be assumed to be the union of a  $P$ -sequence.*

*In particular, we may assume  $H_*(L'; \mathbb{Z}) \rightarrow H_*(X; \mathbb{Z})$  to be injective.*

- (ii) *Conversely, if for each countable complex  $L$  there exists a bigger countable complex  $L'$  such that  $\Omega(C_{L'}, g|_{L'})$  has CW homotopy type and  $(*)$  is a weak homotopy equivalence then  $\Omega(C, g)$  has CW homotopy type.*

**PROOF.** We need only discuss the statement about  $H_*(L'; \mathbb{Z}) \rightarrow H_*(X; \mathbb{Z})$  of (i).

Let  $Q(K, K')$  if  $\tilde{H}_*(K') \rightarrow \tilde{H}_*(X)$  is injective on the image of  $\tilde{H}_*(K) \rightarrow \tilde{H}_*(K')$ . Then  $Q$  is a directed order preserving property by (i) of Lemma 2.2.4. Now apply (ii) of Lemma 2.2.4 together with Theorem 1.3.12.  $\square$

Theorem 1.3.9 translates into

**THEOREM 2.2.6.** *Let  $X$  be a CW complex with uncountably many cells, and let  $Y$  be an arbitrary CW complex. Let  $Z \in \mathcal{F}(X, Y)$  and let  $C$  be a set of path components of  $Z$ . For each subcomplex  $L$  of  $X$  denote by  $C_L$  the image of  $C$  under  $Z \rightarrow Z(L)$ .*

*Let  $\Gamma$  be a topological space and assume that there exists a map of  $\Gamma$  into the inverse system  $\{C_L | L\}$ .*

*Let  $\mathcal{L}$  be the set of those subcomplexes  $L$  of  $X$  for which  $C_L$  has CW homotopy type and  $\Gamma \rightarrow C_L$  is a weak equivalence.*

*Assume that for each countable subcomplex  $L$  there exists a countable subcomplex  $L' \in \mathcal{L}$  containing  $L$ .*

*Then for any ordinal  $\eta$  and every subcomplex  $L$  of  $X$  with  $\aleph_\eta$  cells there exists a larger subcomplex  $L' \in \mathcal{L}$  with  $\aleph_\eta$  cells.*

*In particular  $C$  has CW homotopy type, and  $\Gamma \rightarrow C$  is a weak equivalence.  $\square$*

COROLLARY 2.2.7. *Let  $X$  be a CW complex with uncountably many cells such that for each countable subcomplex  $L$  there exists a countable subcomplex  $L'$  containing  $L$  so that  $(Y, *)^{(L', *)}$  is contractible. Then  $(Y, *)^{(X, *)}$  is contractible.*

PROOF. Note that  $(Y, *)^{(L', *)}$  is contractible if and only if the section  $Y \rightarrow Y^{L'}$  is a homotopy equivalence. By Theorem 2.2.6 the space  $Y^X$  has CW type and  $Y \rightarrow Y^X$  is a (weak and hence genuine) homotopy equivalence.  $\square$

LEMMA 2.2.8. *Let  $X$  and  $Y$  be connected CW complexes. Assume that the path component  $C$  of  $g \in (Y, *)^{(X, *)}$  has CW homotopy type. Then there exists a finite subcomplex  $K$  of  $X$  such that for any  $L \geq K$  and every  $k \geq 2$  the sequence of groups and homomorphisms*

$$(*) \quad 0 \rightarrow \pi_k((Y, *)^{(X, *)}, g) \rightarrow \pi_k((Y, *)^{(L, *)}, g|_L) \rightarrow \pi_{k-1}(F_L, g) \rightarrow 0$$

*is exact. Here  $F_L$  denotes the fibre of  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  over  $g|_L$ . In addition, the morphism*

$$(**) \quad \pi_1((Y, *)^{(X, *)}, g) \rightarrow \pi_1((Y, *)^{(L, *)}, g|_L)$$

*is injective. If  $C$  is open in  $(Y, *)^{(X, *)}$  then for  $k = 1$ ,  $(*)$  is an exact sequence of pointed sets, and so is  $* \rightarrow [X, Y]_* \rightarrow [L, Y]_*$  at the base point  $g$ .*

*If  $g \equiv *$  then  $(*)$  transforms as  $0 \rightarrow [S^k \wedge X, Y]_* \rightarrow [S^k \wedge L, Y]_* \rightarrow [S^{k-1} \wedge (X/L), Y]_* \rightarrow 0$ , and similarly also  $(**)$ .*

PROOF. Follows from (i) and (ii) of Proposition 1.1.1 by applying the long homotopy exact sequence of fibration together with the observation that  $\pi_k(F_L, g) \cong \pi_k(F_L \cap C, g)$  for  $k \geq 1$ . For  $g$  constant apply adjunction  $\pi_k((Y, y_0)^{(L, x_0)}, \text{const}_{y_0}) \cong [S^k \wedge L, Y]_*$ .  $\square$

REMARK. If the function  $[X, Y]_* \rightarrow [L, Y]_*$  is naturally equivalent to a group morphism, the conclusion of Lemma 2.2.8 in case  $C$  is the path component of the constant map (and is open) implies it is an injection.

COROLLARY 2.2.9. *Let  $Y$  be an Eilenberg-MacLane space  $K(G, n)$ , and let  $X$  be an arbitrary CW complex. If the path component  $C$  of the constant map in  $(Y, *)^{(X, *)}$  has CW type then there exists a finite subcomplex  $K$  of  $X$  such that for any  $L \geq K$  and every  $k$  with  $1 \leq k \leq n - 1$  the inclusion induced morphism*

$$(*) \quad \tilde{H}^k(X; G) \rightarrow \tilde{H}^k(L; G)$$

*is injective. If, in addition,  $C$  is open, then  $(*)$  is also injective for  $k = n$ .*  $\square$

### 2.3. Counting path components

Here we focus on the restrictions that we may impose on path components of a function space of CW type, an issue which is delicate since the set of path components in general does not carry a group structure. We begin with the case when it does.

**PROPOSITION 2.3.1.** *Let  $X$  and  $Y$  be CW complexes such that the space  $Y^X$  has CW type. If either  $X$  is an  $H$ -cogroup or  $Y$  is an  $H$ -group, then for all large enough finite  $K$ , the functions*

$$\pi_0((Y, *)^{(X, *)}) \rightarrow \pi_0((Y, *)^{(K, *)}) \text{ and } \pi_0(Y^X) \rightarrow \pi_0(Y^K)$$

*are injective.*

**PROOF.** We prove the pointed version first. If  $Y$  is an  $H$ -group then the contravariant functor  $C(-, Y)_*, (L, *) \mapsto (Y, *)^{(L, *)}$  maps into the category of  $H$ -groups and  $H$ -morphisms, hence this case is an immediate consequence of Theorem 1.4.4.

Assume that  $X$  is an  $H$ -cogroup and denote comultiplication  $\nu: X \rightarrow X \vee X$  and coinverse  $\iota: X \rightarrow X$ . The path component  $C$  of the constant map in  $(Y, *)^{(X, *)}$  is open and has CW type hence by Theorem 1.4.4 there exists a finite subcomplex  $K$  of  $X$  such that the fibre of

$$(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$$

contracts in  $C$  for every  $L \geq K$ . In particular, if for some  $f: (X, *) \rightarrow (Y, *)$  and  $L \geq K$  the restriction  $f|_L$  is nullhomotopic, then also  $f$  is nullhomotopic.

The image  $\nu(K)$  is a compact subset of  $X \vee X$ , hence there exists a finite subcomplex  $L$  of  $X$  such that  $L \geq K$  and  $\nu(K) \subset L \vee L$ .

Let  $M$  be a finite subcomplex that contains  $L$ . Let  $f, g: (X, *) \rightarrow (Y, *)$  be such that  $f|_M$  and  $g|_M$  are homotopic. Then so also are  $f|_L, g|_L$ . Pick a (pointed) homotopy  $h: L \times I \rightarrow Y$ , and define  $H: K \times I \rightarrow Y$  as follows

$$(*) \quad K \times I \xrightarrow{\nu} (L \vee L) \times I = L \times I \sqcup_{* \times I} L \times I \xrightarrow{h \sqcup g \circ \iota \circ \text{pr}_L} Y \vee Y \xrightarrow{\nabla} Y.$$

Then  $H_0$  equals the composite  $K \rightarrow X \xrightarrow{\nu} X \vee X \xrightarrow{f \vee g \circ \iota} Y \vee Y \xrightarrow{\nabla} Y$ , and  $H_1$  equals the composite  $K \rightarrow X \xrightarrow{\nu} X \vee X \xrightarrow{g \vee g \circ \iota} Y \vee Y \xrightarrow{\nabla} Y$ . This is to say that  $H_0$  is the restriction of  $f - g$  to  $K$  and  $H_1$  is the restriction of  $g - g$  to  $K$ . Since  $g - g$  is nullhomotopic, so is  $H_1$ . Thus  $(f - g)|_K$  is nullhomotopic. But then  $f - g$  is also nullhomotopic which is to say that  $f$  and  $g$  are homotopic, using the  $H$ -cogroup structure. Thus  $\pi_0((Y, *)^{(X, *)}) \rightarrow \pi_0((Y, *)^{(M, *)})$  is injective for all  $M \geq L$ .

The homotopy ladder with exact rows which is induced by the morphism of evaluation fibrations  $(Y^X \rightarrow Y) \rightarrow (Y^M \rightarrow Y)$  implies that  $\pi_0(Y^X) \rightarrow \pi_0(Y^M)$  is injective. (See (i) of Lemma 2.3.3.)  $\square$

**DEFINITION.** A space  $Y$  has finite homotopy groups (respectively of cardinality at most  $\aleph_\eta$ ) if it has finitely many (respectively at most  $\aleph_\eta$ ) path components and the homotopy groups of each path component are finite (respectively at most  $\aleph_\eta$ ).

An  $n$ -ad  $(Y; \mathbb{B})$  has finite homotopy groups if  $Y$  and all possible intersections  $\cap_i B_i$  have finite homotopy groups.

Similarly, an  $n$ -ad has homotopy groups of cardinality at most  $\aleph_\eta$  if  $Y$  and all possible intersections have cardinality at most  $\aleph_\eta$ .

An  $n$ -ad  $(Y; \mathbb{B})$  is said to be an  $r$ -Postnikov section if  $Y$  and all possible intersections  $\cap_i B_i$  are  $r$ -Postnikov sections.

**PROPOSITION 2.3.2.** (i) *Let  $(L; \mathbb{A})$  be an  $n$ -ad with  $L$  a finite complex. Let  $(Y; \mathbb{B})$  have finite homotopy groups (respectively of cardinality  $\aleph_\eta$ ). Then  $(Y; \mathbb{B})^{(L; \mathbb{A})}$  has finite homotopy groups (respectively of cardinality  $\aleph_\eta$ ).*

(ii) *Let  $(Y; \mathbb{B})$  be an  $r$ -Postnikov  $n$ -ad. Let  $(L'; \mathbb{A})$  be an  $n$ -ad with  $L'$  arbitrary. Suppose  $(L', L)$  is an adjunction of one cell of dimension  $d$ . If  $d \geq r + 1$  then the restriction fibration*

$$(Y; \mathbb{B})^{(L'; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$$

*is a homotopy equivalence onto its image, and if  $d \geq r + 2$ , then it is also surjective.*

**LEMMA 2.3.3.** (i) *Let  $p: Y \rightarrow X$  be the principal fibration obtained by pulling back  $f: X \rightarrow B$  over the evaluation  $PB \rightarrow B$ . Assume  $b_0$  is the base point of  $B$ . Then there exists a left action of  $\pi_1(B, b_0)$  on  $\pi_0(Y)$  such that  $p_\#: \pi_0(Y) \rightarrow \pi_0(X)$  collapses precisely the orbits of this action. Moreover for  $g \in \pi_1(B, b_0)$  and  $[y] \in \pi_0(Y)$  we have  $g[y] = [y]$  if and only if there exists  $\tilde{g} \in \pi_1(X, p(y))$  such that  $f_\#(\tilde{g}) = g$ .*

(ii) *Let  $E \rightarrow B$  be a fibration. If  $E$  and  $B$  have finite (resp.  $\aleph_\eta$ ) homotopy groups, then so has each fibre. Conversely, if  $B$  and all fibres of  $E \rightarrow B$  have finite (resp.  $\aleph_\eta$ ) homotopy groups, then so has  $E$ .*  $\square$

**PROOF OF PROPOSITION 2.3.2.** Denote for convenience  $Y = B_0$ ,  $X = A_0$ . Suppose  $L = \{x_1, \dots, x_r\}$ , a finite discrete set. Let  $K_j$  denote the set of indices  $i$  for which  $x_j \in A_i$ . Then  $(Y; \mathbb{B})^{(L; \mathbb{A})} = \prod_{j=1}^r (\cap_{i \in K_j} B_i)$  which has finite ( $\aleph_\eta$ ) homotopy groups.

Thus we can make an induction on the number of cells of dimension  $\geq 1$ . In other words, it suffices to prove the following. If  $(L'; \mathbb{A})$  is an  $n$ -ad where  $L'$  is obtained from a subcomplex  $L$  by attaching a cell  $e$ , and  $(Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$  has finite  $(\aleph_\eta)$  homotopy groups, then so has  $(Y; \mathbb{B})^{(L'; \mathbb{A})}$ .

Let  $x_0$  be an appropriate point (not necessarily a 0-cell) in  $L'$ . Let  $J$  denote the set of  $j \geq 0$  for which  $x_0 \in A_j$ . We can consider the fibration (see the remark after Lemma 2.1.1)

$$(\dagger) \quad (Y; \mathbb{B})^{(L', \mathbb{A})} \xrightarrow{\varepsilon} (Y, \mathbb{B})^{\{\{x_0\}, \mathbb{A} \cap \{x_0\}\}} = \bigcap_{j \in J} B_j = B''.$$

By Lemma 2.3.3 it suffices to show that for any  $y_0 \in B''$  the fibre of  $\varepsilon$  over  $x_0$  has finite  $(\aleph_\eta)$  homotopy groups. The fibre of  $(\dagger)$  over  $y_0$  is the space  $(Y, y_0 \oplus \mathbb{B})^{(L', x_0 \oplus \mathbb{A})}$ .

Suppose  $x_0 \in L$ . By inductive assumption and Lemma 2.3.3 also the space  $(Y, y_0 \oplus \mathbb{B})^{(L, x_0 \oplus \mathbb{A} \cap L)}$  has finite  $(\aleph_\eta)$  homotopy groups.

Let  $K$  denote the set of indices  $i \geq 0$  for which  $e \subset A_i$ , and let  $A' = \bigcap_{i \in K} A_i$ . Let  $\varphi: S^{d-1} \rightarrow A' \cap L \leq L$  be the attaching map, and  $\Phi: B^d \rightarrow A' \leq L'$  the characteristic map for  $e$ . Pick a base point  $\zeta_0 \in S^{d-1}$  and set  $x_0 = \varphi(\zeta_0)$ . Note that since  $\bar{e} \subset A'$ , the set  $K$  is a subset of  $J$ . Thus  $y_0 \in \bigcap_{j \in J} B_j \subset \bigcap_{i \in K} B_i = B'$ .

Consider the following diagram.

(•)

$$\begin{array}{ccccc} (Y; y_0 \oplus \mathbb{B})^{(L', x_0 \oplus \mathbb{A})} & \xrightarrow{\text{restriction}} & (B', y_0)^{(A', x_0)} & \xrightarrow{\Phi^\#} & (B', y_0)^{(B^d, \zeta_0)} \\ \downarrow & & \downarrow & & \downarrow \\ (Y; y_0 \oplus \mathbb{B})^{(L, x_0 \oplus \mathbb{A} \cap L)} & \xrightarrow{\text{restriction}} & (B', y_0)^{(A' \cap L, x_0)} & \xrightarrow{\varphi^\#} & (B', y_0)^{(S^{d-1}, \zeta_0)} \end{array}$$

The left-hand square is a pullback square by construction while the right-hand square is a pullback induced by the pushout diagram of the adjunction of  $e$ . By ‘horizontal composition’, the large rectangle is also a pullback. This exhibits

$$(\ddagger) \quad (Y; y_0 \oplus \mathbb{B})^{(L', x_0 \oplus \mathbb{A})} \rightarrow (Y; y_0 \oplus \mathbb{B})^{(L, x_0 \oplus \mathbb{A} \cap L)}$$

as a principal fibration with fibre  $(B', y_0)^{(S^d, \zeta_0)}$ . By inductive hypothesis and another application of Lemma 2.3.3 we conclude the proof of (i).



Consider the following morphism of fibrations over  $B''$ .

$$\begin{array}{ccc} (Y; \mathbb{B})^{(L'; \mathbb{A})} & \longrightarrow & (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)} \\ & \searrow \varepsilon_{x_0} & \swarrow \varepsilon_{x_0} \\ & & B'' \end{array}$$

By Theorem 6.3 of Dold [8], the map  $(Y; \mathbb{B})^{(L'; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$  is a homotopy equivalence onto image if and only if for every  $y_0 \in B''$  the induced map of fibres over  $y_0$  is a homotopy equivalence onto image. The induced map of fibres is  $(\ddagger)$ .

Since  $d \geq r+1$  and  $B'$  is an  $r$ -Postnikov section, the space  $(B', y_0)^{(S^{d-1}, \zeta_0)}$  is homotopy discrete. In particular, the path component of the constant map is contractible, and the pullback rectangle of diagram  $(\bullet)$  implies that  $(\ddagger)$  is a homotopy equivalence onto image. If  $d \geq r+2$  then  $(B', y_0)^{(S^{d-1}, \zeta_0)}$  is contractible hence  $(\ddagger)$  is surjective. This concludes the proof of **(ii)**.  $\square$

The following is a very rough estimate which we do by simply counting.

**PROPOSITION 2.3.4.** *Let  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  have CW homotopy type. Assume that the homotopy groups of  $Y$  are of cardinality at most  $\aleph_\eta$ , and that  $X$  has  $\aleph_\xi$  cells. Then  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  has at most  $\aleph_\xi \times \aleph_\eta = \aleph_{\max\{\xi, \eta\}}$  path components.*

*In particular, if  $Y$  is a countable complex, then  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  has at most  $\aleph_\xi$  path components.*

**PROOF.** Let  $\mathcal{K}$  be the set of finite subcomplexes of  $X$ . Note that the cardinality of  $\mathcal{K}$  is also  $\aleph_\xi$ . By Theorem 1.4.4 there exists for each  $[g] \in [(X; \mathbb{A}), (Y; \mathbb{B})]$  an element  $K = K(g) \in \mathcal{K}$  such that the preimage of  $[g|_K]$  under  $[(X; \mathbb{A}), (Y; \mathbb{B})] \rightarrow [(K; \mathbb{A} \cap K), (Y; \mathbb{B})]$  is exactly  $[g]$ . By making a choice over all path components we define a function  $[(X; \mathbb{A}), (Y; \mathbb{B})] \rightarrow \mathcal{K}$ . Evidently for each  $K \in \mathcal{K}$  the function

$$\Phi^{-1}(K) \rightarrow [(K; \mathbb{A} \cap K), (Y; \mathbb{B})]$$

is an injection. By **(i)** of Proposition 2.3.2 the cardinality of  $\Phi^{-1}(K)$  is at most  $\aleph_\eta$ . Since  $[(X; \mathbb{A}), (Y; \mathbb{B})] = \coprod_{K \in \mathcal{K}} \Phi^{-1}(K)$ , the assertion follows.  $\square$

**DEFINITION.** For a topological space  $Z$  let  $\text{Compon}(Z)$  denote the quotient space of  $Z$  obtained by collapsing path components.

**LEMMA 2.3.5.** *Let  $\cdots \rightarrow Z_3 \xrightarrow{p_3} Z_2 \xrightarrow{p_2} Z_1$  be an inverse sequence of spaces of CW homotopy type, and let  $\zeta = \{\zeta_i\}$  be an element of the limit space  $Z_\infty$ . If  $\lim^1 \pi_1(Z_i, \zeta_i)$  vanishes then  $\text{Compon } Z_\infty$  is homeomorphic to  $\lim \text{Compon } Z_i$ .*

PROOF. For each  $j$  let  $\bar{p}_j: \text{Compon}(Z_j) \rightarrow \text{Compon}(Z_{j-1})$  denote the map induced by  $p_j: Z_j \rightarrow Z_{j-1}$ . By assumption on  $\lim^1$  and Proposition 1.4.1 the natural function  $\text{Compon}(Z) \rightarrow \lim_j \text{Compon}(Z_j)$  is a continuous bijection. We claim that it is a homeomorphism. Since the spaces  $Z_j$  have CW type, the spaces  $\text{Compon}(Z_j)$  are discrete and by a trivial induction we may construct left inverses  $\vartheta_j: \text{Compon}(Z_j) \rightarrow Z_j$  for the quotient maps  $q_j: Z_j \rightarrow \text{Compon}(Z_j)$  so that  $p_j \circ \vartheta_j = \vartheta_{j-1} \circ \bar{p}_j$  for all  $j$ . The maps  $\vartheta_j$  then furnish a continuous map  $\lim(\text{Compon } Z_j) \rightarrow Z$  which we compose with the quotient  $Z \rightarrow \text{Compon}(Z)$  to obtain the desired inverse  $\lim(\text{Compon } Z_j) \rightarrow \text{Compon}(Z)$ .  $\square$

PROPOSITION 2.3.6. *Let  $(X; \mathbb{A})$  be an  $n$ -ad where  $X$  is countable. Let  $(Y; \mathbb{B})$  be an  $n$ -ad with finite homotopy groups. If  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  has CW homotopy type, it has at most finitely many path components.*

PROOF. Take any filtration  $L_1 \leq L_2 \leq \dots$  of finite subcomplexes for  $X$  (i.e. a countable cofinal subset of  $\mathcal{K}$ ) and consider the associated inverse sequence  $Z_j = (Y; \mathbb{B})^{(L_j; \mathbb{A} \cap L_j)}$ . Denote also  $Z = (Y; \mathbb{B})^{(X; \mathbb{A})}$ . By Proposition 1.4.4 and Lemma 2.3.5  $\text{Compon } Z$  is homeomorphic to  $\lim \text{Compon } Z_j$ . Since the  $Z_j$  have CW homotopy types, the spaces  $\text{Compon } Z_j$  are discrete, and by (i) of Proposition 2.3.2 they are finite. Hence the limit space is compact. Since it is discrete by assumption, it must be finite.  $\square$

## 2.4. Phantom maps

If a path component of a function space has CW type and is *open* then it does not form a nontrivial phantom pair by Theorem 1.4.4. Indeed, Theorem 1.4.4 says much more but we will apply it in full strength in later chapters. Here we list a few classical examples of function spaces with phantom maps which are consequently examples of function spaces not of CW type. We make a brief introduction to phantom maps so as to identify them with our previously defined concept of phantom path components from section 1.4. The reader is invited to consult McGibbon's survey article [36] on phantom maps for a comprehensive treatment.

A (pointed) map  $f: X \rightarrow Y$  between CW complexes is called a phantom map if the restriction  $f|_K: K \rightarrow Y$  to any finite subcomplex  $K$  of  $X$  is nullhomotopic. This may be restated in the following homotopy invariant manner as (cf. Roitberg [50])

DEFINITION. A pointed map  $f: X \rightarrow Y$  between connected CW complexes is a *phantom map* with respect to the class  $\mathcal{K}$  of finite connected CW complexes if for any  $W \in \mathcal{K}$  and any map  $j: W \rightarrow X$  the composition  $W \xrightarrow{j} X \xrightarrow{f} Y$  is nullhomotopic.

This definition agrees with that studied by, for example, Meier [39] and Zabrodsky [65]. The other standard definition (see McGibbon [36]) takes instead of  $\mathcal{K}$  the class of all finite dimensional spaces.

For investigations involving the compact open topology, it is natural to consider phantom maps with respect to  $\mathcal{K}$ . It is convenient to generalize the concept to functions between  $n$ -ads as follows.

**DEFINITION.** Let  $X$  be a CW complex and let  $Y$  have CW type. Further let  $\mathbb{A} \in \mathcal{S}_X$  and  $\mathbb{B} \in \mathcal{S}_Y$ . We say that maps  $f, g: (X; \mathbb{A}) \rightarrow (Y; \mathbb{B})$  form a phantom pair if for every finite subcomplex  $K$  of  $X$  the restrictions  $f|_K, g|_K$  are homotopic as maps  $(K; \mathbb{A} \cap K) \rightarrow (Y; \mathbb{B})$ . This is to say that the restrictions  $f|_K, g|_K$  belong to the same path component of the function space  $(Y; \mathbb{B})^{(K; \mathbb{A} \cap K)}$ .

The definition depends only on the path components of the maps involved, whence

**LEMMA 2.4.1.** *If two maps  $f, g$  form a phantom pair then their path components form a phantom pair of path components as defined in section 1.4, with respect to the inverse system  $\{(Y; \mathbb{B})^{(L; \mathbb{A} \cap L)} \mid L\}$  indexed by the set  $\mathcal{X}$  of all subcomplexes of  $X$  with set of finite elements  $\mathcal{K}$  the subset of finite subcomplexes.*  $\square$

As before we denote by  $\text{Ph}[g]$  the set of phantom components that form a phantom pair with the path component  $[g]$  of  $g: (X; \mathbb{A}) \rightarrow (Y; \mathbb{B})$ . Denoting the natural function

$$\Psi: \pi_0((Y; \mathbb{B})^{(X; \mathbb{A})}) \rightarrow \lim_K \pi_0((Y; \mathbb{B})^{(K; \mathbb{A} \cap K)}).$$

it is clear that  $\Psi^{-1}(\Psi[g]) = \text{Ph}[g]$ .

**PROPOSITION 2.4.2.** *Let  $X$  be a countable CW complex. Let  $(X; \mathbb{A})$  be a CW  $n$ -ad and let  $(Y; \mathbb{B})$  have the type of a CW  $n$ -ad. Let  $L_1 \leq L_2 \leq L_3 \leq \dots$  be a filtration of finite subcomplexes for  $X$ . Let  $g: (X; \mathbb{A}) \rightarrow (Y; \mathbb{B})$  be a map of  $n$ -ads. Then the following natural identification can be made*

$$\text{Ph}[g] = \lim_j^1 \pi_1((Y; \mathbb{B})^{(L_j; \mathbb{A} \cap L_j)}, g|_{L_j}).$$

**PROOF.** Follows directly from Proposition 1.4.1.  $\square$

**EXAMPLE 5.** *Let  $m < n$ . We consider the space of maps  $M(\mathbb{Z}[\frac{1}{p}], m) \rightarrow K(\mathbb{Z}, n)$ .*

- (i) *The space  $K(\mathbb{Z}, n)^{M(\mathbb{Z}[\frac{1}{p}], m)}$  has the weak homotopy type of a product  $K(\hat{\mathbb{Z}}_p/\mathbb{Z}, n - m - 1) \times K(\mathbb{Z}, n)$ , but does not have the type of a CW complex.*

In particular, for  $m = n - 1$  the space in question has uncountably many homotopy equivalent path components that have the weak homotopy type of  $K(\mathbb{Z}, n)$  but do not have CW homotopy type.

- (ii) The space  $(K(\mathbb{Z}, n), *)^{(M(\mathbb{Z}[\frac{1}{p}], m), *)}$  of pointed maps has the weak homotopy type of a  $K(\hat{\mathbb{Z}}_p/\mathbb{Z}, n - m - 1)$  but does not have CW homotopy type.

In particular, for  $m = n - 1$ , the space in question has uncountably many homotopy equivalent path components that are weakly contractible but not contractible.

If we take for  $M(\mathbb{Z}[\frac{1}{p}], m)$  the usual telescoping construction (see for example Hatcher [26], §3.F) and denote the finite stages of the telescope by  $L_k$ , say, then it can be easily verified that the sequence  $\{\pi_{n-m}(K(\mathbb{Z}, n)^{L_k}) \mid k\}$  fails to be Mittag-Leffler.

In case  $m = n - 1$  this implies the existence of phantom components. In particular, the space of pointed maps has weakly contractible path components none of which has the homotopy type of a CW complex. The path components are those of an  $H$ -group and hence homotopy equivalent. See Proposition 3.3.1 for a detailed proof of a more general result.

Here we focus on a more geometric representation of case  $m = n - 1$ . First consider the short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\frac{1}{p}] \rightarrow \mathbb{Z}_{p^\infty} \rightarrow 0$ . This we can realize as a cofibration sequence

$$(*) \quad S^{n-1} \xrightarrow{\varphi} M(\mathbb{Z}[\frac{1}{p}], n-1) \xrightarrow{\leq} M(\mathbb{Z}_{p^\infty}, n-1).$$

Suppose we construct  $M(\mathbb{Z}[\frac{1}{p}], n-1)$  as the reduced mapping telescope of the sequence

$$(**) \quad S^{n-1} \xrightarrow{p} S^{n-1} \xrightarrow{p} \dots$$

The first stage of the telescope  $L_1$  may be identified with the sphere  $S^{n-1}$  and the map  $\varphi: S^{n-1} \rightarrow M(\mathbb{Z}[\frac{1}{p}], n-1)$  is then also a degree  $p$  map  $S^{n-1} \rightarrow L_1$  composed by the inclusion into the telescope.

Set  $Y = K(\mathbb{Z}, n)$ . The associated fibrations  $(Y, *)^{(L_i, *)} \rightarrow (Y, *)^{(L_{i-1}, *)}$  may be identified as  $\Omega^{n-1}K(\mathbb{Z}, n) \xrightarrow{p} \Omega^{n-1}K(\mathbb{Z}, n)$  where  $p$  denotes the  $H$ -space multiplication. An application of Proposition 1.2.1 yields a homotopy equivalence

$$(K(\mathbb{Z}, n), *)^{(M(\mathbb{Z}[\frac{1}{p}], n-1), *)} \simeq T_p$$

where  $T_p$  denotes the  $p$ -adic solenoid of Example 2 on page 20. Moreover the induced map

$$\varphi^\#: (Y, *)^{(M(\mathbb{Z}[\frac{1}{p}], n-1), *)} \rightarrow (Y, *)^{(S^{n-1}, *)}$$

may be identified with the canonical projection  $T_p \rightarrow S^1$ .

By Lemma 0.0.8 the restriction fibration

$$(Y, *)^{(M(\mathbb{Z}_p^\infty, n-1), *)} \rightarrow (Y, *)^{(M(\mathbb{Z}[\frac{1}{p}], n-1), *)}$$

is the principal fibration obtained by taking the homotopy fibre of  $\varphi^\#$ . Since  $T_p \rightarrow S^1$  is a fibration,  $(Y, *)^{(M(\mathbb{Z}_p^\infty, n-1), *)}$  is homotopy equivalent to the fibre of  $T_p \rightarrow S^1$ , the group  $\hat{\mathbb{Z}}_p$  of  $p$ -adic integers with its profinite topology.

Thus while  $(Y, *)^{(M(\mathbb{Z}[\frac{1}{p}], n-1), *)} \simeq T_p$  has uncountably many weakly contractible path components none of which is contractible,  $(Y, *)^{(M(\mathbb{Z}_p^\infty, n-1), *)} \simeq \hat{\mathbb{Z}}_p$  has every finite set of path components (homotopy) discrete, contains countable sets of path components which are not homotopy discrete (hence not of CW type), and countable sets of path components which are homotopy discrete (hence of CW type).  $\square$

We apply Theorem 1.4.4 to give an alternative proof of a theorem of Meier [39].

**COROLLARY 2.4.3** (Theorem 2. (a) of Meier [39]). *Let  $X$  be a nilpotent CW complex of finite type. If there exists an integer  $n$  such that  $H_i(X) = 0$  for  $i \geq n + 1$  then the natural function  $[X, Y]_* \rightarrow \lim[K, Y]_*$  is bijective. Here  $K$  ranges over the finite subcomplexes of  $X$ .*

**PROOF.** The hypotheses on  $X$  imply that  $X$  is dominated by a finite complex  $L$  (see Mislin [44]). Hence  $Y^X$  and  $(Y, *)^{(X, *)}$  have CW type by Corollary 0.0.4. By Theorem 1.4.4 the space  $(Y, *)^{(X, *)}$  does not have phantom path components, and the result follows.  $\square$

Function spaces with essential phantom maps exist in abundance. In particular, we recall the following Zabrodsky's generalization of Miller's theorem on the Sullivan conjecture.

**THEOREM** (Zabrodsky [65]). *Let  $X$  and  $Y$  be nilpotent connected CW complexes of finite type with finite fundamental groups. If  $X$  is a Postnikov space, or an iterated suspension of such a space, and if  $Y$  has the homotopy type of a finite complex, or an iterated loop space of such a space, then all maps  $X \rightarrow Y$  are phantom maps.*  $\square$

The following is a trivial consequence.

**COROLLARY 2.4.4** (of Zabrodsky's theorem and Theorem 1.4.4). *Let  $X$  and  $Y$  be nilpotent connected CW complexes of finite type with finite fundamental groups. Assume that  $X$  is a Postnikov space, or an iterated suspension of such a space, and that  $Y$  has the homotopy type of a finite complex, or an iterated loop space of such a space.*

Then if the space  $(Y, *)^{(X, *)}$  is not contractible, it does not have the homotopy type of a CW complex. Consequently the space  $Y^X$  has CW type if and only if the evaluation  $Y^X \rightarrow Y$  is a homotopy equivalence.

