

# A uniform study on the cyclicity of period annulus of the reversible quadratic Hamiltonian systems\*

Chengzhi Li<sup>1)</sup> and Jaume Llibre<sup>2)</sup>

<sup>1)</sup>*School of Mathematical Science, Peking University, Beijing, 100871, P.R.China E-mail: licz@math.pku.edu.cn*

<sup>2)</sup>*Department of Matemàtiques, Universitat Autònoma de Barcelona, 08193, Bellaterra, Barcelona Spain E-mail: jllibre@mat.uab.es*

*Dedicated to Professor Shui-Nee Chow  
on the occasion of his 60th birthday*

## Abstract

The cyclicity of the period annulus of reversible quadratic Hamiltonian systems under quadratic perturbations was studied by several authors for different cases by using different methods. In this paper, we study this problem in a uniform way.

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## 1 Introduction

Following [15] and [11] the quadratic centers are classified into four classes: Hamiltonian ( $Q_3^H$ ), reversible ( $Q_3^R$ ), Lotka–Volterra ( $Q_3^{LV}$ ) and codimension four ( $Q_4$ ). It was pointed out by Ilive in [9] that if  $X_H \in Q_3^H \cap Q_3^R$ , then except one case (the Bogdanov–Takens system, which has been extensively

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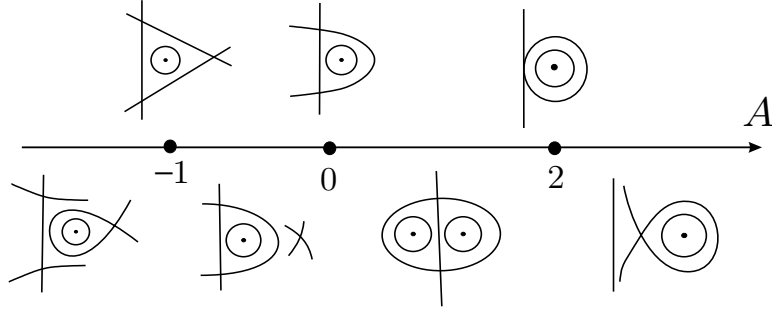


Figure 1. The phase portraits of  $X_{H_A}$  for the different values of  $A$ .

studied in many papers, see [1, 13, 12, 9]), the cubic Hamiltonian  $H(x, y)$  can be transformed into the following one-parameter family

$$H_A(x, y) = x[3(A - 2) - 3(A - 1)x + Ax^2 + y^2], \quad A \in (-\infty, \infty). \quad (1.1)$$

The phase portraits of the corresponding vector fields  $X_{H_A} = \frac{\partial H_A}{\partial y} \frac{\partial}{\partial x} - \frac{\partial H_A}{\partial x} \frac{\partial}{\partial y}$  are shown in figure 1. We consider the quadratic perturbations of  $X_{H_A}$  as follows.

$$\begin{aligned} \dot{x} &= 2xy + \epsilon f(x, y, \epsilon), \\ \dot{y} &= -(3(A - 2) - 6(A - 1)x + 3Ax^2 + y^2) + \epsilon g(x, y, \epsilon), \end{aligned} \quad (1.2)$$

where  $\epsilon$  is a small parameter,  $f$  and  $g$  are quadratic polynomials in  $x, y$  with coefficients depending analytically on  $\epsilon$ . A natural question is: *how many limit cycles can appear for small  $\epsilon$  from the period annulus (or annuli if  $A \in (0, 2)$ ), surrounding the center (or centers if  $A \in (0, 2)$ )?* Here the limit cycles that emerge from the center (or centers) are included. The sharp upper bound of the number of such limit cycles is called the *cyclicity* of the period annulus (or annuli).

Ilive [11] investigated the perturbations of quadratic centers, he pointed out, among other, that  $X_{H_{-1}} \in Q_3^H \cap Q_3^R \cap Q_3^{LV} \setminus Q_4$  (the Hamiltonian triangle, see figure 1), and the cyclicity of  $X_{H_{-1}}$  is determined by a third order Milnikov function (in fact, he already proved in [10] that the cyclicity of the period annulus is equal to three in this case); if  $A \in (-\infty, \infty) \setminus \{-1\}$  then  $X_{H_A} \in Q_3^H \cap Q_3^R \setminus \{Q_3^{LV} \cup Q_4\}$ , and the cyclicity of the period annulus of  $X_{H_A}$  is equal to the sharp upper bound of the number of zeros of the second order Milnikov function (see Theorem 3 of [11])

$$J_A(h) = \iint_{\text{Int}(\delta_A(h))} (\alpha + \beta x + \gamma x^{-1}) dx dy, \quad (1.3)$$

for all constants  $\alpha$ ,  $\beta$  and  $\gamma$ , and  $\delta_A(h)$  is an oval contained in the level curves  $\{H_A = h \text{ with } h \in (h_c, h_s)\}$ , where  $h_c$  and  $h_s$  denote the corresponding values of the level curves which shrinks to a center  $C_A$ , or expands to the homoclinic or heteroclinic loop connected to one or two saddles respectively if  $A \neq 2$ , and to a Non-Morsean point if  $A = 2$ .

Based on this formula and on the Picard–Fuchs equation, and using different techniques, the papers [17], [9] and [16] investigated the cyclicity of the period annulus of  $X_{H_A}$  under quadratic perturbations for the cases of  $A \in (-1, 0)$ ,  $A = 0$ , and  $A = 2$ , respectively, and [4] studied the same problem for the cases of  $A \in (-\infty, -1)$  and  $A \in (0, 2)$ . On the other hand, using the Picard–Fuchs equation and the Picard–Lefschetz formula (in complex domain), [7] studied the cyclicity of  $X_{H_A}$ ,  $A \in (2, \infty)$ , under degree  $n$  polynomial perturbations. Combining the above results together, and restricting them to quadratic perturbations, one has the following conclusion.

**Theorem 1.1** *If  $A \in (-\infty, \infty) \setminus \{-1\}$ , then for small  $\varepsilon$  the maximum number of limit cycles for system (1.2) which emerge from the center and the period annulus altogether is equal to two.*

**Remark 1.1** *If  $A \in (-\infty, -1) \cup (2, \infty)$ , then the period annulus is bounded by a homoclinic loop. By the result of [14] the maximal number of limit cycles in theorem 1.1 also includes the limit cycles bifurcating from the loop.*

In this paper, borrowing some techniques from [3] and using some results of [4], we will give a uniform and easier proof of theorem 1.1. To explain the idea of the proof, we reformulate theorem 1.1 in a geometric way. As mentioned above, the maximal number of limit cycles in theorem 1.1 is equal to the sharp upper bound of the number of zeros of the integral (1.3). We define

$$I_k(h) = \iint_{\text{Int}(\delta_A(h))} x^k dx dy, \quad k = 0, 1, -1. \quad (1.4)$$

and

$$p(h) = \frac{I_1(h)}{I_0(h)}, \quad q(h) = \frac{I_{-1}(h)}{I_0(h)}, \quad \omega(h) = \frac{I'_1(h)}{I'_0(h)}, \quad \nu(h) = \frac{I''_{-1}(h)}{I''_0(h)}. \quad (1.5)$$

Note that  $I_0(h)$  is the area of  $\text{Int}(\delta_A(h))$  and  $I'_0(h)$  is the period of  $\delta_A(h)$ , so  $I_0(h) > 0$  and  $I'_0(h) > 0$  for  $h \in (h_c, h_s)$ . Since the period function of the quadratic Hamiltonian vector field  $X_{H_A}$  is monotone (see [2]), we also have  $I''_0(h) \neq 0$ . Hence  $p, q, \omega, \nu \in C^\infty(h_c, h_s)$ , and (1.3) can be written as

$$J_A(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_{-1}(h) = I_0(h)(\alpha + \beta p(h) + \gamma q(h)), \quad (1.6)$$

and we also have

$$J_A''(h) = \alpha I_0''(h) + \beta I_1''(h) + \gamma I_{-1}''(h) = I_0''(h)(\alpha + \beta\omega(h) + \gamma\nu(h)). \quad (1.7)$$

We define two curves in the  $(p, q)$ -plane and in the  $(\omega, \nu)$ -plane, respectively, as follows

$$\Sigma_A = \{(p, q)(h) : h \in (h_c, h_s)\}, \quad \Omega_A = \{(\omega, \nu)(h) : h \in (h_c, h_s)\}. \quad (1.8)$$

Then the number of zeros of  $J_A(h)$  for  $h > h_c$  is equal to the number of intersection points (counting their multiplicities) of  $\Sigma_A$  and the straight line  $L_{\alpha\beta\gamma} : \{\alpha + \beta p + \gamma q = 0\}$  in the  $(p, q)$ -plane.

**Definition 1.1** *A plane curve is called sectorial, if it is smooth, and when running it, the tangential vector rotates an angle less than  $\pi$ .*

Therefore, if  $X_{H_A}$  has only one period annulus, then Theorem 1.1 follows from

**Theorem 1.2** *For any  $A \in (-\infty, \infty) \setminus \{-1\}$  the curve  $\Sigma_A$  is sectorial and strictly convex with non-zero curvature.*

It is hard to study  $\Sigma_A$  directly, but it is easier if we link it with  $\Omega_A$ . From (1.7) we see that in the  $(\omega, \nu)$ -plane the intersection of  $\Omega_A$  with the straight line  $L'_{\alpha\beta\gamma} : \{\alpha + \beta\omega + \gamma\nu = 0\}$  corresponds to a zero of the function  $J_A''(h) = 0$ . We identify the  $(p, q)$ -plane with the  $(\omega, \nu)$ -plane, then the two straight lines  $L_{\alpha\beta\gamma}$  and  $L'_{\alpha\beta\gamma}$  are identified (we will make this identification throughout the paper). We will prove that both curves  $\Sigma_A$  and  $\Omega_A$  are smooth, and that the following basic lemma holds.