PROOF. Let  $(Y, *)^{(X, *)}$  have the homotopy type of a CW complex that is not contractible, and let  $k$  denote the least nonnegative integer for which  $\pi_k((Y, *)^{(X, *)}, *)$  is non-trivial. Since iterated loop spaces of CW type spaces have CW type, the space  $Z = \Omega^k((Y, *)^{(X, *)}, *) \approx (\Omega^k Y, *)^{(X, *)}$  has CW type. By Zabrodsky's theorem, every map in  $Z$  is a phantom map, and since  $Z$  is not path-connected, it contains an essential phantom map, a contradiction.  $\square$

EXAMPLE 6. The space of (pointed) maps  $K(\mathbb{Z}, 2n) \rightarrow S^{2n+1}$  has  $\text{Ext}(\mathbb{Q}, \mathbb{Z})$  phantom components and does not have CW homotopy type.  $\square$

If  $X$  is a Postnikov section and  $Y$  is a simply connected finite complex, then Zabrodsky's theorem conveniently implies that the function space  $(Y, *)^{(X, *)}$  cannot have CW homotopy type – if it has a nontrivial homotopy group.

If  $(Y, *)^{(X, *)}$  is weakly contractible, then we cannot say whether it has CW homotopy type or not. In particular, this is the case with ‘Miller function spaces’  $(Y, *)^{(K(G, 1), *)}$  for a locally finite group  $G$  and a finite dimensional complex  $Y$ . We conjecture that these spaces do not have CW homotopy type, having in mind the following observation. Let  $L_1 \leq L_2 \leq \dots$  be a filtration for  $X = K(G, 1)$ . If  $(Y, *)^{(X, *)}$  is contractible, then Corollary 1.2.7 yields a subsequence  $\{L_{\varphi(i)} \mid i\}$  such that for each  $i$  the restriction fibration  $(\Omega Y, *)^{(L_{\varphi(i+1)}, *)} \rightarrow (\Omega Y, *)^{(L_{\varphi(i)}, *)}$  is nullhomotopic. In particular, the induced morphisms on homotopy groups are all trivial. By adjunction, this means that the morphisms

$$[S^k \wedge L_{\varphi(i+1)}, Y]_* \rightarrow [S^k \wedge L_{\varphi(i)}, Y]_*$$

are trivial for all  $k \geq 1$ .

However, refining Miller's proof of the Sullivan conjecture for the case  $X = \mathbb{R}P^\infty$ , Lesh [31] exhibits a function  $j(k, i)$ , going to infinity with  $k$ , such that  $[S^k \mathbb{R}P^{j(k, i)}, Y]_* \rightarrow [S^k \mathbb{R}P^i, Y]_*$  is trivial. The nature of dependence of  $j$  on  $k$  gives reason to believe that this cannot be done uniformly with respect to  $k$ .

## 2.5. Spaces of maps into Postnikov $n$ -ads

THEOREM 2.5.1. Let  $(Y, \mathbb{B})$  be an  $r$ -Postnikov  $n$ -ad and let  $(X, \mathbb{A})$  be a CW  $n$ -ad. Let  $g: (X; \mathbb{A}) \rightarrow (Y; \mathbb{B})$  be a map.

Let for a subcomplex  $L$  of  $X$  the symbol  $F_L$  denote the fibre of the restriction fibration  $(Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}$  over  $g|_L$ .

The path component  $Z$  of  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  containing  $g$  is open in  $(Y; \mathbb{B})^{(X; \mathbb{A})}$  and has CW homotopy type if and only if for each finite subcomplex  $L$  there exists a larger finite subcomplex  $L'$  such that the inclusion  $F_{L'} \hookrightarrow F_L$  induces trivial morphisms on all homotopy groups.

REMARK. Let  $\mathcal{K}$  denote the set of finite subcomplexes of  $X$ , and let  $\mathcal{I}$  denote the subset of  $\mathcal{K}$  consisting of those finite subcomplexes  $L$  for which the restriction induced morphism

$$\pi_k((Y; \mathbb{B})^{(X; \mathbb{A})}, g) \rightarrow \pi_k((Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}, g|_L)$$

is injective for all  $k \geq 0$  (for  $k = 0$  in the usual sense).

The condition in the theorem is then equivalent to the following two.

- (i)  $\mathcal{I}$  is cofinal in  $\mathcal{K}$ , and
- (ii) for each  $L$  in  $\mathcal{K}$  there exists a larger  $L' \in \mathcal{K}$  such that for any subcomplex  $M$  of  $X$  containing  $L'$  and any  $k \geq 1$ , the image of

$$\pi_k((Y; \mathbb{B})^{(M; \mathbb{A} \cap M)}, g|_M) \rightarrow \pi_k((Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}, g|_L)$$

coincides with that of

$$\pi_k((Y; \mathbb{B})^{(L'; \mathbb{A} \cap L')}, g|_{L'}) \rightarrow \pi_k((Y; \mathbb{B})^{(L; \mathbb{A} \cap L)}, g|_L).$$

Note that if  $\mathcal{I}$  is cofinal in  $\mathcal{K}$ , this implies that  $Z$  is open.

PROOF. The necessity part is clear by Theorem 1.4.4; we have to prove sufficiency.

Let  $L$  be a countable subcomplex of  $X$  and let  $K_1 \leq K_2 \leq \dots$  be a filtration for  $L$  of finite subcomplexes. There exists a sequence of finite subcomplexes of  $X$

$$L_1 \leq L_2 \leq L_3 \leq \dots$$

such that  $L_1 \in \mathcal{I}$  (hence  $L_i \in \mathcal{I}$  for all  $i$ ),  $L_i \supseteq K_i$ , and for each  $i$ , the image of  $\pi_k((Y; \mathbb{B})^{(L_i; \mathbb{A} \cap L_i)}, g|_{L_i}) \rightarrow \pi_k((Y; \mathbb{B})^{(L_{i-1}; \mathbb{A} \cap L_{i-1})}, g|_{L_{i-1}})$  equals that of  $\pi_k((Y; \mathbb{B})^{(M; \mathbb{A} \cap M)}, g|_M) \rightarrow \pi_k((Y; \mathbb{B})^{(L_{i-1}; \mathbb{A} \cap L_{i-1})}, g|_{L_{i-1}})$  for any  $M$  containing  $L_{i-1}$  (including infinite  $M$ ). In particular  $\pi_*(F_{L_i}, g) \rightarrow \pi_*(F_{L_{i-1}}, g)$  is trivial.

Let  $L_\infty = \cup_i L_i$ . Note that the fibre of  $(Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(L_\infty; \mathbb{A} \cap L_\infty)}$  over  $g|_{L_\infty}$  is weakly contractible. In particular, it is path-connected and is equal to the fibre of  $Z \rightarrow Z_\infty$  where  $Z_\infty$  is the image of  $Z$ . Let  $Z_i$  denote the path component of  $g|_{L_i}$  in  $(Y; \mathbb{B})^{(L_i; \mathbb{A} \cap L_i)}$ . By Theorem 1.5.1 the space  $\mathfrak{R}(Z_\infty)$  has the homotopy type of a CW complex. But  $L_\infty$  is a countable complex, hence the natural map  $\mathfrak{R}((Y; \mathbb{B})^{(L_\infty; \mathbb{A} \cap L_\infty)}) \rightarrow (Y; \mathbb{B})^{(L_\infty; \mathbb{A} \cap L_\infty)}$  is a homotopy equivalence (see Lemma 3.4 of [56]). Since compactly generated refinement preserves path components and inclusions, also the natural map  $\mathfrak{R}(Z_\infty) \rightarrow Z_\infty$  is a homotopy equivalence. In particular,  $Z_\infty$  has CW homotopy type. By Theorem 2.2.6,  $Z$  has CW homotopy type.  $\square$

For countable domain complexes  $X$  the statement of Theorem 2.5.1 can be somewhat simplified.

**COROLLARY 2.5.2.** *Let  $(X; \mathbb{A})$  be a CW  $n$ -ad with  $X$  countable and let  $(Y; \mathbb{B})$  be an arbitrary CW  $n$ -ad. Let  $L_1 \leq L_2 \leq \dots$  be a filtration of finite subcomplexes for  $X$ . Further let  $C$  be the path component of some  $g \in (Y; \mathbb{B})^{(X; \mathbb{A})}$ .*

*Then  $C$  is open and has CW type if and only if*

- (i) *for each  $k \geq 1$  the sequence  $\{\pi_k((Y; \mathbb{B})^{(L_i; \mathbb{A} \cap L_i)}, g|_{L_i}) \mid i\}$  satisfies the Mittag-Leffler condition, and*
- (ii) *there exists  $j \geq 0$  such that for each  $k \geq 0$  (for  $k = 0$  in the usual sense) the ‘morphism’  $\pi_k((Y; \mathbb{B})^{(X; \mathbb{A})}, g) \rightarrow \pi_k((Y; \mathbb{B})^{(L_j; \mathbb{A} \cap L_j)}, g|_{L_j})$  is injective.*

*In particular,  $C$  is contractible if and only if for each  $k \geq 1$  and each  $i \geq 1$  there exists  $s(i)$  such that  $\pi_k((Y; \mathbb{B})^{(L_{s(i)}; \mathbb{A} \cap L_{s(i)})}, g|_{L_{s(i)}}) \rightarrow \pi_k((Y; \mathbb{B})^{(L_i; \mathbb{A} \cap L_i)}, g|_{L_i})$  is trivial, and, in addition, exists  $j$  such that  $[(X; \mathbb{A}), (Y; \mathbb{B})] \rightarrow [(L_j; \mathbb{A} \cap L_j), (Y; \mathbb{B})]$  is ‘injective’ at the base point  $g$ .  $\square$*

**COROLLARY 2.5.3.** *If  $X$  and  $Y$  are countable CW complexes and  $Y$  has only finitely many nontrivial homotopy groups then  $(Y, *)^{(X, *)}$  is contractible if and only if it is weakly contractible.*

**PROOF.** Let  $L_1 \leq L_2 \leq \dots$  be a filtration of finite subcomplexes for  $X$ . By (i) of Lemma 2.3.2 the spaces  $(Y, *)^{(L_i, *)}$  have countable homotopy groups. By Proposition 1.4.1 the groups  $\lim^1 \pi_k((Y, *)^{(L_i, *)}, *)$  vanish for all  $k \geq 1$ , hence by Proposition 1.4.2 the sequences  $\{\pi_k((Y, *)^{(L_i, *)}, *) \mid i\}$  satisfy the Mittag-Leffler condition.  $\square$

Corollary 2.5.3 is in sharp contrast with the case of general  $Y$ , see Example 8 on page 56.

## 2.6. General constructions of function spaces of CW type

**PROPOSITION 2.6.1.** *Let  $(Y; \mathbb{B})$  be an  $r$ -Postnikov  $n$ -ad, and let  $(X; \mathbb{A})$  be a CW  $n$ -ad. Let  $T$  be a subcomplex of  $X$ . If the pair  $(X, T)$  is an adjunction of cells of dimension  $\geq r + 1$ , then the fibration*

$$(\bullet) \quad (Y; \mathbb{B})^{(X; \mathbb{A})} \rightarrow (Y; \mathbb{B})^{(T; \mathbb{A} \cap T)}$$

*is a homotopy equivalence onto image. In particular, this is true for  $T = X^{(r)}$ .*

*If  $(X, T)$  is an adjunction of cells of dimension  $\geq r + 2$ , then the fibration  $(\bullet)$  is a homotopy equivalence. In particular,  $T$  may be  $X^{(r+1)}$ .*



PROOF. Let  $\{L_\lambda \mid \lambda \in W(\alpha)\}$  be a good filtration for the pair  $(X, T)$  with each consecutive step the adjunction of a single cell. Set  $Z_\lambda = (Y; \mathbb{B})^{(L_\lambda; \mathbb{A} \cap L_\lambda)}$ . Then  $\{Z_\lambda \mid \lambda\}$  is a restricted inverse systems of fibrations indexed by  $W(\alpha)$ . The limit space of this system is  $(Y; \mathbb{B})^{(X; \mathbb{A})} = Z$ . For each  $\lambda$  denote by  $C_\lambda$  the image of  $Z$  under  $Z \rightarrow Z_\lambda$ . By (ii) of Proposition 2.3.2 for each  $\lambda$  the fibration  $Z_{\lambda+1} \rightarrow Z_\lambda$  is a homotopy equivalence onto image. Hence  $C_{\lambda+1} \rightarrow C_\lambda$  is a homotopy equivalence. By (ii) of Lemma 1.3.8 the system  $\{C_\lambda \mid \lambda\}$  is a system of homotopy equivalences with limit  $Z$ . In particular  $Z \rightarrow C_0$  is a homotopy equivalence, and  $C_0$  is the image of the map  $(\bullet)$ .

If  $(X, T)$  is an adjunction of cells of dimension  $\geq r + 2$ , then each  $Z_{\lambda+1} \rightarrow Z_\lambda$  is a homotopy equivalence, and the assertion follows.  $\square$

COROLLARY 2.6.2. *Given the hypothesis of Proposition 2.6.1, if  $(Y; \mathbb{B})^{(T; \mathbb{A} \cap T)}$  has CW type then so has  $(Y; \mathbb{B})^{(X; \mathbb{A})}$ . In particular, if the  $r$ -skeleton  $X^{(r)}$  of  $X$  is finite, we may take  $T = X^{(r)}$ .  $\square$*

PROPOSITION 2.6.3. *Let  $X$  be a simply connected, and  $Y$  an arbitrary CW complex. Assume that for each  $m$  the function space  $(Y, *)^{(M(H_m(X), m), *)}$  has CW type. Then  $(Y, *)^{(X, *)}$  has CW type in the following three cases.*

- (i) *The complex  $X$  is finite dimensional.*
- (ii) *The complex  $Y$  has finitely many nontrivial homotopy groups.*
- (iii) *The spaces  $(Y, *)^{(M(H_m(X), m), *)}$  are contractible for all  $m$ .*

PROOF. By possibly replacing  $X$  with a homotopy equivalent complex we may assume a homology decomposition for  $X$ , i.e. a filtration of sub-complexes  $X_2 \leq X_3 \leq X_4 \leq \dots$  where  $X_2$  is of type  $M(H_2(X), 2)$  and for each  $i \geq 3$  the subcomplex  $X_i$  is the mapping cone of a (based) cellular map  $M(H_i(X), i-1) \rightarrow X_{i-1}$ . (For a particular choice of homology decomposition see Chapter 4.)

Note that (i) implies (ii) since by Proposition 2.6.1 the fibrations  $(Y, *)^{(X_i, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$  are homotopy equivalences for all large enough  $i$ , and we may apply Corollary 1.2.2.

We prove by induction that the space  $(Y, *)^{(X_i, *)}$  has CW type for each  $i$ . The basis  $i = 2$  holds trivially. Assume that  $(Y, y_0)^{(X_{i-1}, x_0)}$  has CW type.

By Corollary 0.0.8 the fibration  $R: (Y, y_0)^{(X_i, x_0)} \rightarrow (Y, y_0)^{(X_{i-1}, x_0)}$  is a principal fibration with fibres either empty or homotopy equivalent to  $(Y, y_0)^{(M(H_i(X), i), *)}$ . The latter has CW type by assumption, and therefore  $(Y, *)^{(X_i, *)}$  has CW type by Stasheff's theorem.

If  $X$  is finite-dimensional then  $X = X_N$  for some  $N$  and (i) follows. In case the spaces  $(Y, *)^{(M(H_i(X), i), *)}$  are contractible for all  $i$ , the above

induction shows that also the spaces  $(Y, *)^{(X_i, *)}$  are contractible for all  $i$ , hence **(iii)** follows from Proposition 1.2.4.  $\square$

Proposition 2.6.3 has the dual

**PROPOSITION 2.6.4.** *Let  $Y_N \rightarrow Y_{N-1} \rightarrow \cdots \rightarrow Y_2 \rightarrow Y_1$  be a sequence of fibrations where  $Y_1 = K(G_1, n_1)$  with  $G_1$  abelian and for each  $i \geq 2$  the fibration  $Y_i \rightarrow Y_{i-1}$  is a principal fibration obtained by taking the homotopy fibre of a pointed map  $(Y_{i-1}, y_{i-1}) \rightarrow (K(G_i, n_i+1), *)$  of well-pointed spaces (where  $G_i$  is abelian). Let  $(X, x_0)$  be an arbitrary pointed CW complex.*

*If for each  $i$  the space  $K(G_i, n_i)^X$  has the homotopy type of a CW complex then so has the space  $Y_N^X$ .*

**PROOF.** By Lemma A.0.1 and Proposition A.0.2 the map  $(Y_i, y_i)^{(X, x_0)} \rightarrow (Y_{i-1}, y_{i-1})^{(X, x_0)}$  is a principal fibration with fibres either empty or homotopy equivalent to  $(K(G_i, n_i), *)^{(X, x_0)}$  which has CW homotopy type by hypothesis. Hence we may proceed inductively using Stasheff's theorem, the basis of the induction holding trivially.  $\square$

**COROLLARY 2.6.5.** *If  $Y$  is a simply connected CW complex with  $\pi_k(Y) = 0$  for all  $k \geq n+1$  and  $X$  is a CW complex such that the space  $K(\pi_i Y, i)^X$  has CW type for each  $i$ , then also the space  $Y^X$  has CW type.*  $\square$

## Localization

### 3.1. A genuine Zabrodsky lemma

In this section our aim is to prove

**PROPOSITION 3.1.1.** *Let  $E \rightarrow B$  be a fibration with fibre  $F$  where  $B$  has the homotopy type of a connected CW complex. Let  $X$  be a CW complex. If the section  $X \rightarrow X^F$  is a homotopy equivalence, then so is  $X^B \rightarrow X^E$ .*

**REMARK.** Let  $y_0$  be a base point of  $F$ . Since the evaluation map  $X^F \rightarrow X$ ,  $f \mapsto f(y_0)$ , is a left inverse for the section  $X \rightarrow X^F$ , one of the maps is a homotopy equivalence if and only if the other is. If  $y_0$  is a nondegenerate base point in  $F$ , then the evaluation at  $y_0$  is a fibration. In this case by Theorem 6.3 of Dold [8] for any path-connected Dold space  $X$  (in particular any connected space of CW type)

- (i)  $(X, *)^{(F,*)}$  is contractible if and only if the evaluation  $X^F \rightarrow X$  is a homotopy equivalence, and also
- (ii)  $X^B \rightarrow X^E$  is a homotopy equivalence if and only if  $(X, *)^{(B,*)} \rightarrow (X, *)^{(E,*)}$  is.

In ‘most’ cases we are able to assume that  $F$  has a nondegenerate base point. Notably, if  $(Y, y_0) \xrightarrow{g} (B, b_0)$  is a map of well-pointed spaces, then the base point  $(y_0, \text{const}_{b_0})$  is nondegenerate in the homotopy fibre of  $g$  (formed as a subspace of  $Y \times B^I$ ).

The proposition analogous to 3.1.1 that assumes *weak* contractibility, and infers *weak* homotopy equivalence is known as ‘the Zabrodsky lemma’ (see Miller [40], §9 or McGibbon [36], Lemma 5.5).

Our proof of Proposition 3.1.1 is essentially that of Lemma 1.5 of Zabrodsky [66] modulo some care about genuine homotopy type; whence our name.

Note that we do not impose any particular restriction on the homotopy types of  $F$  and  $E$ .

PROOF. There exists a homotopy equivalence  $B' \xrightarrow{f} B$  where  $B'$  is a simplicial complex. Consider the pullback diagram

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \bar{p} \downarrow & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

By Corollary 1.4 of Brown and Heath [3] the map  $\bar{f}: E' \rightarrow E$  is a homotopy equivalence, and the proposition follows from Lemma 3.1.2 below.  $\square$

LEMMA 3.1.2. *Let  $B$  be a connected simplicial complex and  $p: E \rightarrow B$  a fibration with fibre  $F$ . Let  $X$  be a connected CW complex. If the function space  $(X, *)^{(F, *)}$  is contractible, then  $X^B \rightarrow X^E$  is a homotopy equivalence.*

REMARK. By Theorem 6.3 of Dold [8] for any connected CW complex  $X$

- (i)  $X^B \rightarrow X^E$  is a homotopy equivalence if and only if  $(X, *)^{(B, *)} \rightarrow (X, *)^{(E, *)}$  is, and
- (ii)  $(X, *)^{(F, *)}$  is contractible if and only if the evaluation  $X^F \rightarrow X$  (and hence its section  $X \rightarrow X^F$ ) is a homotopy equivalence.

PROOF. Let  $\mathcal{B}$  be the set of subcomplexes of  $B$ . By Lemma 2.1.4 there exists a limit ordinal  $\alpha$  and an order preserving injection  $U: W(\alpha) \rightarrow \mathcal{B}$  such that for a limiting  $\mu < \alpha$ ,  $U(\mu) = \cup_{\lambda < \mu} U(\lambda)$ . Moreover  $\cup_{\lambda < \alpha} U(\lambda) = B$ , and we may assume that  $U(\lambda + 1) = U(\lambda) \cup \sigma_\lambda$  where  $\sigma_\lambda$  is a simplex, for each  $\lambda$ . Clearly  $U(0)$  is a 0-simplex. We write  $B_\lambda = U(\lambda)$ , and set  $E_\lambda = p^{-1}(B_\lambda)$ . Let  $\lambda < \lambda'$ . Since  $B_\lambda$  is cofibered in  $B_{\lambda'}$ , and  $E_{\lambda'} \rightarrow B_{\lambda'}$  is a fibration,  $E_\lambda$  is cofibered in  $E_{\lambda'}$ . Thus  $\{X^{B_\lambda} | \lambda\}$  and  $\{X^{E_\lambda} | \lambda\}$  are restricted inverse systems of fibrations and the map  $p^*: X^B \rightarrow X^E$  is the limit map of maps  $p_\lambda^*: X^{B_\lambda} \rightarrow X^{E_\lambda}$ .

In light of Theorem 1.3.4 it is therefore enough to prove the following. Suppose  $L$  and  $L'$  are subcomplexes of  $B$  such that  $L' = L \cup \sigma$  where  $\sigma$  is a simplex. If  $X^L \rightarrow X^{E|_L}$  is a homotopy equivalence, then so is  $X^{L'} \rightarrow X^{E|_{L'}}$ .

Note  $E|_{L'} = E|_L \cup E|_\sigma$ . Since  $\sigma$  is contractible, there exists a fibre homotopy equivalence  $\sigma \times F \rightarrow E|_\sigma$  (over  $\sigma$ ). In particular this equivalence restricts to a (fibre) homotopy equivalence  $\partial\sigma \times F \rightarrow E|_{\partial\sigma}$ .

Thus there exists the following commutative diagram.

$$\begin{array}{ccccc}
& & X^{E|_{L'}} & \longrightarrow & X^{E|_{\sigma}} & \xrightarrow{\simeq} & X^{\sigma \times F} \\
& \nearrow & \downarrow & & \downarrow & & \downarrow \\
X^{L'} & \longrightarrow & X^{\sigma} & \longrightarrow & X^{E|_{\sigma}} & & X^{\sigma \times F} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & X^{E|_L} & \longrightarrow & X^{E|_{\partial\sigma}} & \xrightarrow{\simeq} & X^{\partial\sigma \times F} \\
\downarrow & \nearrow & \downarrow & & \downarrow & & \downarrow \\
X^L & \longrightarrow & X^{\partial\sigma} & \longrightarrow & X^{E|_{\partial\sigma}} & & X^{\partial\sigma \times F}
\end{array}$$

Since  $\sigma \times F \rightarrow E|_{\sigma}$  is a map over  $\sigma$ , the composite  $X^{\sigma} \rightarrow X^{E|_{\sigma}} \rightarrow X^{\sigma \times F}$  is induced by projection  $\sigma \times F \rightarrow \sigma$ . Similarly for the composite  $X^{\partial\sigma} \rightarrow X^{E|_{\partial\sigma}} \rightarrow X^{\partial\sigma \times F}$ .

Since the projection induced maps  $X^{\sigma} \rightarrow X^{\sigma \times F}$ , respectively  $X^{\partial\sigma} \rightarrow X^{\partial\sigma \times F}$ , are homotopy equivalences, so are the maps  $X^{\sigma} \rightarrow X^{E|_{\sigma}}$ , respectively  $X^{\partial\sigma} \rightarrow X^{E|_{\partial\sigma}}$ . By assumption,  $X^L \rightarrow X^{E|_L}$  is a homotopy equivalence. The cube in the diagram is a morphism of pullback diagrams with vertical arrows fibrations, so by coglueing homotopy equivalences [3] also the map (of pullback spaces)  $X^{L'} \rightarrow X^{E|_{L'}}$  is a homotopy equivalence.

(We note that expressing the sphere  $S^n$  as the union of two disks meeting in a common boundary it follows by a similar finite inductive argument that  $X^{S^n} \rightarrow X^{S^n \times F}$  is a homotopy equivalence for all  $n$ .)  $\square$

We immediately infer

**COROLLARY 3.1.3.** *Assume  $Y$  is a connected CW complex, and  $X$  is a connected CW complex. If  $(Y, *)^{(\Omega X, *)}$  is contractible, so is  $(Y, *)^{(X, *)}$ .*  $\square$

### 3.2. Local target complex

In the theory of homotopy localization with respect to a map (see Dror Farjoun [10]) function spaces play a central role. Given a map of CW complexes  $f: A \rightarrow B$ , a CW complex  $Y$  is called  $f$ -local if the induced map  $Y^B \rightarrow Y^A$  is a weak homotopy equivalence. In particular, in case  $f$  is the map  $W \rightarrow *$  then  $f$ -local CW complexes  $Y$  are called  $W$ -null.

There exist the notions of strongly  $f$ -local and strongly  $W$ -null obtained by replacing weak homotopy equivalence above for genuine homotopy equivalence. It would be of some importance in homotopy theory to be able to characterize the difference between  $f$ -local and strongly  $f$ -local (see V. Halperin [25]). However, this seems to be a difficult task in general.

In Section 3.3 below (see Theorem 3.3.8 and Example 8 on page 56) we exhibit pairs  $Y, W$  such that  $Y$  is weakly  $W$ -null but not strongly  $W$ -null.

For any map  $f: A \rightarrow B$  there exists a ‘localization functor’  $L_f$ , that is a homotopy idempotent functor equipped with a natural transformation  $\text{Id} \rightarrow L_f$  so that for every space  $X$  of CW type the map  $X \rightarrow L_f X$  is a homotopically universal map to  $f$ -local spaces.

If  $Y$  is an  $f$ -local space then the natural map  $X \rightarrow L_f X$  induces a weak homotopy equivalence (see [10])

$$(\star) \quad Y^{L_f X} \rightarrow Y^X.$$

So a problem related to the one above would be to investigate when the map  $(\star)$  is a genuine equivalence. We show below that this is the case for localization with respect to a set of primes which is a particular case of localization with respect to a map (see [10]).

Our basic reference for localization with respect to a set of primes is Hilton, Mislin, Roitberg [27].

If  $P$  is a set of primes then

$$\mathbb{Z}_{(P)} = \left\{ \frac{a}{b} \mid \text{prime divisors of } b \text{ belong to the complement of } P \right\} \subseteq \mathbb{Q}.$$

For an abelian group  $A$  we will denote  $A_{(P)} = A \otimes \mathbb{Z}_{(P)}$  the localization of  $A$  at the set  $P$  (or just  $A_{(p)}$  in case of a single prime  $P = \{p\}$ ), and for a (nilpotent) CW complex  $X$  we will denote by  $X_{(P)}$  the localization of  $X$  at the set  $P$ .

**THEOREM 3.2.1.** *Let  $P$  be a set of primes and let  $X$  be a simply connected CW complex. Let  $Y$  be a  $P$ -local complex. Then the map  $(Y, *)^{(X_{(P)}, *)} \rightarrow (Y, *)^{(X, *)}$  induced by localization  $X \rightarrow X_{(P)}$  is a homotopy equivalence.*

We list first a few corollaries and an example, and later prove the theorem.

**COROLLARY 3.2.2.** *Let  $\mathbb{P} = P_1 \cup P_2$  be a partition of the primes. Let  $X$  and  $Y$  be simply connected CW complexes. There exists a pullback square*

$$\begin{array}{ccc} (Y, *)^{(X, *)} & \longrightarrow & (Y_{(P_2)}, *)^{(X_{(P_2)}, *)} \\ \downarrow & & \downarrow \\ (Y_{(P_1)}, *)^{(X_{(P_1)}, *)} & \longrightarrow & (Y_{(0)}, *)^{(X_{(0)}, *)}. \end{array}$$

**PROOF.** We construct the square by first constructing a rationalization  $Y_{(0)}$ , followed by localizations  $Y_{(P_1)}, Y_{(P_2)}$  so that the maps  $Y_{(P_i)} \rightarrow Y_{(0)}$  are

fibrations. The topological pullback  $Y'$  of  $Y_{(P_1)} \rightarrow Y_{(0)} \leftarrow Y_{(P_2)}$  is then homotopy equivalent to  $Y$ , and  $(Y', *)^{(X,*)}$  is the pullback of  $(Y_{(P_1)}, *)^{(X,*)} \rightarrow (Y_{(0)}, *)^{(X,*)} \leftarrow (Y_{(P_2)}, *)^{(X,*)}$ .  $\square$

**COROLLARY 3.2.3.** *Let  $X$  be a simply connected CW complex with torsion homology  $\tilde{H}_*$ . Then for any partition of the primes  $\mathbb{P} = P_1 \cup P_2$ , the space of pointed maps  $(Y, *)^{(X,*)}$  is homotopy equivalent to the product*

$$(Y_{(P_1)}, *)^{(X_{(P_1)}, *)} \times (Y_{(P_2)}, *)^{(X_{(P_2)}, *)}.$$

*In particular if  $Y^X$  has the homotopy type of a CW complex, so has the space  $Y_{(p)}^X \cong Y_{(p)}^{X^{(p)}}$  for any prime  $p$ .*  $\square$

It is natural to ask for some kind of a converse. The following example shows it is not possible.

**EXAMPLE 7.** *Set  $X = \bigvee_{p \in \mathbb{P}} M(\mathbb{Z}_p, p)$  and let  $Y$  be a CW complex with  $\pi_{p^2}(Y) = \mathbb{Z}_p$  for  $p \in \mathbb{P}$  and  $\pi_k(Y) = 0$  for  $k \notin \mathbb{P}$ . The localization  $Y_{(p)}$  at each prime  $p$  is an Eilenberg-MacLane space  $K(\mathbb{Z}_p, p^2)$ , hence  $Y_{(p)}^X$  has the homotopy type of a CW complex by Proposition 2.6.1.*

*However, since  $(Y, *)^{(M(\mathbb{Z}_p, p), *)}$  is not contractible for any  $p$ ,  $Y^X$  does not have the homotopy type of a CW complex, by Lemma 2.1.6.*

The rest of this section is devoted to the proof of Theorem 3.2.1.

**PROPOSITION 3.2.4.** *Let  $\mathbb{P} = P \cup P'$  be a partition of the primes with  $P' \neq \emptyset$  and let  $X$  be a simply connected CW complex with  $\tilde{H}_*(X)$  torsion  $P'$ -primary. Further let  $Y$  be a  $P$ -local complex. Then  $(Y, *)^{(X,*)}$  is contractible.*

**LEMMA 3.2.5.** *Proposition 3.2.4 holds for countable CW complexes  $X$ .*

**PROOF.** In light of Proposition 2.6.3 it is enough to prove the special case  $X = M(A, m)$  where  $A$  is a  $P'$ -torsion group and  $m \geq 2$ . Since  $A$  is a countable torsion group, its  $q$ -primary components are totally  $q$ -projective (see [21]), and consequently  $A$  is the union of a possibly infinite ascending sequence of finite subgroups  $A_0 \leq A_1 \leq \dots$  where  $A_0 \cong \mathbb{Z}_{q_0}$  and  $A_i/A_{i-1} \cong \mathbb{Z}_{q_i}$  where  $q_i \in P'$  for each  $i$ . Then  $X$  may be obtained as the union of an ascending sequence of subcomplexes  $M(A_0, m) \leq M(A_1, m) \leq \dots$ . Consequently  $(Y, *)^{(X,*)}$  is the limit of  $(Y, *)^{(M(A_i, *), *)}$ . Since the fibre of  $(Y, *)^{(M(A_i, *), *)} \rightarrow (Y, *)^{(M(A_{i-1}, *), *)}$  over the constant map is  $(Y, *)^{(M(\mathbb{Z}_{q_i}, m), *)}$  it suffices for a proof by induction to prove that  $(Y, *)^{(M(\mathbb{Z}_q, m), *)}$  is contractible for each  $q \in P'$ . Since  $Y$  is  $P$ -local,  $(Y, *)^{(M(\mathbb{Z}_q, m), *)}$  is weakly contractible. But  $M(\mathbb{Z}_q, m)$  is a finite complex, and hence  $(Y, *)^{(M(\mathbb{Z}_q, m), *)}$  has CW type by Milnor's theorem.  $\square$

PROOF OF PROPOSITION 3.2.4. Let  $K_\infty$  be a countable subcomplex of  $X$ . By Lemma 2.2.4 and Lemma 1.3.11 (see also the proof of Theorem 2.2.5) there exists a countable subcomplex  $L_\infty$  of  $X$  containing  $K_\infty$  such that the inclusion induced morphism

$$H_*(L_\infty) \rightarrow H_*(X)$$

is injective. In particular,  $\tilde{H}_*(L_\infty)$  is torsion  $P'$ -primary. By Lemma 3.2.5 the space  $(Y, *)^{(L_\infty, *)}$  is contractible. By Corollary 2.2.7, also  $(Y, *)^{(X, *)}$  is contractible.  $\square$

PROOF OF THEOREM 3.2.1. Let  $F$  denote the homotopy fibre of the localization map  $X \rightarrow X_{(P)}$ . Then  $F$  is a connected space of CW type with homotopy groups  $\pi_*(F)$  abelian torsion  $P'$ -primary. Moreover  $\pi = \pi_1(F)$  is divisible. Let  $f: F \rightarrow K(\pi, 1)$  denote a map inducing the identity on the fundamental group, and let  $p: \tilde{F} \rightarrow F$  be the homotopy fibre of  $f$ . Then  $\tilde{F}$  has the homotopy type of a simply connected CW complex with torsion  $P'$ -primary homotopy groups. Since these form a Serre class, also  $\tilde{H}_*(\tilde{F})$  is  $P'$ -primary torsion. Then by Proposition 3.2.4, the space  $(Y, *)^{(\tilde{F}, *)}$  is contractible. (Note that  $F$  has nondegenerate base points.) By Proposition 3.1.1, the map  $(Y, *)^{(K(\pi, 1), *)} \rightarrow (Y, *)^{(F, *)}$  is a homotopy equivalence. We claim that  $(Y, *)^{(K(\pi, 1), *)}$  is contractible.

Since  $\pi$  is divisible  $P'$ -torsion, it is isomorphic to the direct sum of some copies of the quasicyclic groups  $\mathbb{Z}_{q^\infty}$ , for various  $q \in P'$ . This is to say  $\pi$  is the direct sum of totally projective  $q$ -groups (for various  $q$ ), and as such admits a composition series (see the discussion preceding Lemma 1.3.6 on page 10, and Fuchs [21])

$$N_0 \leq N_1 \leq \dots \leq N_\lambda \leq \dots \quad (\lambda < \alpha)$$

where the successive quotients  $N_{\lambda+1}/N_\lambda$  are cyclic of prime power  $q_\lambda$  with  $q_\lambda \in P'$ .

Let  $S$  denote the set of those  $\lambda < \alpha$  for which  $(Y, *)^{(K(N_\lambda, 1), *)}$  is contractible. Suppose that for some  $\eta$ , the initial segment  $W(\eta)$  is contained in  $S$ . Assume first that  $\eta$  has a predecessor  $\eta - 1$ . The extension  $0 \rightarrow N_{\eta-1} \rightarrow N_\eta \rightarrow \mathbb{Z}_q \rightarrow 0$  (where  $q = q_{\eta-1}$ ) gives rise to a fibration

$$K(N_{\eta-1}, 1) \rightarrow K(N_\eta, 1) \rightarrow K(\mathbb{Z}_q, 1).$$

By hypothesis,  $(Y, *)^{(K(N_{\eta-1}, 1), *)}$  is contractible, hence by Proposition 3.1.1, the map  $(Y, *)^{(K(\mathbb{Z}_q, 1), *)} \rightarrow (Y, *)^{(K(N_\eta, 1), *)}$  is a homotopy equivalence. Since  $q \in P'$ , Example 1 of [56] shows that  $(Y, *)^{(K(\mathbb{Z}_q, 1), *)}$  is contractible. Hence  $\eta \in S$ .

If  $\eta$  is limiting, write  $\eta = \mu + \omega$  where  $\mu$  is the largest limit ordinal smaller than  $\eta$ . Then  $N_\eta$  is the colimit of the sequence  $N_\mu \leq N_{\mu+1} \leq \dots$ ,



and  $K(N_\eta, 1)$  may be viewed as the union of an expanding sequence of subcomplexes

$$K(N_\mu, 1) \leq K(N_{\mu+1}, 1) \leq K(N_{\mu+2}, 1) \leq \dots$$

Consequently  $(Y, *)^{(K(N_\eta, 1), *)}$  is the limit of  $\{(Y, *)^{(K(N_{\mu+i}, 1), *)} \mid i\}$  which is a sequence of contractible spaces. By Proposition 1.2.4, so is the limit. Hence  $\eta \in S$ .

By transfinite induction  $S = W(\alpha)$  and by reapplying the above argument for  $\eta = \alpha$ , it follows that  $(Y, *)^{(K(\pi, 1), *)}$  is contractible. Thus also  $(Y, *)^{(F, *)}$  is contractible, hence the localization induced map  $(Y, *)^{(X(P), *)} \rightarrow (Y, *)^{(X, *)}$  is a homotopy equivalence by another application of Proposition 3.1.1.  $\square$

### 3.3. Local domain complex and relation to the Moore conjecture

Let  $K$  be the suspension of a finite complex and  $P$  a set of primes. The  $H$ -cogroup structure allows us to form a sequence

$$(\star) \quad K \xrightarrow{q_1} K \xrightarrow{q_2} \dots$$

where  $q_i$  denotes  $H$ -cogroup multiplication. If  $\{q_i\}$  is a sequence of primes in  $\mathbb{P} \setminus P$  with each member occurring infinitely many times, then the telescope of the sequence  $(\star)$  is the  $P$ -localization of  $X$ . Then for a space  $Y$  the function space  $(Y, *)^{(K(P), *)}$  can be viewed as the inverse limit of

$$\dots \rightarrow (Y, *)^{(K, *)} \xrightarrow{q_2} (Y, *)^{(K, *)} \xrightarrow{q_1} (Y, *)^{(K, *)}$$

where now  $q_i$  denotes the induced  $H$ -group multiplication by  $q_i$ . This construction is particularly suitable for application of Theorem 1.4.4 and we do so in this section.

For the particular case when  $K$  is a sphere, we obtain the following result.

**PROPOSITION 3.3.1.** *Let  $\mathbb{P} = P \cup P'$  be a partition of the primes and let  $Y$  be a simply connected finite complex. Further let  $R$  be a set of primes with  $R \cap P \neq \emptyset$  and  $m$  a positive integer.*

- (i) *If the space  $(Y_{(R)}, *)^{(S_{(P')}, *)}^m$  has the homotopy type of a CW complex then for  $k \geq 1$  the homotopy groups  $\pi_{m+k}(Y_{(P \cap R)})$  are finite abelian groups bounded by a common bound. Moreover, the loop space  $\Omega^{m+1}Y_{(P \cap R)}$  has an  $H$ -space exponent, the space  $(Y_{(P \cap R)}, *)^{(S_{(0)}, *)}^m$  is contractible and  $(Y_{(R)}, *)^{(S_{(P')}, *)}^m \simeq \Omega^m Y'$  where  $Y'$  denotes the homotopy fibre of the localization  $Y_{(R)} \rightarrow Y_{(P \cap R)}$ . In particular,  $Y$  is an elliptic complex.*
- (ii) *If, in addition, the homology group  $H_k(Y)$  is infinite for some positive  $k$ , then  $P \cap R$  must be finite.*

- (iii) Conversely, if for some  $l$  the space  $\Omega^l Y_{(P \cap R)}$  admits an H-space exponent, then  $(Y_{(R)}, *)^{(S^l_{(P')}, *)}$  has CW homotopy type, and is homotopy equivalent to  $\Omega^l Y'$ .

Recall that a simple space  $Z$  is said to have a homotopy exponent at prime  $p$  if there exists  $k$  such that  $p^k$  annihilates the  $p$ -primary component of  $\pi_l(Z)$  for all  $l$ .

If  $Z$  is a homotopy associative H-space then  $Z$  is said to have an H-space (or *geometric*) exponent if there exists a number  $b$  such that the map  $b: Z \rightarrow Z$ , sending  $z$  to  $z^b$ , is nullhomotopic. Note that if  $Z$  has H-space exponent  $b$ , then also  $\Omega Z$  has H-space exponent  $b$ .

We refer the reader to the survey article of Neisendorfer and Selick [48] for an introduction to (geometric) exponents in homotopy theory.

A simply connected finite complex  $Y$  is *elliptic* if  $\pi_k(Y)$  is torsion for all but finitely many  $k$ . Otherwise it is *hyperbolic*. See Felix, Halperin, Thomas [18].

We will prove Proposition 3.3.1 as a corollary to the more general Theorem 3.3.8 below.

EXAMPLE 8. *The function space  $(S^n, *)^{(M(\mathbb{Q}, m), *)}$  does not have CW type for any  $m$ . In particular, for  $m > n$  the space is weakly contractible but not contractible. Thus for  $m > n$  the sphere  $S^n$  is weakly  $M(\mathbb{Q}, m)$ -null but not strongly so.*

*Since  $\Omega((S^n, *)^{(M(\mathbb{Q}, m), *)}, *) \approx (S^n, *)^{(M(\mathbb{Q}, m+1), *)}$  the situation does not ‘improve’ after looping.*  $\square$

Until the end of this section let  $X_m$  denote the localization of the  $m$ -sphere away from  $p$  where  $p$  is a prime understood from the context.

By exploiting results from the theory of homotopy exponents and H-space exponents (see [48]) we give the following rather nontrivial

EXAMPLE 9. *For all large enough  $m$ , the function space  $(S^n, *)^{(X_m, *)}$  has the homotopy type of a CW complex (and is homotopy equivalent to  $\Omega^m X_n$ ).*

*This follows from the fact that  $\Omega^l S^n_{(p)}$  has an H-space exponent for all large enough  $l$  (the case  $n$  even is due to James [28], the case  $n, p$  odd is due to Cohen, Moore, Neisendorfer [5], and the case  $n$  odd,  $p = 2$  is attributed to an unpublished result of Moore).*

*Let  $W$  denote the localization at  $p$  of the  $n$ -connected cover of  $S^n$ . It was shown by Neisendorfer and Selick [48] that  $\Omega^{n-3}W$  does not have an H-space exponent. However,  $\Omega^{n-2}W$  does, see Neisendorfer [47].*

*In particular, the space  $(W, *)^{(X_{n-4}, *)}$  does not have the type of a CW complex, and the space  $(W, *)^{(X_{n-2}, *)}$  does.*  $\square$

As a consequence we infer

**PROPOSITION 3.3.2.** *Let  $Y$  be a rationally elliptic simply connected finite complex. For large enough  $m$  and almost all primes  $p$  the function space  $(Y, *)^{(X_m, *)}$  has the homotopy type of a CW complex.*

**PROOF.** By McGibbon and Wilkerson [38] the loop space  $\Omega Y$  is  $p$ -equivalent to a finite product of spheres and loop spaces of spheres, for almost all primes  $p$ . Our assertion now follows from Proposition 3.3.1 in conjunction with Example 9 on page 56.  $\square$

We recall the following conjecture due to John C. Moore (see Selick [54]).

**CONJECTURE 3.3.3 (Moore).** *Let  $Y$  be the localization at  $p$  of a simply connected finite complex. Then the following are equivalent.*

- (1) *There exists a number  $l$  such that  $\Omega^l Y$  has an  $H$ -space exponent.*
- (2) *The complex  $Y$  has a homotopy exponent.*
- (3) *The rational homotopy groups  $\pi_*(Y) \otimes \mathbb{Q}$  are totally finite-dimensional.*

This conjecture can now be rephrased in the following manner.

**CONJECTURE 3.3.4.** *Let  $Y$  be the localization at  $p$  of a simply connected finite complex. Then the following are equivalent.*

- (1\*) *There exists a number  $m$  such that the function space  $(Y, *)^{(X_m, *)}$  has the homotopy type of a CW complex.*
- (2) *The complex  $Y$  has a homotopy exponent.*
- (3\*) *There exists a number  $m$  such that the function space  $(Y, *)^{(X_m, *)}$  is weakly contractible.*

We turn to the proof of Proposition 3.3.1.

Consider the space of maps  $K_{(P)} \rightarrow Y$  where  $K$  is a simply connected finite complex,  $P$  is a set of primes (possibly empty), and  $Y$  is any simply connected complex. Let  $P' = \mathbb{P} \setminus P$ .

Corollary 3.2.2 yields the pullback diagram

$$\begin{array}{ccc} (Y, *)^{(K_{(P)}, *)} & \longrightarrow & (Y_{(P)}, *)^{(K_{(P)}, *)} \\ \downarrow & & \downarrow \\ (Y_{(P')}, *)^{(K_{(0)}, *)} & \longrightarrow & (Y_{(0)}, *)^{(K_{(0)}, *)} \end{array}$$

Note that reapplying Theorem 3.2.1 we deduce  $(Y_{(P)}, *)^{(K_{(P)}, *)} \simeq (Y_{(P)}, *)^{(K, *)}$  and  $(Y_{(0)}, *)^{(K_{(0)}, *)} \simeq (Y_{(0)}, *)^{(K, *)}$  whence by Milnor's theorem we infer

PROPOSITION 3.3.5. (i) *The space  $(Y, *)^{(K(P), *)}$  has CW type if and only if the image of  $(Y, *)^{(K(P), *)} \rightarrow (Y_{(P')}, *)^{(K(P), *)} \simeq (Y_{(P')}, *)^{(K(0), *)}$  has.*

*In particular, if  $Y$  is  $R$ -local where  $R$  is a set of primes contained in  $P$ , then  $(Y, *)^{(K(P), *)}$  has CW homotopy type.*

(ii) *If  $Y$  is a loop space, then  $(Y, *)^{(K(P), *)}$  has CW type if and only if  $(Y_{(P')}, *)^{(K(P), *)} \simeq (Y_{(P')}, *)^{(K(0), *)}$  has.*

PROOF. Only (ii) needs be discussed. Suppose  $Y \simeq \Omega(Z, *)$ . Let  $F$  be the homotopy fibre of  $Z \rightarrow Z_{(P')}$ , and  $F'$  the homotopy fibre of  $F \rightarrow Z$ . Then  $F'$  is homotopy equivalent to  $\Omega(Z_{(P')}, *) \simeq Y_{(P')}$ , and  $F' \rightarrow F$  is a principal fibration with fibre homotopy equivalent to  $\Omega(Z, *) \simeq Y$ . Then

$$(F', *)^{(K(P), *)} \rightarrow (F, *)^{(K(P), *)}$$

is a (principal) fibration with fibre (over the constant map) homotopy equivalent to  $(Y, *)^{(K(P), *)}$ . Note that  $F$  is  $P$ -local. Thus by Theorem 3.2.1,  $(F, *)^{(K(P), *)}$  is homotopy equivalent to  $(F, *)^{(K, *)}$  which has CW type. Thus if  $(Y, *)^{(K(P), *)}$  has CW type, so has  $(F', *)^{(K(P), *)} \simeq (Y_{(P')}, *)^{(K(P), *)}$ , by Stasheff's theorem.  $\square$

COROLLARY 3.3.6. *If  $(Y_{(P')}, *)^{(K(0), *)}$  happens to be contractible, then the space  $(Y, *)^{(K(P), *)}$  is homotopy equivalent to  $(Y_{\tau P}, *)^{(K, *)}$  where  $Y_{\tau P}$  denotes the homotopy fibre of the localization  $Y \rightarrow Y_{(P')}$ .*  $\square$

LEMMA 3.3.7. *Let  $P$  and  $R$  be sets of primes with  $P \cap R \neq \emptyset$ , and let  $Y$  be a simply connected complex of finite type over  $\mathbb{Z}_{(R)}$ . Further let  $A$  be a finite complex, and denote  $\Gamma = [SA, Y]_*$ . Let  $p_1, p_2, \dots$  be a sequence of primes in  $P$  with each prime in  $P$  occurring infinitely many times. The limit group of the inverse sequence  $\dots \rightarrow \Gamma \xrightarrow{\xi \rightarrow \xi^{p_2}} \Gamma \xrightarrow{\xi \rightarrow \xi^{p_1}} \Gamma$  is trivial.*

PROOF. Since  $A$  is finite,  $[SA, Y]_* \cong [SA, Y_i]_*$  for some Postnikov section  $Y_i$  of  $Y$ , and the lemma follows easily by induction, applying left-exactness of the functor  $\lim$ , and taking into account that no element of a finitely generated  $\mathbb{Z}_{(R)}$ -module is infinitely divisible by a prime belonging to  $P \cap R \subset R$ .  $\square$

THEOREM 3.3.8. *Let  $P$  be a set of primes. Let  $Y$  be a simply connected CW complex and let  $K$  be the suspension of a finite complex. Denote  $\Gamma_k = \pi_k((Y, *)^{(K, *)}, *)$ .*

a. *If the function space  $(Y, *)^{(K(\mathbb{F} \setminus P), *)}$  has the homotopy type of a CW complex, then there exists an integer  $b \in (P)$  so that for all  $k \geq 1$  the group  $D_k = b \cdot \Gamma_k$  is  $P$ -divisible and contains no  $P$ -torsion summands. The short exact sequence  $0 \rightarrow D_k \rightarrow \Gamma_k \rightarrow$*

$\Gamma_k \otimes \mathbb{Z}_b \rightarrow 0$  induces the split short exact sequence

$$0 \rightarrow D_{k(P)} \rightarrow \Gamma_{k(P)} \rightarrow \Gamma_k \otimes \mathbb{Z}_b \rightarrow 0$$

where  $D_{k(P)}$  is a rational group.

Moreover, the divisible part of  $\Gamma_0$  does not contain  $P$ -torsion, and therefore the divisible part of  $\Gamma_{0(P)}$  is rational.

- b. If, in addition to the assumption of **a.**,  $Y$  is of finite type over  $\mathbb{Z}_{(R)}$  where  $P \cap R$  is nonempty then for all  $k \geq 1$  the localization  $\Gamma_{k(P)}$  is a finite abelian  $b$ -bounded group. Consequently  $(Y_{(P)}, *)^{(K_{(\mathbb{F} \setminus P)}, *)}$  is contractible and the abelian  $H$ -space  $(\Omega Y_{(P)}, *)^{(K, *)}$  admits an  $H$ -space exponent.
- c. Conversely, if the  $H$ -space  $(Y_{(P)}, *)^{(K, *)}$  admits an  $H$ -space exponent, then  $(Y_{(P)}, *)^{(K_{(\mathbb{F} \setminus P)}, *)}$  is contractible.

PROOF. We construct localization  $X = K_{(\mathbb{F} \setminus P)}$  as the reduced infinite mapping telescope of the sequence

$$K \xrightarrow{p_1} K \xrightarrow{p_2} K \xrightarrow{p_3} \dots$$

where  $p_1, p_2, \dots$  is a sequence of primes in  $P$  with each prime in  $P$  occurring infinitely many times.

More precisely, we assume  $X$  filtered by a based sequence  $L_0 \leq L_1 \leq L_2 \leq \dots$  of finite subcomplexes each of which is homotopy equivalent to  $K$ , and for all  $i$ , the inclusion  $L_{i-1} \leq L_i$  corresponds to  $H$ -cogroup multiplication by  $p_i$ . Then  $(Y, *)^{(X, *)}$  is the limit of the associated inverse sequence  $(Y, *)^{(L_i, *)}$  where the bonding map  $(Y, *)^{(L_i, *)} \rightarrow (Y, *)^{(L_{i-1}, *)}$  is equivalent to  $H$ -group multiplication by  $p_i$  on  $(Y, *)^{(K, *)}$ .

In particular, the morphisms  $\pi_k((Y, *)^{(L_i, *)}, *) \rightarrow \pi_k((Y, *)^{(L_{i-1}, *)}, *)$  correspond to multiplication by  $p_i$  on  $\Gamma_k$  (the  $p_i$ -th power map on  $\Gamma_0$ ). Since  $(Y, *)^{(X, *)}$  has the homotopy type of a CW complex, it follows by Theorem 1.4.4 that there exists a number  $i$  such that for all  $j \geq i$  and for all  $k \geq 1$ , the image of the morphism  $\pi_k((Y, *)^{(L_j, *)}, *) \rightarrow \pi_k((Y, *)^{(L_0, *)}, *)$  equals that of  $\pi_k((Y, *)^{(L_i, *)}, *) \rightarrow \pi_k((Y, *)^{(L_0, *)}, *)$ . Set  $b = p_1 \dots p_i$ . We have shown

$$(*) \quad \forall j > i \quad \forall k \geq 1 : p_{i+1} \dots p_j (b\Gamma_k) = b\Gamma_k.$$

Denote  $D_k = b\Gamma_k$ . Property  $(*)$  shows that  $D_k$  is  $P$ -divisible, hence  $D_{k(P)}$  is divisible and consequently

$$(**) \quad 0 \rightarrow D_{k(P)} \rightarrow \Gamma_{k(P)} \rightarrow \Gamma_k \otimes \mathbb{Z}_b \rightarrow 0$$

is a split exact sequence. In particular,  $r(\Gamma_{k(P)}) \cong \Gamma_k \otimes \mathbb{Z}_b$ .

For  $k = 0$  let  $D_0$  be the divisible part of  $\Gamma_0$ . We may also record this in a short (split) exact sequence  $0 \rightarrow D_0 \rightarrow \Gamma_0 \rightarrow R_0 \rightarrow 0$ .

The sequence of fundamental groups  $\pi_1((Y, *)^{(L_i, *)}, *)$  satisfies the Mittag-Leffler property, and by Proposition 1.4.1 the set of path components  $[X, Y]_*$  is the limit of

$$(\star) \quad \cdots \rightarrow [L_i, Y]_* \rightarrow [L_{i-1}, Y]_* \rightarrow \cdots$$

Hence  $[X, Y]_*$  has a group structure with which the restrictions  $[X, Y]_* \rightarrow [L_i, Y]_*$  are homomorphisms. Thus Theorem 1.4.4 implies that for some  $N > 0$ , the restriction induced morphisms

$$(\dagger) \quad \pi_k((Y, *)^{(X, *)}, *) \rightarrow \pi_k((Y, *)^{(L_i, *)}, *)$$

are injective for all  $k$  and all  $i \geq N$ .

Let  $k \geq 0$ . Let for any group  $G$  the symbol  $\lim_{p_i} G$  denote the limit group of the sequence  $\cdots \rightarrow G \xrightarrow{p_2} G \xrightarrow{p_1} G$ . Functoriality of  $\lim$  guarantees commutativity of

$$\begin{array}{ccc} \lim_{p_i} D_k & \longrightarrow & \lim_{p_i} \Gamma_k \\ \downarrow & & \downarrow \\ D_k & \longrightarrow & \Gamma_k \end{array}$$

Here the vertical arrows denote canonical projections. Left-exactness of  $\lim$  implies that the morphism  $\lim_{p_i} D_k \rightarrow \lim_{p_i} \Gamma_k$  is injective (actually bijective for  $k \geq 1$ ). If  $D_k$  contains a  $\mathbb{Z}_{p^\infty}$  summand for some prime  $p \in P$ , then the kernel of any canonical projection  $\lim_{p_i} D_k \rightarrow D_k$  is uncountable. In particular, the kernel is nontrivial. Hence the same holds for  $\lim_{p_i} \Gamma_k \rightarrow \Gamma_k$ , contradicting injectivity of  $(\dagger)$ .

Thus for all  $k \geq 0$ ,  $D_k$  does not contain  $\mathbb{Z}_{p^\infty}$ -summands for  $p \in P$ . For  $k \geq 1$  it follows that  $D_{k(P)}$  is torsion-free, and since it is divisible, it is a rational group. For a reduced group  $R$ , the divisible part of localization  $R_{(P)}$  is always rational, hence since  $D_0 = d(\Gamma_0)$  does not contain  $P$ -torsion summands,  $d(\Gamma_{0(P)}) \cong D_{0(P)} \oplus d(R_{0(P)})$  is rational. This proves **a**.

If  $Y$  is of finite type over  $\mathbb{Z}_{(R)}$  then the groups  $\Gamma_k$  are finitely generated as modules over  $\mathbb{Z}_{(R)}$ , and  $\Gamma_{k(P)}$  finitely generated as modules over  $\mathbb{Z}_{(P \cap R)}$ . Since  $P \cap R$  is nonempty the rational group  $D_{k(P)}$  must be trivial and  $\Gamma_{k(P)}$  a finite abelian group bounded by  $b$ . We form a sequence

$$i = \varphi(0) < \varphi(1) < \varphi(2) < \cdots$$

such that for each  $j$ , the product  $p_{\varphi(j-1)+1} p_{\varphi(j-1)+2} \cdots p_{\varphi(j)}$  is a multiple of  $b$ . Then the restriction  $(Y, *)^{(L_{\varphi(j)}, *)} \rightarrow (Y, *)^{(L_{\varphi(j-1)}, *)}$  induces trivial

morphisms on homotopy groups  $\Gamma_k$  for  $k \geq 1$  and consequently all corresponding homotopy groups of the limit space  $(Y, *)^{(X, *)}$  are trivial. We have to show that  $(Y, *)^{(X, *)}$  is also path-connected.

As discussed above, the group  $[X, Y]_*$  is the limit of the sequence  $(\star)$  which is isomorphic to

$$\cdots \rightarrow [K, Y]_* \xrightarrow{\xi \mapsto \xi^{p^2}} [K, Y]_* \xrightarrow{\xi \mapsto \xi^{p^1}} [K, Y]_*.$$

The limit vanishes by Lemma 3.3.7. Hence  $(Y, *)^{(X, *)}$  is contractible. Corollary 1.2.7 yields the rest of **b.** while Theorem 1.2.6 gives **c.**  $\square$

**PROOF OF PROPOSITION 3.3.1.** Statements **(i)** and **(iii)** follow from Theorem 3.3.8 in conjunction with Proposition 3.3.5 and Corollary 3.3.6 by taking  $K = S^m$ .

In order to prove **(ii)** we note that by the generalized Serre's theorem (see McGibbon and Neisendorfer [37]) the group  $\pi_k(Y)$  contains a subgroup of order  $p$  for infinitely many  $k$ , and each prime  $p$ .  $\square$

## Maps to Eilenberg-MacLane spaces

The purpose of this chapter is to investigate the space of pointed maps from an arbitrary CW complex  $X$  to an Eilenberg-MacLane space  $K(G, n)$ , where  $G$  is an abelian group.

LEMMA 4.0.1 (Minimal decomposition). *Given a simply connected CW complex  $X$  and a specific free presentation of each of its homology groups  $H_n(X) \cong \langle S_n; \Sigma_n \rangle$  ( $n \geq 2$ ), there is a CW complex  $Z$  and a homotopy equivalence  $f: Z \rightarrow X$  such that each cell of  $Z$  is either*

- a.** a ‘generator’  $n$ -cell  $e_\alpha^n$  which is a cycle in cellular homology mapped by  $f$  to a cellular cycle representing the specified generator  $\alpha \in S_n$ ;  
or
- b.** a ‘relator’  $(n + 1)$ -cell  $e_\beta^{n+1}$  with cellular boundary corresponding to the specified relator  $\beta \in \Sigma_{n+1}$ .

*In addition, we may assume that all cells are attached along based maps of spheres, and that*

- a’.** every generator  $n$ -cell  $e_\alpha^n$  is attached along a (based) map  $\varphi_\alpha^n: S^{n-1} \rightarrow Z^{(n-2)}$  of the  $(n - 1)$ -sphere into the  $(n - 2)$ -skeleton.
- b’.** The closure of a relator  $(n + 1)$ -cell  $e_\beta^{n+1}$  meets an  $n$ -cell  $e_\alpha^n$  if and only if the incidence number  $[e_\beta^{n+1} : e_\alpha^n]$  is non-zero.

PROOF. Theorem 4C.1 of Hatcher [26] states **a.** and **b.** for  $X$  of finite type. The finite type restriction is unnecessary, and the generalization to **b.’** is straightforward. For **a.’**, use Whitehead’s certain exact sequence [64] as done in Rutter [51], Lemma 2.1.  $\square$

DEFINITION. A CW decomposition with the properties of  $Z$  as in Lemma 4.0.1 is called *minimal*.

To a minimal decomposition of a complex  $X$  we may associate a homology filtration

$$\{*\} = X_1 \leq X_2 \leq X_3 \leq \dots$$

by letting  $X_i$  be the union of the  $i$ -skeleton  $X^{(i)}$  of  $X$  and all the relator  $(i + 1)$ -cells. Then the inclusion induced morphism  $H_j(X_i) \rightarrow H_j(X)$  is



bijjective for  $j \leq i$ , and  $H_j(X_i) = 0$  for  $j > i$ . Moreover, if we set  $H_i = H_i(X)$ , then the quotient  $X_i/X_{i-1}$  is a Moore complex  $M(H_i, i)$ , and there exists a map  $\varphi: M(H_i, i-1) \rightarrow X_{i-1}$  which induces a homotopy equivalence  $C_\varphi \rightarrow X_i$ . Here  $C_\varphi$  denotes the homotopy cofibre of  $\varphi$ .

**4.1. Homotopy type of  $K(G, n)^X$  depends only upon homology of  $X$**

Let  $Y$  be a CW complex of type  $K(G, n)$  for  $G$  abelian and let  $X$  be any CW complex. Then  $Y$  and consequently  $(Y, *)^{(X, *)}$  are topological abelian monoids (if  $G$  is uncountable then in the category of compactly generated spaces; see also Appendix B). As has been observed by Thom [61], it follows that all Postnikov invariants of (a CW approximation of)  $(Y, *)^{(X, *)}$  vanish, and  $(Y, *)^{(X, *)}$  is a *weak* product of Eilenberg-MacLane spaces. Thus the weak homotopy type of  $(Y, *)^{(X, *)}$  only depends upon homology groups of  $X$ . We show that the actual homotopy type of  $(Y, *)^{(X, *)}$  only depends upon homology groups of  $X$ .

**THEOREM 4.1.1.** *Let  $X$  be a connected CW complex and let  $Y$  be a CW complex of type  $K(G, n)$  where  $G$  is abelian. Then  $(Y, *)^{(X, *)}$  has the homotopy type of the product*

$$(Y, *)^{(M_1, *)} \times \dots \times (Y, *)^{(M_n, *)},$$

where  $M_i = M(H_i X, i)$ . In particular,  $(Y, *)^{(X, *)}$  has the homotopy type of a CW complex if and only if the spaces  $(Y, *)^{(M_i, *)}$ , for  $i \leq n$ , have.

**REMARK.** Since  $Y$  is homotopy equivalent to  $\Omega K(G, n+1)$ , the function space  $(Y, *)^{(X, *)}$  is homotopy equivalent to  $(\Omega K(G, n+1), *)^{(X, *)} \approx (K(G, n+1), *)^{(SX, *)}$ , and it suffices to consider both  $X$  and  $Y$  simply connected. In view of this we define  $M(A, 1)$  to be a CW complex  $M$  with  $H_1(M) \cong A$  and  $H_k(M) \cong 0$  for  $k \geq 2$ .

Note that by Proposition B.1.3 we may arrange for a pointed homotopy equivalence in Theorem 4.1.1, by choosing an appropriate space  $Y$  of type  $K(G, n)$ .

The rest of this section will be devoted to proving Theorem 4.1.1.

**LEMMA 4.1.2.** *Let  $Y$  be an Eilenberg-MacLane space  $K(G, n)$  and let  $X$  be a simply connected CW complex with homology filtration  $X_2 \leq X_3 \leq \dots$  associated to a minimal decomposition. Then  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(X_n, *)}$  is a homotopy equivalence.*

**PROOF.** By Proposition 2.6.1 we know that  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(X^{(n+1)}, *)}$  is a homotopy equivalence, and that

$$(*) \quad (Y, *)^{(X^{(n+1)}, *)} \rightarrow (Y, *)^{(X_n, *)}$$

is a homotopy equivalence onto image. It suffices to prove that  $(*)$  is surjective. By assumption  $X^{(n+1)}$  is obtained from  $X_n$  by attaching generator  $(n+1)$ -cells, that is  $X^{(n+1)}$  is the cofibre of a map

$$\varphi: \bigvee_{\lambda} S^n \rightarrow X^{(n-1)} \hookrightarrow X_n.$$

Then the restriction  $(*)$  is the homotopy fibre of the induced map  $\varphi^\# : (Y, *)^{(X_n, *)} \rightarrow (Y, *)^{(\bigvee_{\lambda} S^n, *)}$  which factors through  $(Y, *)^{(X^{(n-1)}, *)}$ . Since  $Y$  is a  $K(G, n)$ , on the group of path components  $\varphi^\#$  transforms as

$$\tilde{H}^n(X_n; G) \rightarrow \tilde{H}^n(X^{(n-1)}; G) \rightarrow \tilde{H}^n(\bigvee_{\lambda} S^n; G).$$

But  $\tilde{H}^n(X^{(n-1)}; G)$  is trivial hence the homotopy fibre  $(*)$  is surjective.  $\square$

PROOF OF THEOREM 4.1.1. Thus we may assume a minimal decomposition for  $X$  and take the associated homology filtration  $X_2 \leq X_3 \leq \dots$ . By Lemma 4.1.2 we know that the restriction  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(X_n, *)}$  is a homotopy equivalence, and we may replace  $X$  by its  $n$ -th homology stage  $X_n$ . For convenience concerning connectedness, we may view  $Y$  as the triple loop space of the  $H$ -group  $K(G, n+3)$ . Thus it suffices to show that  $(K(G, n+3), *)^{(X_n, *)}$  has the homotopy type of  $\prod_{i=2}^n (K(G, n+3), *)^{(M_i, *)}$ . By renumbering, it suffices to prove Lemma 4.1.3 below.  $\square$

LEMMA 4.1.3. *Let  $Y = K(G, n)$  and  $3 \leq i \leq n-3$ . The fibration  $(Y, *)^{(X_i, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$  is trivial.*

The following step is crucial.