**Lemma 1.1** *Curves  $\Sigma_A$  and  $\Omega_A$  have no common tangent line for  $A \in (-\infty, \infty) \setminus \{-1\}$ .*

**The idea of the proof of theorem 1.2** We denote by  $\xi_t$  the tangent line of  $\Sigma_A$  at the point  $(p, q)(t) \in \Sigma_A$ ,  $t \in (h_c, h_s)$ . Note that  $J_A(h_c) = 0$  for any  $A$ , hence if we take  $\xi_t$  to be  $L_{\alpha\beta\gamma}$ , then  $J_A(h)$  has a double zero at  $h = t$  plus a zero at  $h = h_c$ . By Rolle's Theorem this implies that  $\xi_t$  intersect  $\Omega_A$  at some point  $(\omega, \nu)(t')$  with  $t' \in (h_c, t)$ . It is easy to prove that if  $0 < t - h_c \ll 1$  then  $\xi_t$  intersect  $\Omega_A$  at a unique point transversally. On the other hand, we will show that the two endpoints of  $\Omega_A$  keep on the different sides of  $\xi_t$  for  $t \in (h_c, h_s)$ . Therefore, lemma 1.1 implies  $\xi_t \cap \Omega_A$  consists of a unique point for all  $t \in (h_c, h_s)$ , counting the multiplicity. This proves the convexity and non-zero curvature property of  $\Sigma_A$ . Since  $p'(h) \neq 0$  for all  $A$  (see lemma 3.1 of [4]), the curve  $\Sigma_A$  is obviously sectorial. Hence, theorem 1.2 follows.

**Remark 1.2** *If  $A \in (0, 2)$ , then  $X_{H_A}$  has two period annuli, hence there are two curves  $\Sigma_A$  and  $\Sigma'_A$ . Once we prove both of them are convex with non-zero curvature, then it is easy to prove that any straight line intersect  $\Sigma_A \cup \Sigma'_A$  at most in two points, counting their multiplicities. In fact each of these two curves is located in one of the two opposite sectors, sharing one vertex and having two crossing straight lines as boundary, and the tangent line at  $h = h_c$  and the asymptotic line for  $h \rightarrow h_s$  of each curve intersect at a point inside the same sector (see figure 4 of [4]). Therefore, in this paper we will only prove theorem 1.2.*

The proof of lemma 1.1 depends on the Picard–Fuchs equation and some techniques, among them the use of a Riccati equation satisfied by  $\omega(h)$  is important. A Riccati equation was used in the study of the cyclicity problem, probably for the first time, by Iliev in [10].

The paper is organized as follows. We make some preliminaries and recall some known results in section 2, study the properties of the curves  $\Sigma_A$  and  $\Omega_A$  in section 3 (we will use the equivalent forms  $\bar{\Sigma}_A$  and  $\bar{\Omega}_A$ ), and prove theorem 1.2 in section 4.

## 2 Preliminaries

### 2.1 Phase portraits of $X_{H_A}$

The next result is easily obtained from system (1.2).

**Lemma 2.1** *The following statements hold for  $X_{H_A}$  (i.e.  $\epsilon = 0$  of (1.2)):*

- (1) *For any  $A$ ,  $X_{H_A}$  has an invariant straight line  $\{(x, y) : x = 0\}$ , and has a center at  $(x, y) = (1, 0)$ , corresponding to the level value of  $h = h_1 = A - 3$ .*
- (2) *For any  $A \neq 0$ ,  $X_{H_A}$  has a singularity at  $((A-2)/A, 0)$ , corresponding to  $h = h_2 = (A+1)(A-2)^2/A^2$ , which is a saddle if  $A < 0$  or  $A > 2$ , or a center if  $0 < A < 2$ .*
- (3) *For  $A < 2$ ,  $X_{H_A}$  has two saddles on the invariant straight line at  $(0, \pm\sqrt{3(2-A)})$ , and has a heteroclinic loop, corresponding to  $h = 0$  if  $-1 < A < 2$ .*
- (4) *As  $A \rightarrow 0$ , the singularity  $((A-2)/A, 0)$  goes to infinity. As  $A \rightarrow 2$ , the three singularities  $((A-2)/A, 0)$  and  $(0, \pm\sqrt{3(2-A)})$  combine together and form a degenerate singularity at the origin.*

The 7 phase portraits of  $X_{H_A}$  are shown in figure 1, depending on the values of  $A \in (-\infty, -1), -1, (-1, 0), 0, (0, 2), 2$  and  $(2, \infty)$ , respectively. Hence, to study the cyclicity of the period annulus surrounding the center  $(1, 0)$  we need to take

$$(h_c, h_s) = \begin{cases} (h_1, h_2), & \text{if } A < -1 \text{ or } A > 2, \\ (h_1, 0), & \text{if } A \in (-1, 2], \end{cases} \quad (2.1)$$

where  $h_1 = A - 3$  and  $h_2 = (A + 1)(A - 2)^2/A^2$ . In the rest of the paper, we always take the value of  $(h_c, h_s)$  as in (2.1), if no other values are specified. Note that  $0 \in (h_c, h_s)$  only in the case  $A \in (2, 3)$ .

## 2.2 The Picard–Fuchs equations

The Picard–Fuchs equation for  $I_0(h)$ ,  $I_1(h)$  and  $I_{-1}(h)$  was deduced in [4] as follows:

$$G(h) \frac{d}{dh} \begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_{-1} \end{pmatrix}, \quad (2.2)$$

where

$$\begin{aligned} G(h) &= 6h(h - h_1)(A^2h - (A + 1)(A - 2)^2), \\ a_{00} &= 4A^2h^2 - 2(A - 1)(2A^2 - 4A - 9)h, \\ a_{01} &= -12Ah, \\ a_{02} &= A(A - 1)h^2 - (A + 1)(A - 2)(A - 3)h, \\ a_{10} &= -A(A - 1)h^2 + (A - 2)(A^2 - 2A + 9)h, \\ a_{11} &= 6A^2h^2 - 6A(A - 1)(A - 2)h, \\ a_{12} &= 2h^2, \\ a_{20} &= A(9A^2 - 18A + 1)h - 9(A^2 - 1)(A - 2)(A - 3), \\ a_{21} &= -6A^2(A - 1)h + 6A(A + 1)(A - 2)(A - 3), \\ a_{22} &= 2A^2h^2 - 2(A^2 - 1)(A - 3)h. \end{aligned}$$

From the definition of  $p(h)$  and  $q(h)$  (see (1.5)), we have

$$p'(h) = \frac{I_1'(h)}{I_0(h)} - \frac{I_0'(h)}{I_0(h)}p(h), \quad q'(h) = \frac{I_{-1}'(h)}{I_0(h)} - \frac{I_0'(h)}{I_0(h)}q(h).$$

Combining this with (2.2), we obtain a system of equations for  $(h, p, q)$ :

$$\begin{aligned} \dot{h} &= G(h), \\ \dot{p} &= a_{10} + a_{11}p + a_{12}q - p(a_{00} + a_{01}p + a_{02}q), \\ \dot{q} &= a_{20} + a_{21}p + a_{22}q - q(a_{00} + a_{01}p + a_{02}q). \end{aligned} \quad (2.3)$$

Note that the curve  $\Sigma_A$ , defined in (1.8), is the projection onto  $(p, q)$ -plane of an orbit of system (2.3), which is the one-dimensional stable or unstable manifold (depending on the value of  $A$ ) of the singularity  $(h_1, 1, 1)$ . From (2.2) we can deduce the following equality (see (3.7) of [4])

$$I''_{-1}(h) = c_1(h, A)I''_0(h) + c_2(h, A)I''_1(h), \quad (2.4)$$

where

$$\begin{aligned} c_1(h, A) &= \frac{6(A-2)}{h} - \frac{4A(A-1)}{(A+1)(A-3)}, \\ c_2(h, A) &= -\frac{9(A-1)}{h} + \frac{8A^2}{(A+1)(A-3)}. \end{aligned}$$

From (2.4) and (1.5) we obtain

$$\nu(h) = c_1(h, A) + c_2(h, A)\omega(h). \quad (2.5)$$

From (2.2) we also obtain the following 2-dimensional system for  $(h, \omega)$  (see (3.10) of [4])

$$\dot{h} = T(h), \quad \dot{\omega} = -b_{01}\omega^2 + (b_{11} - b_{00})\omega + b_{10} = \phi(h, \omega), \quad (2.6)$$

where

$$\begin{aligned} T(h) &= (A+1)(A-3)G(h), \\ b_{00} &= -4A^2(3A^2 - 6A - 5)h^2 + 6(A^2 - 1)(A-3)(3A^2 - 6A - 5)h - \\ &\quad 6(A+1)^2(A-2)^2(A-3)^2, \\ b_{01} &= 8A^3(A-1)h^2 - A(A+1)(A-3)(17A^2 - 34A - 3)h + \\ &\quad 9(A-1)(A-2)(A+1)^2(A-3)^2, \\ b_{10} &= -A(A-1)(A^2 - 2A + 5)h^2 + (A-2)(A+1)^2(A-3)^2h, \\ b_{11} &= -2A^2(3A^2 - 6A - 17)h^2 + 6(A-1)(A+1)^2(A-3)^2h. \end{aligned}$$

**Remark 2.1** *From (2.4) and (2.5) we have that if  $A = 3$  then  $\omega(h) \equiv 1/3$  for  $h \in (h_c, h_s)$ . If  $A = 2$ , then the two equations of system (2.6) have a common factor  $2h$ . Since we consider  $h \in (h_1, 0)$  for  $A = 2$ , we could, for simplicity, eliminate this factor, and (2.6) becomes*

$$\dot{h} = -36h(h+1), \quad \dot{\omega} = -(32h-9)\omega^2 + (28h-18)\omega - 5h.$$

*In the following, we will use this system instead (2.6) in the case of  $A = 2$ .*

### 2.3 Some related results

By using the existence of the invariant straight line and from (2.3), we have the following results (see Lemmas 3.1 and 4.1 of [4]).

**Lemma 2.2** For  $A \in (-\infty, \infty) \setminus \{-1\}$  we have

- (1)  $p'(h) \neq 0$  for  $h \in (h_c, h_s)$ .
- (2) At its endpoint  $(p, q)(h_1) = (1, 1)$  of  $\Sigma_A$ , we have that  $\frac{dq}{dp} = -\frac{A+3}{A+1}$ ,  
 $\frac{d^2q}{dp^2} = \frac{20}{A+1}$ .