LEMMA 4.1.4. *Let  $(L', L)$  be the adjunction of an  $i$ -cell  $e$  attached along a based map*

$$\varphi: (S^{i-1}, *) \rightarrow (L^{(i-2)}, x_0)$$

*of the  $(i-1)$ -sphere to the  $(i-2)$ -skeleton of  $L$ . Denote by  $K$  the smallest subcomplex of  $L$  that contains the image of  $\varphi$ . Note that  $K$  is finite and contained in  $L^{(i-2)}$ . Set  $K' = K \cup e$ .*

*Then the induced map  $\varphi^\# : (Y, *)^{(K, *)} \rightarrow (Y, *)^{(S^{i-1}, *)}$  is nullhomotopic as a pointed map. Hence the restriction fibration  $(Y, *)^{(K', *)} \rightarrow (Y, *)^{(K, *)}$  is fibre-homotopy trivial and consequently so is  $(Y, *)^{(L', *)} \rightarrow (Y, *)^{(L, *)}$ . If  $\Phi$  denotes the fibre of  $(Y, *)^{(B^i, *)} \rightarrow (Y, *)^{(S^{i-1}, *)}$  (over the constant), the canonical fibre-homotopy equivalence  $(Y, *)^{(L, *)} \times \Phi \rightarrow (Y, *)^{(L', *)}$  restricts to  $(Y, *)^{(K, *)} \times \Phi \rightarrow (Y, *)^{(K', *)}$ . In particular, the canonical sections give*

rise to a commutative diagram as follows.

$$\begin{array}{ccc} (Y, *)^{(L', *)} & \longrightarrow & (Y, *)^{(K \cup e, *)} \\ \uparrow s & & \uparrow s_K \\ (Y, *)^{(L, *)} & \longrightarrow & (Y, *)^{(K, *)} \end{array}$$

In other words, if  $f, g: L \rightarrow Y$  are functions with  $f|_K = g|_K$ , the extensions  $s(f)$  and  $s(g)$  restrict to the same function on  $K \cup e$ .

REMARK. We use the term ‘canonical section’ according to definition following Lemma A.0.3.

PROOF. Lemma 0.0.8 yields the following pullback.

$$\begin{array}{ccc} (Y, *)^{(L', *)} & \longrightarrow & (Y, *)^{(B^i, *)} \\ \downarrow & & \downarrow \\ (Y, *)^{(L, *)} & \xrightarrow{\varphi^\#} & (Y, *)^{(S^{i-1}, *)} \end{array}$$

This is to say that the fibration  $(Y, *)^{(L', *)} \rightarrow (Y, *)^{(L, *)}$  is the principal fibration obtained as the homotopy fibre of  $\varphi^\#$ . The latter factors as

$$(Y, *)^{(L, *)} \rightarrow (Y, *)^{(K, *)} \xrightarrow{\varphi^\#} (Y, *)^{(S^{i-1}, *)}.$$

Since  $K$  is finite,  $(Y, *)^{(K, *)}$  is globally well-pointed (see Corollary B.2.5 and has the homotopy type of a CW complex (hence has the pointed homotopy type of a CW complex). Thus we compute

$$[(Y, *)^{(K, *)}, (Y, *)^{(S^{i-1}, *)}]_* \cong \tilde{H}^{n-i+1}((Y, *)^{(K, *)}; G).$$

But  $\pi_k((Y, *)^{(K, *)}, \text{const}) \cong \tilde{H}^{n-k}(K; G)$  is trivial for  $n - k \geq (i - 2) + 1$  since  $K$  is  $(i - 2)$ -dimensional. By the Hurewicz theorem the group  $[(Y, *)^{(K, *)}, (Y, *)^{(S^{i-1}, *)}]_*$  is trivial, therefore  $\varphi^\#$  is nullhomotopic as a pointed map. Let  $h: (Y, *)^{(K, *)} \times I \rightarrow (Y, *)^{(S^{i-1}, *)}$  denote a homotopy between the constant map and  $\varphi^\#$ .

The canonical section  $(Y, *)^{(K, *)} \rightarrow (Y, *)^{(K', *)}$  is homotopic as a pointed map to the section

$$s_K: (Y, *)^{(K, *)} \rightarrow (Y, *)^{(K', *)}, \quad s_K(f)|_K = f, \quad s_K(f)(\phi[\zeta, t]) = h(f, t)(\zeta).$$

Here  $[-, -]$  denotes the quotient map  $S^{i-1} \times I \rightarrow S^{i-1} \times I / S^{i-1} \times 0 = B^i$ , while  $\phi: B^i \rightarrow K \cup e$  denotes the characteristic map.

The pre-composition of  $h$  with restriction  $(Y, *)^{(L, *)} \rightarrow (Y, *)^{(K, *)}$  gives a pointed trivialization of  $\varphi^\# : (Y, *)^{(L, *)} \rightarrow (Y, *)^{(S^{i-1}, *)}$  and the induced fibre homotopy equivalence  $(Y, *)^{(L, *)} \times \Phi \rightarrow (Y, *)^{(L', *)}$  clearly lifts  $(Y, *)^{(K, *)} \times \Phi \rightarrow (Y, *)^{(K', *)}$ .

In particular, the section  $s : (Y, *)^{(L, *)} \rightarrow (Y, *)^{(L', *)}$ , defined by

$$s(f)|_L = f, \quad s(f)(\phi[\zeta, t]) = h(f|_K, t)(\zeta)$$

(is homotopic to the canonical one) and lifts  $s_K$ .  $\square$

PROOF OF LEMMA 4.1.3. We construct  $X_i$  from  $X_{i-1}$  in two stages. First, we attach all the generator  $i$ -cells via an attaching map

$$\varphi = \vee_\lambda \varphi_\lambda : \bigvee_\lambda S^{i-1} \rightarrow X_{i-1}^{(i-2)} \hookrightarrow X_{i-1}$$

to obtain  $X^{(i)}$ . By Lemma 0.0.8 the map  $\varphi$  induces the pullback diagram below.

$$\begin{array}{ccccc} (Y, *)^{(X^{(i)}, *)} & \longrightarrow & (Y, *)^{(\vee_\lambda B^i, *)} & \xrightarrow{=} & \prod_\lambda (Y, *)^{(B^i, *)} \\ \downarrow & & \downarrow & & \downarrow \\ (Y, *)^{(X_{i-1}, *)} & \xrightarrow{\varphi^\#} & (Y, *)^{(\vee_\lambda S^{i-1}, *)} & \xrightarrow{=} & \prod_\lambda (Y, *)^{(S^{i-1}, *)} \end{array}$$

The induced map  $\varphi^\#$  maps into the product  $\prod_\lambda (Y, *)^{(S^{i-1}, *)}$ . The composition with the  $\lambda$ -th projection  $\text{pr}_\lambda \circ \varphi^\#$  equals  $\varphi_\lambda^\#$  which is nullhomotopic as a pointed map by Lemma 4.1.4. Hence so is  $\varphi^\#$ , by a product homotopy. Let  $K_\lambda$  denote the smallest subcomplex of  $X$  that contains  $\text{im } \varphi_\lambda$  and fix homotopies  $h_\lambda : (Y, *)^{(K_\lambda, *)} \times I \rightarrow (Y, *)^{(S^{i-1}, *)}$  between  $*$  and  $\varphi_\lambda^\#$ . Then the ‘canonical section’  $s : (Y, *)^{(X_{i-1}, *)} \rightarrow (Y, *)^{(X^{(i)}, *)}$  has the form

$$s(f)|_{X_{i-1}} = f, \quad s(f)(\phi_\lambda[\zeta, t]) = h_\lambda(f|_{K_\lambda}, t)(\zeta).$$

Now we attach the relator  $(i+1)$ -cells to  $X^{(i)}$  via an attaching map

$$\psi = \vee_\mu \psi_\mu : \bigvee_\mu S^i \rightarrow X^{(i)}.$$

We obtain a pullback diagram similar to the one above

$$\begin{array}{ccc} (Y, *)^{(X_i, *)} & \longrightarrow & (Y, *)^{(\bigvee_{\mu} B^{i+1}, *)} \\ \downarrow & & \downarrow \\ (Y, *)^{(X^{(i)}, *)} & \xrightarrow{\psi^{\#}} & (Y, *)^{(\bigvee_{\mu} S^i, *)} \end{array}$$

We claim that we can lift  $s$  to a map (necessarily a section)  $s': (Y, *)^{(X_{i-1}, *)} \rightarrow (Y, *)^{(X_i, *)}$ . To prove this we observe projections of the composite

$$(Y, *)^{(X_{i-1}, *)} \xrightarrow{s} (Y, *)^{(X^{(i)}, *)} \xrightarrow{\psi^{\#}} (Y, *)^{(\bigvee_{\mu} S^i, *)} = \prod_{\mu} (Y, *)^{(S^i, *)}$$

onto the factors. Let  $e_{\mu}$  be an  $(i+1)$ -relator cell with attaching map  $\psi_{\mu}: S^i \rightarrow X^{(i)}$ , and let  $L$  denote the smallest subcomplex containing the image of  $\psi_{\mu}$ . Note that  $L$  contains only *generator*  $i$ -cells; by property **b.**' of Lemma 4.0.1. (On  $L \cap X^{(i-1)}$  there are no particular restrictions.) We number the  $i$ -cells  $e_1, \dots, e_r$  and let  $K$  denote  $L \setminus \{e_1, \dots, e_r\}$ . Then  $K$  is a finite subcomplex contained in  $X^{(i-1)} \subset X_{i-1}$ . By construction, the following diagram is commutative.

$$\begin{array}{ccccc} (Y, *)^{(X^{(i)}, *)} & \longrightarrow & (Y, *)^{(L, *)} & \xrightarrow{\psi_{\mu}^{\#}} & (Y, *)^{(S^i, *)} \\ \uparrow s & & \uparrow s_K & & \\ (Y, *)^{(X_{i-1}, *)} & \longrightarrow & (Y, *)^{(K, *)} & & \end{array}$$

We compute

$$[(Y, *)^{(K, *)}, (Y, *)^{(S^i, *)}]_* \cong \tilde{H}^{n-i}((Y, *)^{(K, *)}; G).$$

But  $\pi_k((Y, *)^{(K, *)}, \text{const}) \cong \tilde{H}^{n-k}(K; G)$  is trivial for  $n-k \geq (i-1)+1$ , i.e.  $k \leq n-i$ , since  $K$  is  $(i-1)$ -dimensional. By the Hurewicz theorem, the group  $[(Y, *)^{(K, *)}, (Y, *)^{(S^i, *)}]_*$  is trivial, and therefore the composite  $\psi_{\mu}^{\#} \circ s_K$  is nullhomotopic (as a pointed map). Hence so is the pre-composite with restriction  $(Y, *)^{(X_{i-1}, *)} \rightarrow (Y, *)^{(K, *)}$ . By commutativity, this equals  $\psi_{\mu}^{\#} \circ s: (Y, *)^{(X_{i-1}, *)} \rightarrow (Y, *)^{(S^i, *)}$ . Hence also the map into the product  $\psi^{\#} \circ s$  is nullhomotopic. Therefore  $s$  may be lifted to  $\sigma: (Y, *)^{(X_{i-1}, *)} \rightarrow (Y, *)^{(X_i, *)}$ , and  $\sigma$  is automatically a section. But there exists a (pointed cellular) map  $k: M(H_i X, i-1) \rightarrow X_{i-1}$  such that the inclusion  $X_{i-1} \rightarrow X_i$  extends to a homotopy equivalence  $F: C_k \rightarrow X_i$  with  $C_k$  the mapping cone

of  $k$ . Lemma 0.0.8 gives the following commutative diagram where the square is a pullback and  $F^\#$  is a fibre homotopy equivalence.

$$\begin{array}{ccccc}
 (Y, *)^{(X_i, *)} & \xrightarrow{F^\#} & (Y, *)^{(C_k, *)} & \longrightarrow & (Y, *)^{(CM(H_i X, i-1), *)} \\
 & \searrow & \downarrow & & \downarrow \\
 & & (Y, *)^{(X_{i-1}, *)} & \xrightarrow{k^\#} & (Y, *)^{(M(H_i X, i-1), *)}
 \end{array}$$

The section of  $(Y, *)^{(X_i, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$  trivially defines a section of  $(Y, *)^{(C_k, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$ . Since  $(Y, *)^{(CM(H_i X, i-1), *)}$  is contractible, the existence of a section implies that  $k^\#$  is nullhomotopic. Hence  $(Y, *)^{(C_k, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$  is fibre homotopy equivalent to a product fibration. Since  $F^\#$  is a fibre homotopy equivalence also  $(Y, *)^{(X_i, *)} \rightarrow (Y, *)^{(X_{i-1}, *)}$  is fibre homotopy equivalent to a product fibration.  $\square$

#### 4.2. Spaces of maps $M(A, m) \rightarrow K(G, n)$

The purpose of this section is to strengthen Theorem 2.2.5 in the particular case of function spaces  $K(G, n)^{M(A, m)}$ , see Corollary 4.2.3 below. We apply the latter in the following section where we give necessary and sufficient conditions for  $K(G, n)^{M(A, m)}$  to have CW homotopy type.

The following delooping proposition is an immediate consequence of Theorem 2.5.1.

**PROPOSITION 4.2.1.** *Let  $X = M(A, m)$  and  $Y = K(G, n)$  where  $m \leq n - 2$ . If  $(Y, *)^{(X, *)}$  has CW homotopy type, then so also has  $(Y', *)^{(X, *)}$  where  $Y' = K(G, n + 1)$ .  $\square$*

The case  $m = n - 1$  can, by a different method, be obtained in a weaker form.

**PROPOSITION 4.2.2.** *Let  $X = M(A, m)$  and  $Y = K(G, n)$  where  $m \leq n - 1$ . Assume that  $(Y, *)^{(X, *)}$  is contractible. Then so also is  $(Y', *)^{(X, *)}$  where  $Y' = K(G, n + 1)$ .*

**PROOF.** We assume that  $X$  has a minimal decomposition. Let  $K_\infty$  be a countable subcomplex of  $X$ , and let  $K_1 \leq K_2 \leq \dots$  be a filtration of finite subcomplexes for  $K_\infty$ . By Theorem 2.2.5 and Proposition 1.1.2 there exists a sequence of finite subcomplexes  $L_0 \leq L_1 \leq L_2 \leq \dots$  such that  $L_i \geq K_i$  for all  $i$  and, moreover

- (i) the inclusion  $H_*(L_i) \rightarrow H_*(X)$  is injective on the image of  $H_*(L_{i-1}) \rightarrow H_*(L_i)$ .

- (ii) The restriction  $\Omega((Y, *)^{(L_i, *)}, *) \rightarrow \Omega((Y, *)^{(L_{i-1}, *)}, *)$  is nullhomotopic.

The subcomplexes  $L_i$  are Moore spaces of type  $M(H_m L_i, m)$ . In particular,

$$\pi_{n-m}((Y, *)^{(L_i, *)}, *) \rightarrow \pi_{n-m}((Y, *)^{(L_{i-1}, *)}, *)$$

transforms naturally as

$$(\dagger) \quad \text{Hom}(H_m L_i, G) \rightarrow \text{Hom}(H_m L_{i-1}, G)$$

which is trivial by property (ii) since  $n-m \geq 1$ . For each  $i$  let  $B_i$  denote the image of  $H_m L_i \rightarrow H_m X = A$ . By property (i) actually  $B_{i-1} \leq H_m L_i$ , for all  $i \geq 1$ . For  $i \geq 1$  the induced morphism  $\text{Hom}(B_{i+1}, G) \rightarrow \text{Hom}(B_{i-1}, G)$  factors as

$$\text{Hom}(B_{i+1}, G) \rightarrow \text{Hom}(H_m L_{i+1}, G) \rightarrow \text{Hom}(H_m L_i, G) \rightarrow \text{Hom}(B_{i-1}, G)$$

and is therefore trivial.

Set  $L_\infty = \cup_i L_i$ . By (i) (see Lemma 2.2.4) the morphism  $H_* L_\infty \rightarrow H_* X$  is injective. Denote  $H_m L_\infty = B_\infty$ . The group  $B_\infty$  is the union of groups  $B_i$ . We may inductively form an expanding sequence of finite CW complexes

$$M(B_1, m) \hookrightarrow M(B_2, m) \hookrightarrow M(B_3, m) \hookrightarrow \dots$$

the union of which is a CW complex of type  $M(B_\infty, m)$ . Set  $M(B_i, m) = M_i$ ,  $1 \leq i \leq \infty$ . Since  $B_i \leq A$  for all  $i$ , the exact sequence  $\text{Ext}(A, G) \rightarrow \text{Ext}(B_i, G) \rightarrow 0$  shows that  $\text{Ext}(B_i, G) = 0$ . Therefore the space  $(Y, *)^{(M_i, *)}$  has the type of  $K(\text{Hom}(B_i, G), n-m)$ , and  $(Y', *)^{(M_i, *)}$  has the type of  $K(\text{Hom}(B_i, G), n-m+1)$ . Then since the restriction  $(Y', *)^{(M_{i+1}, *)} \rightarrow (Y', *)^{(M_{i-1}, *)}$  induces the trivial morphism  $\text{Hom}(B_{i+1}, G) \rightarrow \text{Hom}(B_{i-1}, G)$  on the single non-zero homotopy group, it is nullhomotopic. By Proposition 1.2.4, the space  $(Y', *)^{(M_\infty, *)}$  is contractible. Since  $L_\infty \simeq M_\infty$ , also  $(Y', *)^{(L_\infty, *)}$  is contractible.

This shows that every countable subcomplex  $K_\infty$  of  $X$  is contained in a bigger countable subcomplex  $L_\infty$  such that  $(Y', *)^{(L_\infty, *)}$  is contractible. Hence by Corollary 2.2.7 also  $(Y', *)^{(X, *)}$  is contractible which completes the proof.  $\square$

**COROLLARY 4.2.3.** *Let  $X = M(A, m)$ ,  $Y = K(G, n)$  where  $m \leq n-1$ . If  $(Y, *)^{(X, *)}$  has CW homotopy type, there exists a countable subcomplex  $L$  (containing a given countable subcomplex) of  $X$  such that the restriction  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is a homotopy equivalence. In addition, we may assume  $L = M(A_L, m)$  with  $A_L \rightarrow A$  injective.*

**PROOF.** Since  $(Y, *)^{(X, *)} \simeq (K(G, n+i), *)^{(M(A, m+i), *)}$  for all  $i \geq 0$  we may assume  $m \geq 3$  with no loss of generality.

By Theorem 2.2.5 there exists a countable subcomplex  $L$  (containing a given countable subcomplex) of  $X$  such that the homology of  $L$  injects into that of  $X$  and the space  $(Y, *)^{(X/L, *)}$  is contractible. Since  $B = H_m L \rightarrow H_m X = A$  is injective,  $X/L$  is a Moore complex of type  $M(A/B, m)$ . By Proposition 4.2.2 also  $(Y', *)^{(X/L, *)}$  is contractible where  $Y' = K(G, n+1)$ .

Since all spaces in question are simply connected, there exists a pointed cellular map  $\varphi: M(A/B, m-1) \rightarrow L$  such that the mapping cone of  $\varphi$  is homotopy equivalent to  $X$ . By Lemma 0.0.8 the fibration  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is (fibre homotopically equivalent to) the principal fibration obtain by taking the homotopy fibre of  $\varphi^\#: (Y, *)^{(L, *)} \rightarrow (Y, *)^{(M(A/B, m-1), *)}$ . Since  $(Y, *)^{(M(A/B, m-1), *)} \simeq (Y', *)^{(M(A/B, m), *)}$  is contractible,  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is a homotopy equivalence.  $\square$

Case  $m = n$  is particular.

LEMMA 4.2.4. *Let  $W$  be an  $(n-1)$ -connected CW complex. Let  $C = H_n(W)$  and assume that  $\text{Hom}(C, G) = 0$ . Then  $(K(G, n), *)^{(W, *)}$  is contractible.*

PROOF. By Lemma 4.1.2 we may assume that  $W = M(C, n)$ . Set  $Y = K(G, n)$ . Note that  $Z = (Y, *)^{(W, *)}$  is path-connected. Let  $K_\infty$  be a countable subcomplex of  $W$ , and let  $K_1 \leq K_2 \leq \dots$  be a filtration of finite complexes for  $K_\infty$ . For any subcomplex  $L$  of  $W$  denote by  $Z_L$  the image of  $Z = (Y, *)^{(W, *)} \rightarrow (Y, *)^{(L, *)}$ . Note that  $Z_L$  is the path component of the constant map in  $(Y, *)^{(L, *)}$ . Then  $\{Z_{K_i}\}$  is an inverse sequence of contractible spaces. By (i) of Lemma 1.3.8 its inverse limit is  $Z_{K_\infty}$  and is contractible.  $\square$

PROPOSITION 4.2.5. *Let  $X = M(A, n)$  and  $Y = K(G, n)$ . Then  $(Y, *)^{(X, *)}$  has CW homotopy type if and only if there exists a finitely generated subgroup  $B$  of  $A$  such that  $\text{Hom}(A/B, G) = 0$ . In this case, if the decomposition of  $X$  is minimal, there exists a finite subcomplex  $L$  of  $X$  such that  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is a homotopy equivalence onto image.*

PROOF OF PROPOSITION 4.2.5. Assume that  $X$  has a minimal decomposition. If  $(Y, *)^{(X, *)}$  has CW type, then by Theorem 1.4.4 and Proposition 2.3.1 there exists a finite subcomplex  $L$  of  $X$  such that the morphisms  $\pi_k((Y, *)^{(X, *)}, *) \rightarrow \pi_k((Y, *)^{(L, *)}, *)$  are injective for all  $k \geq 0$ . All higher homotopy groups vanish, so automatically  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is an equivalence onto image. On  $\pi_0$  the condition says that  $\tilde{H}^n(X; G) \rightarrow \tilde{H}^n(L; G)$  is injective. Since  $L$  is a complex of type  $M(\Lambda, n)$ , the morphism  $\text{Hom}(A, G) \rightarrow \text{Hom}(\Lambda, G)$  is injective. Let  $B$  denote the image of  $\Lambda \rightarrow A$ . Then also  $\text{Hom}(A, G) \rightarrow \text{Hom}(B, G)$  is injective which shows  $\text{Hom}(A/B, G) = 0$ .



Conversely, if  $\text{Hom}(A/B, G) = 0$  for a finitely generated subgroup  $B$  of  $A$ , then  $(Y, *)^{(M(A/B, n), *)}$  is contractible by Lemma 4.2.4, hence  $(Y, *)^{(X, *)}$  has CW homotopy type by Lemma 4.3.7.  $\square$

**COROLLARY 4.2.6.** *Let  $X = M(A, n)$ ,  $Y = K(G, n)$ . Assume that  $A$  admits a finitely generated subgroup  $B$  such that  $\text{Hom}(A/B, G) = 0$ . Assume  $X$  is in a minimal decomposition and let  $K_\infty$  be a countable subcomplex of  $X$ . Then there exists a countable subcomplex  $L_\infty$  of  $X$  such that  $L_\infty \supseteq K_\infty$ , the morphism  $H_*(L_\infty) \rightarrow H_*(X)$  is injective and the restriction fibration  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L_\infty, *)}$  is a homotopy equivalence onto image.*

**PROOF.** There exists a finite subcomplex  $L_0 = M(A_0, n)$  of  $X$  such that the image  $B_0$  of  $H_n L_0 \rightarrow H_n X = A$  contains  $B$ . By Lemma 2.2.4 and Lemma 1.3.11 we may construct an ascending chain  $L_1 \leq L_2 \leq \dots$  such that  $L_0 \leq L_1$ ,  $K_\infty \leq L_\infty$  and  $H_*(L_\infty) \rightarrow H_*(X)$  is injective. Note that for all  $i$  the image  $B_i$  of  $H_n L_i \rightarrow H_n X$  contains  $B$ . Consequently  $\text{Hom}(A/B_i, G) = 0$  for all  $i$  and  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L_i, *)}$  is a homotopy equivalence onto image which we denote by  $\Gamma_i$ . Let  $\Gamma_\infty$  denote the image of  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L_\infty, *)}$ . Evidently then  $\Gamma_i$  is the image of  $\Gamma_\infty$  under  $(Y, *)^{(L_\infty, *)} \rightarrow (Y, *)^{(L_i, *)}$ . Since for each  $i$ , restriction  $\Gamma_i \rightarrow \Gamma_{i-1}$  is a homotopy equivalence,  $\Gamma_\infty \rightarrow \Gamma_i$  is a homotopy equivalence for all  $i$ , by Lemma 1.3.8.  $\square$

### 4.3. Explicit determinations

Here we give necessary and sufficient conditions on abelian groups  $A$  and  $G$  for CW homotopy type of  $K(G, n)^{M(A, m)}$  and that of  $K(G, n)^{K(A, m)}$  for  $m \geq 2$ .

Before stating Theorem 4.3.1 we discuss some properties of abelian groups.

Let  $P$  be a set of primes and  $A$  an abelian group. Then  $A$  is called  $P$ -divisible if it is divisible by each prime belonging to  $P$ , i.e. if for each  $y \in A$  and  $p \in P$  there exists  $x \in A$  with  $px = y$ . If  $A$  is divisible by all primes, it is called a *divisible* group. Divisible abelian groups are exactly injective abelian groups.

It is well known that every divisible group is isomorphic with a direct sum of some copies of the rationals  $\mathbb{Q}$  and some copies of the quasicyclic groups  $\mathbb{Z}_{p^\infty}$  for various primes  $p$ , say  $A \cong (\oplus_{\Lambda_0} \mathbb{Q}) \oplus \oplus_{p \in \mathbb{P}} (\oplus_{\Lambda_p} \mathbb{Z}_{p^\infty})$ . The cardinalities of  $\Lambda_0$  and  $\Lambda_p$  for  $p \in \mathbb{P}$  form a complete system of invariants of  $A$  (see Fuchs [20]). Moreover in this case the rank of  $A$  equals  $\Lambda_0 + \sum_{p \in \mathbb{P}} \Lambda_p$ .

We say that an abelian group  $G$  is bounded if there exists a number  $b$  such that  $b \cdot x = 0$  for all  $x \in G$ .

THEOREM 4.3.1. *Let  $A$  and  $G$  be abelian groups. Let  $T(G)$  denote the torsion subgroup of  $G$ , and  $F$  the quotient  $G/T(G)$ . Define*

$$P = \{p \in \mathbb{P} \mid G * \mathbb{Z}_p \neq 0\}, \quad S = \left\{ p \in \mathbb{P} \mid \frac{F \otimes \mathbb{Q}}{F}_{(p)} \neq 0 \right\}.$$

The function space  $K(G, n)^{M(A, m)}$  has CW homotopy type if and only if

- (i)  $m > n$ , or
- (ii)  $m = n$ , and there exists a finitely generated subgroup  $B$  of  $A$  such that  $\text{Hom}(A/B, G) = 0$ , or
- (iii)  $m < n$ , and there exists a short exact sequence  $0 \rightarrow B \rightarrow A \rightarrow A' \rightarrow 0$  such that  $B$  is finitely generated,  $A'$  is  $P$ -divisible, the divisible  $T(A')_{(P)}$  has finite rank, and, in addition, either
  - $G$  is a  $P$ -bounded group, or
  - $A'$  is a torsion group such that for  $R = \{p \in P \mid A'_{(p)} \neq 0\}$ 
    - a.  $T(G)_{(R)}$  is bounded and  $F_{(R)}$  is rational (if and only if  $R \cap S = \emptyset$ ),
    - b.  $A'_{(S)}$  is a finite group.

COROLLARY 4.3.2. *Let  $A$  be a torsion, and  $G$  an arbitrary abelian group. Let  $P$  and  $S$  be defined as above, and let  $m < n$ . The space  $K(G, n)^{M(A, m)}$  has CW homotopy type if and only there exists  $R \subset P$  such that  $T(G)_{(R)}$  is bounded,  $F_{(R)}$  is rational,  $A_{(R)}$  is isomorphic with the direct sum of finitely many cocyclic groups, and  $A_{(P \cup S \setminus R)}$  is a finite group.  $\square$*

THEOREM 4.3.3. *Let  $A$  and  $G$  be abelian groups, and  $m, n \geq 2$ . Then the function space  $K(G, n)^{K(A, m)}$  has CW type if and only if  $K(G, n)^{M(A, m)}$  has, and consequently if and only if the conditions of Theorem 4.3.1 hold.*

In subsequent sections we will frequently use some well known results on abelian groups, and for convenience we summarize a few in proposition Proposition 4.3.4. We refer to Fuchs [20],[21] for all results on abelian groups. We recall some more definitions. An abelian group is called *reduced* if contains no nontrivial divisible subgroups. For an abelian group  $A$  we may define  $d(A)$  to be the unique maximal divisible subgroup of  $A$ . Then  $A/d(A)$  is a reduced group and since  $d(A)$  is injective,  $A$  is isomorphic to  $d(A) \oplus A/d(A)$ . Although  $A/d(A)$  is unique only up to isomorphism, we may still define the ‘*reduced part*’ of  $A$  as a representative of the isomorphism class of  $A/d(A)$ . This makes sense when we are only interested in the isomorphism class.

A subgroup  $S$  of  $A$  is *pure* in  $A$  if  $nS = S \cap nA$  for every integer  $n$ .

An abelian group is called *cocyclic* if it is isomorphic with a finite cyclic group  $\mathbb{Z}_{p^k}$  for some prime  $p$  and  $k \geq 1$  or with a quasicyclic group  $\mathbb{Z}_{p^\infty}$ .

- PROPOSITION 4.3.4. (1) (*Prüfer and Baer.*) *A bounded abelian group splits into the direct sum of finite cyclic groups (bounded by a common bound).*
- (2) (*Kulikov.*) *A bounded pure subgroup is a direct summand. In particular if the  $P$ -torsion part of a group  $A$  is bounded then it is a direct summand.*
- (3) (*Kulikov.*) *If an abelian group contains elements of finite order, then it contains a cocyclic direct summand. In particular, if a reduced torsion abelian group is unbounded, it contains finite cyclic direct summands of arbitrarily large orders.*

PROOF. See Theorem 17.2, Theorem 27.5, and Corollary 27.3 of Fuchs [20].  $\square$

We are now ready to pursue the proof of Theorem 4.3.1.

PROPOSITION 4.3.5. *Let  $G$  be an abelian group and let  $Y$  denote the Eilenberg-MacLane space  $K(G, n)$ . Decompose  $G$  as  $G \cong D \oplus R$  where  $D$  is divisible and  $R$  a reduced group. Further let  $X$  be an  $(r - 1)$ -connected rational CW complex where  $r \geq 2$ . Set  $Q_k = H_k(X)$ . Assume  $Q_r \neq 0$ . The space  $(Y, *)^{(X, *)}$  has CW homotopy type if and only if one of the following requirements is met.*

- (i)  $r > n$ .
- (ii)  $r = n$ ,  $D$  is torsion free, and  $\dim_{\mathbb{Q}}(Q_r)$  is finite if  $D$  is nontrivial.
- (iii)  $r < n$ , the group  $D$  is torsion free,  $R$  is bounded, and for  $r \leq k \leq n$ , the dimension  $\dim_{\mathbb{Q}}(Q_k)$  is finite if  $D$  is nontrivial.

Moreover when  $D$  is trivial in any of the three cases the space  $(Y, *)^{(X, *)}$  is contractible.

PROOF. By Theorem 4.1.1 it suffices to consider  $X = M(Q, m)$  where  $Q$  is a rational group. More precisely,  $Q$  is isomorphic to the direct sum  $\bigoplus_{\Lambda} \mathbb{Q}$  for some indexing set  $\Lambda$ . Then  $X = \bigvee_{\Lambda} M(\mathbb{Q}, m) = \bigvee_{\Lambda} S_{(0)}^m$ , and  $(Y, *)^{(X, *)} \approx \prod_{\Lambda} (Y, *)^{(S_{(0)}^m, *)}$ . Thus (see Example 1 on page 4) the space  $(Y, *)^{(X, *)}$  has CW type if and only if  $(Y, *)^{(S_{(0)}^m, *)}$  has CW type and is either contractible or else  $\Lambda$  is finite.

If  $m > n$ , then  $S_{(0)}^m$  may be assumed to have trivial  $n$ -skeleton, hence  $(Y, *)^{(S_{(0)}^m, *)}$  is contractible by Proposition 2.6.1. In case  $m \leq n$  the stated conditions are necessary by **a.** of Theorem 3.3.8.

Let  $m \leq n$ . Set  $Y_D = K(D, n)$  and  $Y_R = K(R, n)$ . Then  $Y \simeq Y_D \times Y_R$  and  $(Y, *)^{(S_{(0)}^m, *)} \simeq (Y_D, *)^{(S_{(0)}^m, *)} \times (Y_R, *)^{(S_{(0)}^m, *)}$ . If  $D$  is torsion free then by Theorem 3.2.1 the space  $(Y_D, *)^{(S_{(0)}^m, *)}$  is equivalent to  $(Y_D, *)^{(S^m, *)}$ , and has nontrivial homotopy group  $\pi_{n-m}$  if  $D$  is nontrivial.

Since there are no morphisms from a divisible to a reduced group, the space  $(Y_R, *)^{(S_{(0)}^m, *)}$  is contractible by Lemma 4.2.4.

If  $m < n$  and  $R$  is bounded, then the H-spaces  $Y_R$  and consequently also  $(Y_R, *)^{(S^m, *)} = \Omega^m(Y_R, *)$  admit H-space exponents. Thus  $(Y_R, *)^{(S_{(0)}^m, *)}$  is contractible by **c.** of Theorem 3.3.8.  $\square$

**PROPOSITION 4.3.6.** *Let  $T$  be a divisible torsion group, and write  $T \cong \bigoplus_{p \in P} \bigoplus_{\lambda \in \Lambda_p} \mathbb{Z}_{p^\infty}$  where  $\Lambda_p$  is nonempty for each  $p \in P$ . Set  $X = M(T, m)$  and  $Y = K(G, n)$ . Then  $(Y, *)^{(X, *)}$  has CW type if and only if one of the following is true.*

- (i)  $m > n$ ,
- (ii)  $m = n$ , and  $d(G_{(P)})$  is rational,
- (iii)  $m < n$ ,  $d(G_{(P)})$  is rational,  $r(G_{(P)})$  is bounded, and for

$$R = \{p \in P \mid r(G_{(P)})_{(p)} \neq 0\}$$

the set  $\bigcup_{p \in R} \Lambda_p$  is finite.

**PROOF.** Suppose  $(Y, *)^{(X, *)}$  has CW type. Note that  $Y$  is homotopy equivalent to the homotopy fibre  $Y'$  of a map  $Y_{(P)} \rightarrow F$  where  $F$  is the homotopy fibre of  $K(G, n+1) \rightarrow K(G_{(P)}, n+1)$ . Thus  $(Y', *)^{(X, *)} \rightarrow (Y_{(P)}, *)^{(X, *)}$  is a principal fibration with fibre homotopy equivalent to  $(\Omega F, *)^{(X, *)}$ . But  $F$  is a  $\mathbb{P} \setminus P$ -local (torsion) space and  $X$  is a  $P$ -local torsion space, hence by Proposition 3.2.4 the space  $(\Omega F, *)^{(X, *)}$  is contractible, and thus  $(Y, *)^{(X, *)}$  is homotopy equivalent to  $(Y_{(P)}, *)^{(X, *)}$ .

Set  $T' = \bigoplus_{p \in P} \mathbb{Z}_{p^\infty}$  and  $X' = M(T', m)$ . Then  $X'$  is a wedge summand of  $X$ , and by Lemma 0.0.5 the space  $(Y_{(P)}, *)^{(X, *)}$  dominates  $(Y_{(P)}, *)^{(X', *)}$ . Since  $T'$  is the cokernel of the monomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}_{(\mathbb{P} \setminus P)}$ , the map  $(Y_{(P)}, *)^{(S_{(\mathbb{P} \setminus P)}^m, *)} \rightarrow (Y_{(P)}, *)^{(S^m, *)}$  is a principal fibration (see Lemma 0.0.8) with fibres either empty or equivalent to  $(Y_{(P)}, *)^{(X', *)}$ . Hence the latter has CW type if and only if  $(Y_{(P)}, *)^{(S_{(\mathbb{P} \setminus P)}^m, *)}$  has. By Theorem 3.2.1, the space  $(Y_{(P)}, *)^{(S_{(\mathbb{P} \setminus P)}^m, *)}$  is homotopy equivalent to  $(Y_{(P)}, *)^{(S_{(0)}^m, *)}$ .

Thus the necessity of the stated conditions follows immediately from Proposition 4.3.5, with the exception of the condition on the set  $\bigcup_{p \in R} \Lambda_p$ .

Now assume that  $m < n$ ,  $d(G_{(P)})$  is rational, and  $r(G_{(P)})$  is bounded. Since  $X$  splits as  $X \simeq \bigvee_{p \in P} \bigvee_{\lambda \in \Lambda_p} M(\mathbb{Z}_{p^\infty}, m)$ ,  $(Y_{(P)}, *)^{(X, *)}$  factors as

$$(Y_{(P)}, *)^{(X, *)} \simeq \prod_{p \in P} \prod_{\lambda \in \Lambda_p} (Y_{(P)}, *)^{(M(\mathbb{Z}_{p^\infty}, m), *)}.$$

Note that

$$\pi_{n-m-1}((Y_{(P)}, *)^{(M(\mathbb{Z}_{p^\infty}, m), *)}, *) \cong \text{Ext}(\mathbb{Z}_{p^\infty}, r(G_{(P)})) \cong r(G_{(P)})_{(p)},$$

since  $r(G_{(P)})$  is bounded. A space of CW type only admits factorizations into finitely many non-contractible factors (see Example 1 on page 4), and  $\cup_{p \in R} \Lambda_p$  must be a finite set.

Conversely, if  $\cup_{p \in R} \Lambda_p$  is finite, then  $(Y_{(P)}, *)^{(X, *)}$  is homotopy equivalent to a product of finitely many factors each of which is dominated by  $(Y_{(P)}, *)^{(X', *)}$ , and by sufficiency part of Proposition 4.3.5 the stated conditions evidently suffice here as well.  $\square$

EXAMPLE 10. Let  $X = M(\mathbb{Z}_{p^\infty}, m)$  and  $Y = K(\mathbb{Z}_p, n)$ . By Proposition 4.3.6 the space  $(Y, *)^{(X, *)}$  has the type of a CW complex (of type  $K(\mathbb{Z}_p, n - m - 1)$ ). Contrary to examples provided in Kahn [29] and in [56], this function space of CW type is not covered by Milnor's theorem in the sense that if  $X$  is any CW complex of type  $M(\mathbb{Z}_{p^\infty}, m)$  and  $L$  is a finite subcomplex of  $X$ , then  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(L, *)}$  is not a homotopy equivalence.  $\square$

LEMMA 4.3.7. Let  $B$  be a finitely generated subgroup of  $A$ , and let  $Y$  be an arbitrary CW complex. Then  $(Y, *)^{(M(A, m), *)}$  has CW homotopy type if and only if  $(Y, *)^{(M(A/B, m), *)}$  has if  $m \geq 3$ .

REMARK. If  $Y$  is an infinite loop space, for example an Eilenberg-MacLane space, then the restriction on  $m$  is unnecessary, since then  $(Y, *)^{(M(A, m), *)}$  is homotopy equivalent to  $(Y', *)^{(M(A, m+2), *)}$ , and  $(Y, *)^{(M(A/B, m), *)}$  is homotopy equivalent to  $(Y', *)^{(M(A/B, m+2), *)}$ .

PROOF. There is a cellular map  $M(B, m-1) \rightarrow M(A, m-1)$  which realizes the inclusion  $B \leq A$  in homology, since  $m-1 \geq 2$ . Expanding this to the Puppe sequence yields a subcomplex inclusion  $M(B, m) \leq M(A, m)$  which is the cofibre of a map  $\varphi: M(A/B, m-1) \rightarrow M(B, m)$ .

Thus by Lemma 0.0.8 the map  $(Y, *)^{(M(A, m), *)} \rightarrow (Y, *)^{(M(B, m), *)}$  is a principal fibration with fibres either empty or of the type of  $(Y, *)^{(M(A/B, m), *)}$ , and the assertion follows by Stasheff's theorem.  $\square$

LEMMA 4.3.8. Let  $\Phi$  be a nontrivial countable torsion-free abelian group, and let  $G$  be an arbitrary abelian group. Assume  $m < n$ . If  $(K(G, n), *)^{(M(\Phi, m), *)}$  is contractible then  $G$  is bounded by a number  $b$ , and  $\Phi$  is  $b$ -divisible.

PROOF. The group  $\Phi$  may be represented as the union group of an expanding sequence of subgroups

$$(\dagger) \quad 0 = \Phi_0 \leq \Phi_1 \leq \Phi_2 \leq \Phi_3 \leq \dots$$

where for each  $i$ , the group  $\Phi_i$  is a free group of finite rank  $r_i$ , and  $r_{i-1} \leq r_i \leq i$ , for all  $i \geq 1$ . Inductively we can form a sequence of inclusions

$$M(\Phi_1, m) \leq M(\Phi_2, m) \leq M(\Phi_3, m) \leq \dots$$

with union space a CW complex homotopy equivalent to  $M(\Phi, m)$ . Set  $L_i = M(\Phi_i, m)$  and  $Y = K(G, n)$ . Note that  $(Y, *)^{(L_i, *)}$  is a space of type  $K(G^{r_i}, n - m)$ .

By Proposition 4.2.2 and Theorem 1.2.6 the space  $(K(G, n), *)^{(M(\Phi, m), *)}$  is contractible if and only if there exists a sequence  $i_1 < i_2 < i_3 < \dots$  such that for each  $j$ , the restriction

$$(*) \quad (Y, *)^{(L_{i_j}, *)} \rightarrow (Y, *)^{(L_{i_{j-1}}, *)}$$

is nullhomotopic.

We replace the original sequence  $\{L_i \mid i\}$  by the subsequence  $\{L_{i_j} \mid j\}$ .

We fix isomorphisms  $\mathbb{Z}^{r_i} \cong \Phi_i$  and rewrite  $(\dagger)$  as

$$(\dagger) \quad \mathbb{Z}^{r_1} \xrightarrow{A_2} \mathbb{Z}^{r_2} \xrightarrow{A_3} \mathbb{Z}^{r_3} \xrightarrow{A_4} \dots$$

where the  $A_i$  are matrices with integral entries. The map  $(Y, *)^{(L_i, *)} \rightarrow (Y, *)^{(L_{i-1}, *)}$  is nullhomotopic if and only if the  $\pi_{n-m}$ -induced morphism

$$(**) \quad \text{Hom}(\mathbb{Z}^{r_i}, G) \cong G^{r_i} \xrightarrow{A_i^T} G^{r_{i-1}} \cong \text{Hom}(\mathbb{Z}^{r_{i-1}}, G)$$

is trivial. Here  $A_i^T$  denotes the transpose of  $A_i$ , that is, the morphism

$$(g_1, \dots, g_{r_i}) \rightarrow (h_1, \dots, h_{r_{i-1}}), \quad h_k = \sum_{l=1}^{r_i} (A_i)_{l,k} g_l.$$

If this is the trivial morphism, then  $(A_i)_{l,k} g = 0$ , for all  $l$ , all  $k$ , and all  $g$ .

Since this is valid for all  $i$ , either  $A_i = 0$  for all  $i$  which is a contradiction, or else  $G$  is bounded by a number  $b$ . Supposing  $b$  minimal it follows that  $A_i = b \cdot A'_i$  for all  $i$ . Thus  $\Phi$  is  $b$ -divisible.  $\square$

**PROPOSITION 4.3.9.** *Let  $A$  be a nontrivial torsion free abelian group, and let  $G$  be an arbitrary abelian group. Assume  $m < n$ . Then  $(K(G, n), *)^{(M(A, m), *)}$  is contractible if and only if  $G$  is bounded by a number  $b$ , and  $A$  is  $b$ -divisible.*

**PROOF.** Set  $Y = K(G, n)$ . By Corollary 4.2.3 there exists a countable subcomplex  $L$  of  $M(A, m)$  with  $H_*(L) \rightarrow H_*(M(A, m))$  injective, so that  $(Y, *)^{(M(A, m), *)} \rightarrow (Y, *)^{(L, *)}$  is a homotopy equivalence. Then  $L$  is of type  $M(\Phi, m)$  with  $\Phi \leq A$ , and by Lemma 4.3.8  $G$  is bounded by a number  $b$ . Assume a minimal  $b$ .

By Proposition 4.2.2 and Theorem 2.2.5 the space  $(Y, *)^{(M(A, m), *)}$  is contractible if and only if for each countable subgroup  $\Phi$  of  $A$  there exists a bigger countable subgroup  $\Phi'$  of  $A$  such that  $(Y, *)^{(M(\Phi', m), *)}$  is contractible. By the lemma and minimality of  $b$ , the group  $\Phi'$  is  $b$ -divisible. But this implies that  $A$  is  $P$ -divisible for  $P$  the set of prime divisors of  $b$ .

Conversely, assume that  $A$  is  $P$ -divisible and  $G$  is  $P$ -bounded. Then

$$(Y, *)^{(M(A_{(P)}, m), *)} \rightarrow (Y, *)^{(M(A, m), *)}$$

is a homotopy equivalence by Theorem 3.2.1. Since  $A$  is torsion-free,  $A_{(P)}$  is rational, and Proposition 4.3.5 shows that  $(Y, *)^{(M(A_{(P)}, m), *)}$  has CW type.  $\square$

LEMMA 4.3.10. *Assume the notation of the theorem, and let  $T$  be a torsion group with trivial  $(P \cup S)$ -part. Then  $(K(G, n), *)^{(M(T, m), *)}$  is contractible.*

PROOF. The rationalization  $K(G, n) \rightarrow K(G_{(0)}, n)$  may be viewed as a principal fibration with fibre  $\Phi$  such that  $\pi_n(\Phi) \cong T(G)$  and  $\pi_{n-1}(\Phi) \cong (F \otimes \mathbb{Q})/F$ . Thus  $\Phi$  is a  $P \cup S$ -local torsion space, so Theorem 3.2.1 implies that  $(\Phi, *)^{(M(T, m), *)}$  is homotopy equivalent to  $(\Phi, *)^{(M(T, m)_{(P \cup S)}, *)}$ . The latter is contractible since  $M(T, m)_{(P \cup S)}$  is. By Theorem 3.2.1 also  $(K(G_{(0)}, n), *)^{(M(T, m), *)}$  is contractible. Hence so is  $(K(G, n), *)^{(M(T, m), *)}$ , as claimed.  $\square$

PROOF OF THEOREM 4.3.1. Assume a minimal decomposition for  $X = M(A, m)$ , and set  $Y = K(G, n)$ . Case  $m > n$  follows from Proposition 2.6.1, and case  $m = n$  by Proposition 4.2.5, so we assume  $m \leq n$ . Let  $Y^X$  have CW homotopy type. By Corollary 2.2.9 there exists a finite subcomplex  $L$  of  $X$  such that for every  $k \in \{0, \dots, n\}$  the inclusion induced morphism

$$(*) \quad \tilde{H}^k(X; G) \rightarrow \tilde{H}^k(L; G)$$

is injective.

Minimal decomposition of  $X$  implies that  $L$  is of type  $M(A_L, m)$ . Denote by  $\varphi: A_L \rightarrow A$  the morphism induced by inclusion  $L \leq X$ , and set  $B = \text{im } \varphi$ .

Injectivity of  $(*)$  says that the morphisms  $\varphi^*: \text{Hom}(A, G) \rightarrow \text{Hom}(A_L, G)$  and  $\varphi^*: \text{Ext}(A, G) \rightarrow \text{Ext}(A_L, G)$  are injective. Consequently also

$$\text{Hom}(A, G) \rightarrow \text{Hom}(B, G) \text{ and, if } m < n, \text{ Ext}(A, G) \rightarrow \text{Ext}(B, G)$$

are injective.

Plugging this into the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(A/B, G) \rightarrow \text{Hom}(A, G) \rightarrow \text{Hom}(B, G) \\ \rightarrow \text{Ext}(A/B, G) \rightarrow \text{Ext}(A, G) \rightarrow \text{Ext}(B, G) \rightarrow 0 \end{aligned}$$

it follows that  $\text{Hom}(A/B, G)$  is trivial and

$$\text{Ext}(A, G) \cong \text{Ext}(B, G) \cong T(B) \otimes G \text{ is a bounded abelian group.}$$

Set  $A' = A/B$ . By Lemma 4.3.7 the space  $K(G, n)^{M(A', m)}$  also has CW type. Reapplying the above with  $A'$  in place of  $A$  yields a finitely generated subgroup  $B'$  of  $A'$  with

$$\text{Ext}(A', G) \cong \text{Ext}(B', G) \cong T(B') \otimes G \text{ a bounded abelian group.}$$

Set  $X' = M(A', m)$ . We proceed with investigating  $Y^{X'}$  of CW type with  $\text{Hom}(A', G)$  trivial and  $\text{Ext}(A', G)$  a bounded abelian group.

Let  $p \in P$ . The epimorphism  $A' \rightarrow A' \otimes \mathbb{Z}_p$  induces the monomorphism  $\text{Hom}(A' \otimes \mathbb{Z}_p, G) \rightarrow \text{Hom}(A', G)$ . Furthermore the monomorphism  $G * \mathbb{Z}_p \rightarrow G$  induces the monomorphism  $\text{Hom}(A' \otimes \mathbb{Z}_p, G * \mathbb{Z}_p) \rightarrow \text{Hom}(A' \otimes \mathbb{Z}_p, G)$ . Since  $\text{Hom}(A', G)$  is trivial and  $G * \mathbb{Z}_p$  is nontrivial,  $A' \otimes \mathbb{Z}_p$  must be trivial. In other words,  $A'$  is  $p$ -divisible.

Consider the exact sequence

$$\begin{aligned} (**) \quad 0 &\rightarrow \text{Hom}(A'/T(A'), H) \rightarrow \text{Hom}(A', H) \rightarrow \text{Hom}(T(A'), H) \\ &\rightarrow \text{Ext}(A'/T(A'), H) \rightarrow \text{Ext}(A', H) \rightarrow \text{Ext}(T(A'), H) \rightarrow 0 \end{aligned}$$

Since  $\text{Hom}(A', G) = 0$ , substituting  $H = G$  simplifies  $(**)$  to the exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Hom}(T(A'), G) \xrightarrow{\delta} \text{Ext}(A'/T(A'), G) \xrightarrow{\alpha} \\ &\text{Ext}(A', G) \rightarrow \text{Ext}(T(A'), G) \rightarrow 0. \end{aligned}$$

It follows that  $\text{Hom}(T(A'), G) \cong \ker \alpha$ , and that  $\text{coker } \delta \rightarrow \text{Ext}(A', G)$  is injective. However, since  $A'/T(A')$  is torsion-free,  $\text{Ext}(A'/T(A'), G)$  is divisible, and hence so is its quotient group  $\text{coker } \delta$ . Since no divisible group is bounded,  $\text{coker } \delta = 0$ , and consequently

$$\text{Ext}(A', G) \xrightarrow{\cong} \text{Ext}(T(A'), G)$$

and  $\text{Hom}(T(A'), G) \xrightarrow{\cong} \text{Ext}(A'/T(A'), G)$  are isomorphisms. Since  $T(A')$  is a torsion group,  $\text{Hom}(T(A'), G)$  is a reduced group (see Theorem 46.1 of Fuchs [20]). The only possibility is

$$\text{Hom}(T(A'), G) = \text{Ext}(A'/T(A'), G) = 0.$$

Applying  $\text{Hom}(T(A'), \_)$  to  $0 \rightarrow T(G) \rightarrow G \rightarrow F \rightarrow 0$  yields the isomorphism  $\text{Hom}(T(A'), G) \cong \text{Hom}(T(A'), T(G))$  and the exact sequence

$$(\dagger) \quad 0 \rightarrow \text{Ext}(T(A'), T(G)) \rightarrow \text{Ext}(T(A'), G) \rightarrow \text{Ext}(T(A'), F) \rightarrow 0.$$

Since

$$\text{Hom}(T(A'), G) \cong \text{Hom}(T(A'), T(G)) \cong \prod_{p \in P} \text{Hom}(T(A')_{(p)}, T(G)_{(p)})$$

it follows immediately that for  $p \in P$ , either  $T(A')_{(p)}$  is trivial or else  $T(G)_{(p)}$  is reduced.



The sequence (†) shows that both  $\text{Ext}(T(A'), T(G))$  and  $\text{Ext}(T(A), F)$  are bounded.

Consider

$$\text{Ext}(T(A'), T(G)) \cong \prod_{p \in P} \text{Ext}(T(A')_{(p)}, T(G)_{(p)}),$$

and let  $R = \{p \in P \mid T(A')_{(p)} \neq 0\}$ . Take  $p \in R$ . Then  $T(G)_{(p)}$  is reduced. If it is unbounded, then it admits cyclic direct summands of arbitrarily large orders, and consequently so does  $\text{Ext}(T(A')_{(p)}, T(G)_{(p)})$ , contradiction. Furthermore, if  $R$  is infinite, then  $\prod_{p \in P} \text{Ext}(T(A')_{(p)}, T(G)_{(p)})$  contains elements of arbitrarily high orders which again contradicts boundedness of  $\text{Ext}(T(A'), T(G))$ . Thus  $R$  is finite, and we conclude that

$$T(G)_{(R)} \text{ is a bounded group.}$$

In the short exact sequence

$$0 \rightarrow T(G)_{(R)} \rightarrow G_{(R)} \rightarrow F_{(R)} \rightarrow 0$$

the group  $T(G)_{(R)}$  is a bounded pure subgroup of  $G_{(R)}$ , hence it is a direct summand. It follows that the divisible part  $d(G_{(R)})$  of  $G_{(R)}$  is isomorphic to the divisible part of  $F_{(R)}$ , and as such is rational. Furthermore,

$$r(G_{(R)}) \cong T(G)_{(R)} \oplus r(F_{(R)}).$$

Since  $T(A')_{(R)}$  is divisible, it is a direct summand of  $A'$ ; we decompose  $A' \cong T(A')_{(R)} \oplus A_1$ . Then  $M(A', m) \simeq M(T(A')_{(R)}, m) \vee M(A_1, m)$ , and by Lemma 0.0.5, the space  $(Y, *)^{(M(T(A')_{(R)}, m), *)}$  has CW homotopy type. By Proposition 4.3.6 it follows that  $r(F_{(R)})$  is trivial, and that  $T(A')_{(R)}$  only admits finitely many nontrivial direct summands.

Since  $r(F_{(R)})$  is trivial,  $F_{(R)}$  is a rational group. In particular,  $F \otimes \mathbb{Q} \cong F_{(R)}$ , and consequently  $(F \otimes \mathbb{Q})/F$  is torsion away from  $R$ . In other words  $R \cap S = \emptyset$ .

Now we consider

$$\text{Ext}(T(A'), F) \cong \text{Hom}(T(A'), \frac{F \otimes \mathbb{Q}}{F}) \cong \prod_{p \in S} \text{Hom}(T(A')_{(p)}, (F_{(R)}/F)_{(p)}).$$

(Note that from boundedness of  $\text{Ext}(T(A'), F)$  it also follows that  $R \cap S = \emptyset$ .)

Since for  $p \in S$  the torsion group  $(F_{(R)}/F)_{(p)}$  is divisible, and  $\text{Ext}(T(A'), F)$  is bounded,  $T(A')_{(p)}$  must be reduced, and bounded as well. Furthermore,  $T(A')_{(p)}$  can be nontrivial for at most finitely many  $p \in S$ . Hence  $T(A')_{(S)}$  itself is bounded, and as such a direct summand of  $A'$ . As above, it follows that  $(Y, *)^{(M(T(A')_{(S)}, m), *)}$  has CW type. As a bounded group,  $T(A')_{(S)}$  splits into a direct sum of finite cyclic groups, say  $T(A')_{(S)} \cong$

$\bigoplus_{\lambda \in \Lambda} C_\lambda$ . Consequently  $(Y, *)^{(M(T(A')_{(S)}, m), *)}$  is homotopy equivalent to the product

$$\prod_{\lambda \in \Lambda} (Y, *)^{(M(C_\lambda, m), *)}.$$

Since all factors are noncontractible,  $\Lambda$  must be finite by Example 1 on page 4. In particular,  $T(A')_{(S)}$  is a finite group.

We decompose

$$T(A') \cong T(A')_{(R)} \oplus T(A')_{(S)} \oplus T''.$$

Since  $T''$  has trivial  $(P \cup S)$ -torsion, the space  $(Y, *)^{(M(T'', m), *)}$  is contractible by Lemma 4.3.10. By Milnor's theorem also  $(Y, *)^{(M(T(A')_{(S)}, m), *)}$  has CW type, hence  $(Y, *)^{(M(T(A'), m), *)}$  has CW homotopy type by Lemma 0.0.5. By the above we already know that

$$(Y, *)^{(M(A', m), *)} \rightarrow (Y, *)^{(M(T(A'), m), *)}$$

is a weak homotopy equivalence, hence it is a genuine homotopy equivalence, and the fibres are contractible. The fibre over the constant map is homotopy equivalent to

$$(Y, *)^{(M(A'/T(A'), m), *)}.$$

If  $A'/T(A')$  is nontrivial then by Proposition 4.3.9,  $G$  is bounded. Thus  $G = T(G) = T(G)_{(P)}$ , and the bound of  $G$  is a number whose prime divisors equal  $P$ . This concludes the proof of necessity.

Conversely, if  $G$  is  $P$ -bounded,  $A'$  is  $P$ -divisible and  $T(A')_{(P)}$  only admits finitely many direct summands, then  $K(G, n)^{(M(A', m), *)}$  which is equivalent to  $K(G, n)^{(M(A'_{(P)}, m), *)}$  by Theorem 3.2.1 has CW homotopy type by Proposition 4.3.5 and Proposition 4.3.6.

If  $A'$  is a torsion group with  $A'_{(P)} = A'_{(R)}$  divisible with finitely many summands,  $A'_{(S)}$  finite,  $T(G)_{(R)}$  bounded and  $F_{(R)}$  rational, then  $K(G, n)^{(M(A'_{(R)}, m), *)}$  has CW type by Proposition 4.3.6,  $K(G, n)^{(M(A'_{(S)}, m), *)}$  has CW type by Milnor's theorem, and  $K(G, n)^{(M(A'_{(P \setminus (P \cup S))}, m), *)}$  has CW type by Lemma 4.3.10. Then  $K(G, n)^{(M(A', m), *)}$  has CW type by Lemma 0.0.5.

An application of Lemma 4.3.7 concludes also the proof of sufficiency.  $\square$

Now we turn to the proof of Theorem 4.3.3.

By Theorem 4.1.1 the necessity part is clear, and we have to prove sufficiency.

**DEFINITION.** Let  $\mathcal{C}$  be a class of abelian groups. Then  $\mathcal{C}$  is a Serre class if

- for a short exact sequence  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  the group  $A$  belongs to  $\mathcal{C}$  if and only if both  $A'$  and  $A''$  do,
- if  $A$  and  $B$  belong to  $\mathcal{C}$  then so do  $A \otimes B$  and  $A * B$ , and
- for  $A \in \mathcal{C}$  also  $\tilde{H}_i(K(A, 1); \mathbb{Z}) \in \mathcal{C}$ , for all  $i$ .

LEMMA 4.3.11. *Let  $R$  be a (finite) set of primes, and let  $Q$  be a set of primes (possibly empty) disjoint from  $R$ . Let  $\mathcal{C}$  be the class of torsion abelian groups  $A$  such that*

- $A_{(R)}$  is the direct sum of finitely many cocyclic groups, and
- $A_{(Q)}$  is finite.

*Then  $\mathcal{C}$  is a Serre class with the additional property that if  $A \in \mathcal{C}$  and  $B$  is finitely generated, then  $A \otimes B$  and  $A * B$  belong to  $\mathcal{C}$ .*

PROOF. It suffices to show that the required properties hold for the class  $\mathcal{C}_p$  of torsion abelian groups that are isomorphic with a direct sum of finitely many cocyclic  $p$ -groups.

Let  $D$  be a divisible torsion  $p$ -group of finite rank, i.e.  $D$  is isomorphic to the direct sum of finitely many quasicyclic groups  $\mathbb{Z}_{p^\infty}$ . It follows easily by induction on the rank of  $D$  that

- if  $E$  is a quotient group of  $D$  it is divisible with  $\text{rank } E \leq \text{rank } D$ ,
- if  $C$  is a subgroup of  $D$  then  $C$  is the direct sum of cocyclic  $p$ -groups (and  $\text{rank } C \leq \text{rank } D$ ).

Consequently if  $A \in \mathcal{C}_p$  and

$$(*) \quad 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

is exact, then  $A' \in \mathcal{C}_p$  and  $A'' \in \mathcal{C}_p$ .

Conversely, assume that in  $(*)$  the groups  $A'$  and  $A''$  belong to  $\mathcal{C}$ . We may decompose  $A' \cong D \oplus F$  where  $F$  a finite  $p$ -group, and  $D$  is a divisible  $p$ -group of finite rank. Since  $D$  is injective, it also splits  $A$ , say  $A \cong D \oplus \Gamma$  where  $\Gamma$  is an extension of  $F$  by  $A''$ , i.e.

$$0 \rightarrow F \rightarrow \Gamma \rightarrow A'' \rightarrow 0.$$

There exists a bound  $p^m$  for  $F$ , and since  $A''$  is also isomorphic to the direct sum of a finite  $p$ -group and a divisible  $p$ -group of finite rank,  $p^n A''$  is divisible for some  $n$ . Consequently also  $p^{m+n} \Gamma$  is divisible. Considering a decomposition  $\Gamma \cong d(\Gamma) \oplus r(\Gamma)$  with  $d(\Gamma)$  divisible and  $r(\Gamma)$  reduced it follows that  $r(\Gamma)$  is bounded, and hence isomorphic to the direct sum of cyclics. Rank considerations imply that  $\Gamma$  belongs to  $\mathcal{C}_p$ , and hence so does  $A$ .

For tensor and torsion products we note that if  $T$  is any torsion group,  $\mathbb{Z}_{p^\infty} \otimes T = 0$ , and for any group  $G$ , the torsion product  $\mathbb{Z}_{p^\infty} * G$  is isomorphic to  $T(G)_{(p)}$ .

Since  $K(\mathbb{Z}_{p^\infty}, 1)$  is the colimit of lens spaces, we know that  $\tilde{H}_i(K(\mathbb{Z}_{p^\infty}, 1); \mathbb{Z})$  is isomorphic to  $\mathbb{Z}_{p^\infty}$  for  $i$  odd and zero for  $i$  even.

If  $A \in \mathcal{C}_p$  then  $\tilde{H}_i(K(A, 1); \mathbb{Z}) \in \mathcal{C}_p$  by the Künneth theorem, combined with what we have shown above.  $\square$

LEMMA 4.3.12. *Let  $\mathcal{C}$  be as in Lemma 4.3.11 and let  $0 \rightarrow B \rightarrow A \rightarrow A' \rightarrow 0$  be an exact sequence with  $B$  finitely generated and  $A' \in \mathcal{C}$ . Then for each  $i \geq m$ , the group  $H_i(K(A, m); \mathbb{Z})$  is given by an extension*

$$0 \rightarrow B_i \rightarrow H_i(K(A, m); \mathbb{Z}) \rightarrow A'_i \rightarrow 0$$

with  $B_i$  finitely generated and  $A'_i \in \mathcal{C}$ .

PROOF. We consider the Serre homology spectral sequence of the fibration  $K(B, m) \rightarrow K(A, m) \rightarrow K(A', m)$ . By Lemma 4.3.11, for the  $E^2$ -term

$$E_{p,q}^2 = H_p(K(A', m); H_q(K(B, m)))$$

we have  $E_{0,q}^2$  a finitely generated group and  $E_{p,q}^2 \in \mathcal{C}$  for  $p \geq 1$ . Since finitely generated groups and  $\mathcal{C}$  are Serre classes, also  $E_{0,q}^\infty$  is finitely generated, and  $E_{p,q}^\infty \in \mathcal{C}$  for  $p \geq 1$ . This implies that for each  $i$ , the group  $H_i(K(A, m))$  has a filtration  $F_0 \leq F_1 \leq \dots \leq F_n = H_i(K(A, m))$  with  $F_0$  finitely generated, and  $F_j/F_{j-1} \in \mathcal{C}$  for positive  $j$ . Set  $B_i = F_0$  and suppose that for some  $j$ , the group  $F_{j-1}/B_i$  belongs to  $\mathcal{C}$ . Then also  $B_i \leq F_j$ , and  $F_j/B_i$  is given by the extension

$$0 \rightarrow F_{j-1}/B_i \rightarrow F_j/B_i \rightarrow F_j/F_{j-1} \rightarrow 0.$$

By inductive hypothesis,  $F_{j-1}/B_i \in \mathcal{C}$ , and since  $F_j/F_{j-1} \in \mathcal{C}$ , also  $F_j/B_i \in \mathcal{C}$ , and our assertion follows.  $\square$

PROOF OF THEOREM 4.3.3. By Theorem 4.1.1 the case  $m \geq n$  is already contained in Theorem 4.3.1. So we assume that  $m < n$ .

Assume that  $G$  is a  $P$ -bounded group (with  $G * \mathbb{Z}_p \neq 0$  for all  $p \in P$ ) and that  $A$  is given by an extension  $0 \rightarrow B \rightarrow A \rightarrow A' \rightarrow 0$  where  $A'$  is  $P$ -divisible and  $T(A')_{(P)}$  is the sum of finitely many quasicyclic groups. Let  $T' = T(A')$  and  $\Phi = A'/T'$ . The kernel  $\Gamma$  of the composite  $A \rightarrow A' \rightarrow \Phi$  is given by the extension  $B \rightarrow \Gamma \rightarrow T'$ .

Consider the fibration  $K(\Gamma_{(P)}, m) \rightarrow K(A_{(P)}, m) \rightarrow K(\Phi_{(P)}, m)$ . Since  $\Phi$  is  $P$ -divisible and torsion-free,  $\Phi_{(P)}$  is a rational group. Hence  $K(\Phi_{(P)}, m-1)$  is a rational space, and thus  $(K(G, n), *)^{(K(\Phi_{(P)}, m-1), *)} \simeq (K(G, n+1), *)^{(SK(\Phi_{(P)}, m-1), *)}$  is contractible by Proposition 4.3.5. Since  $K(\Phi_{(P)}, m-1)$  is the homotopy fibre of  $K(\Gamma_{(P)}, m) \rightarrow K(A_{(P)}, m)$ , the map

$$K(G, n)^{K(A_{(P)}, m)} \rightarrow K(G, n)^{K(\Gamma_{(P)}, m)}$$

is a homotopy equivalence by Proposition 3.1.1. By Theorem 3.2.1,  $K(G, n)^{K(A(P), m)}$  is homotopy equivalent to  $K(G, n)^{K(A, m)}$ , so it suffices to prove that the space  $K(G, n)^{K(\Gamma(P), m)} \simeq K(G, n)^{K(\Gamma, m)}$  has CW homotopy type.

By Lemma 4.3.12, for each  $i$  the homology group  $H_i = \tilde{H}_i(K(\Gamma, m))$  is the extension of a finitely generated group by a torsion group whose  $P$ -part is isomorphic with the direct sum of finitely many cocyclic groups, say  $0 \rightarrow B_i \rightarrow H_i \rightarrow T_i \rightarrow 0$ . Set  $Y = K(G, n)$ . Since  $B_i$  is finitely generated,  $(Y, *)^{(M(H_i, i), *)}$  has CW type if and only if  $(Y, *)^{(M(T_i, i), *)}$  has, by Lemma 4.3.7. We split

$$T_i = T_1 \oplus T_2 \oplus T_3$$

where  $T_1 = T_{i(\mathbb{P} \setminus P)}$ ,  $T_2$  is the direct sum of finitely many quasicyclic groups and  $T_3$  is a finite  $P$ -group. By Corollary 0.0.5 the space  $(Y, *)^{(M(T_i, i), *)}$  has CW type if and only if the  $(Y, *)^{(M(T_j, i), *)}$ , for  $1 \leq j \leq 3$ , have. The space  $(Y, *)^{(M(T_1, i), *)}$  is contractible by Proposition 3.2.4, and  $(Y, *)^{(M(T_3, i), *)}$  has CW type by Milnor's theorem. Furthermore,  $(Y, *)^{(M(T_2, i), *)}$  has CW type by Proposition 4.3.6.