It is easy to see that if  $A \neq 0, 1$  and  $2$ , then system (2.6) has six distinct singularities: three saddles  $(h_1, \omega_1), (0, \tilde{\omega}), (h_2, \omega'_2)$  and three improper nodes  $(h_1, 1), (0, 0), (h_2, \omega_2)$ , where

$$\begin{aligned} \omega_1 &= \frac{5A^2 - 6A - 3}{5A^2 + 6A + 9}, & \tilde{\omega} &= \frac{2(A-2)}{3(A-1)}, \\ \omega_2 &= \frac{A-2}{A}, & \omega'_2 &= \frac{(A-2)(5A^2 - 14A + 5)}{A(5A^2 - 26A + 41)}. \end{aligned} \quad (2.7)$$

We recall that an improper node is a node such that all the orbits arrive to or exit from it in one direction. Let

$$C_\omega = \{(h, \omega(h)) : \omega(h) \text{ is defined in (1.5), } h \in (h_c, h_s)\}.$$

**Lemma 2.3** In the  $(h, \omega)$ -plane, the curve  $C_\omega$  is the stable (or unstable) one-dimensional manifold of the saddle  $(h_1, \omega_1)$ , and it connects to another singularity as  $h \rightarrow h_s$  as follows (see figure 2, where the dashed curves show different possible positions of  $C_\omega$ ):

- (1) to the improper node  $(h_2, \omega_2)$  with slope  $-\infty$  if  $A \in (-\infty, -1) \cup [2, 3)$ , and with slope  $+\infty$  if  $A \in (3, \infty)$  (when  $A \in (2, 3)$ ,  $C_\omega$  passes through the saddle  $(0, \tilde{\omega})$  and coincides with the unstable manifold of this saddle, with negative slope);
- (2) to the improper node  $(0, 0)$  with slope  $+\infty$  if  $A \in (-1, 2)$ .

**Remark 2.2** We do not show the cases  $A = 1$  and  $A = 3$  in Figure 2. In fact the behavior of  $C_\omega$  for case  $A = 1$  is the same as for the case  $0 < A < 1$  (or the case  $1 < A < 2$ ), but the value  $\tilde{\omega}$  tends to  $\infty$  (or  $-\infty$ ). In the case  $A = 3$ ,  $C_\omega$  is just a line segment  $\{\omega = 1/3\}$ , see Remark 2.1.



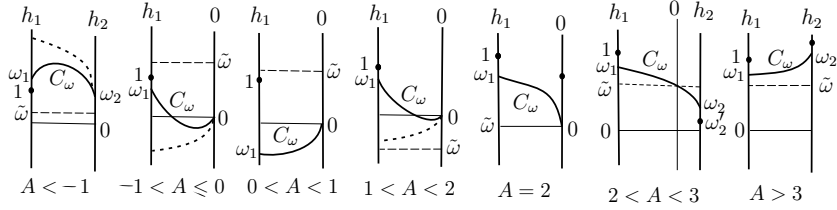


Figure 2. The behavior of  $C_\omega$  for different values of  $A$ .

**Proof.** From (4.6) of [4] we see that  $C_\omega$  is the stable or unstable manifold of the saddle  $(h_1, \omega_1)$ , going to the direction  $h > h_c = h_1$ . We need to study its limit as  $h \rightarrow h_s$ . For this purpose, we can use the same arguments as in section 5 of [4]. So, we list the main points below, and give details for one case (the little complicated case  $A \in (2, 3)$ ).

- (1) The function  $\phi(h, \omega)$  in (2.6) is a polynomial of degree two in  $h$  and in  $\omega$ . Hence each branch of the 0-clines of (2.6) (i.e. the curves  $(h, \omega_0(h))$  in  $(h, \omega)$ -plane defined by  $\phi(h, \omega) = 0$ ) is monotone, or has at most one extreme point.
- (2) From the coefficient of  $\omega^2$  in  $\phi(h, \omega)$  of (2.6), it is not difficult to find that only for  $A \in (0, 2)$  the orbits of system (2.6) can have a vertical asymptotic line in  $(h_c, h_s)$ , and this case was studied in section 5 of [4]. In the other cases, each branch of the 0-clines connects two singularities on the invariant lines  $h = h_c$  and  $h = h_s$ , respectively, and the two branches of the 0-clines do not intersect for  $h \in (h_c, h_s)$ .
- (3) At the saddles  $(h_1, \omega_1)$  and  $(0, \tilde{\omega})$  the 0-clines of (2.6) have slopes

$$\omega'_{01} = 2\omega'(h_1), \quad \omega'_{00} = 2\tilde{\omega}'(0), \quad (2.8)$$

respectively, where

$$\omega'(h_1) = \frac{5(A+1)(A-3)g_1(A)}{24(5A^2+6A+9)^2}, \quad \tilde{\omega}'(0) = \frac{5(A+1)(A-3)}{108(A-2)(A-1)^2}, \quad (2.9)$$

with  $g_1(A) = 7A^3 + 21A^2 - 27A - 9$  (having zeros approximately at  $-3.90$ ,  $-0.278$  and  $1.18$ ),  $\omega'(h_1)$  is the slope of  $C_\omega$  at  $(h_1, \omega_1)$ , and  $\tilde{\omega}'(0)$  is the slope of the stable (or unstable) manifold of system (2.6) at the saddle  $(0, \tilde{\omega})$ .

As an example, we study the case  $A \in (2, 3)$  in detail. In this case,  $h_c = h_1 < 0 < h_2 = h_s$ ,  $1 > \omega_1 > \tilde{\omega} > 0$  and  $\tilde{\omega} > \omega_2 > \omega'_2 > 0$ . One

branch of the 0-clines,  $C_0^1$ , passes through three singularities  $(h_1, 1)$ ,  $(0, \tilde{\omega})$  and  $(h_2, \omega_2)$ , and has a unique maximum point between the points  $(h_1, 1)$  and  $(0, \tilde{\omega})$ . Another branch of the 0-clines,  $C_0^2$ , passes through other three singularities  $(h_1, \omega_1)$ ,  $(0, 0)$  and  $(h_2, \omega'_2)$ , and has a unique minimum between the points  $(0, 0)$  and  $(h_2, \omega'_2)$ . By (2.8), the stable manifold of the saddle  $(h_1, \omega_1)$ , i.e.  $C_\omega$ , and the unstable manifold of the saddle  $(0, \tilde{\omega})$ , denoted by  $C'_\omega$ , must stay in the region between  $C_0^1$  and  $C_0^2$  if  $0 < h - h_1 \ll 1$  or  $0 < -h \ll 1$ . Since  $\delta(0)$  is an ordinary oval of  $X_{H_A}$  for  $2 < A < 3$ , we have that  $|I_0''(0)|$  and  $|I_1''(0)|$  are bounded while  $I_0''(0) \neq 0$  (see [2]). On the other hand, if we take  $h = 0$  in (3.6) of [4], then we find  $I_1''(0) \neq 0$ ; otherwise, we would get  $I_0'(0) = 0$ , which contradicts the fact that  $I_0'(0)$  is the period of  $\delta(0)$ . Hence, by (1.5),  $\omega(h)$  must tend to a finite non-zero value as  $h \rightarrow 0$ . This implies that, when  $h$  increases from  $h_1$  to 0,  $C_\omega$  cannot intersect the 0-cline  $C_0^1$ , then tends to  $\infty$ , and cannot go to the improper node  $(0, 0)$ . Therefore, the only possibility is that  $C_\omega$  coincides with  $C'_\omega$ . When  $h$  increases from 0 to  $h_2$ ,  $C_\omega$  is above the 0-cline  $C_0^1$  (using (2.8) again), hence  $C_\omega$  decreases and goes to the improper node  $(h_2, \omega_2)$  with slope  $-\infty$ .  $\square$

We denote by  $S_{\tilde{\omega}}$  the segment  $\{(h, \omega) : \omega = \tilde{\omega}, h \in (h_c, h_s)\}$ .

**Lemma 2.4**  $C_\omega \cap S_{\tilde{\omega}} = \emptyset$  if  $A \in (-\infty, \infty) \setminus (\{-1\} \cup (2, 3))$ ; and  $C_\omega \cap S_{\tilde{\omega}} = \{(0, \tilde{\omega})\}$  if  $A \in (2, 3)$ .

**Proof.** If  $A = 1$ , then  $S_{\tilde{\omega}}$  tends to  $\infty$  (or  $-\infty$ , see Remark 2.2), hence the statement is obviously true. The case of  $A \in (2, 3)$  was studied in the proof of lemma 2.3. In other cases, the two endpoints of  $C_\omega$  are located on the same side of  $S_{\tilde{\omega}}$ , see figure 2. Hence, if  $C_\omega \cap S_{\tilde{\omega}} \neq \emptyset$ , then there is at least one point on  $S_{\tilde{\omega}}$ , at which the vector field (2.6) would be tangent to  $S_{\tilde{\omega}}$ . We prove that this is impossible. If  $A = 0$ , then from (2.7) and (2.6) we have  $\tilde{\omega} = 4/3$  and  $\phi(h, 4/3) = 30h \neq 0$  for  $h \in (h_c, h_s)$ . If  $A \in (-\infty, \infty) \setminus (\{-1\} \cup \{0\} \cup \{1\} \cup (2, 3))$ , then

$$\phi(h, \tilde{\omega}) = -\frac{5A(A-1)(A+1)^2(A-3)^2h(h-h^*)}{9(A-1)^2},$$

where  $h^* = (A+1)(A-2)(A-3)/(A(A-1))$ , and

$$(h_1 - h^*)(h_2 - h^*) = \frac{4(A+1)(A-2)(A-3)}{A^3(A-1)^2} > 0, \text{ if } A < -1 \text{ or } A > 3;$$

and

$$(h_1 - h^*)(0 - h^*) = \frac{2(A+1)(2-A)(A-3)^2}{A^2(A-1)^2} > 0, \text{ if } A \in (-1, 2) \setminus (\{0\} \cup \{1\}).$$

Hence,  $h^*$  does not belong to  $(h_c, h_s)$ , and the lemma is proved.  $\square$

### 3 Properties of the curves $\bar{\Sigma}_A$ and $\bar{\Omega}_A$

Instead of  $\Sigma_A$  and  $\Omega_A$  (see (1.8)), for simplicity, we will defined the curves  $\bar{\Sigma}_A$  and  $\bar{\Omega}_A$ . For  $A \neq -1$  we let

$$\bar{q}(h) = q(h) + \frac{A+3}{A+1}p(h) - \frac{2(A+2)}{A+1}. \quad (3.1)$$

Correspondingly, let

$$\bar{\nu}(h) = \nu(h) + \frac{A+3}{A+1}\omega(h) - \frac{2(A+2)}{A+1}. \quad (3.2)$$

We define the two curves

$$\bar{\Sigma}_A = \{(p, \bar{q})(h) : h \in (h_c, h_s)\}, \quad \bar{\Omega}_A = \{(\omega, \bar{\nu})(h) : h \in (h_c, h_s)\}. \quad (3.3)$$

**Lemma 3.1** For  $A \in (-\infty, \infty) \setminus \{-1\}$  the following statements hold.

- (1) As  $h$  increasing from  $h_c = h_1$  to  $h_s$ , the curve  $\bar{\Omega}_A$  goes from its endpoint  $(\omega, \bar{\nu})(h_1) = (w_1, 0)$  to the endpoint  $(\omega, \bar{\nu})(h_2) = (w_2, \bar{\nu}_2)$  if  $A \in (-\infty, -1) \cup (2, \infty)$ , where  $\bar{\nu}_2 = 12/(A(A+1)(A-2))$ .
- (2) Along  $\bar{\Omega}_A$  we have

$$\lim_{h \rightarrow h_1} \frac{d\bar{\nu}}{d\omega} = \frac{72(5A^2 + 6A + 9)}{(A-3)g_1(A)},$$

$$\lim_{h \rightarrow h_1} \frac{d^2\bar{\nu}}{d\omega^2} = -\frac{28(5A^2 + 6A + 9)^3 g_2(A)}{5(A+1)(A-3)(g_1(A))^3},$$

where  $g_1(A)$  is defined below (2.9), and  $g_2(A) = 55A^4 + 204A^3 + 162A^2 - 324A - 81$  (having two real roots approximately at  $-0.2306$  and  $0.9803$ ).

**Proof.** Statement (1) is easily deduced from (3.2), (2.5) and lemma 2.3. Statement (2) can be obtained from system (2.6) by using (3.2) and the method of lemma 2 from [5].  $\square$

**Lemma 3.2** For  $A \in (-\infty, \infty) \setminus \{-1\}$  the following statements hold.

- (1) For  $0 < h - h_1 \ll 1$  we have the following expansions

$$p(h) = 1 - \frac{1}{24}(A+1)(h-h_1) + \dots, \quad \bar{q}(h) = \frac{5}{288}(A+1)(h-h_1)^2 + \dots.$$

- (2) Along  $\bar{\Sigma}_A$  we have  $\lim_{h \rightarrow h_1} \frac{d\bar{q}}{dp} = 0$  and  $\lim_{h \rightarrow h_1} \frac{d^2\bar{q}}{dp^2} = \frac{20}{A+1}$ .
- (3) Along  $\bar{\Sigma}_A$  we have  $\lim_{h \rightarrow h_2} \frac{d\bar{q}}{dp} = \frac{\bar{q}(h_2) - \bar{v}(h_2)}{p(h_2) - \omega(h_2)}$   
if  $A \in (-\infty, -1) \cup (2, \infty)$ .

**Proof.** Statement (1) can be obtained from system (2.3) by using (3.1) and the method of lemma 2 from [5]. Statement (2) follows from statement (1). To prove statement (3), we use the following formula (see the end of section 1 of [8]). Suppose that  $f(x, y)$  is a polynomial. Then near the value  $h_s$ , corresponding to a saddle loop with a saddle at  $(x_s, y_s)$ , any integral  $F(h) = \iint_{\text{Int}(\delta_A(h))} f(x, y) dx dy$ , has the expansion

$$F(h) = F(h_s) + f(x_s, y_s)(h - h_s) \ln |h - h_s| + c_3(h - h_s) + \dots,$$

where  $c_3$  is some coefficient which we do not need in our discussion. If  $A \in (-\infty, -1) \cup (2, \infty)$ , then by lemma 2.1(2),  $(x_s, y_s) = ((A-2)/A, 0)$  is a saddle with a saddle loop, corresponding to the level value  $h_s = h_2$ . From this formula and definition (1.4) we have that

$$I_k(h) = I_k(h_2) + (x_s)^k (h - h_s) \ln |h - h_s| + c_k(h - h_s) + \dots, \quad k = 0, 1, -1.$$

Hence, by using the above formula and definition (1.5) we have

$$\lim_{h \rightarrow h_2} \frac{dq}{dp} = \lim_{h \rightarrow h_2} \frac{I'_{-1}(h) - I'_0(h)q(h)}{I'_1(h) - I'_0(h)p(h)} = \frac{(x_s)^{-1} - q(h_2)}{x_s - p(h_2)},$$

where  $x_s = (A-2)/A$ . Combining this with (3.1), we obtain statement (3).  
□

Consider the following transformation from the  $(h, \omega)$ -plane to the  $(\omega, \bar{v})$ -plane

$$\omega = \omega, \quad \bar{v} = \left( c_1(h, A) - \frac{2(A+2)}{A+1} \right) + \left( c_2(h, A) + \frac{A+3}{A+1} \right) \omega.$$

By (2.4), if  $A \neq 3$ , the transformation has the form

$$\omega = \omega, \quad \bar{v} = \frac{3(h - h_1)}{h(A-3)} [3(A-1)\omega - 2(A-2)]. \quad (3.4)$$

It is easy to see that (3.4) maps the straight line  $\{(h, \omega) : h = h_0\}$  ( $h_0 \in [h_c, h_s]$ ) of the  $(h, \omega)$ -plane into a straight line of the  $(\omega, \bar{v})$ -plane. In particular, maps  $\{(h, \omega) : h = h_1\}$  into  $L_1 = \{(\omega, \bar{v}) : \bar{v} = 0\}$ , and maps  $\{(h, \omega) : h = h_s\}$  into  $L_2$ , where

$$L_2 = \begin{cases} \left\{ (\omega, \bar{\nu}) : \bar{\nu} = \frac{36(A-1)(\omega - \tilde{\omega})}{(A+1)(A-3)(A-2)^2} \right\}, \\ \quad \text{if } A \in (-\infty, -1) \cup (2, \infty) \setminus \{3\}, \\ \{(\omega, \bar{\nu}) : \omega = \tilde{\omega}\}, \quad \text{if } A \in (-1, 2] \setminus \{1\}. \end{cases} \quad (3.5)$$

We note that if  $A \neq 1$  then  $L_1 \cap L_2 = \{(\tilde{\omega}, 0)\}$ . From (2.5) and (3.2) we see that the transformation (3.4) maps the curve  $C_\omega$  of the  $(h, \omega)$ -plane into the curve  $\bar{\Omega}$  of the  $(\omega, \bar{\nu})$ -plane.

Let

$$D_A = \begin{cases} \{ (h, \omega) : h \in [h_1, h_2], \omega > \tilde{\omega} \} \text{ if } A \in (-\infty, -1) \cup (3, \infty), \\ \{ (h, \omega) : h \in [h_1, 0], \omega < \tilde{\omega} \} \text{ if } A \in (-1, 1), \\ \{ (h, \omega) : h \in [h_1, 0], -\infty < \omega < \infty \} \text{ if } A = 1, \\ \{ (h, \omega) : h \in [h_1, 0], \omega > \tilde{\omega} \} \text{ if } A \in (1, 2], \\ \{ (h, \omega) : h \in [h_1, 0] \cup (0, h_2], \omega > \tilde{\omega} \} \text{ if } A \in (2, 3). \end{cases} \quad (3.6)$$

Correspondingly, let  $D'_A$  be the region in the  $(\omega, \bar{\nu})$ -plane, which is the half-plane  $\bar{\nu} \geq 0$  if  $A = 1$ , and is the sector limited by the two straight lines  $L_1$  and  $L_2$  with vertex at  $(\tilde{\omega}, 0)$  (containing  $\bar{\Omega}$  inside) if  $A \neq -1, 1, 3$ . Note that  $L_1$  is included in  $D'_A$  and the vertex  $(\tilde{\omega}, 0)$  is not; and  $L_2$  is included only when  $A < -1$  and  $A > 2$ . If  $A \in (2, 3)$ , then both  $D_A$  and  $D'_A$  are divided into two sub-regions.

The Jacobian of transformation (3.4)

$$\frac{D(\omega, \bar{\nu})}{D(h, \omega)} = -\frac{3[2(A-1)\omega - 2(A-2)]}{h^2}$$

is well defined and non-zero if  $(h, \omega) \in D_A$  (see the definitions of (3.6) and (2.7)). Hence, we immediately have the following result.

**Lemma 3.3** *For any  $A \in (-\infty, \infty) \setminus (\{-1\} \cup \{3\})$  the transformation (3.4) from  $D_A$  to  $D'_A$  is a smooth diffeomorphism. Hence, system (2.6) in  $D_A$  becomes the smooth system*

$$\dot{\omega} = \varphi_1(\omega, \bar{\nu}), \quad \dot{\bar{\nu}} = \varphi_2(\omega, \bar{\nu}), \quad (3.7)$$

in  $D'_A$ .

**Lemma 3.4** *For  $A \in (-\infty, \infty) \setminus (\{-1\} \cup \{3\})$  we have*

- (1) *Any orbit of system (2.6), especially  $C_\omega$ , is transversal to all straight lines  $\{h = h_0, h_0 \in [h_1, h_s]\}$  in  $D_A$ .*

- (2) Any orbit of system (3.7), especially  $\bar{\Omega}_A$ , is transversal to all straight lines between  $L_1$  and  $L_2$  in  $D'_A$  ( $L_1$  is included but  $L_2$  is not), the lines are parallel if  $A = 1$ , or are in a sector region with vertex  $(\tilde{\omega}, 0)$  if  $A \neq 1$ .

**Proof.** Statement (1) follows from the first equation of (2.6), i.e.  $\dot{h} \neq 0$ , if  $h \in (h_1, h_s)$  and  $A \in (-\infty, \infty) \setminus (\{-1\} \cup (2, 3])$ , and by using the first expression of (2.9). The case  $A \in (2, 3)$  was discussed in the proof of lemma 3.3. Statement (2) follows from statement (1) and lemma 3.3.  $\square$

We denote by  $L_{\alpha\beta\gamma}^*$  the part of the straight line  $L'_{\alpha\beta\gamma} = \{\alpha + \beta\omega + \gamma\bar{\nu} = 0\}$  in the  $(\omega, \bar{\nu})$ -plane, which is contained in  $D'_A$ . Let  $C_U = \{(h, \omega) : h_1 \leq h \leq h_s, \omega = U(h)\}$  where

$$U(h) = U(h; A, \alpha, \beta, \gamma) = \frac{M(h)}{N(h)}, \quad (3.8)$$

with

$$\begin{aligned} M(h) &= [-(A-3)\alpha + 6(A-2)\gamma]h - 6(A-2)(A-3)\gamma, \\ N(h) &= [(A-3)\beta + 9(A-1)\gamma]h - 9(A-1)(A-3)\gamma. \end{aligned}$$

**Lemma 3.5** For  $A \in (-\infty, \infty) \setminus (\{-1\} \cup \{3\})$  and any constants  $\alpha, \beta$  and  $\gamma$ ,  $L_{\alpha\beta\gamma}^*$  is tangent with order  $k$  to an orbit of system (3.7) (in particular to  $\Omega_A$ , at a point  $(\omega, \bar{\nu})(h_0)$  for  $h_0 \in (h_1, h_s)$ ), if and only if  $C_U$  is tangent with order  $k$  to the corresponding orbit of system (2.6) (in particular to  $C_\omega$ , at  $(h_0, \omega(h_0))$ ).