Thus  $(K(G, n), *)^{(K(\Gamma, m), *)}$  has CW type by Theorem 4.1.1, and consequently  $K(G, n)^{K(A, m)}$  has CW type, as claimed.

Assume the notation of the statement of Theorem 4.3.1. Our second case is that  $A'$  is torsion with  $A'_{(R)}$  the sum of finitely many cocyclic groups,  $A'_{(S \cup P \setminus R)}$  is a finite group,  $T(G)_{(R)}$  is bounded, and  $F_{(R)}$  is rational.

By Lemma 4.3.12 every  $H_i = \tilde{H}_i(K(A, m))$  is given by an exact sequence  $0 \rightarrow B_i \rightarrow H_i \rightarrow A'_i \rightarrow 0$  where  $B_i$  is finitely generated, and  $A'_i$  is torsion such that  $A'_{i(R)}$  is isomorphic with a direct sum of finitely many cocyclic groups, and  $A'_{i(S \cup P \setminus R)}$  is finite.

Since  $B_i$  is finitely generated,  $(Y, *)^{(M(H_i, i), *)}$  has CW homotopy type if and only if  $(Y, *)^{(M(A'_i, i), *)}$  has. Split

$$A'_i \cong A'_1 \oplus A'_2 \oplus A'_3$$

where  $A'_1$  is a  $\mathbb{P} \setminus (S \cup P)$ -group,  $A'_2$  is isomorphic with a direct sum of finitely many  $R$ -quasicyclic groups, and  $A'_3$  is finite. Then  $(Y, *)^{(M(A'_1, i), *)}$  and  $(Y, *)^{(M(A'_2, i), *)}$  have CW type by Corollary 4.3.2, and  $(Y, *)^{(M(A'_3, i), *)}$  has CW type by Milnor's theorem. Consequently  $(Y, *)^{(M(A'_i, i), *)}$  has CW type by Corollary 0.0.5, hence so has  $(Y, *)^{(M(H_i, i), *)}$ , and hence so has  $(Y, *)^{(K(A, m), *)}$ , by Theorem 4.1.1.  $\square$

#### 4.4. Spaces of maps into $K(G, 1)$

PROPOSITION 4.4.1. *Let  $X$  be a connected CW complex with base point  $x_0$ . Pick a base point  $y_0$  in  $K(G, 1)$ . Then  $(K(G, 1), y_0)^{(X, x_0)}$  has CW homotopy type if and only if for each  $g: (X, x_0) \rightarrow (K(G, 1), *)$  there exists a finite subcomplex  $L$  of  $X$  (containing  $x_0$ ) such that the induced function*

$$(*) \quad \text{Hom}(\pi_1(X, x_0), G) \rightarrow \text{Hom}(\pi_1(L, x_0), G)$$

*only sends the class  $[g]$  to  $[g|_L]$ .*

PROOF. The condition is necessary by Lemma 2.2.8.

For sufficiency, note that ‘injectivity’ of  $(*)$  implies that the path component of  $g$  in  $(K(G, 1), y_0)^{(X, x_0)}$  is open, and Lemma 4.4.3 below concludes the proof.  $\square$

We note the easy

LEMMA 4.4.2. *Let  $Y$  be an aspherical space of CW type with base point  $y_0$  and let  $X$  be a connected CW complex with base point  $x_0$ . Then for any  $g: (X, x_0) \rightarrow (Y, y_0)$  the path component of  $g$  in  $(Y, y_0)^{(X, x_0)}$  is weakly contractible.*  $\square$

LEMMA 4.4.3. *Let  $(X, x_0)$  be a based connected CW complex and let  $Y = K(G, 1)$ . The space  $(Y, y_0)^{(X, x_0)}$  has contractible path components, for any  $y_0 \in Y$ .*

PROOF. With no loss of generality assume that  $X$  has a single 0-cell which is  $x_0$ . Let  $g: (X, x_0) \rightarrow (Y, y_0)$  be a map, and let  $C$  denote the path component of  $g$  in  $(Y, y_0)^{(X, x_0)}$ . Let  $K_\infty$  be any countable subcomplex of  $X$  and let  $K_1 \leq K_2 \leq \dots$  be a filtration of finite subcomplexes for  $K_\infty$ . For  $L \leq X$  let  $C_L$  denote the image of  $C$  under  $(Y, y_0)^{(X, x_0)} \rightarrow (Y, y_0)^{(L, x_0)}$ .

By Lemma 4.4.2 and Milnor’s theorem the spaces  $C_{K_i}$  are contractible. By (i) of Lemma 1.3.8 so is  $C_{K_\infty} = \lim_i C_{K_i}$ . Then by Theorem 2.2.6 also  $C$  is contractible, as claimed.  $\square$

#### 4.5. CW complexes $X$ such that $Y^X$ has CW type for ‘all’ $Y$

We close this chapter by showing that Milnor’s theorem cannot be improved in the sense that if for a space  $X$  the function space  $Y^X$  has CW homotopy type for *all* CW complexes  $Y$ , then  $X$  is ‘almost finitely dominated’. We begin with a few preliminary results.

LEMMA 4.5.1. *Let  $A$  be an arbitrary abelian group.*

- (i) *The torsion-free rank of  $\text{Hom}(A, \mathbb{Z})$  is at least  $d$  if and only if  $A$  admits a direct summand which is a free group of rank  $d$ .*

- (ii) If the torsion-free rank of  $\text{Hom}(A, \mathbb{Z})$  is exactly  $d$  then there exists a decomposition  $A \cong \mathbb{Z}^d \oplus A'$  with  $\text{Hom}(A', \mathbb{Z}) = 0$ .  $\square$

PROOF. See Fuchs [20], §43, Exercise 10.  $\square$

PROPOSITION 4.5.2. Let  $X$  be a connected CW complex. Then  $K(\mathbb{Z}, n)^X$  has CW homotopy type if and only if the groups  $H_i(X)$  are finitely generated for  $1 \leq i \leq n-1$ , and there exists a decomposition  $H_n(X) \cong \mathbb{Z}^d \oplus H'$  with  $\text{Hom}(H', \mathbb{Z}) = 0$ .

PROOF. By Theorem 4.1.1 and Theorem 4.3.1 combined with Milnor's theorem the sufficiency part is clear, and so is the necessity part except for the splitting of  $H_n(X)$ .

If  $K(\mathbb{Z}, n)^X$  has CW type, then by Theorem 4.3.1 there exists a finitely generated subgroup  $B$  of  $H_n(X)$  such that  $\text{Hom}(H_n(X), \mathbb{Z}) \rightarrow \text{Hom}(B, \mathbb{Z})$  is injective. Since  $\text{Hom}(B, \mathbb{Z})$  is free of finite rank, the desired splitting follows from Lemma 4.5.1.  $\square$

Recall that a CW complex  $X$  is called quasifinite (see Mislin [44]) if  $\bigoplus_n H_n(X; \mathbb{Z})$  is a finitely generated abelian group.

THEOREM 4.5.3. **a.** Let  $X$  be a connected CW complex. Then the following are equivalent.

- (i) For each  $i \geq 1$ , the group  $H_i(X)$  is finitely generated.
  - (ii) For each nilpotent CW complex  $Y$  with finitely many nontrivial homotopy groups, the space  $Y^X$  has CW homotopy type.
  - (iii) For each  $n \geq 2$  the path component  $C$  of the constant map in  $K(\mathbb{Z}, n)^X$  has CW homotopy type.
- b.** Let  $X$  be a connected CW complex and let  $n \geq 2$ . Then the following are equivalent.
- (i) For each  $1 \leq i \leq n$ , the group  $H_i(X)$  is finitely generated.
  - (ii) For each nilpotent CW complex  $Y$  with  $\pi_k(Y) = 0$  for  $k \geq n+1$  the space  $Y^X$  has CW homotopy type.
  - (iii) For each abelian group  $G$  the space  $K(G, n)^X$  has CW homotopy type.
- c.** Let  $X$  be a nilpotent connected CW complex. Then the following are equivalent.
- (i) The space  $Y^X$  has CW homotopy type for every CW complex  $Y$ .
  - (ii) The path component  $C$  of  $Y^X$  has CW homotopy type for every simply connected CW complex  $Y$  of finite type.
  - (iii)  $X$  is dominated by a finite CW complex.
- d.** Let  $X$  be any connected CW complex. Then the following are equivalent.

- (i) For every CW complex  $Y$  the path component  $C$  of the constant map in  $Y^X$  is open and has CW homotopy type.
- (ii) There exists a finite subcomplex  $L$  of  $X$  such that the quotient  $X/L$  is homotopy equivalent to a finite CW complex.
- e. Let  $X$  be a connected CW complex and  $n \geq 2$ . Then  $X$  is homotopy equivalent to a complex with finite  $n$ -skeleton if and only if  $\pi_1(X)$  is finitely presented and  $Y^X$  has CW type for all CW complexes  $Y$  with  $\pi_k(Y) = 0$  for  $k \geq n + 1$ .
- f. Let  $X$  be a connected CW complex. Then  $X$  is finitely dominated if and only if  $\pi_1(X)$  is finitely presented and  $Y^X$  has CW type for all CW complexes  $Y$ .

PROOF. **a.** Implication (i)  $\implies$  (ii) follows by Proposition 2.6.4, Theorem 4.1.1 and Milnor's theorem.

Let  $n \geq 2$  and assume that the path component  $C$  of the constant map in  $(K(\mathbb{Z}, n), *)^{(X, *)}$  (if and only if the path component of the constant map in  $K(\mathbb{Z}, n)^X$ ) has CW homotopy type. By Corollary 2.2.9 it follows that the cohomology groups  $H^i(X; \mathbb{Z})$  are finitely generated for  $i \leq n - 1$ . Hence the homology groups  $H_i(X)$  are finitely generated for  $i \leq n - 2$ . If this holds for all  $n \geq 2$ , it follows that the  $H_i(X)$  are finitely generated for all  $i$ . This shows (iii)  $\implies$  (i).

- b. We need only show (iii)  $\implies$  (i). As above, by taking  $G = \mathbb{Z}$  it follows from Corollary 2.2.9 that the  $H_i(X)$  are finitely generated for  $i \leq n - 1$ . Let  $\mathcal{B}$  be the set of all finitely generated subgroups of  $H_n(X)$ , and define  $G = \prod_{B \in \mathcal{B}} H_n(X)/B$ . By Corollary 2.2.9 it follows that  $H_n(X) = B$  for  $B$  the image of  $H_n(L) \rightarrow H_n(X)$  for some finite subcomplex  $L$  of  $X$ .
- c. Assume that the path component  $C$  of  $Y^X$  has CW homotopy type for every simply connected CW complex  $Y$  of finite type. By **a.** it follows that  $H_*(X)$  is of finite type. Let  $Y$  be a simply connected CW complex with  $\pi_k(Y) \cong \mathbb{Z}$  for all  $k \geq 2$  and all  $k$ -invariants trivial. In other words,  $Y$  is weakly equivalent to the product  $\prod_{k \geq 2} K(\mathbb{Z}, k)$ . By Lemma 2.2.8 there exists a finite complex  $L$  such that for all  $k \geq 1$  the morphism

$$[S^k X, Y]_* \rightarrow [S^k L, Y]_*$$

is injective. By choice of  $Y$  this morphism is the product morphism

$$\prod_{j \geq k+1} (\tilde{H}^{j-k}(X; \mathbb{Z}) \rightarrow \tilde{H}^{j-k}(L; \mathbb{Z})).$$



It follows that there exists a number  $N$  with  $H^i(X; \mathbb{Z}) = 0$  for  $i \geq N$  and consequently  $H_i(X) = 0$  for  $i \geq N$ . Thus  $X$  quasifinite, and nilpotent by assumption. Hence it is finitely dominated by Mislin [44]. This shows (ii)  $\implies$  (iii).

- d. Assume (i). The proof of (ii)  $\implies$  (iii) of c. shows that  $X$  is quasifinite. Let  $L$  be any finite subcomplex of  $X$ . Consider the fibration  $Y^X \rightarrow Y^L$ . Let  $C$ , respectively  $C_L$ , denote the path component of the constant map in  $Y^X$ , respectively  $Y^L$ . Then  $C \rightarrow C_L$  is a fibration, and since  $C$  has CW type, so has the fibre  $F_L$  over the constant map. Note that  $F_L = C \cap (Y, *)^{(X, L)}$  is open  $(Y, *)^{(X, L)}$ . The path component  $C'$  of the constant map in  $(Y, *)^{(X, L)}$  is a path component of  $F_L$ , and since  $F_L$  has CW type,  $C'$  is open in  $F_L$ , and has CW type. Hence  $C'$  is open in  $(Y, *)^{(X, L)}$  which is homeomorphic with  $(Y, *)^{(X/L, *)}$ .

By Lemma 4.5.4 there exists a finite subcomplex  $K$  of  $X$  such that the function

$$[X, K(\pi_1(X/M, *), 1)]_* \rightarrow [L, K(\pi_1(X/M, *), 1)]_*$$

is injective for all finite subcomplexes  $M$  and all finite subcomplexes  $L$  containing  $K$ . It follows that  $\pi_1(X/L, *)$  is trivial for all finite  $L$  containing  $K$ . Thus  $X/L$  is simply connected, and is quasifinite, hence is homotopy equivalent to a finite complex.

Conversely, assume that for some finite subcomplex  $L$  of  $X$  the quotient  $X/L$  is homotopy equivalent to a finite complex. Consider the fibration  $Y^X \rightarrow Y^L$ . By Milnor's theorem the fibre over the constant map has CW homotopy type, hence by Stasheff's theorem the preimage  $\tilde{C}$  of the path component of the constant map  $C_L$  in  $Y^L$  has CW type, and is open since  $C_L$  is open in  $Y^L$ . Since  $\tilde{C}$  is a union of path components of  $Y^X$  which contains  $C$ ,  $C$  has CW type and is open in  $\tilde{C}$ , and hence in  $Y^X$ .

- e. Let  $\pi$  be a group and let  $M$  be a left  $\pi$ -module; that is we are given a morphism  $h: \pi \rightarrow \text{Aut}(M)$ . Further let  $k \geq 2$ . By Gitler [23] there exists a (pointed) space  $L_\pi(M, k)$  and a 'fundamental class'  $u \in H^k(L_\pi(M, k); h, M)$  such that the assignment

$$(*) \quad [X, L_\pi(M, k)]_\alpha \rightarrow H^k(X; h\alpha, M), \quad [f] \mapsto f^*(u),$$

is a bijection. Here  $[X, L_\pi(M, k)]_\alpha$  denotes those pointed classes  $[f]$  for which  $f_\#: \pi_1(X) \rightarrow \pi_1(L_\pi(M, k)) = \pi$  realizes a given morphism  $\alpha: \pi_1(X, x_0) \rightarrow \pi$ .

Let now  $\pi = \pi_1(X, x_0)$ . By our assumption and Lemma 4.5.4 there exists a finite subcomplex  $K$  of  $X$  such that the restriction

induced function

$$(\dagger) \quad [X, L_\pi(M, k)]_* \rightarrow [K, L_\pi(M, k)]_*$$

is an injection for all  $k$  with  $2 \leq k \leq n$  and all  $(h, M)$  belonging to a given set  $\mathcal{M}$  of left  $\pi$ -modules.

Denote the inclusion  $i: K \rightarrow X$ . Note that  $(\dagger)$  maps the set  $[X, L_\pi(M, k)]_1$  to  $[K, L_\pi(M, k)]_{i\#}$ . Hence the natural bijection  $(*)$  together with  $(\dagger)$  show that

$$(\ddagger) \quad H^k(X; h, M) \rightarrow H^k(K; hi\#, M)$$

is a monomorphism for  $2 \leq k \leq n$  and all  $M \in \mathcal{M}$ .

Assume that  $\mathcal{M} = \{(h_\lambda, M_\lambda) \mid \lambda \in \Lambda\}$  is a system of left  $\pi$ -modules indexed by a directed set  $\Lambda$  and let  $(h_\infty, M_\infty) = \text{colim}_\lambda (h_\lambda, M_\lambda)$ .

Consider the following commutative diagram.

$$\begin{array}{ccc} \text{colim}_\lambda H^k(X; h_\lambda, M_\lambda) & \longrightarrow & H^k(X; h_\infty, M_\infty) \\ \downarrow & & \downarrow \\ \text{colim}_\lambda H^k(K; h_\lambda i\#, M_\lambda) & \longrightarrow & H^k(K; h_\infty i\#, M_\infty) \end{array}$$

Since  $K$  is a finite complex, the bottom horizontal arrow is an isomorphism by Theorem 1 of K. Brown [2]. Since  $H^k(X; h_\lambda, M_\lambda) \rightarrow H^k(K; h_\lambda i\#, M_\lambda)$  is injective for all  $\lambda$  so is the induced map of colimits. By commutativity also

$$(\star) \quad \text{colim}_\lambda H^k(X; h_\lambda, M_\lambda) \rightarrow H^k(X; h_\infty, M_\infty)$$

is injective. In particular if  $M_\infty = 0$ , also  $\text{colim}_\lambda H^k(X; h_\lambda, M_\lambda) = 0$ , for  $2 \leq k \leq n$ .

Let  $\tilde{X}$  denote the universal cover of  $X$  at  $x_0$ . If  $\pi_1(X)$  is finitely generated, then we may assume  $X$  to have a single 0-cell  $x_0$  and finitely many 1-cells. Thus  $C_i(\tilde{X})$ , for  $i = 0, 1$ , are finitely generated (free)  $\mathbb{Z}\pi$ -modules. By Theorem 2 of [2] it follows that if  $M_\infty = 0$  then  $\text{colim}_\lambda H^k(X; h_\lambda, M_\lambda) = 0$  for  $k = 0, 1$ . Reapplying Theorem 2 of [2] it follows that  $(\star)$  is an isomorphism for any direct system  $\{(h_\lambda, M_\lambda) \mid \lambda\}$  with colimit  $(h_\infty, M_\infty)$  and all  $k \leq n$ . Our claim follows by Wall [63], Theorem A.

- f. Assume  $Y^X$  has CW type for all CW complexes  $Y$ . Proceeding as above we may assume that, in addition,  $(\ddagger)$  is a monomorphism for all  $k \geq 2$ . Thus  $X$  has finite cohomological dimension and

is finitely dominated by [63], Theorem E. (See also Mislin [45], Theorem 3.4).  $\square$

We say that a space  $Z$  is  $n$ -coconnected if  $\pi_k(Z) = 0$  for all  $k \geq n + 1$ .

LEMMA 4.5.4. *Let  $X$  be a connected CW complex such that  $Y^X$  has CW homotopy type for every CW complex  $Y$  (respectively every CW complex  $Y$  with  $\pi_k(Y) = 0$  for  $k \geq n + 1$ ). Then for any set*

$$\{Y_\lambda \mid \lambda \in \Lambda\}$$

*of arbitrary CW complexes (respectively of  $n$ -coconnected CW complexes) there exists a finite subcomplex  $K$  of  $X$  such that for any  $L \geq X$  and every  $\lambda$ , the restriction induced function*

$$[X, Y_\lambda]_* \rightarrow [L, Y_\lambda]_*$$

*is injective.*

PROOF. Let  $Y$  be a connected CW complex. Pick a choice function  $[X, Y]_* \rightarrow (Y, *)^{(X, *)}$ , say  $C \mapsto g(C)$ . Set

$$Z = \prod_{C \in [X, Y]_*} Y$$

and let  $\omega: W \rightarrow Z$  be a CW approximation for  $Z$ . By contracting a path in  $W$  if necessary we may assume that for some  $w_0 \in W$  we have  $\omega(w_0) = z_0 = \{y_0\}$ . Note that  $Z$  is path-connected, and since  $(Z, z_0)^{(S^k, *)} \approx \prod_C (Y, y_0)^{(S^k, *)}$ , also  $Z$  is  $n$ -coconnected if  $Y$  is. The same properties hold for  $W$ , and thus by assumption,  $(W, w_0)^{(X, x_0)}$  has CW homotopy type.

Note that  $\omega$  induces a natural equivalence of functors  $\omega_\#: [-, W]_* \rightarrow [-, Z]_*$ .

Define  $\Gamma: X \rightarrow Z$  by  $\Gamma = \{\Gamma_C\}_C$  where  $\Gamma_C = g(C)$ , and let  $G: X \rightarrow W$  be a representative for  $\omega_\#^{-1}[\Gamma]$ .

There exists a finite subcomplex  $K$  of  $X$  such that for any  $L$  containing  $K$  the restriction induced function  $[X, W]_* \rightarrow [L, W]_*$  is ‘injective at  $G$ ’, that is, if for some function  $F: X \rightarrow W$  the restriction  $F|_L$  is homotopic to  $G|_L$ , then  $F$  and  $G$  are homotopic.

In view of the natural equivalence the function  $[X, Z]_* \rightarrow [L, Z]_*$  is ‘injective at  $\Gamma$ ’. Assume that for  $f, g: X \rightarrow Y$  the restriction  $f|_L: L \rightarrow Y$  and  $g|_L: L \rightarrow Y$  are homotopic. With no loss of generality we may assume  $g = g(C_0)$  for some  $C_0$ . We define  $\Phi = \{\Phi_C\}_C: X \rightarrow Z$  by

$$\Phi_C = g(C) \text{ if } C \neq C_0 \text{ and } \Phi_{C_0} = f.$$

Then clearly  $\Phi|_L$  and  $\Gamma|_L$  are homotopic. Therefore  $\Phi$  and  $\Gamma$  are homotopic. By projecting the homotopy, it follows that  $f$  and  $g$  are homotopic. We have

shown that  $[X, Y]_* \rightarrow [L, Y]_*$  is injective for any  $L$  containing  $K$ . Applying this with  $Y = \prod_{\lambda} Y_{\lambda}$  we prove the lemma.  $\square$

## Spaces of maps into finite type Postnikov sections

For a CW complex  $Y$  let  $Y\langle k \rangle$  denote the  $k$ -connected cover of  $Y$ . In this chapter our aim is to prove the following result. In Example 11 on page 102 at the end we illustrate one of the reasons for the discrepancy between the necessary conditions and the sufficient conditions given.

**THEOREM 5.0.1.** *Let  $X$  be a simply connected CW complex and let  $Y$  be a simply connected complex of finite type with  $\pi_k(Y) = 0$  for all  $k \geq n + 1$ , and  $\pi_n(Y) \neq 0$ .*

*Set  $r = \sup \{k \mid \pi_k(Y) \otimes \mathbb{Q} \neq 0\}$  and let  $r_p = \sup \{k \mid \pi_k(Y) * \mathbb{Z}_p \neq 0\}$  for a prime  $p$ . As usual, we understand  $\sup \emptyset = -\infty$ .*

*If  $Y^X$  has CW type then the following hold.*

- (1) *For  $i \leq r - 2$  the group  $H_i(X)$  is finitely generated, and there exists a decomposition  $H_{r-1}(X) \cong \mathbb{Z}^{d_{r-1}} \oplus H'_{r-1}$  with  $\text{Hom}(H'_{r-1}, \mathbb{Z}) = 0$ .*
- (2) *For each  $i \leq n$  there exists a finitely generated subgroup  $B_i \leq H_i(X)$  (we may assume  $B_{r-1} \geq \mathbb{Z}^{d_{r-1}}$  and  $B_i = H_i(X)$  for  $i \leq r - 2$ ) so that*
  - a. *For  $i \leq r_p$  the quotient  $H_i(X)/B_i$  is  $p$ -divisible, and moreover*
  - b. *for  $i \leq r_p - 1$  the divisible  $p$ -localization  $(H_i(X)/B_i)_{(p)}$  admits a direct sum decomposition into  $Q \oplus (\oplus_j \mathbb{Z}_{p^\infty})$  where  $Q$  is a rational group and  $j$  ranges over a finite set.*

*If, in addition, either  $X$  is an  $H$ -cogroup or  $Y\langle r - 1 \rangle$  is an  $H$ -group or  $n \leq 2r - 5$  or  $r \geq n - 1$  then in fact*

- (1\*) *for  $i \leq r - 1$  the group  $H_i(X)$  is finitely generated and there exists a decomposition  $H_r(X) \cong \mathbb{Z}^{d_r} \oplus H'_r$  with  $\text{Hom}(H'_r, \mathbb{Z}) = 0$ , and we may assume  $B_r \geq \mathbb{Z}^{d_r}$ .*

*Conversely, conditions (1\*) and (2) are sufficient for  $Y^X$  to have CW type.*

We record a few auxiliary results. See Section 4.3 for a recollection of some definitions and results from the theory of abelian groups.

LEMMA 5.0.2. *Let  $P$  be a finite set of primes and let  $A$  be a  $P$ -divisible abelian group.*

- (i) *If  $A$  has an element of infinite order, then for each  $p \in P$  there exists a subgroup of  $A$ , isomorphic to  $\mathbb{Z}[\frac{1}{p}]$ .*
- (ii) *If  $A$  has an element of order  $p \in P$  then  $A$  contains  $\mathbb{Z}_{p^\infty}$  as a direct summand.  $\square$*

PROPOSITION 5.0.3. *Let  $A$  be an arbitrary abelian group.*

- a. *If  $\text{Hom}(A, \mathbb{Z}_{p^n})$  is finitely generated, then there exists a finitely generated subgroup  $B$  of  $A$  so that  $A/B$  is  $p$ -divisible, and consequently  $(A/B)_{(p)} = A_{(p)}/B_{(p)}$  is a divisible  $p$ -local group which is trivial if and only if  $A/B$  is  $\mathbb{P} \setminus \{p\}$ -torsion. In particular  $\text{Hom}(A, \mathbb{Z}_{p^n})$  is finite and  $\text{Ext}(A, \mathbb{Z}_{p^n})$  is bounded, hence a direct sum of (bounded) cyclic groups. If  $(A/B)_{(p)}$  is nontrivial, then  $B$  can be so chosen that  $A/B$  contains  $\mathbb{Z}_{p^\infty}$  as a direct summand. Moreover,  $B$  may be replaced with any bigger finitely generated subgroup without affecting the properties stated.*
- b. *If also  $\text{Ext}(A, \mathbb{Z}_{p^n})$  is finitely generated, then in the splitting  $(A/B)_{(p)} \cong (\oplus_i \mathbb{Q}) \oplus (\oplus_j \mathbb{Z}_{p^\infty})$  the index  $j$  must range over a finite family and  $\text{Ext}(A, \mathbb{Z}_{p^n})$  is finite.*

PROOF. Since  $A/pA$  is a vector space over  $\mathbb{Z}_p$ , it is isomorphic to a direct sum  $A/pA \cong \oplus_N \mathbb{Z}_p$ . Apply  $\text{Hom}(-, \mathbb{Z}_{p^n})$  to the exact sequence  $0 \rightarrow p \cdot A \rightarrow A \rightarrow A/pA \rightarrow 0$  to note that  $\text{Hom}(\oplus_N \mathbb{Z}_p, \mathbb{Z}_{p^n}) \cong \prod_N \mathbb{Z}_p$  is a subgroup of  $\text{Hom}(A, \mathbb{Z}_{p^n})$ . Then  $N$  must be finite if  $\text{Hom}(A, \mathbb{Z}_{p^n})$  is finitely generated.

Choose a finite subset  $S$  of  $A$  whose image generates  $A/pA$  and let  $B = \langle S \rangle$ . Thus  $B/pB \rightarrow A/pA$  is onto which shows that  $A/B$  is divisible by  $p$ . By Lemma 5.0.2 the localization  $(A/B)_{(p)}$  is trivial if and only if  $A/B$  is  $\mathbb{P} \setminus \{p\}$ -torsion. If this is not the case, then  $A/B$  contains a subgroup isomorphic to either  $\mathbb{Z}_{p^\infty}$  or  $\mathbb{Z}[\frac{1}{p}]$ . In the latter case enlarge  $S$  to kill the subgroup  $\mathbb{Z}$  of  $\mathbb{Z}[\frac{1}{p}]$  in  $A/B$  to obtain  $\mathbb{Z}_{p^\infty}$ .

Let  $B'$  be a finitely generated subgroup of  $A$  that contains  $B$ . The sequence  $0 \rightarrow B'/B \rightarrow A/B \rightarrow A/B' \rightarrow 0$  is exact. This is to say that if  $A/B$  is  $p$ -divisible, so is  $A/B'$ , and if  $A/B$  is  $\mathbb{P} \setminus \{p\}$ -torsion, so is  $A/B'$ . Moreover if  $A/B$  contains a subgroup isomorphic to  $\mathbb{Z}_{p^\infty}$  then  $A/B'$  contains a subgroup isomorphic to a quotient of  $\mathbb{Z}_{p^\infty}$  modulo a finitely generated subgroup, which is again isomorphic to  $\mathbb{Z}_{p^\infty}$ .

Since  $(A/B)_{(p)}$  is a divisible  $p$ -local group it is isomorphic to the direct sum of a rational group  $R$  and some copies of  $\mathbb{Z}_{p^\infty}$ , say of cardinality  $M$ .

Then  $\text{Hom}(A/B, \mathbb{Z}_{p^n})$  is trivial and

$$\text{Ext}(A/B, \mathbb{Z}_{p^n}) \cong \text{Ext}((A/B)_{(p)}, \mathbb{Z}_{p^n}) \cong \prod_M \mathbb{Z}_{p^n}.$$

An application of  $\text{Hom}(-, \mathbb{Z}_{p^n})$  to  $0 \rightarrow B \rightarrow A \rightarrow A/B \rightarrow 0$  concludes the proof.  $\square$

LEMMA 5.0.4. *Let  $R$  be an infinite set of primes and let  $A$  be an  $R$ -torsion group. If  $A$  is not finitely generated, then  $\text{Ext}(A, \mathbb{Z})$  contains an uncountable  $R$ -local subgroup.*

PROOF. We split  $A$  into the direct sum of  $r$ -primary components  $A \cong \bigoplus_{r \in R} A_{(r)}$ . Since  $A_{(r)}$  is torsion, the sequence  $0 \rightarrow \text{Ext}(A_{(r)} \otimes \mathbb{Z}_r, \mathbb{Z}) \rightarrow \text{Ext}(A_{(r)}, \mathbb{Z})$  is exact. If  $A_{(r)} \otimes \mathbb{Z}_r$  is infinite then  $\text{Ext}(A_{(r)}, \mathbb{Z})$  contains as a subgroup an infinite product  $\prod \mathbb{Z}_r$ .

If  $A_{(r)} \otimes \mathbb{Z}_r$  is finite then for a finitely generated subgroup  $B$  of  $A_{(r)}$  the quotient  $A_{(r)}/B$  is  $r$ -divisible. Since  $A_{(r)}/B$  is  $r$ -torsion, it contains  $\mathbb{Z}_{r^\infty}$  as a direct summand, if it is nontrivial. Again  $0 \rightarrow \text{Ext}(A_{(r)}/B, \mathbb{Z}) \rightarrow \text{Ext}(A_{(r)}, \mathbb{Z})$  is exact, hence  $\text{Ext}(A_{(r)}, \mathbb{Z})$  contains a subgroup isomorphic to  $\hat{\mathbb{Z}}_r$ .

If for all  $r$  the group  $A_{(r)}$  is finitely generated, and  $A$  itself is not finitely generated, then  $A$  splits into a direct sum  $\bigoplus_{\lambda \in \Lambda} C_\lambda$  where  $\Lambda$  is infinite and for each  $\lambda \in \Lambda$ , the summand  $C_\lambda$  is a cyclic group of order a power of some element of  $R$ . Then  $\text{Ext}(A, \mathbb{Z}) \cong \prod_\lambda C_\lambda$ , and the result follows.  $\square$

LEMMA 5.0.5. *Let  $H$  be an abelian group with a finitely generated subgroup  $B$  such that the quotient  $H/B$  is  $p$ -divisible for some prime  $p$ . Then  $\text{Hom}(H, \mathbb{Z})$  is finitely generated hence  $H$  admits a splitting  $H \cong \mathbb{Z}^d \oplus H'$  with  $\text{Hom}(H', \mathbb{Z}) = 0$ .*

PROOF. The exact sequence  $0 \rightarrow B \rightarrow H \rightarrow H/B \rightarrow 0$  induces the exact sequence  $0 \rightarrow \text{Hom}(H/B, \mathbb{Z}) \rightarrow \text{Hom}(H, \mathbb{Z}) \rightarrow \text{Hom}(B, \mathbb{Z}) \rightarrow \dots$ . Since  $H/B$  is  $p$ -divisible,  $\text{Hom}(H/B, \mathbb{Z}) = 0$ , hence  $\text{Hom}(H, \mathbb{Z})$  is finitely generated by Lemma 4.5.1.  $\square$

For convenience we record another easy lemma.