**Proof.** Under transformation (3.4) the line  $L'_{\alpha\beta\gamma}$  becomes

$$\alpha + \beta\omega + \gamma\bar{\nu} = \frac{N(h)\omega - M(h)}{(A-3)h}, \quad (3.9)$$

where  $\beta^2 + \gamma^2 \neq 0$ , and the linear functions  $N(h)$  and  $M(h)$  are defined in (3.8). If  $N(h_0) \neq 0$ , then for  $h$  near  $h_0$  we can rewrite the above equality as

$$\alpha + \beta\omega + \gamma\bar{\nu} = \frac{N(h)}{(A-3)h}[\omega - U(h)].$$

This means that transformation (3.4) maps the straight line  $L_{\alpha\beta\gamma}^*$  to the curve  $C_U$ , and the lemma is proved for  $h$  near  $h_0$  by lemma 3.3. Next, we show that we can skip the zero point of  $N(h)$  for  $h \in [h_1, h_s]$ . In fact, if  $N(h_0) = 0$  but  $M(h_0) \neq 0$ , then the equation (3.9) is not satisfied, and we

do not need to consider it. If  $N(h_0) = M(h_0) = 0$ , then the resultant of  $N(h)$  and  $M(h)$  must be zero, i.e.

$$(A - 3)^2[3(A - 1)\alpha + 2(A - 2)\beta]\gamma = 0.$$

If  $\gamma = 0$ , then  $N(h) = (A - 3)h\beta$  and  $M(h) = (A - 3)h\alpha$ , the right hand side of (3.9) becomes  $\beta\omega + \alpha$ , and the above discussion is still valid. Note that in this case  $U(h)$  is a constant  $-\alpha/\beta$  ( $\beta \neq 0$  when  $\gamma = 0$ ). If  $3(A - 1)\alpha + 2(A - 2)\beta = 0$ , then  $L_{\alpha\beta\gamma}^* \subset D'_A$  is parallel to  $L_1$  when  $A = 1$ , or passes through the vertex  $(\tilde{\omega}, 0)$  of the sector  $D'_A$  when  $A \neq 1$ . By lemma 3.4, there is no orbit of system (3.7) tangent to  $L_{\alpha\beta\gamma}^*$ , and the assumption of the lemma is not satisfied.  $\square$

**Lemma 3.6** *For  $A \in (-\infty, \infty) \setminus (\{-1\} \cup \{3\})$  and for any constants  $\alpha, \beta$ , and  $\gamma$ , there exist at most three points on  $L_{\alpha\beta\gamma}^*$ , counting their multiplicities, such that at each of these points the vector field (3.7) is tangent to  $L_{\alpha\beta\gamma}^*$ .*

**Proof.** By lemma 3.5 we only need to consider the number of tangent points on  $C_U$  (corresponding to  $L_{\alpha\beta\gamma}^*$ ) with respect to the vector field (2.6) in the  $(h, \omega)$ -plane. By using (2.6) and (3.8) we obtain

$$\dot{\omega} - U'(h)\dot{h}\big|_{\omega=U(h)} = \phi(h, U(h)) - U'(h)T(h) = \frac{h(A - 3)^2 F(h)}{N^2(h)}, \quad (3.10)$$

where  $F(h) = F(h; A, \alpha, \beta, \gamma)$  is a polynomial in all its arguments, and of degree 3 in  $h$ . Besides,  $F(h)$  has the factor  $(h - h_1)$  or  $(h - h_2)$  if  $L_{\alpha\beta\gamma}^*$  has the endpoint  $(\omega, \bar{\nu})(h_1)$  or  $(\omega, \bar{\nu})(h_2)$ , respectively. Note that we may suppose  $N(h) \neq 0$  for  $h \in [h_1, h_s]$ , see the proof of lemma 3.5, and the straight line  $\{h = 0\}$  is not included in  $D_A$ , see (3.6).  $\square$

**Lemma 3.7** *In the identified  $(p, q)$ - and  $(\omega, \bar{\nu})$ -plane, if a straight line  $L_{\alpha\beta\gamma}$  intersects  $\bar{\Sigma}_A$  at least in two points (counting their multiplicities, but the left endpoint  $(1, 0)$  of  $\bar{\Sigma}_A$  is not included), then  $L_{\alpha\beta\gamma}$  must intersect  $\bar{\Omega}_A$  at some point  $(\omega, \bar{\nu})(h)$  with  $h \in (h_c, h_s)$ .*

**Proof.** For this  $(\alpha, \beta, \gamma)$ , the Abelian integral  $I_A(h) = I_0(h)(\alpha + \beta p(h) + \gamma \bar{q}(h))$  has at least 3 zeros for  $h \in [h_c, h_s]$ , since  $I(h_c) = 0$ . Hence  $I_A''(h) = I_0''(h)(\alpha + \beta\omega(h) + \gamma\bar{\nu}(h))$  has at least one zero for  $h \in (h_c, h_s)$ . This gives the desired conclusion.  $\square$

**Lemma 3.8** *The following statements hold.*

- (1) *For any  $A \neq -1$ ,  $\bar{\Sigma}_A \setminus \{(1, 0)\}$  stays on one side of the tangent line at its endpoint  $(p, \bar{q})(h_1) = (1, 0)$  (i.e. the line  $L_1$ ). Hence, for  $h > h_1$  we have that  $\bar{q}(h) < 0$  if  $A < -1$ , and  $\bar{q}(h) > 0$  if  $A > -1$ .*

- (2) For any  $A \neq -1$ ,  $\bar{\Omega}_A \setminus \{(\omega_1, 0)\}$  stays on one side of the tangent line, denoted by  $L_3$ , at its endpoint  $(\omega, \bar{\nu})(h_1) = (\omega_1, 0)$ .
- (3) For any  $A \in (-\infty, -1) \cup (2, \infty) \setminus \{3\}$ ,  $\bar{\Omega}_A \setminus \{(\omega_1, 0), (\omega_2, \bar{\nu}_2)\}$  stays on one side of the straight line, denoted by  $L_4$ , passing through its two endpoints  $(\omega, \bar{\nu})(h_1) = (\omega_1, 0)$  and  $(\omega, \bar{\nu})(h_2) = (\omega_2, \bar{\nu}_2)$ , and stay on one side of  $L_2$  and is tangent to  $L_2$  at  $(\omega_2, \bar{\nu}_2)$ .
- (4) For any  $A \neq -1$ , if a straight line  $L$  intersect  $\bar{\Omega}_A$  at least in four points (counting their multiplicities), then it is impossible that the intersection point of  $L \cap L_1$  is located on  $L_1$  in the same side of  $(\omega_1, 0)$  with respect to the point  $(1, 0)$ .

**Proof.** By lemma 3.2(1) and (2), we have  $(p, \bar{q})(h_1) = (1, 0)$ ,  $\lim_{h \rightarrow h_1} d\bar{q}/dp = 0$ , and  $\lim_{h \rightarrow h_1} d^2\bar{q}/dp^2 = 20/(A+1)$ . Hence  $\bar{\Sigma}_A$  is tangent to  $L_1 = \{\bar{q} = 0\}$ , and  $q(h)(A+1) > 0$  for  $0 < h - h_1 \ll 1$ . If  $\bar{q}(h) = 0$  for some  $h > h_1$ , then  $\bar{\Sigma}_A$  intersects the straight line  $L_1$  at list at three points (counting their multiplicities), hence by the same argument as in the proof of lemma 3.7,  $L_1$  must intersect  $\bar{\Omega}_A$  at a point  $(\omega, \bar{\nu})(h)$  with  $h \in (h_1, h_s)$ , and this contradicts lemma 3.4(2). Hence, statement (1) is proved.

We note that for  $h$  near  $h_1$  and near  $h_s$ ,  $\bar{\Omega}_A$  is located on the same side of  $L_3$ . Hence, if  $\bar{\Omega}_A \cap L_3 \neq \emptyset$  for  $h > h_1$ , then there are at least two points on  $L_2$  (counting their multiplicities), such that the vector field (3.7) is tangent to  $L_3$  in  $D'_A$ . By lemma 3.5, this is equivalent to a similar tangency property of vector field (2.6) with  $C_\omega$ . Along  $L_3$ , we have that  $\alpha = \omega_1 \gamma(d\bar{\nu}/d\omega)(h_1)$ ,  $\beta = -\gamma(d\bar{\nu}/d\omega)(h_1)$ . By using (2.6), lemma 3.1(2) and lemma 3.6, we obtain that the function  $F(h)$  of (3.10) has the form

$$\gamma^2(A+1)^2(A-3)^2(h-h_1)^2\psi_1(h, A)/(g_1(h))^2,$$

where  $\psi_1(h, A)$  is a linear function in  $h$ . This proves statement (2).

To prove statement (3), we use the same argument as above. We take the straight line  $L_4$  to be  $L_{\alpha\beta\gamma}$ , i.e.  $\alpha = \gamma\omega_1\bar{\nu}_2/(\omega_2 - \omega_1)$ ,  $\beta = -\gamma\bar{\nu}_2/(\omega_2 - \omega_1)$ . Hence the function  $F(h)$  of (3.10) has the form

$$\gamma^2(A-3)^2(h-h_1)(h-h_2)\psi_2(h, A)/(A+3)^2(A-2)^2,$$

where  $\psi_2(h, A)$  is also a linear function in  $h$ . On the other hand, it is easy to verify that for  $h$  near  $h_1$  and near  $h_2$ ,  $\bar{\Omega}_A$  is located on the same side of  $L_4$ . Since  $L_2$  is contained in the boundary of  $D'_A$ ,  $\bar{\Omega}_A \setminus \{(\omega_2, \bar{\nu}_2)\}$  certainly stay on one side of  $L_2$ , and by lemmas 2.3 and 3.3  $\bar{\Omega}_A$  is tangent to  $L_2$  at the point  $(\omega_2, \bar{\nu}_2)$ .