LEMMA 5.0.6. *Let  $Q, P \subset \mathbb{P}$  where  $\mathbb{P}$  denotes the set of all primes. Set  $P' = \mathbb{P} \setminus P$ . Let  $\pi$  be a finite  $Q$ -torsion group. Let  $H$  be an arbitrary abelian group with a finitely generated subgroup  $B$ .*

- (i) *Suppose that the quotient  $H/B$  is  $P$ -divisible. Then  $\text{Hom}(H, \pi)$  is isomorphic to a direct sum of a finite  $Q \cap P$ -group and a  $Q \cap P'$ -bounded group.*

- (ii) Suppose that the quotient  $H/B$  is  $Q$ -divisible, and the (divisible) localization  $(H/B)_{(Q)}$  is isomorphic to the direct sum of some copies of the rationals  $\mathbb{Q}$ , some copies of groups  $\mathbb{Z}_{q^\infty}$  with  $q \in Q \setminus P$  and at most finitely many copies of groups  $\mathbb{Z}_{q^\infty}$  with  $q \in Q \cap P$ . Then  $\text{Ext}(H, \pi)$  is an extension of a  $Q \cap P'$ -bounded group by a finite  $Q \cap P$ -group.  $\square$

LEMMA 5.0.7. Let  $B$  be a bounded abelian group. Let  $\hat{\mathbb{Z}}_p$  denote the group of  $p$ -adic integers,  $\hat{\mathbb{Z}}_p = \varprojlim \mathbb{Z}_{p^i}$ . Any morphism  $\hat{\mathbb{Z}}_p \rightarrow B$  factors through a canonical projection  $\pi_n: \hat{\mathbb{Z}}_p \rightarrow \mathbb{Z}_{p^n}$ ; hence the kernel is isomorphic to  $\hat{\mathbb{Z}}_p$ .

PROOF. With no loss of generality we may assume that  $B$  is a bounded  $p$ -group. By the theorem of Prüfer and Baer (see Proposition 4.3.4)  $B$  decomposes as  $B \cong \bigoplus_{\mu} C_{\mu}$  where the  $C_{\mu}$  are cyclic  $p$ -groups bounded by a common bound, say  $p^n$ . Set  $G_r = \mathbb{Z}_{p^r}$ .

First we determine  $\text{Hom}(\hat{\mathbb{Z}}_p, G_r)$ . The quotient  $\hat{\mathbb{Z}}_p/\mathbb{Z}$  is divisible,  $G_r$  is a reduced group, and  $\mathbb{Z}$  is free, hence the exact sequence obtained by applying  $\text{Hom}(-, G_r)$  to  $0 \rightarrow \mathbb{Z} \rightarrow \hat{\mathbb{Z}}_p \rightarrow \hat{\mathbb{Z}}_p/\mathbb{Z} \rightarrow 0$  simplifies to

$$0 \rightarrow \text{Hom}(\hat{\mathbb{Z}}_p, G_r) \rightarrow G_r \rightarrow \text{Ext}(\hat{\mathbb{Z}}_p/\mathbb{Z}, G_r) \rightarrow \text{Ext}(\hat{\mathbb{Z}}_p, G_r) \rightarrow 0.$$

The natural projection  $\pi_r: \hat{\mathbb{Z}}_p \rightarrow G_r$  is clearly nontrivial, hence  $\text{Hom}(\hat{\mathbb{Z}}_p, G_r) \cong G_r$  and every morphism  $\hat{\mathbb{Z}}_p \rightarrow G_r$  is obtained as the composite of  $\pi_r$  followed by an endomorphism  $G_r \rightarrow G_r$ .

Evidently then for  $r \leq n$  every morphism  $\hat{\mathbb{Z}}_p \rightarrow G_r$  factors through  $G_n$  since  $G_r \leq G_n$ .

As for the general case, since  $\bigoplus_{\mu} C_{\mu} \leq \prod_{\mu} C_{\mu}$ , a morphism  $\hat{\mathbb{Z}}_p \rightarrow B$  may be viewed as a morphism  $\varphi: \hat{\mathbb{Z}}_p \rightarrow \prod C_{\mu}$ . Since  $C_{\mu} \cong G_{r_{\mu}}$  with  $r_{\mu} \leq n$ , it follows by the above that  $\varphi$  factors as a composite  $\hat{\mathbb{Z}}_p \xrightarrow{\pi_n} \mathbb{Z}_{p^n} \xrightarrow{\psi} \prod C_{\mu}$ . Since the projection  $\pi_n$  is surjective and  $\varphi$  maps to  $\bigoplus C_{\mu}$ , so does  $\psi$ .  $\square$

LEMMA 5.0.8. Let  $\mathbb{P} = P \cup P'$  be a partition of the primes. Let  $A$  be a  $P'$ -local group, and let  $G$  be a  $P$ -bounded group. Then  $\text{Hom}(A, G) = 0$ .

PROOF. Since  $G$  is  $P$ -local,  $\text{Hom}(A, G)$  is isomorphic to  $\text{Hom}(A_{(P)}, G)$ . Since  $A$  is  $P'$ -local, its  $P$ -localization  $A_{(P)}$  is a rational group. In particular, it is divisible. As it is bounded,  $G$  is a reduced group, hence  $\text{Hom}(A_{(P)}, G) = 0$ .  $\square$

CONVENTION. As discussed in section 4.3, the torsion-free rank and  $p$ -ranks for primes  $p$  are invariants of a divisible abelian group  $A$ . Instead of saying that the  $p$ -rank of a divisible group  $A$  we will also say that  $A$  admits only finitely many  $p$ -torsion (or quasicyclic) summands.



LEMMA 5.0.9. *Assume that  $\pi_k(Y) = 0$  for  $k \geq n+1$ , and that for some  $m \leq n$ , the skeleton  $T = X^{(m-1)}$  is compact. If  $Y^X$  has the type of a CW complex, then there exists a finite subcomplex  $L$  containing  $T$  so that the morphisms*

$$\begin{aligned} (\dagger) \quad & H^m(X, T; \pi_n(Y)) \rightarrow H^m(L, T; \pi_n(Y)), \text{ and} \\ (\ddagger) \quad & H^{m+1}(X, T; \pi_n(Y)) \rightarrow H^{m+1}(L, T; \pi_n(Y)) \end{aligned}$$

*induced by inclusion are injective.*

PROOF. By Stasheff's theorem the fibre  $(Y, *)^{(X/T, *)}$  of  $(Y, *)^{(X, *)} \rightarrow (Y, *)^{(T, *)}$  over the constant map also has CW type. We appeal to Lemma 2.2.8 with  $X/T$  in place of  $X$  and infer the existence of a finite subcomplex  $L$  of  $X$  containing  $T$  such that the functions

$$\beta_k: [S^k \wedge (X/T), Y]_* \rightarrow [S^k \wedge (L/T), Y]_*$$

are 'monomorphisms'. We are interested in cases  $k = n-m$  and  $k = n-m-1$ . Let  $W_i$  denote the  $i$ -connected cover of  $Y$ . For connectivity reasons we may replace  $Y$  with  $W_{n-1}$  in case  $k = n-m$ , respectively  $W_{n-2}$  in case  $k = n-m-1$ . Since  $W_{n-1}$  is an Eilenberg-MacLane space  $K(\pi_n(Y), n)$ ,  $\beta_{n-m}$  is equivalent to the natural morphism  $H^m(X/T; \pi_n(Y)) \rightarrow H^m(L/T; \pi_n(Y))$  and the assertion follows if  $k = n-m$ .

If  $k = n-m-1$  then we may assume a fibration  $F \hookrightarrow W_{n-2} \rightarrow B$  with  $F$  and  $B$  Eilenberg-MacLane spaces. Then the induced map  $(W_{n-2}, *)^{(A, *)} \rightarrow (B, *)^{(A, *)}$  is a fibration for any CW complex  $A$  (see Proposition A.0.2). The fibre over the constant map is  $(F, *)^{(A, *)}$ . Using naturality of this construction in  $A$  (applied to inclusion  $L/T \rightarrow X/T$ ) and naturality of the arising long homotopy exact sequence, the assertion follows easily also in this case. □

PROOF OF SUFFICIENCY OF THEOREM 5.0.1. By Proposition 2.6.4 it suffices to show that for each  $j$  the space  $(K(\pi_j Y, j), *)^{(X, *)}$  has CW type. Then by Proposition 2.6.3 it suffices to show that for each  $i$  the space  $(K(\pi_j Y, j), *)^{(M(H_i X, i), *)}$  has CW type.

Let  $\tau_j$  denote the torsion subgroup of  $\pi_j Y$ . Then  $\pi_j Y \cong \tau_j \oplus \mathbb{Z}^{\nu_j}$  where  $\nu_j$  is the rank of the free abelian group  $\pi_j Y / \tau_j$ . Hence  $K(\pi_j Y, j) \simeq K(\tau_j, j) \times K(\mathbb{Z}, j)^{\nu_j}$  and

$$K(\pi_j Y, j)^{M(H_i X, i)} \simeq K(\tau_j, j)^{M(H_i X, i)} \times [K(\mathbb{Z}, j)^{M(H_i X, i)}]^{\nu_j}.$$

Conditions (2) and (1\*) now justify the application of Theorem 4.3.1 and Proposition 4.5.2 to conclude the proof. □

We turn to necessity.

First note that if  $r < n$ , and (2) of Theorem 5.0.1 holds then the quotient  $H_r(X)/B_r$  is divisible by some prime, and Lemma 5.0.5 implies the existence of a decomposition  $H_r(X) \cong \mathbb{Z}^{d_r} \oplus H'_r$  with  $\text{Hom}(H'_r, \mathbb{Z}) = 0$ .

STEP 1. *There exists a finite subcomplex  $L$  of  $X$  so that the space  $F_L = (Y, *)^{(X/L, *)}$  has finitely generated abelian homotopy groups  $\pi_k$  for  $k \geq 1$  and at most a countable set of path components.*

PROOF. By Proposition 1.1.2 there exists a finite subcomplex  $K$  of  $X$  such that for all  $L \geq K$  the fibre  $F_L$  of  $Y^X \rightarrow Y^L$  over the constant map deforms in  $Y^X$  to a point. Since  $F_L$  is homeomorphic to  $(Y, *)^{(X/L, *)}$ , we infer immediately from the fibration homotopy exact sequence that for  $k \geq 1$  the group  $\pi_k(F_L, *)$  is finitely generated abelian and that the set of path components  $\pi_0(F_L)$  is at most countable. Moreover  $F_L$  is an H-space (not necessarily homotopy associative or with inverse). We may assume a minimal decomposition for  $X$  (in the sense of Lemma 4.0.1) and hence for some  $m \geq 2$  the skeleton  $X^{(m-1)}$  is finite. Therefore we may choose  $L$  so that it contains  $X^{(m-1)}$  and consequently that the  $(m-1)$ -skeleton of  $X/L$  is trivial.  $\square$

STEP 2. *Assume that  $r = n$ , i.e. the top group  $\pi_n(Y)$  contains a  $\mathbb{Z}$  summand. Then the groups  $H_i(X)$  are finitely generated for  $i \leq n-1$ , and there exists a decomposition  $H_n(X) \cong \mathbb{Z}^d \oplus H'$  with  $\text{Hom}(H', \mathbb{Z}) = 0$ . If the torsion subgroup  $\tau_n$  of  $\pi_n$  is nonempty, then there exists a finitely generated subgroup  $B' \leq H'$  with the quotient  $H'/B'$  divisible by any prime  $t$  with  $\tau_n \otimes \mathbb{Z}_t$  nontrivial.*

PROOF. Assume inductively that  $X$  has finite  $(m-1)$ -skeleton for some  $m \geq 2$ . Assume also  $m \leq n-1$ . It follows from Lemma 5.0.9 that both  $H^m(X; \pi_n Y)$  and  $H^{m+1}(X; \pi_n Y)$  are finitely generated. In particular,  $H^m(X; \mathbb{Z})$  and  $H^{m+1}(X; \mathbb{Z})$  are finitely generated, and hence  $H_m(X)$  is finitely generated. The decomposition of  $X$  is assumed minimal and therefore  $X^{(m)}$  is finite. This may be continued until  $m = n-1$ , as asserted. For  $m = n-1$  we can still go on and apply Lemma 5.0.9 to conclude that  $H^n(X; \pi_n)$  is finitely generated, and hence that  $\text{Hom}(H_n(X), \mathbb{Z})$  and  $\text{Hom}(H_n(X), \tau_n)$  are finitely generated. The asserted decomposition  $H_n(X) \cong \mathbb{Z}^d \oplus H'$  follows immediately from Lemma 4.5.1. Now  $\text{Hom}(H_n(X), \pi_n)$  is isomorphic with a direct sum of  $\text{Hom}(H', \tau_n)$  and a finitely generated group. This implies the existence of  $B'$  as in the statement by Proposition 5.0.3.  $\square$

CONVENTION. Let  $A$  be an abelian group and  $G, H$  two finite abelian groups. Let  $P$  be the (finite) set of primes  $p$  for which  $G \otimes \mathbb{Z}_p \neq 0$  and  $Q$  the set of primes  $q$  for which  $H \otimes \mathbb{Z}_q \neq 0$ .

By abuse of language, saying that  $A$  is a  $G$ -group we mean it is a torsion  $P$ -primary group; saying it is  $G$ -divisible, we mean it is divisible by primes belonging to  $P$ ; saying it is  $G$ -bounded we mean that it is bounded by a number all of whose prime divisors belong to  $P$ ; saying it is a  $G \cap H$ -group we mean it is a torsion  $P \cap Q$ -primary group; saying it is  $G \setminus H$ -bounded we mean it is bounded by a number all of whose prime divisors belong to  $P \setminus Q$ .

In particular we note that if  $B$  is  $G \setminus H$ -bounded and  $A$  is  $H$ -bounded then  $\text{Hom}(A, B) = 0$ .

STEP 3. Assume that  $r < n$ . There exist finitely generated subgroups  $B_i \leq H_i(X)$  for  $2 \leq i \leq n$  so that the following hold.

- (i) If  $r = -\infty$  then for each  $i \leq n$ 
  - ( $D_i$ ) the quotient  $H_i(X)/B_i$  is  $T(\pi_{i+1}) \cup \pi_{i+2} \cup \dots \cup \pi_n$ -divisible, and for  $t \in T(\pi_{i+1}) \cup \pi_{i+2} \cup \dots \cup \pi_n$  the divisible localization  $(H_{i-1}(X)/B_{i-1})_{(t)}$  contains at most finitely many torsion summands.
- (ii) If  $r$  is positive then for  $i \leq r - 2$  the groups  $H_i(X)$  are finitely generated, and there exists a decomposition  $H_{r-1}(X) \cong \mathbb{Z}^{d_{r-1}} \oplus H'_{r-1}$  with  $\text{Hom}(H'_{r-1}, \mathbb{Z})$  trivial. (Therefore it makes sense to assume  $r - m > 1$ .) Moreover, the property ( $D_i$ ) can be established for  $i \geq r - 1$ .

PROOF. Set  $\Xi = X/L$ . We consider the Federer spectral sequence converging to the homotopy groups of  $Y^\Xi$ . This is a second quadrant homology type spectral sequence with  $E_{-p,q}^2 \cong H^p(\Xi; \pi_q Y)$  for  $q - p \geq 1$  and  $E_{-p,q}^2 \leq H^p(\Xi; \pi_q Y)$  for  $q - p = 0$ . For details see Federer [17].

Set  $H_i = H_i(\Xi)$  and  $\pi_j = \pi_j(Y)$ .

Assume for a decreasing induction on  $k$  that for  $q - p = k$  (with  $q \geq r$  and  $p \geq 2$ ) there exists a finitely generated subgroup  $B_p$  of  $H_p$  such that

- (\*) the quotient group  $H_p/B_p$  is  $T(\pi_q) \cup \pi_{q+1} \cup \dots \cup \pi_n$ -divisible,
- (\*\*) for a prime  $t$  with  $t \in \pi_{q+1} \cup \dots \cup \pi_n$  the divisible  $t$ -local group  $(H_p/B_p)_{(t)}$  admits only a finite number of torsion (i.e.  $\mathbb{Z}_{t^\infty}$ ) summands.

Using Lemma 5.0.6 it is easy to see that (\*) and (\*\*) imply that  $H^p(\Xi; \pi_q)$  is a finite  $\pi_q$ -group and  $E_{-(p+1),q}^2 \cong H^{p+1}(\Xi; \pi_q)$  is an extension of a  $\pi_q \setminus (\pi_{q+1} \cup \dots \cup \pi_n)$ -bounded group by a finite  $\pi_q \cap (\pi_{q+1} \cup \dots \cup \pi_n)$ -group, say

$$0 \rightarrow I_{-(p+1),q}^2 \rightarrow H^{p+1}(\Xi; \pi_q) \rightarrow F_{-(p+1),q}^2 \rightarrow 0.$$

Evidently then for each  $u \geq 3$  there exists an exact sequence

$$0 \rightarrow I_{-(p+1),q}^u \rightarrow E_{-(p+1),q}^u \rightarrow F_{-(p+1),q}^u \rightarrow 0$$

where  $I_{-(p+1),q}^u$  is a homology group of  $I_{-(p+1),q}^{u-1}$ , and  $F_{-(p+1),q}^u$  is a homology group of  $F_{-(p+1),q}^{u-1}$ . In particular,  $I_{-(p+1),q}^u$  is a  $\pi_q \setminus (\pi_{q+1} \cup \dots \cup \pi_n)$ -bounded group and  $F_{-(p+1),q}^u$  is a finite  $\pi_q \cap (\pi_{q+1} \cup \dots \cup \pi_n)$ -group.

For  $k = n - m + 1$  there is nothing to prove. Assume that the inductive hypothesis holds for  $k > \max\{r - m, 1\}$  and consider the diagonal  $q - p = k - 1$ . We induct again on  $q$ . For  $q \geq n + 1$  there is nothing to prove. Assume that (\*) and (\*\*) hold for  $q - p = k - 1$  with  $q > s$ . Let  $s - p = k - 1$ . Further assume that for some  $u \geq 2$  the subgroup  $I_{-p,s}^u$  of  $E_{-p,s}^u$  is not finitely generated. Since  $I_{-p,s}^u$  is  $\pi_s \setminus (\pi_{s+1} \cup \dots \cup \pi_n)$ -bounded, and by hypothesis of the second induction  $E_{-p-u,s+u-1}^u$  is  $\pi_{s+u-1}$ -bounded (or even trivial), the kernel of  $d^u: E_{-p,s}^u \rightarrow E_{-p-u,s+u-1}^u$  contains  $I_{-p,s}^u$ . Since by inductive hypothesis on  $k$  the diagonal  $j - i = k$  has  $E_{-i,j}^u$  a finitely generated group,  $I_{-p,s}^{u+1}$  contains a quotient of  $I_{-p,s}^u$  by a finitely generated subgroup and hence cannot be finitely generated. Hence  $I_{-p,s}^\infty$  is not finitely generated. The contradiction implies that  $I_{-p,s}^2$  is finite, and Proposition 5.0.3 establishes (\*) and (\*\*) for the pair  $(p, s)$ .

The above works as stated unless  $k - 1 = r - m$  or  $k - 1 = 0$ . Note that  $k - 1 = 0$  occurs if  $r = -\infty$ .

- If  $k - 1 = 0$ , then we have shown that the groups  $H^{i-1}(\Xi; \pi_i)$  are finite for all  $i$ . Let  $W_i$  denote the  $i$ -connected cover of  $Y$  for  $i \geq 1$ . We may assume that  $K(\pi_n, n) = W_{n-1} \rightarrow W_{n-2} \rightarrow \dots \rightarrow W_2 \rightarrow W_1 = Y$  is a sequence of principal fibrations where for each  $i$  the fibre of  $W_i \rightarrow W_{i-1}$  is a  $K(\pi_i, i - 1)$ . Therefore since  $[\Xi, W_i]_* \rightarrow [\Xi, W_{i-1}]_*$  collapses precisely the orbits of the action of  $H^{i-1}(\Xi; \pi_i)$  on  $[\Xi, W_i]_*$ , it follows by a trivial induction that if  $[\Xi, W_{n-1}]_* \cong H^n(\Xi; \pi_n)$  is uncountable then so is  $[\Xi, Y]_*$  which is a contradiction. Hence  $H^n(\Xi; \pi_n)$  is finitely generated, there exists a finitely generated subgroup  $B_n$  of  $H_n$  such that  $H_n/B_n$  is  $\pi_n$ -divisible, and  $B_{n-1}$  may be enlarged to assume that the divisible  $(H_{n-1}/B_{n-1})_{(\pi_n)}$  contains at most finitely many  $\mathbb{Z}_{t^\infty}$  summands if  $\pi_n \otimes \mathbb{Z}_t \neq 0$ . In particular,  $H^n(\Xi; \pi_n)$  is finite.
- If  $k - 1 = r - m$ , we proceed just as above, establishing (\*) and (\*\*) for pairs  $(p, s)$  with  $s > r$ . Now consider the term  $E_{-m,r}^2 \cong H^m(\Xi; \pi_r) \cong \text{Hom}(H_m, \pi_r)$ . Since  $H_m/B_m$  is  $\pi_{r+1} \cup \dots \cup \pi_n$ -divisible the restriction  $\text{Hom}(H_m, \mathbb{Z}) \rightarrow \text{Hom}(B_m, \mathbb{Z})$  is an injection which by Lemma 4.5.1 implies the existence of a decomposition  $H_m \cong \mathbb{Z}^{d_m} \oplus H'_m$  with  $\text{Hom}(H'_m, \mathbb{Z}) = 0$ . Since  $B_m$  may be

assumed to contain the summand  $\mathbb{Z}^{d_m}$  there exists finitely generated  $B'_m \leq H'_m$  with  $H'_m/B'_m$  isomorphic to  $H_m/B_m$ .

Using Lemma 5.0.6 we note that  $\text{Hom}(H_m, \pi_r)$  is an extension of a  $\tau_r \setminus (\pi_{r+1} \cup \dots \cup \pi_n)$ -bounded group by a finitely generated group. Reasoning as above it follows that we may enlarge  $B_m$  so that  $H_m/B_m$  is  $\tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$ -divisible,  $\text{Hom}(H_m, \tau_r)$  is finite, and  $\text{Ext}(H_m, \tau_r)$  is an extension of a  $\tau_r \setminus (\pi_{r+1} \cup \dots \cup \pi_n)$ -bounded group by a finite group.

We have assumed that  $r - m > 1$  hence also  $k - 2 = r - m - 1 \geq 1$ . On diagonal  $k - 1$  there appear now only finitely generated groups and by the above we may again proceed to establish (\*) and (\*\*) for  $s - p = k - 2$  with  $s > r$ . Now we consider  $E_{-(m+1),r}^2 \cong H^{m+1}(\Xi; \pi_r)$ . Split

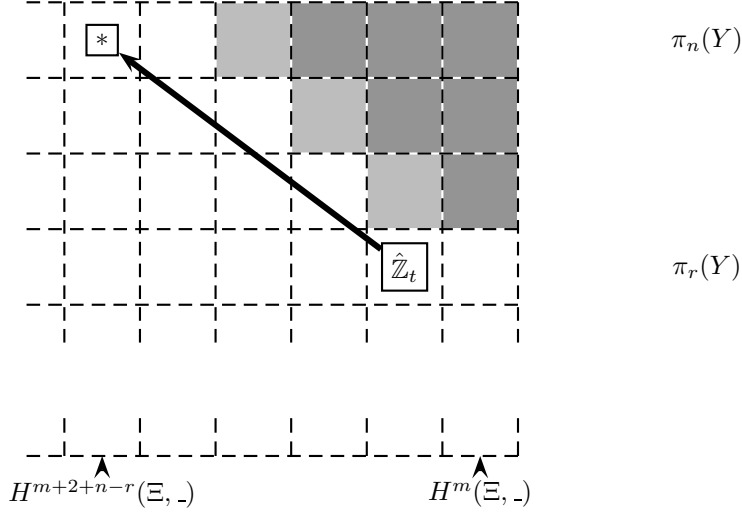
$$\begin{aligned} H^{m+1}(\Xi; \pi_r) &\cong \text{Hom}(H_{m+1}, \mathbb{Z}^\rho) \oplus \text{Ext}(H_m, \mathbb{Z}^\rho) \\ &\oplus \text{Hom}(H_{m+1}, \tau_r) \oplus \text{Ext}(H_m, \tau_r). \end{aligned}$$

Note that  $\text{Hom}(H_{m+1}, \tau_r) \oplus \text{Ext}(H_m, \tau_r)$  is an extension of a  $\tau_r \setminus (\pi_{r+1} \cup \dots \cup \pi_n)$ -bounded group, say  $I_{-(m+1),r}^2$ , by a finite group. Then also  $E_{-(m+1),r}^2$  is an extension of  $I_{-(m+1),r}^2$  and we proceed as above, concluding that  $I_{-(m+1),r}^2$  is finite which implies that we may assume  $H_{m+1}/B_{m+1}$  to be  $\tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$ -divisible, and that for  $t \in \tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$  the divisible group  $(H_m/B_m)_{(t)}$  contains at most finitely many torsion summands.

By Lemma 5.0.5 the divisibility properties of  $H_{m+1}/B_{m+1}$  imply a decomposition  $H_{m+1} \cong \mathbb{Z}^{d_{m+1}} \oplus H'_{m+1}$  with  $\text{Hom}(H'_{m+1}, \mathbb{Z})$  trivial (and  $H_{m+1}/B_{m+1} \cong H'_{m+1}/B'_{m+1}$  for a suitable  $B'_{m+1}$ ).

Next we take care of  $\text{Ext}(H_m, \mathbb{Z}^\rho)$ . Set  $P = \tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$  and let  $R = \mathbb{P} \setminus \{P\}$ . Since  $H_m/B_m$  is  $P$ -divisible, it is either an  $R$ -torsion group or we may assume that it admits a  $\mathbb{Z}_{t^\infty}$ -summand for some  $t \in P$ . Thus if  $H_m/B_m$  is not finitely generated,  $\text{Ext}(H_m, \mathbb{Z})$  either contains a subgroup isomorphic to  $\hat{\mathbb{Z}}_t$  modulo a finitely generated group or it contains an uncountable  $(R)$ -local subgroup  $\Gamma_R$  (see Lemma 5.0.4).

On diagonal  $k - 3$  there are only  $P$ -bounded groups strictly above row  $r$ . Hence if  $\Gamma_R \leq E_{-(m+1),r}^2$ , then by Lemma 5.0.8 also  $\Gamma_R \leq E_{-(m+1),r}^u$  for  $u \geq 3$ . If  $E_{-(m+1),r}^2$  contains a quotient of  $\hat{\mathbb{Z}}_t$  by a finitely generated subgroup, then by Lemma 5.0.7 each  $E_{-(m+1),r}^u$  for  $u \geq 3$  admits a subgroup isomorphic to a quotient of  $\hat{\mathbb{Z}}_t$  by a finitely generated group. See the figure below.



Both cases lead to contradiction hence  $H_m/B_m$  is finitely generated, and hence so is  $H_m$ . The exact sequence  $H_m(L) \rightarrow H_m(X) \rightarrow H_m(X, L) = H_m$  implies that also  $H_m(X)$  is finitely generated, hence  $X^{(m)}$  is finite and we may restart with Step 1 in case still  $m < r - 2$ . Therefore with no loss of generality we may assume  $m = r - 2$ .

We have shown that for  $q - p \geq r + 1 - m$  as well for  $q - p \in \{r - m, r - m - 1\}$  with  $q \geq r + 1$  the groups  $H^p(\Xi; \pi_q)$  are finite, and that for  $q - p \in \{r - m, r - m - 1\}$  with  $q = r$  the groups  $H^p(\Xi; \pi_q)$  are finitely generated. By considering exact sequences

$$\tilde{H}^p(\Xi; \pi_q) \cong H^p(X, L; \pi_q) \rightarrow H^p(X; \pi_q) \rightarrow H^p(L; \pi_q)$$

we note that exactly the same properties hold for  $H^p(X; \pi_q)$ . Hence we may apply Proposition 5.0.3 and Lemma 4.5.1 to deduce the asserted properties of groups  $H_i(X)$ .  $\square$

STEP 4. Let  $Y \rightarrow \bar{Y}$  denote Zabrodsky's integral approximation with homotopy fibre  $Y_t$ . Then  $Y_t^X$  has the homotopy type of a CW complex, and (2) of Theorem 5.0.1 holds.

PROOF. The map  $Y_t \rightarrow Y$  is a principal fibration with fibre  $\Omega\bar{Y}$  hence the fibre of the associated fibration  $(Y_t, *)^{(X, *)} \rightarrow (Y, *)^{(X, *)}$  over any point is the space  $(\Omega\bar{Y}, *)^{(X, *)}$ .

According to Zabrodsky [65],  $\Omega\bar{Y}$  is homotopy equivalent to the product

$$K(\mathbb{Z}^{d_r}, r - 1) \times \cdots \times K(\mathbb{Z}^{d_2}, 1)$$

where  $d_i$  denotes the torsion-free rank of  $\pi_i(Y)$ . Then  $Y_t^X$  has CW type by Proposition 4.5.2.

For  $j \geq r + 1$ , the morphism  $\pi_j(Y_t) \rightarrow \pi_j(Y)$  is bijective, and  $\pi_r(Y_t) \rightarrow \pi_r(Y)$  is a monomorphism onto the torsion subgroup  $\tau_r$  of  $\pi_r(Y)$ . In general there exist short exact sequences

$$0 \rightarrow \text{coker}(\pi_{j+1}Y \rightarrow \pi_{j+1}\bar{Y}) \rightarrow \pi_j Y_t \rightarrow T(\pi_j Y) \rightarrow 0.$$

Since the groups  $\text{coker}(\pi_{j+1}Y \rightarrow \pi_{j+1}\bar{Y})$  are finite, the following implication holds

$$(*) \quad \pi_j Y * \mathbb{Z}_p \neq 0 \implies \pi_j Y_t * \mathbb{Z}_p \neq 0.$$

Note that  $Y_t \simeq \prod_{p \in P} Y_{t(p)}$  where  $P$  is a finite set of primes. This implies that for each  $p \in P$  the space  $(Y_{t(p)}, *)^{(X,*)}$  has CW type and (i) of Step 3 applies to establish (2) of Theorem 5.0.1 by virtue of (\*), as claimed.

In particular, for  $i \geq r + 1$  the groups  $H^i(X; \pi_i)$  are finite. □

STEP 5. *Let  $Y\langle r - 1 \rangle$  denote the  $(r - 1)$ -connected cover of  $Y$ . Then  $Y\langle r - 1 \rangle^X$  has CW homotopy type.*

PROOF. The map  $Y\langle r - 1 \rangle \rightarrow Y$  is the principal fibration obtained by taking the homotopy fibre of  $Y \rightarrow Y_{r-1}$ . The fibres of the induced principal fibration  $(Y\langle r - 1 \rangle, *)^{(X,*)} \rightarrow (Y, *)^{(X,*)}$  are either empty or are homotopy equivalent to  $(\Omega Y_{r-1}, *)^{(X,*)}$ . Since  $\pi_k(\Omega Y_{r-1}) = 0$  for  $k \geq r - 1$  and  $X$  has finite  $(r - 2)$ -skeleton, the assertion follows. □

STEP 6. *By the previous step we may assume that  $Y$  is  $(r - 1)$ -connected. Assume that  $X$  is an H-cogroup or that  $Y$  is an H-group or that  $n \leq 2r - 5$ . Then also  $H^r(X; \pi_r)$  is finitely generated.*

PROOF. Note that in case  $X$  is an H-cogroup we cannot afford to replace it by a (homologically more convenient) quotient  $X/L$  for suitable  $L$ . Hence we proceed with  $X$  as it is.

We consider the Postnikov tower of principal fibrations  $Y_n \rightarrow Y_{n-1} \rightarrow \dots \rightarrow Y_r$  for  $Y$ . We know that for  $i \geq r + 1$  the groups  $H^i(X; \pi_i)$  are finite, and the groups  $H^{i+1}(X; \pi_i)$  are  $\pi_i$ -bounded.

We know that  $H_r/B_r$  is  $\pi_{r+1} \cup \dots \cup \pi_n$ -divisible, and therefore  $\text{Hom}(H_r, \mathbb{Z}) \rightarrow \text{Hom}(B_r, \mathbb{Z})$  is an injection. This guarantees  $H_r \cong \mathbb{Z}^{d_r} \oplus H'_r$  with  $\text{Hom}(H'_r, \mathbb{Z}) = 0$ .

We also know that  $H_{r-1}/B_{r-1}$  is  $\tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$ -divisible and that for  $t \in \pi_{r+1} \cup \dots \cup \pi_n$  the localization  $(H_{r-1}/B_{r-1})_{(t)}$  contains at most finitely many torsion summands. Set  $P = \tau_r \cup \pi_{r+1} \cup \dots \cup \pi_n$ . If  $H_{r-1}$  itself is not finitely generated, we may assume that either  $H_{r-1}/B_{r-1}$  contains a  $\mathbb{Z}_{t^\infty}$  summand for some  $t \in P$  or that  $H_{r-1}/B_{r-1}$  is an infinitely generated  $\mathbb{P} \setminus P$ -torsion group. It follows that  $\text{Ext}(H_{r-1}, \mathbb{Z})$  contains either a subgroup

isomorphic to a quotient of  $\hat{\mathbb{Z}}_t$  modulo a finitely generated subgroup or an uncountable  $\mathbb{P} \setminus P$ -local group, the latter by Lemma 5.0.4. This leads to contradiction as follows.