The proof of statement (4) is similar. If  $L \cup \bar{\Omega}_A$  consists of at least 4 points, then there are at least 3 points on  $L$  at which the vector field (3.7)



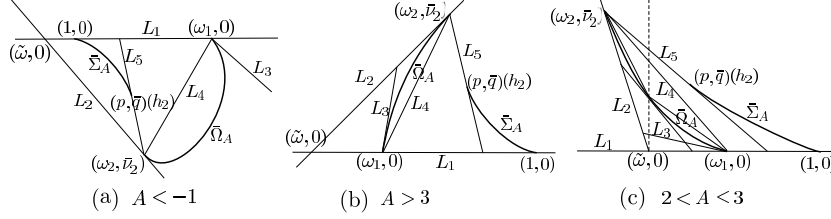


Figure 3. The relative positions of  $\bar{\Sigma}_A$  and  $\bar{\Omega}_A$  for  $A \in (-\infty, -1) \cup (2, \infty)$ .

is tangent to  $L$ . Hence, by lemma 3.5, the vector field (2.6) is tangent to  $C_U$ , corresponding to  $L$ , at least at 3 points. If in the  $(\omega, \bar{v})$ -plane the point  $L \cap L_1$  is located on  $L_1$  in the same side of  $(\omega_1, 0)$  with respect to the point  $(1, 0)$ , then in the  $(h, \omega)$ -plane,  $C_U$  intersects the invariant line  $\{h = h_1\}$  at a point  $M$  on this line in the same side of the saddle  $(h_1, \omega_1)$  with respect to the other singularity  $(h_1, 1)$ . Note that  $C_\omega$  is the stable or unstable manifold of the saddle  $(h_1, \omega_1)$ , hence the directions of the vector field (2.6) with respect to  $C_U$  at the point  $M$  and at the nearest intersection point of  $C_U \cap C_\omega$  are opposite, hence between these two points there is at least one more tangent point on  $C_U$  with respect to the vector field (2.6), and this contradicts lemma 3.6. Note that by the same reason  $M$  cannot coincide with the saddle point  $(h_1, \omega_1)$ . Therefore, the proof of the lemma is finished.  $\square$

As we defined above, in the identified  $(p, \bar{q})$ - and  $(\omega, \bar{v})$ -plane, the sector  $D'_A$  is formed by the two straight lines  $L_1$  and  $L_2$  (if  $A \neq -1, 1$ ),  $L_3$  is the tangent line of  $\bar{\Omega}_A$  at its endpoint  $(\omega_1, 0)$ , and  $L_4$  is the line joining the two endpoints of  $\bar{\Omega}_A$ . Now we let  $L_5$  be the tangent line of  $\bar{\Sigma}_A$  at its endpoint  $(p, \bar{q})(h_2)$  if  $A < -1$  or  $A > 2$ . Besides, we denote by  $D_\Sigma$  the region in  $D'_A$ , which is formed by  $L_1, L_2$  and  $L_5$ , denote by  $D_\Omega$  the region in  $D'_A$ , which is formed by  $L_3, L_4$  and  $L_2$ . See figure 3.

**Lemma 3.9** *If  $A \in (-\infty, -1) \cup (2, \infty) \setminus \{3\}$ , then we have : (1)  $\bar{\Omega}_A \subset D_\Omega$ . (2)  $D_\Sigma \cap D_\Omega$  consists of a unique point  $(\omega, \bar{v})(h_2) = (\omega_2, \bar{v}_2)$ . (3)  $\bar{\Sigma}_A \subset D_\Sigma$ . (4) If  $A > 2$ , then the boundary of  $D_\Omega$  is a triangle. In particular, if  $A \in (2, 3)$ , then each of the two pieces of  $\bar{\Omega}_A$  (contained in each of the two parts of  $D'_A$ , respectively) is also located in the triangle region formed by the two tangent lines at its two endpoints and the straight line passing through them, see figure 3(c).*

**Proof.** Statement (1) follows from lemma 3.8(2),(3) and the fact that  $L_2$  is contained in the boundary of  $D'_A$ .

By lemma 3.2(3),  $L_5$  and  $L_4$  pass through the same point  $(\omega, \bar{\nu})(h_2) = (\omega_2, \bar{\nu}_2)$ . Hence, to prove statement (2), it is enough to prove that in  $D'_A$  the line  $L_5$  is on the left of  $L_4$  if  $A < -1$ , and  $L_5$  is on the right of  $L_4$  if  $A > 2$ , and this fact follows from the following claims (see figure 3):

- (i)  $p(h_2) < \omega_2$  if  $A < -1$  and  $p(h_2) > \omega_2$  if  $A > 2$ .
- (ii)  $\bar{q}(h_2) > \bar{\nu}_2$  if  $A < -1$  and  $\bar{q}(h_2) < \bar{\nu}_2$  if  $A > 2$ .
- (iii)  $p(h_2) < \omega_1$  if  $A < -1$ , and  $p(h_2) > \omega_1$  if  $A > 2$ .

Now we prove these claims. From lemma 2.1(2) we see that if  $A \in (-\infty, -1) \cup (2, \infty)$  then  $\text{Int}(\delta_A(h_2))$  is the region bounded by the homoclinic loop of the saddle  $((A-2)/A, 0)$ , and  $\text{Int}(\delta_A(h_2))$  is on the left of the saddle if  $A < -1$ , and on the right of the saddle if  $A > 2$  (see figure 1). From (2.7) we have  $\omega(h_2) = \omega_2 = (A-2)/A$ . Hence, by the definition of  $p(h)$  in (1.5), we obtain claim (i). If claim (ii) is not true, then by lemma 3.2(3) we would find a straight line  $L$ , such that  $L \cap D_\Omega = \emptyset$  and  $L$  intersects  $\Sigma_A$  at least in two points, and this contradicts lemma 3.7. By using the definition of  $p(h)$  in (1.5) and (1.4), claim (iii) is equivalent to  $\xi(A) = \iint_{\text{Int}(\delta_A(h_2))} (x - \omega_1) dx dy < 0$  if  $A < -1$ , and  $\xi(A) > 0$  if  $A > 2$ . In polar coordinates  $(r, \theta)$  given by  $x = \omega_2 + r \cos \theta$ ,  $y = r \sin \theta$ , the oval  $\delta_A(h_2)$  is given by

$$r = r(\theta) = \frac{3A \cos^2 \theta - (A-2) \sin^2 \theta}{A \cos \theta (\sin^2 \theta + A \cos^2 \theta)},$$

where  $\theta \in \tan^{-1}(-t_A, t_A)$ ,  $t_A = \sqrt{3A/(A-2) \cos \theta}$ . Let  $t = \tan \theta$ , then

$$\xi(A) = \int_{-t_A}^{t_A} \left( \frac{1}{3} f^3(t, A) + \frac{\omega_2 - \omega_1}{2} f^2(t, A) \right) dt,$$

where  $f(t, A) = ((2-A)t^2 + 3A)/(A(t^2 + A))$ . A calculation shows that if  $A < -1$ , then

$$\xi(A) = \frac{(A+1)[(3-A)\sqrt{3(2-A)}\eta(A) - \zeta(A) \operatorname{arctanh}(\sqrt{3/(2-A)})]}{4(-A)^{5/2}(5A^2 + 6A + 9)} < 0,$$

where  $\eta(A) = 25A^2 - 12A + 27$ , and  $\zeta(A) = 25A^4 - 112A^3 + 130A^2 + 120A + 45$ . And if  $A > 2$  then

$$\xi(A) = \frac{(A+1)[(3-A)\sqrt{3(A-2)}\eta(A) + \zeta(A) \operatorname{arctan}(\sqrt{3/(A-2)})]}{4A^{5/2}(5A^2 + 6A + 9)} > 0.$$

In both cases, the numerator of  $\xi(A)$  has fixed sign, it can be shown by a method, for example, as in the proof of lemma 3.4 of [6]. Thus, claim (iii), and consequently statement (2), are proved.

To prove statement (3), we first note that  $\bar{\Sigma}_A$  stays in one side of  $L_1$  by lemma 3.8(1). We claim that  $\bar{\Sigma}_A$  also stays in one side of  $L_5$ . Otherwise,  $L_5$  intersects  $\bar{\Sigma}_A$  at least in two points, this contradicts lemma 3.7, because by statements (1) and (2)  $L_5 \cap \bar{\Omega}_A = \{(\omega, \bar{\nu})(h_2)\}$ . The same argument implies that  $\bar{\Sigma}_A$  also stays in one side of  $L_2$ . Otherwise, we would find a straight line, which intersects  $\bar{\Sigma}_A$  at least in two points, and has no intersection with  $D_\Omega$ .

To prove statement (4), we first note that

$$\omega_1 - \tilde{\omega} = \frac{5(A+1)(A-3)^2}{3(5A^2+6A+9)(A-1)}, \quad 1 - \omega_1 = \frac{12(A+1)}{5A^2+6A+9}.$$

Hence,  $\tilde{\omega} < \omega_1 < 1$  if  $A > 2$ , and  $\bar{\Omega}_A$  must be located inside the triangle region bounded by  $L_1$ ,  $L_2$  and  $L_4$ , see figures 3(b) and 3(c). On the other hand, by lemma 3.8(2),  $\bar{\Omega}_A$  is located on one side of  $L_3$ , so  $L_3$  must intersect  $L_2$  at a point between the points  $(\tilde{\omega}, 0)$  and  $(\omega_2, \bar{\nu}_2)$ , and  $\bar{\Omega}_A$  is located inside the triangle region bounded by  $L_2$ ,  $L_3$  and  $L_4$ . If  $A \in (2, 3)$ , then  $D_A$  (resp.  $D'_A$ ) is divided by the straight line  $\{h = 0\}$  (resp.  $\{\omega = \tilde{\omega}\}$ ) into two parts. From (3.4) we see that the images of  $\{(h, \omega) : \omega = \tilde{\omega}\}$  and  $\{(h, \omega) : h = 0\}$  are contained in  $\{(\omega, \bar{\nu}) : \omega = \tilde{\omega}\}$ . By taking  $\omega = \omega(h)$  in (3.4), and using (2.6), (2.8) and the L'Hopital Rule we find that for  $2 < A < 3$

$$\lim_{h \rightarrow 0} \bar{\nu}(h) = \frac{5(A+1)(3-A)}{12(A-1)(A-2)} > 0, \quad \lim_{h \rightarrow 0} \frac{\bar{\nu}'(h)}{\omega'(h)} = -\frac{41A^2 - 82A + 21}{8(A-2)^2} < 0.$$

Hence, we can determine the position of  $(\omega, \bar{\nu})(0)$  and the curve  $\bar{\Omega}_A$  is smooth at this point. The proof of the property for each arc of  $\bar{\Omega}_A$ , on the left and on the right of this point, is the same as in the proof of lemma 3.8(2) and (3). Here we also use the fact that the singularity  $(\tilde{\omega}, 0)$  is a saddle of system (2.6).  $\square$

If  $A \in (-1, 2]$ , then we need to consider  $h \in (h_1, 0)$ . In the identified plane, we let  $L_6 = \{(p, \bar{q}) : p = p(0)\}$ , and let  $L_0$  be the identified  $\bar{q}$ - and  $\bar{\nu}$ -axes. From (2.7) it is easy to find the two zeros of  $\omega_1 = 0$  at  $A_1 \approx -0.3798$  and  $A_2 \approx 1.5798$ .

**Lemma 3.10** *If  $A \in (-1, 2]$ , then the following statements hold.*

- (1)  $\bar{\Sigma}_A \setminus \{(1, 0)\}$  is located above  $L_1$  (the tangent line at its endpoint  $(1, 0)$ ), and on the right of its asymptotic line  $L_6$ , which is on the right of  $L_0$ .
- (2)  $\bar{\Omega}_A \setminus \{(\omega_1, 0)\}$  is located above  $L_1$  and on one side of  $L_3$  (the tangent line at its endpoint  $(\omega_1, 0)$ ), and has the asymptotic line  $L_0$ . Besides,  $\bar{\Omega}_A$  has the following properties:

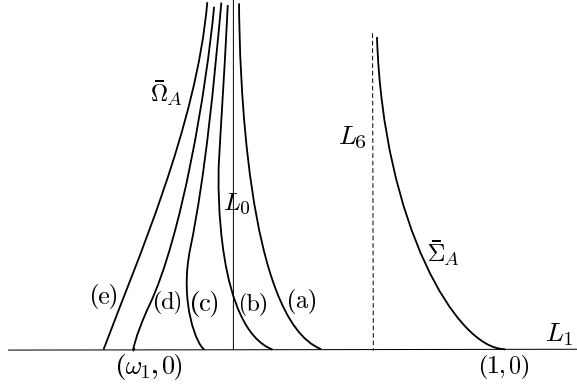


Figure 4. The relative positions and behavior of  $\bar{\Sigma}_A$  and  $\bar{\Omega}_A$  for  $A \in (-1, 2]$ .

- (i) If  $A = 2$ , then  $\bar{\Omega}_2$  is located on the right of  $L_0$ . If  $A \in (-1, A_1] \cup [A_2, 2)$ , then  $\bar{\Omega}_A \cap L_0$  consists of a unique point. In these two cases along  $\bar{\Omega}_A$  near its endpoint  $(\omega_1, 0)$  we have  $d\bar{v}/d\omega < 0$  and  $d^2\bar{v}/d\omega^2 > 0$ , and  $\bar{\Omega}_A \setminus \{(\omega_1, 0)\}$  stays on the left of the vertical straight line  $L_7 = \{\omega = \omega_1\}$ . The behavior of  $\bar{\Omega}_A$  is shown as curves (a) and (b) in figure 4, respectively.
- (ii) If  $A \in (A_1, A_2)$ , then  $\bar{\Omega}_A$  is located on the left of  $L_0$ , and along  $\bar{\Omega}_A$  near its endpoint  $(\omega_1, 0)$ , if  $d\bar{v}/d\omega < 0$  then  $d^2\bar{v}/d\omega^2 > 0$ . See curves (c), (d) and (e) in figure 4.
- (3) Any tangent line of  $\bar{\Omega}_A$  does not pass through the point  $(\tilde{\omega}, 0)$  if  $A \neq 1$ , where  $\tilde{\omega} \in (-\infty, 0) \cup (1, \infty)$  if  $A \neq 2$ , and  $\tilde{\omega} = 0$  if  $A = 2$ . If  $A = 1$ , then any tangent line of  $\bar{\Omega}_1$  is not parallel to  $L_1$ .
- (4)  $\omega_1 < p(0)$ , hence  $\bar{\Sigma}_A \cap \bar{\Omega}_A = \emptyset$ .

**Proof.** Statement (1) follows from lemmas 3.8(1), 2.2(1) and 3.2(1). Since the heteroclinic loop  $\delta(0)$  is located on the right of the invariant line  $\{x = 0\}$  of system  $X_{H_A}$ , from the definition of  $p(h)$  and  $q(h)$  in (1.5) we find  $p(0) > 0$  and  $\lim_{h \rightarrow 0} q(h) = \infty$ . By (3.1),  $\bar{\Sigma}_A$  has the asymptotic line  $L_6$ .

Statement (2) follows from lemmas 3.8(2) and the following two facts: (I)  $L_1$  corresponds to a straight line  $\{\alpha + \beta\omega + \gamma\bar{v} = 0\}$  with  $\alpha = \beta = 0$ . By (3.8), if  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$ , then  $C_U$  tends to  $\{\omega = \tilde{\omega}\} \cup \{h = h_1\}$  if  $A \neq 1$  and tends to  $\{h = h_1\}$  if  $A = 1$ . In any case, by lemmas 2.4 and 3.4,  $\bar{\Omega}_A \setminus \{(\omega_1, 0)\}$  is located above  $L_1$ . (II) By lemma 2.3,  $\lim_{h \rightarrow 0} \omega(h) = 0$ . Hence, by lemma 3.7 and statement (1) ( $\bar{\Sigma}_A$  has the vertical asymptotic line  $L_6$ ),  $\bar{\Omega}_A$  must have the asymptotic line  $L_0$ .

Now we prove the sub-statements (2)(i) and (2)(ii). The behavior of  $\bar{\Omega}_A$  near its endpoint  $(\omega_1, 0)$  is easy to obtain from lemma 3.1(2) and by comparing the values of the zero points of  $\omega_1$  (as a function of  $A$ ):  $g_1(A)$  and  $g_2(A)$ . Note that for  $h$  near  $h_1$  and near 0,  $\bar{\Omega}_A$  stays on the left of  $L_0$  if  $A \in [A_1, A_2]$  ( $\omega_1(A_1) = \omega_1(A_2) = 0$ ), stays between the two lines  $L_0$  and  $L_7$  if  $A = 2$ , and is located, respectively, on the left of  $L_0$  and between  $L_0$  and  $L_7$  if  $A \in (-1, A_1) \cup (A_2, 2)$ . Hence, if we prove that  $\bar{\Omega}_A$  can intersect each line  $L_0$  and  $L_7$  at most in one point (counting the multiplicity), then the sub-statements (i) and (ii) follow. We note that if  $\bar{\Omega}_A$  intersects the line  $L_0$  (resp.  $L_7$ ) at least in two points, then along  $L_0$  (resp.  $L_7$ ) we may find at least two tangent points with respect to the vector fields (3.7), because  $\bar{\Omega}_A$  is the stable or unstable manifold of the saddle  $(\omega_1, 0)$ , hence the directions of the vector fields (3.7) with respect to  $L_0$  (resp.  $L_7$ ) at the point  $(0, 0)$  (resp.  $(\omega_1, \sigma)$ ,  $0 < \sigma \ll 1$ ) and at the nearest intersection point of  $\bar{\Omega}_A \cap L_0$  (resp.  $\bar{\Omega}_A \cap L_7$ ) are different. Thus, it is enough to show that along  $L_0$  (resp.  $L_7$ ) the vector field (3.7) has at most one tangent point for  $h \in (h_1, 0)$ . In fact, if we take  $L_0$  (resp.  $L_7$ ) as the line  $L_{\alpha\beta\gamma}$ , i.e.  $\alpha = 0, \gamma = 0$  (resp.  $\alpha = -\omega_1\beta, \gamma = 0$ ), we obtain the function  $F(h)$  in (3.10) with a linear function in  $h$  as its factor, besides the factors  $h$  and  $h - h_1$ .

Statement (3) follows from lemma 3.4(2) and the fact that from (2.7) we have  $\tilde{\omega} > 1$  if  $-1 < A < 1$ ,  $\tilde{\omega} > 0$  if  $1 < A < 2$ , and  $\tilde{\omega} = 0$  if  $A = 2$ .

If  $A \in [A_1, A_2]$ , then  $\omega_1 < 0$ , and statements (1) and (2)(ii) obviously imply statement (4). If  $A \in (-1, A_1) \cup (A_2, 2]$ , we calculate  $p(0) - \omega_1$  directly. In this case the period annulus is bounded by  $H_A(x, y) = 0$ . Using (1.1), (1.5) and (1.4) we have that  $p(0) - \omega_1 > 0$  if and only if

$$K_A = \int_0^{x_A} (x - \omega_1) \sqrt{-Ax^2 + 3(A-1)x - 3(A-2)} dx > 0,$$

where  $x_A = [3(A-1) + \sqrt{3(3+2A-A^2)}]/(2A)$ . A calculation shows that if  $A \in (-1, 0) \supset (-1, A_1)$ , then

$$K_A = \frac{3(A+1)(3-A)}{48(5A^2+6A+9)} \left\{ 2\eta(A)\sqrt{3A(A-2)} + 3\zeta(A) \ln \mu(A) \right\} > 0,$$

where

$$\begin{aligned} \eta(A) &= 25A^2 - 12A + 27, & \zeta(A) &= 5A^3 + 15A^2 + 15A - 27, \\ \mu(A) &= \frac{\sqrt{3A(A+1)(3-A)}}{2(3(1-A) - 2\sqrt{3A(A-2)})}. \end{aligned}$$

Similarly, if  $A \in (0, 2] \supset (A_2, 2]$ , then

$$K_A = \frac{3(A+1)(3-A)}{48(5A^2+6A+9)} \left\{ 2\eta(A)\sqrt{3A(2-A)} + 3\zeta(A)\tau(A) \right\} > 0,$$

where  $\eta(A)$  and  $\zeta(A)$  are the same as above, and

$$\tau(A) = \left( \frac{\pi}{2} + \arctan \left( \frac{3(A-1)}{2\sqrt{3A(2-A)}} \right) \right).$$

□

## 4 Proof of the main result

Lemma 3.2(2) shows that for any  $A \neq -1$  the curve  $\bar{\Sigma}_A$  is convex near its endpoint  $(1, 0)$ , corresponding to  $h = h_1$ . We need to prove that  $\bar{\Sigma}_A$  is convex for all  $h \in [h_1, h_s]$ . We suppose the contrary: there is a  $h^* \in (h_1, h_s)$  such that the point  $M^* = (p, \bar{q})(h^*)$  is the first zero-curvature point on  $\bar{\Sigma}_A$  from the point  $(1, 0)$ . We denote by  $\bar{\Sigma}_A^*$  the arc of  $\bar{\Sigma}_A$  from  $(1, 0)$  to  $M^*$ . We first prove the following basic lemma.