Our additional hypotheses guarantee that for each  $i \geq r$  the Puppe exact sequence

$$H^{i+1}(X; \pi_{i+1}) \rightarrow [X, Y_{i+1}]_* \rightarrow [X, Y_i]_* \rightarrow H^{i+2}(X; \pi_{i+1})$$

derived from the principal fibration  $Y_{i+1} \rightarrow Y_i$ , is an exact sequence of groups and group-morphisms. Since  $Y_r = K(\pi_r, r)$ , the group  $[X, Y_r]_* \cong H^r(X; \pi_r)$  contains a subgroup isomorphic to a quotient of  $\hat{\mathbb{Z}}_t$  by a finitely generated subgroup. If  $\text{Ext}(H_{r-1}, \mathbb{Z})$  has a subgroup isomorphic to a quotient of  $\hat{\mathbb{Z}}_p$  we apply Lemma 5.0.7, and if  $\text{Ext}(H_{r-1}, \mathbb{Z})$  has an uncountable  $\mathbb{P} \setminus P$ -local subgroup  $\Gamma$  we apply Lemma 5.0.8 in a straightforward induction to show that  $[X, Y_n]_* = [X, Y]_*$  contains a subgroup which is an extension of a finite group by a quotient of  $\hat{\mathbb{Z}}_t$  by a finitely generated group or an extension of a finite group by an uncountable  $\mathbb{P} \setminus P$ -local subgroup. In particular,  $[X, Y]_*$  is uncountable; which contradicts Proposition 2.3.1. This establishes (1\*) for this case, and concludes the proof of Theorem 5.0.1.  $\square$

**EXAMPLE 11.** *Suppose that  $Y$  is the total space of a fibration  $K(\mathbb{Z}_p, n) \rightarrow Y \rightarrow K(\mathbb{Z}, r)$  where  $n > r$ . This is to say that  $Y$  is the homotopy fibre of a single  $k$ -invariant  $k: K(\mathbb{Z}, r) \rightarrow K(\mathbb{Z}_p, n+1)$ . Let  $X$  be a simply connected CW complex. If  $Y^X$  has CW type, then  $H_i(X)$  is finitely generated for  $1 \leq i \leq r-1$ .*

*By Theorem 5.0.1 we know already that for each  $i \leq n$  the group  $H_i(X)$  admits a finitely generated subgroup  $B_i$  such that  $H_i(X)/B_i$  is  $p$ -divisible and  $(H_{i-1}(X)/B_{i-1})_{(p)}$  only admits finitely many quasicyclic summands. Moreover for  $i \leq r-2$  already  $H_i(X)$  is finitely generated.*

*By passing to cohomology, it follows that  $\tilde{H}^i(X; \mathbb{Z}_p)$  is finite for all  $i \leq n$ , and that  $\tilde{H}^{n+1}(X; \mathbb{Z}_p)$  is bounded by  $p$ . If  $L$  is a finite subcomplex of  $X$  then the quotient  $X/L$  has the same cohomological properties and by Proposition 5.0.3 and Lemma 4.5.1 we may recover the homological properties stated.*

*Assume a minimal decomposition of  $X$  with the associated homology decomposition  $\{X_i\}$ . We may pick a finite subcomplex  $L$  of  $X$  such that  $L$  contains  $X_{r-2}$ , the image  $B'_{r-1}$  of  $H_{r-1}(L) \rightarrow H_{r-1}(X)$  contains the subgroup  $B_{r-1}$ , and the restriction  $Y^X \rightarrow Y^L$  is nullhomotopic. Set  $\Xi = X/L$ . Note that  $\tilde{H}_i(\Xi) = 0$  for  $i \leq r-2$  and that  $H_{r-1}(\Xi) \cong H_{r-1}(X)/B'_{r-1}$  is  $p$ -divisible. Thus if  $H_{r-1}(\Xi)$  is not finitely generated, we may assume by Proposition 5.0.3 that either it contains a quasicyclic summand  $\mathbb{Z}_{p^\infty}$  or that*



it is a  $\mathbb{P} \setminus \{p\}$ -torsion group. Moreover, as in Step 1 it follows that the set  $[\Xi, Y]_*$  is countable.

We replace  $X$  by  $\Xi$ .

We are interested in the induced function

$$(*) \quad k_{\#}: [X, K(\mathbb{Z}, r)]_* \rightarrow [X, K(\mathbb{Z}_p, n+1)]_*$$

More precisely we would like to determine (the size of) the preimage of  $k_{\#}^{-1}(0)$ .

The homotopy class of  $k$  is represented by an element  $\hat{k}$  of  $H^{n+1}(K(\mathbb{Z}, r); \mathbb{Z}_p)$ . By results of Cartan and Borel the mod- $p$  cohomology algebra  $H^*(K(\mathbb{Z}, r); \mathbb{Z}_p)$  has a  $p$ -simple system of generators which are transgressive with respect to the Serre cohomology spectral sequence of the fibration

$$(**) \quad K(\mathbb{Z}, r) \rightarrow PK(\mathbb{Z}, r+1) \rightarrow K(\mathbb{Z}, r+1).$$

Using the associated vector-space basis for  $H^*(K(\mathbb{Z}, r); \mathbb{Z}_p)$  we expand  $\hat{k}$  as  $\hat{k} = \sum_i \lambda_i \xi_i$  where  $\lambda_i \in \mathbb{Z}_p$  and  $\xi_i \in H^{n+1}(K(\mathbb{Z}, r); \mathbb{Z}_p)$ . For each  $i$  we may write

$$(\dagger) \quad \xi_i = \alpha_i \smile \beta_i$$

where  $\alpha_i$  is transgressive (and  $\beta_i$  may be 1).

Applying  $H^*(-; \mathbb{Z}_p)$  to  $f: X \rightarrow K(\mathbb{Z}, r)$  we get the induced morphism

$$f^*: H^*(K(\mathbb{Z}, r); \mathbb{Z}_p) \rightarrow H^*(X; \mathbb{Z}_p).$$

Thus we may represent the composite

$$X \xrightarrow{f} K(\mathbb{Z}, r) \xrightarrow{k} K(\mathbb{Z}_p, n+1)$$

with the cohomology class

$$(\ddagger) \quad f^*(\hat{k}) = \sum_i \lambda_i f^*(\alpha_i) \smile f^*(\beta_i).$$

Transgressive elements are in the image of the cohomology suspension with respect to (\*\*), hence for each  $i$  there exists an element  $\eta_i \in H^{n+2}(K(\mathbb{Z}, r+1); \mathbb{Z}_p)$  such that  $\sigma^*(\eta_i) = \alpha_i$  where

$$\sigma^*: [K(\mathbb{Z}, r+1), K(\mathbb{Z}_p, n+2)]_* \rightarrow [\Omega K(\mathbb{Z}, r+1), \Omega K(\mathbb{Z}_p, n+2)]_*$$

is induced by the adjoint of the natural map

$$l: S\Omega K(\mathbb{Z}, r+1) \rightarrow K(\mathbb{Z}, r+1).$$

This implies that  $f^*(\alpha_i)$  may be represented in  $[X, K(\mathbb{Z}_p, n+1)]_*$  as the composite

$$X \xrightarrow{f} K(\mathbb{Z}, r) \xrightarrow{\cong} \Omega K(\mathbb{Z}, r+1) \xrightarrow{\Omega\psi_i} \Omega K(\mathbb{Z}_p, n+2) \xrightarrow{\cong} K(\mathbb{Z}_p, n+1)$$

for some map  $\psi_i: K(\mathbb{Z}, r+1) \rightarrow K(\mathbb{Z}_p, n+1)$ . Now we view  $f^*(\alpha_i)$  rather as  $\Omega\psi_{i\#}([f])$ . Since  $\Omega\psi_i$  is an  $H$ -group morphism, the induced function  $\Omega\psi_{i\#}$  is a group homomorphism. So we consider  $\Omega\psi_{i\#}: [X, K(\mathbb{Z}, r)]_* \rightarrow [X, K(\mathbb{Z}_p, n+1)]_*$ .

We distinguish two possibilities. The first is that  $H_{r-1}(X)$  contains a quasicyclic summand  $\mathbb{Z}_{p^\infty}$ . Then  $[X, K(\mathbb{Z}, r)]_*$  contains  $\text{Ext}(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong \hat{\mathbb{Z}}_p$  as a direct summand. Since  $[X, K(\mathbb{Z}_p, n+1)]_* \cong \tilde{H}^{n+1}(X; \mathbb{Z}_p)$  is  $p$ -bounded, the kernel  $A_i$  of

$$\Omega\psi_{i\#}|_{\hat{\mathbb{Z}}_p}: \hat{\mathbb{Z}}_p \rightarrow [X, K(\mathbb{Z}_p, n+1)]_*$$

is a subgroup of finite index in  $\hat{\mathbb{Z}}_p$ , by Lemma 5.0.7. The intersection  $A = \bigcap_i A_i$  of finitely many subgroups of finite index is also a subgroup of finite index in  $\hat{\mathbb{Z}}_p$ . This is to say that  $\Omega\psi_{i\#}[f] = f^*(\alpha_i) = 0$  for  $[f] \in A \leq \hat{\mathbb{Z}}_p \leq H^r(X; \mathbb{Z})$ . By (†) also  $k_{\#}[f] = 0$  for all  $[f] \in A$ . Hence the function  $k_{\#}$  sends an uncountable set to  $0 \in [X, K(\mathbb{Z}_p, n+1)]_*$ . By exactness of the sequence (of pointed sets)

$$[X, Y]_* \rightarrow [X, K(\mathbb{Z}, r)]_* \rightarrow [X, K(\mathbb{Z}_p, n+1)]_*$$

the set  $[X, Y]_*$  is uncountable.

The other possibility is that  $H_{r-1}(X)$  is a  $\mathbb{P} \setminus \{p\}$ -torsion group which is not finitely generated. By Lemma 5.0.4 the group  $\text{Ext}(H_{r-1}(X), \mathbb{Z})$  and hence  $H^r(X; \mathbb{Z})$  contains an uncountable  $\mathbb{P} \setminus \{p\}$ -local subgroup  $\Gamma$ . By Lemma 5.0.8 every morphism from  $\Gamma$  into a bounded  $p$ -group is trivial, and, proceeding as above, it follows that  $\Gamma \subset k_{\#}^{-1}(0)$ .

Both cases contradict the assumption that  $H_{r-1}(X)$  is not finitely generated. ■

## APPENDIX A

### Pullbacks and lifting functions

LEMMA A.0.1. *Let  $A$  be the pullback of  $B \xrightarrow{\beta} D \xleftarrow{\gamma} C$ , and let  $X$  be any space. Then  $A^X$  is the pullback of  $B^X \xrightarrow{\beta\#} D^X \xleftarrow{\gamma\#} C^X$ . If  $\beta: (B, b_0) \rightarrow (D, d_0)$  and  $\gamma: (C, c_0) \rightarrow (D, d_0)$ , then  $(A, (b_0, c_0))^{(X, x_0)}$  is the pullback of  $(B, b_0)^{(X, x_0)} \xrightarrow{\beta\#} (D, d_0)^{(X, x_0)} \xleftarrow{\gamma\#} (C, c_0)^{(X, x_0)}$ .*

PROOF. The natural map  $F: (B \times C)^X \rightarrow B^X \times C^X$  is a homeomorphism (see Maunder [35], Theorem 6.2.34), and it is trivial to check that  $F((B \sqcap C)^X) = B^X \sqcap C^X$ .  $\square$

DEFINITION. Let  $p: E \rightarrow B$  be a map and let  $\varepsilon_0: B^I \rightarrow B$  be evaluation at 0. Let  $\bar{E}$  denote the pullback of  $E \xrightarrow{p} B \xleftarrow{\varepsilon_0} B^I$ . A lifting function for  $p$  is a map  $\lambda: \bar{E} \rightarrow E^I$  that makes the following diagram commute.

$$\begin{array}{ccc}
 \bar{E} & \xrightarrow{\quad} & B^I \\
 \downarrow & \searrow \lambda & \nearrow p\# \\
 & & E^I \\
 \downarrow \varepsilon_0 & \nearrow & \downarrow \varepsilon_0 \\
 E & \xrightarrow{p} & B
 \end{array}$$

It is well known that  $p$  has a lifting function if and only if it is a fibration. (See Fadell [16].)

The author was unable to find a proof of the following result which should be folklore. (However, for the compactly generated refinement of the compact open topology the analogous result *is* well known, see Fritsch and Piccinini [19].)

PROPOSITION A.0.2. *Let  $p: E \rightarrow B$  be a fibration and let  $X$  be a compactly generated space. Then  $p\#: E^X \rightarrow B^X$  is a fibration.*

*If  $p(e_0) = b_0$  where  $e_0$  and  $b_0$  are nondegenerate base points of their respective spaces, then  $p\#: (E, e_0)^{(X, x_0)} \rightarrow (B, b_0)^{(X, x_0)}$  is a fibration, for any choice of  $x_0$ .*

PROOF. Let  $\lambda: E \sqcap B^I \rightarrow E^I$  be a lifting function for  $p$ . Here  $E \sqcap B^I$  is the pullback of  $E \xrightarrow{p} B \xleftarrow{\varepsilon_0} B^I$ . The left-hand side diagram below commutes.

$$\begin{array}{ccc}
 E \sqcap B^I & \xrightarrow{\quad} & B^I \\
 \downarrow \varepsilon_0 & \searrow \lambda & \nearrow p_{\#} \\
 & & E^I \\
 \downarrow \varepsilon_0 & \nearrow \varepsilon_0 & \downarrow \varepsilon_0 \\
 E & \xrightarrow{p} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 (E \sqcap B^I)^X & \xrightarrow{\quad} & (B^I)^X \\
 \downarrow (\varepsilon_0)_{\#} & \dashrightarrow \Lambda & \nearrow p_{\#\#} \\
 & & (E^I)^X \\
 \downarrow (\varepsilon_0)_{\#} & \nearrow (\varepsilon_0)_{\#} & \downarrow (\varepsilon_0)_{\#} \\
 E^X & \xrightarrow{p_{\#}} & B
 \end{array}$$

We claim that there exists a lifting function  $\Lambda: E^X \sqcap (B^X)^I \rightarrow (E^X)^I$  for  $p_{\#}$ .

Since  $X$  is compactly generated, so is the product  $X \times I$ , and therefore  $(B^X)^I$  is homeomorphic to  $(B^I)^X$  and  $(E^X)^I$  is homeomorphic to  $(E^I)^X$  (see Dugundji [11], Theorem XII.5.3). By Lemma A.0.1 the pullback  $E^X \sqcap (B^I)^X$  is homeomorphic to  $(E \sqcap B^I)^X$ , and the lifting function diagram for  $p_{\#}$  transcribes into the right-hand side diagram above. Thus we may set

$$(\dagger) \qquad \Lambda = \lambda_{\#},$$

and the first assertion follows.

If  $p(e_0) = b_0$  and  $e_0, b_0$  are nondegenerate, then  $\bar{e} = (e_0, \text{const}_{b_0})$  is nondegenerate in  $E \sqcap B^I$ , and consequently  $\lambda$  may be so chosen that  $\lambda(\bar{e}) = \text{const}_{e_0}$ . The function  $\Lambda$  as defined in  $(\dagger)$  is a lifting function for  $p_{\#}: (E, e_0)^{(X, x_0)} \rightarrow (B, b_0)^{(X, x_0)}$ , as is readily verified.  $\square$

Following the proof of Theorem 2.8.14 of Spanier [58] we extract

LEMMA A.0.3. *Let  $p: E \rightarrow B$  be a fibration and let  $f_0, f_1: X \rightarrow B$  be homotopic maps. Set  $E_0 = f_0^*E$ , and  $E_1 = f_1^*E$ . The fibrations  $E_0 \rightarrow X$  and  $E_1 \rightarrow X$  are fibre homotopy equivalent.*

*Let  $\bar{E}$  denote the pullback of  $E \rightarrow B \xleftarrow{\varepsilon_0} B^I$  where  $\varepsilon_0$  denotes evaluation at 0. Let  $\lambda: \bar{E} \rightarrow E^I$  be a lifting function for  $p$ . Let  $\tilde{h}: X \rightarrow B^I$  denote the adjoint of a homotopy between  $f_0$  and  $f_1$ . Then the map*

$$(\star) \qquad E_0 \rightarrow E_1, \quad (x, e) \mapsto (x, [\varepsilon_1 \circ \lambda](e, \tilde{h}(x)))$$

*is a fibre homotopy equivalence with the obvious inverse.*  $\square$

DEFINITION. Let  $p: E \rightarrow B$  be a fibration, and let  $f: X \rightarrow B$  be a map. If  $f$  is homotopic to  $\text{const}_{b_0}$  then by Lemma A.0.3 the pullback fibration  $f^*E \rightarrow X$  is fibre homotopy equivalent to the trivial fibration  $X \times F \rightarrow X$  where  $F$  is the fibre of  $p$  over  $b_0$ .

We will call the fibre homotopy equivalence  $f^*E \rightarrow X \times F$  given by  $(\star)$  the ‘canonical fibre homotopy equivalence’, and the associated section  $X \rightarrow f^*E$  the ‘canonical section’. Note that the equivalence  $(\star)$  depends on the choice of homotopy.

The following is well known. It is contained implicitly for example in Fadell [16].

LEMMA A.0.4. *Let  $p: E \rightarrow B$  be a fibration with  $p(e_0) = b_0$ , and let  $F$  denote the fibre of  $p$  over  $b_0$ . Let  $\Lambda$  be the homotopy fibre of the inclusion  $F \hookrightarrow E$ . More precisely,  $\Lambda$  is the pullback of  $F \hookrightarrow E \xleftarrow{\varepsilon_1} PE$  where  $\varepsilon_1$  is evaluation at 1. We may view  $\Lambda = (E, e_0, F)^{(I,0,1)} \subset E^I$  and  $\Omega(B, b_0) \equiv \{e_0\} \times \Omega(B, b_0) \subset \bar{E}$ . In this way composition with  $p$  yields a map  $\Lambda \rightarrow \Omega(B, b_0)$ , while the lifting function  $\lambda: \bar{E} \rightarrow E^I$  maps  $\Omega(B, b_0)$  to  $\Lambda$ .*

*The maps  $\Lambda \rightarrow \Omega(B, b_0)$  and  $\Omega(B, b_0) \rightarrow \Lambda$  are homotopy inverses of each other.  $\square$*

The following is a particular case of lifting functions for restriction fibrations.

LEMMA A.0.5. *Let  $W$  be a Hausdorff space and  $PW = \{\gamma \mid \gamma: I \rightarrow W, \gamma(0) = w_0\}$  the path space. The pull back  $\overline{PW}$  of  $PW \xrightarrow{\varepsilon_1} W \xleftarrow{\varepsilon_0} W^I$  may be identified with  $(W, w_0)^{(I \times 0 \cup 1 \times I, (0,0))}$ , the space  $PW^I$  may be identified with  $(W, w_0)^{(I \times I, 0 \times I)}$ , and if  $\rho: I \times I \rightarrow I \times 0 \cup 1 \times I$  is any retraction such that  $\rho(0 \times I) = \{(0, 0)\}$ , then*

$$\mu: \overline{PW} \rightarrow PW^I, \quad \mu(\phi) = \phi \circ \rho$$

*is a lifting function for  $\varepsilon_1: PW \rightarrow W$ . In particular,  $\rho$  may be chosen so that  $\rho(s, 1) = (2s, 0)$  for  $s \leq \frac{1}{2}$  and  $\rho(s, 1) = (1, 2s - 1)$  for  $s \geq \frac{1}{2}$ .  $\square$*

Lemmas A.0.3, A.0.4, A.0.5 readily yield

PROPOSITION A.0.6. *Let  $p: E \rightarrow B$  be a fibration and let  $\bar{E}$  denote the pullback of  $E \rightarrow B \xleftarrow{\varepsilon_0} B^I$  where  $\varepsilon_0$  denotes evaluation at 0. Let  $\lambda: \bar{E} \rightarrow E^I$  be a lifting function for  $p$ . Let  $F$  be the fibre of  $p$  over  $b_0 \in B$ .*

*Assume that  $F$  contracts in  $E$  to  $e_0 \in E$ , and let  $k: F \rightarrow E^I$  denote the adjoint of a contracting homotopy. Further let  $\varepsilon_1: E^I \rightarrow E$  denote evaluation at 1.*

*Then the map*

$$\Omega(B, b_0) \rightarrow F \times \Omega(E, e_0), \quad \gamma \mapsto (\varepsilon_1[\lambda(e_0, \gamma)], \lambda(e_0, \gamma) * k(\varepsilon_1[\lambda(e_0, \gamma)]))$$

*is a homotopy equivalence with inverse*

$$F \times \Omega(E, e_0) \rightarrow \Omega(B, b_0), \quad (x, \omega) \mapsto p \circ [\omega * k^{-1}(x)] = (p \circ \omega) * (p \circ k^{-1}(x)). \quad \square$$

## Topological quasi-groups and quasi-LEC spaces

### B.1. Topological quasi-groups

If  $G$  is an uncountable abelian group then the geometric realization  $Y$  of the simplicial Eilenberg-MacLane group  $K(G, n)$  (see Milnor [42]) is not a topological group in its CW topology with the cartesian product. More precisely, there exists a function  $\mu: Y \times Y \rightarrow Y$  which is the multiplication of an abelian group but is continuous only on compact subsets of  $Y \times Y$ .

In general this is still perfectly acceptable since the category of compactly generated spaces is usually suitable enough for homotopy theory. However, if we want to consider function spaces of maps into  $Y$  equipped with the compact open topology to have a good grip on the open sets, we encounter problems which stem from the fact that even for compact spaces  $K$  the function space  $Y^K$  need not be compactly generated.

DEFINITION. Let  $Y$  be a (Hausdorff) topological space with distinguished element  $y_0$  and a function  $\mu: Y \times Y \rightarrow Y$  such that  $(Y, \mu)$  is a monoid with unit  $y_0$ , and for every compact subset  $C \subset Y$  the restrictions

$$(\dagger) \quad \mu|_{Y \times C}: Y \times C \rightarrow Y, \quad \mu|_{C \times Y}: C \times Y \rightarrow Y,$$

are continuous. Then we say that  $Y$  is a *topological quasi-monoid*.

If, in addition, there exists a continuous function  $\text{inv}: Y \rightarrow Y$  such that  $(Y, \mu, \text{inv})$  is a group with unit  $y_0$  then we say that  $Y$  is a *topological quasi-group*.

A morphism of topological quasi-groups is a continuous homomorphism.

Evidently if  $Y$  is compactly generated then  $\mu$  is continuous on compact subsets of  $Y \times Y$  if and only if the restrictions  $(\dagger)$  are continuous for compact  $C$ .

Therefore by [42] for every abelian group  $G$  there exists a CW complex  $Y$  of type  $K(G, n)$  which is an abelian topological quasi-group. (And if  $G$  is countable, ‘quasi’ may be omitted.)

For another example, by Dold and Thom [9] for every connected simplicial complex  $X$ , the infinite symmetric product  $SP(X)$  is an abelian topological quasi-monoid.

Clearly,

LEMMA B.1.1. *If  $Y$  is an abelian topological quasi-monoid or a topological quasi-group then continuity of  $(\dagger)$  for all compact  $C$  is equivalent to the continuity of  $\mu|_{Y \times C}: Y \times C \rightarrow Y$  for all compact  $C$ .  $\square$*

PROPOSITION B.1.2. *Let  $(Y, \mu, \text{inv}, y_0)$  be a topological quasi-group and  $K$  a compactum. Given  $f, g \in Y^K$ , the function  $K \rightarrow Y$ , defined by  $x \mapsto \mu(f(x), g(x))$ , is continuous. Define*

$$M: Y^K \times Y^K \rightarrow Y^K, \quad M(f, g)(x) = \mu(f(x), g(x));$$

$$I: Y^K \rightarrow Y^K, \quad I(f) = \text{inv} \circ f.$$

*Then  $(Y^K, M, I, \text{const}_{y_0})$  is a topological quasi-group. If  $A$  is any subset of  $K$  then  $(Y, y_0)^{(K, A)}$  is a closed sub quasi-group.*

PROOF. Let  $\Gamma$  be a compact subset of  $Y^K$ . The evaluation map  $e: Y^K \times K \rightarrow Y$ ,  $e(f, x) = f(x)$ , is continuous, and hence  $L = e(\Gamma \times K) = \{f(x) \mid f \in \Gamma, x \in K\}$  is compact. Note that  $\Gamma \subset L^K$ . Since  $\mu|_{Y \times L}$  is continuous, so is the composite

$$Y^K \times \Gamma \hookrightarrow Y^K \times L^K \xrightarrow{\cong} (Y \times L)^K \xrightarrow{(\mu|_{Y \times L})^\#} Y^K.$$

The other assertions are trivial.  $\square$

Topological quasi-groups have enough structure to avoid base point problems; the proof carries over from the standard case *mutatis mutandis*.

PROPOSITION B.1.3. (1) *Let  $G$  be topological quasi-group with unit  $e$  and let  $Z$  be a space with  $z_0 \in Z$ . Let  $f: Z \rightarrow G$  be a map. Then  $f$  can be homotoped to a map that sends  $z_0$  to any point in the path component of  $f(z_0)$ .*

*If  $f, g: Z \rightarrow G$  are freely homotopic, then they are homotopic with a homotopy preserving  $z_0$ .*

(2) *If  $G_1$  and  $G_2$  are freely homotopy equivalent topological quasi-groups then for any choice of base points  $g_1 \in G_1, g_2 \in G_2$ , the pairs  $(G_1, g_1)$  and  $(G_2, g_2)$  are homotopy equivalent.  $\square$*

The proof of the following is straightforward.

LEMMA B.1.4. (1) *The cartesian product of topological quasi-groups is a topological quasi-group.*

- (2) If  $B \xrightarrow{\varphi} A \xleftarrow{\varepsilon} P$  is a diagram of topological quasi-groups and morphisms, then the pullback space  $E$  admits a natural topological quasi-group structure so that the pullback square is a diagram of topological quasi-groups and morphisms.  $\square$

Proposition B.1.2 and Lemma B.1.4 yield

COROLLARY B.1.5. Let  $\varphi: B \rightarrow A$  be a morphism of topological quasi-groups. Then the canonical factorization  $B \xrightarrow{\cong} B' \xrightarrow{\varphi'} A$  of  $\varphi$  into the composite of a homotopy equivalence followed by a fibration is a sequence of morphisms of topological quasi-groups.  $\square$

DEFINITION. Let  $G$  be a topological quasi-group and  $E$  a space. Then  $E$  is a left  $G$ -space if there exists a left action  $\Phi: G \times E \rightarrow E$  such that the restrictions  $\Phi|_{C \times E}$  and  $\Phi|_{G \times D}$  are continuous for all compact subsets  $C$  of  $G$  and  $D$  of  $E$ .

PROPOSITION B.1.6. Let  $p: E \rightarrow B$  be a surjective morphism of topological quasi-groups and let  $G = \ker p$ . Then  $G$  is a topological quasi-group and  $E$  is a free left  $G$ -space with the following property.

For any space  $Z$  and continuous maps  $f_1, f_2: Z \rightarrow E$  such that  $p \circ f_1 = p \circ f_2$  there exists a function  $f: Z \rightarrow G$  which is continuous on the compact subsets of  $Z$  such that  $f(z) \cdot f_1(z) = f_2(z)$ , for all  $z \in Z$ .  $\square$

DEFINITION. Let  $G$  be a topological quasi-group. A free left action of  $G$  on  $E$  with the property as in Proposition B.1.6 is *compactly open*. (Compare the definition preceding Lemma 1.5.4 on page 22.)

COROLLARY B.1.7. Let  $\{G_i\}$  be a sequence of topological quasi-groups. Let  $\{p_i\}: \{E_i\} \rightarrow \{B_i\}$  be a level preserving morphism of inverse sequences where for each  $i$ , the space  $E_i$  is a free left  $G_i$ -space and  $p_i: E_i \rightarrow B_i$  is a compactly open fibration. Assume that the maps  $E_i \rightarrow E_{i-1}$  are equivariant given by morphisms  $\omega_i: G_i \rightarrow G_{i-1}$ .

Let  $p_\infty: E_\infty \rightarrow B_\infty$  denote the induced inverse limit map. If the  $\omega_i$  are fibrations then  $p_\infty$  is a fibration for the class of compactly generated spaces.  $\square$

## B.2. Quasi-LEC spaces

A topological space  $Y$  is called *locally equiconnected* (or LEC) (see Milnor [43]) if there exists an open neighbourhood  $V$  of the diagonal  $\Delta$  in  $Y \times Y$  and a continuous function  $\lambda: V \times I \rightarrow Y$  which is *locally equiconnecting*, that is

$$(*) \quad \lambda(y_0, y_1, 0) = y_0, \quad \lambda(y_0, y_1, 1) = y_1, \quad \lambda(y, y, t) = y, \\ \text{for all } y_0, y_1, y \in Y, t \in I.$$



If  $Y$  is an uncountable CW complex then there exists a locally equiconnecting function  $\lambda: V \times I \rightarrow Y$  where  $V$  is open in the compactly generated refinement  $\mathfrak{K}(Y \times Y)$  of  $Y \times Y$ , and  $\lambda$  is continuous on compact subsets of  $V \times I$ .

A slightly more general notion behaves well under forming function spaces.

DEFINITION. Let  $Y$  be a topological space. Suppose there exist a subset  $V$  of the product  $Y \times Y$ , containing the diagonal  $\Delta$ , and a function  $\lambda: V \times I \rightarrow Y$  so that

- (i) for every compact subset  $C$  of  $Y$ , the set  $V_C = V \cap (Y \times C)$  is open in  $Y \times C$ , and the restriction

$$\lambda|_{V_C}: V_C \times I \rightarrow Y$$

is continuous, and

- (ii)  $\lambda$  satisfies (\*), i.e. it is locally equiconnecting.

Then we say that  $Y$  is a quasi-LEC space.

LEMMA B.2.1. *If  $Y$  is a compactly generated regular space, then (i) of the definition holds if and only if  $V$  is open in  $\mathfrak{K}(Y \times Y)$  and  $\lambda$  is continuous on  $\mathfrak{K}(V) \times I = \mathfrak{K}(V \times I)$ .*

PROOF. Since every compact subset of  $Y \times Y$  is contained in a set of the form  $Y \times C$  for a compact set  $C$ , the necessity is clear.

For sufficiency, if  $V \cap A$  is open in  $A$  for every compactum  $A$  in  $Y \times Y$ , then  $V \cap A$  is open in  $A$  for every compactum  $A$  in  $Y \times C$ . But  $Y \times C$  is compactly generated, hence  $V \cap [Y \times C]$  is open in  $Y \times C$ .

Since  $Y$  is regular so is  $Y \times C$ . Hence  $V \cap [Y \times C]$  is an open subset of a compactly generated regular space, and as such is compactly generated (see Steenrod [60], 2.4). Then if  $\lambda|_{A \times I}$  is continuous for each compact subset  $A$  of  $V$ , trivially also  $\lambda|_{A \times I}$  is continuous for each compact subset  $A$  of  $V \cap [Y \times C]$ . Since  $V \cap [Y \times C]$  is compactly generated, the restriction  $\lambda|_{V \cap [Y \times C]}$  is continuous.  $\square$

COROLLARY B.2.2. *Let  $Y$  be a CW complex. Then  $Y$  is a quasi-LEC space. If  $Y$  is countable then it is an LEC space.*

PROOF. See Lewis [32]. If  $Y$  is countable then the cartesian product  $Y \times Y$  is itself compactly generated, and the notions of LEC and quasi-LEC coincide.  $\square$

PROPOSITION B.2.3. *Let  $Y$  be a perfectly normal quasi-LEC space. Then  $Y$  is globally well-pointed.*

PROOF. If  $Y$  is quasi-LEC then every point  $y_0$  has a neighbourhood which deforms in  $Y$  to  $y_0$ , keeping  $y_0$  fixed. Now proceed as in Lemma 3.3 of [56].  $\square$

PROPOSITION B.2.4. *Let  $Y$  be a quasi-LEC space and  $K$  a compactum. Then  $Y^K$  is a quasi-LEC space.*

PROOF. Define  $U := \{(f, g) \mid (f(x), g(x)) \in V, \forall x \in K\}$  and

$$\Lambda: U \times I \rightarrow Y^K, \quad \Lambda(f, g, t)(x) = \lambda(f(x), g(x), t).$$

We claim that  $U$  and  $\Lambda$  define a quasi-LEC structure on  $Y^K$ . In fact we show more. Let  $L$  be any compact subset of  $Y$ . Then  $L^K \subset Y^K$ . We observe that

$$\begin{aligned} U_L &= U \cap [Y^K \times L^K] = \{(f, g) \mid (f(x), g(x)) \in V \cap [Y \times L], \forall x \in K\} \\ &= G(K, V \cap [Y \times L]) \subset (Y \times L)^K. \end{aligned}$$

Thus  $U_L$  is open in  $Y^K \times L^K$ . Set  $V_L = V \cap [Y \times L]$  and let  $\lambda_L$  denote the (continuous) restriction of  $\lambda$  to  $V_L \times I$ . Further let  $\hat{\lambda}_L: V_L \rightarrow Y^I$  denote the adjoint of  $\lambda_L$ . We note that the map

$$\hat{\Lambda}_L = \hat{\lambda}_L\# : U_L \rightarrow (Y^I)^K, \quad \hat{\Lambda}_L(f, g) = \hat{\lambda}_L \circ (f, g),$$

is continuous, and, since  $(Y^I)^K$  is homeomorphic to  $(Y^K)^I$ ,  $\hat{\Lambda}_L$  gives rise to a continuous adjoint  $\Lambda_L: U_L \times I \rightarrow Y^K$  which pointwise coincides with  $\Lambda$ .

Since every compact subset  $\Gamma$  of  $Y^K$  is contained in  $L^K$  for some compact  $L$ , the proof is finished.  $\square$

COROLLARY B.2.5. *Let  $Y$  be a CW complex and  $K$  a compactum. Then  $Y^K$  is a regular quasi-LEC space and since it is stratifiable all of its points are nondegenerate.*

PROOF. That  $Y^K$  is stratifiable (and hence perfectly normal) follows from Cauty [4]. The rest follows from Proposition B.2.4 and Proposition B.2.3.  $\square$

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