**Lemma 4.1** *In the identified  $(p, \bar{q})$ - and  $(\omega, \bar{\nu})$ -plane, for any  $A \neq -1$  the curves  $\bar{\Sigma}_A^*$  and  $\bar{\Omega}_A$  have no common tangent line.*

**Proof.** First we make a proof in the case  $A < -1$ , then we use the same idea for the other cases. We denote by  $T_\Sigma^h$  (resp.  $T_\Omega^h$ ) the tangent line of  $\bar{\Sigma}_A^*$  (resp.  $\bar{\Omega}_A$ ) at the point  $(p, \bar{q})(h)$  (resp.  $(\omega, \bar{\nu})(h)$ ). We suppose that there are  $t \in (h_1, h^*]$  and  $s \in (h_1, h_s)$ , such that  $T_\Sigma^t = T_\Omega^s = T$ .

We claim that the two endpoints of  $\bar{\Omega}_A$ ,  $(\omega_1, 0)$  and  $(\omega_2, \bar{\nu}_2)$ , must keep on different sides of the tangent line  $T_\Sigma^h$  for all  $h \in (h_1, h^*]$ . This is obviously true for the endpoint  $(\omega_1, 0)$  by the convexity of  $\bar{\Sigma}_A^*$ . By lemma 3.8(1)–(3) and lemma 3.9 (see figure 3), if  $T_\Sigma^h$  passes through the point  $(\omega_2, \bar{\nu}_2)$  or leaves this point at the same side as  $(\omega_1, 0)$  does, then  $T_\Sigma^h \cap \bar{\Omega}_A = \{(\omega_2, \bar{\nu}_2)\}$  or  $T_\Sigma^h \cap \bar{\Omega}_A = \emptyset$ , and this contradicts lemma 3.7. We now apply this property to the common tangent line  $T$ , then by lemma 3.8(4), the intersection of  $T \cap \bar{\Omega}_A$  has only two possibilities: either a tangency point with tangency 2 plus a simple point, or a unique point with tangency 3. Besides, the behavior of  $T \cap \bar{\Omega}_A$  has two types, shown in the cases (a) and (b) of figure 5. If the type (a) happens (it may have a tangency 3, or for  $h$  increasing  $\bar{\Omega}_A$  crosses  $T$  at a point, then it meets  $T$  again at a tangency point on the left), then we can find  $s' \in (h_1, s)$ , such that  $T_\Omega^{s'}$  passes through the point  $(\tilde{\omega}, 0)$ , contradicting lemma 3.4(2). If the type (b) happens (it also has two more relative positions of  $T \cap \bar{\Omega}_A$ ), then by lemmas 3.8(2),(3) and 3.9(1) we must find a straight line  $L'$  (see figure 5(b)), such that  $L' \cap \bar{\Omega}_A$  consists of at least 4 point (counting their multiplicities), and  $L' \cap L_1$  at a point on  $L_1$  in the same side of  $(\omega_1, 0)$  with respect to the point  $(1, 0)$ , and this contradicts lemma 3.8(4).

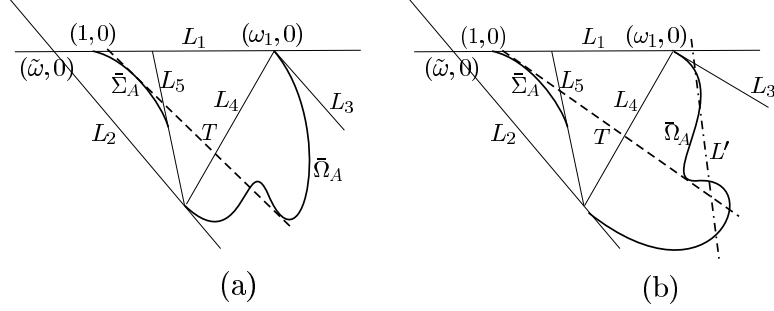


Figure 5. The behavior of  $T \cap \bar{\Omega}_A$  for  $A \in (-\infty, -1)$ .

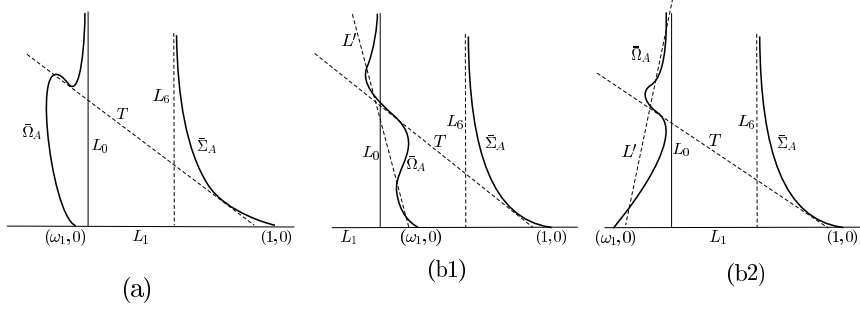


Figure 6. The behavior of  $T \cap \bar{\Omega}_A$  for  $A \in (-1, 2)$ .

The proof in other cases is basically the same, and even more simple. If  $A > 2$ , then by lemma 3.9(4)  $D_\Omega$  is bounded by a triangle (or each piece of  $\bar{\Omega}_A$  is bounded in the triangle region if  $2 < A < 3$ ). Hence, if  $\bar{\Omega}_A$  and  $\bar{\Sigma}_A$  have a common tangent line, then we must find another straight line in  $D'_A$ , such that it intersects  $\bar{\Omega}_A$  at least in 4 points, and it intersects  $L_1$  at the same side as  $(\omega_1, 0)$  with respect to  $(1, 0)$ . We remark that the proof of the case  $A = 2$  is exactly the same. The case  $A = 3$  is special: from remark 2.1 we see that as  $A \rightarrow 3$  then  $\bar{\Omega}_A$  tends to a vertical segment  $\{(\omega, \bar{\nu}) : \omega = 1/3, 0 \leq \bar{\nu} \leq 1\}$ , and the lemma is obviously true.

If  $A \in (-1, 2)$ , then  $\bar{\Omega}_A$  has a vertical asymptotic line  $L_0$ . Hence, it is obvious that the endpoint  $(\omega_1, 0)$  of  $\bar{\Omega}_A$  and the point  $(\omega, \bar{\nu})(h)$  for  $h \sim 0$  are located on different sides of the tangent line  $T_\Sigma^h$  for all  $h \in (h_1, h^*]$ . If  $T_\Sigma^t = T_\Sigma^s = T$  is a common tangent line ( $t \in (h_1, h^*]$  and  $s \in (h_1, 0)$ ), then the behavior of  $T \cap \bar{\Omega}_A$  has two types shown in figure 6. If type (a) of figure

6 happens, then as  $h$  increases from  $s$  to 0, the tangent line  $T_\Omega^h$  moves on  $\bar{\Omega}_A$  in such a way that the intersection of  $T_\Omega^h \cap L_1$  passes through the interval  $\{(\omega, \bar{\nu}) : \bar{\nu} = 0, \omega \in (1, \infty) \cup (-\infty, 0)\}$ , and also it passes a position which is parallel to  $L_1$ . This contradicts lemma 3.10(3). If the type (b1) or (b2) of figure 6 happens, then, by using lemmas 3.10 and 3.8, we may find a straight line, intersecting  $\bar{\Omega}_A$  at least in 4 point, and intersecting  $L_1$  in a point  $(\omega, 0) \in L_1$  with  $\omega < \omega_1 < 1$ . This contradicts lemma 3.8(4).  $\square$

**Proof of Theorem 1.2** We suppose  $A \neq -1$ . By lemma 2.2(1), the curve  $\Sigma_A$  is sectorial. From the definitions of  $\Sigma_A$  and  $\bar{\Sigma}_A$ , it follows that  $\Sigma_A$  is strictly convex with non-zero curvature if and only if  $\bar{\Sigma}_A$  does. If  $\bar{\Sigma}_A$  is not strictly convex with non-zero curvature, then, as we discussed in the beginning of this section, we may find the first zero-curvature point  $M^*$  on it from its endpoint  $(1, 0)$ , and denote by  $\bar{\Sigma}_A^*$  the arc of  $\bar{\Sigma}_A$  from  $(1, 0)$  to  $M^*$ . We let  $T_\Sigma^h$  be the tangent line of  $\bar{\Sigma}_A^*$  at the point  $(p, \bar{q})(h)$ . We take  $T_\Sigma^h$  to be the straight line  $L_{\alpha\beta\gamma}$ , move it on  $\bar{\Sigma}_A$  as  $h$  increases from  $h_1$  to  $h^*$ , and consider the number of intersection points of  $T_\Sigma^h \cap \bar{\Omega}_A$ . If  $0 < h - h_1 \ll 1$ , then by lemmas 3.4 and 3.1(2)  $T_\Sigma^h \cap \bar{\Omega}_A$  consists of a unique point (counting its multiplicity). Since  $\bar{\Sigma}_A$  has zero curvature at the point  $(p, \bar{q})(h^*)$ , the Abelian integral  $I_A(h) = I_0(h)(\alpha + \beta p(h) + \gamma \bar{q}(h))$  has at least a triple zero at  $h^*$  plus a zero at  $h_1$ . Hence  $I_A''(h) = I_0''(h)(\alpha + \beta \omega(h) + \gamma \bar{\nu}(h))$  has at least two zeros for  $h \in (h_1, h^*)$ . This means that  $T_\Sigma^h \cap \bar{\Omega}_A$  consists of at least two points. On the other hand, as we have obtained in the proof of lemma 4.1, the endpoint  $(\omega_1, 0)$  and the point  $(\omega, \bar{\nu})(h)$  for  $h \sim 0$  are located on the different sides of the tangent line  $T_\Sigma^h$  for all  $h \in (h_1, h^*]$ . Therefore, the increasing of the numbers of intersection points of  $T_\Sigma^h \cap \bar{\Omega}_A$ , as  $h$  increases from  $h_1$  to  $h^*$ , can be realized only through a tangent position of  $T_\Sigma^h$  to  $\bar{\Omega}_A$  for a  $h \in (h_1, h^*]$ , and this contradicts lemma 4.1. Thus, the proof of the theorem is finished.  $\square$

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