ANALYTIC NORMAL FORMS FOR NON DEGENERATE SINGULARITIES OF PLANAR VECTOR FIELDS

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In the memory of Michel Grailler

ABSTRACT. After gluing foliated complex manifolds, we derive analytic normal forms for singularities of codimension one foliations and planar vector fields (in the real or complex setting). Our normal forms include those given by Dufour and Zhitomirskii for foliations in dimension 3 (proving in turn the analyticity), by Strozyna and Zoladek for nilpotent planar vector fields and by Écalle for saddle-node foliations in the plane.

1. INTRODUCTION

The first aim of this paper is to present a geometrical proof for

Theorem 1.1. *Let X be a germ of analytic vector field having an isolated singularity at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$). Assume that its linear part is not radial. Then, there exist local analytic (resp. holomorphic) coordinates (x, y) in which the vector field X takes the form*

$$X = (y + f(x))\partial_x + g(x)\partial_y$$

where $f, g \in \mathbb{R}\{x\}$ (resp. $f, g \in \mathbb{C}\{x\}$) vanish at 0.

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Thanks To Reinhard Schäfke for fruitful conversations on normal forms. To Dominique Cerveau and Robert Moussu who drawn my attention on the reference [7]. To Paulo Sad who revealed me the result of [19] simplifying many proofs. To the CRM of Barcelona and Marcel Nicolau for hospitality.

Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the eigenvalues of the vector field X: we have $\lambda_1 + \lambda_2 = f(0)$ and $\lambda_1 \cdot \lambda_2 = -g(0)$. In the nilpotent case $\lambda_1 = \lambda_2 = 0$, Theorem 1.1 was obtained by E. Strozyna and H. Zoladek in [21]. They proved the convergence of an explicit iterative reduction process after long and technical estimates. In the case $\lambda_2/\lambda_1 \notin \mathbb{R}^-$, Theorem 1.1 becomes just useless since H. Poincaré and H. Dulac gave unique and very simple polynomial normal forms (see Section 7). In the remaining case, taking in account the invariant manifolds of the vector field X, we can specify our normal forms as follows

Corollary 1.2. *Let X be as above in the Siegel domain $\lambda_2/\lambda_1 \in \mathbb{R}^-$. Then, there exist local analytic (resp. holomorphic) coordinates in which the vector field X takes the following respective form*

(1) *in the saddle case $\lambda_2/\lambda_1 \in \mathbb{R}^-_*$ (with $\lambda_1, \lambda_2 \in \mathbb{R}$ in the real case)*

$$X = f(x+y)\left\{(\lambda_1 x\partial_x + \lambda_2 y\partial_y) + g(x+y)(x\partial_x + y\partial_y)\right\}$$

(2) *in the real center case $\lambda_2 = -\lambda_1 = i\lambda$, $\lambda \in \mathbb{R}$*

$$X = f(x)\left\{(-\lambda y\partial_x + \lambda x\partial_y) + g(x)(x\partial_x + y\partial_y)\right\}$$

(3) *in the saddle-node case, say $\lambda_2 = 0$, $\lambda_1 \neq 0$*

$$X = f(x)\left\{(\lambda_1 x + y)\partial_x + g(x)y\partial_y\right\}$$

where, in every case, $f(0) = 1$ and $g(0) = 0$.

Orbital normal forms (i.e. normal forms for the induced foliation) can be immediately derived just by setting $f \equiv 1$: coefficient g stands for the moduli of the foliation. Normal forms (2) were derived in [21].

For generic irrational values of λ_2/λ_1 (in the sense of Lebesgue measure on $\mathbb{R}^-_*$), C. L. Siegel proved that the vector field X is actually analytically linearizable (the sharp condition on λ_2/λ_1 was given by A. D. Brjuno in [2]). For remaining exceptional values of $\lambda_2/\lambda_1 \in \mathbb{R}^-$ (in particular for the rational values), the normalizing coordinate is divergent as a rule (see [11]). In the resonant case $\lambda_2/\lambda_1 \in \mathbb{Q}^-$, the analytic classification of the induced foliations, given by J. Martinet and J.-P. Ramis in [13], give rise to functional moduli. The classification of all vector fields inducing a given foliation as well (see [10], [17] and [22]). Therefore, the functional parameters f and g appearing in our normal forms are necessary in many cases.

We derive alternate orbital normal forms given by J. Écalle in [8]

$$X = x\partial_x + y^2\partial_y + yf(x)\partial_x, \quad f \in \mathbb{C}\{x\}$$

for saddle nodes formally orbitally conjugated to $X_0 = x\partial_x + y^2\partial_y$. It follows from the work of P. M. Elizarov in [9] that this family contains no trivial deformation of the formal model X_0: the derivative (in the sense of Gateaux) of Martinet-Ramis' modular map is injective at X_0 (see [9] for details). In this sense, the above normal form is sharp. When X has an analytic invariant curve C in the direction of the vanishing eigenvalue, the counter part of Écalle's Theorem (see [8]) says that we can rarely obtain normal form above with $C : \{y = 0\}$:

one can choose the orbital normal form above with $f(0) = 0$ if, and only if, the analytic invariants $\mathbb{A}_\omega$ of the holonomy map $\varphi(y) = y + \cdots$ of C all vanish while ω runs over $2i\pi\mathbb{N}^*$.

We explain, prove and generalize the two statements above for all saddle-nodes induced by a perturbation of X_0 (other formal invariants) and simultaneously derive alternate orbital normal forms for saddles

$$X = (f(z) + w)\lambda_1 x\partial_x + \lambda_2 y\partial_y + (x + f(y))y\partial_y, \quad f(0) = 0.$$

Similarly to Corollary 1.2, the nilpotent case splits into 3 alternate analytic normal forms depending on the invariant curve of X. They also appear in [21] and [20]. For instance, when the quadratic part of X is generic, we derive new normal forms

$$X = f(x)\{(2y\partial_x + 3x^2\partial_y) + xg(x)(2x\partial_x + 3y\partial_y)\}, \quad f(0) = g(0) = 1$$

underlying the cuspidal trajectory $\{y^2 - x^3 = 0\}$. In this situation, it is proved in [12] and [21] that the vector field X is formally orbitally equivalent to a unique normal form as above where $f(x) \equiv 1$ and $g(x) = \tilde{g}(x^3)$ for a formal power series $\tilde{g}(x)$. Nevertheless, this final reduction seems to be divergent following computer experiments from [4]: precisely, starting with several explicit examples, the authors derive Gevrey estimates of order ~ 2 for the hundred first coefficients of $\tilde{g}(x)$. It is not clear for me whether if one can normalize a finite jet of X, say $g(x) = \tilde{g}(x^3) + o(x^N)$ for a fixed $N \in \mathbb{N}$, without introducing divergence. Perhaps method of [21] is more adapted for this kind of problem. Alternate unique formal orbital normal forms are also given in [12] by replacing variable x^3 by $y^2 - x^3$ in the functional coefficient $\tilde{g}$. In this situation, the generic divergence of the normal form $\tilde{g}(y^2 - x^3)$ has been recently proved by M. Canalis-Durand and R. Schäfke in [5].

When we want to reduce an analytic object at the origin of $\mathbb{C}^n$ (or $\mathbb{R}^n$) into a simple form by mean of analytic changes of coordinates, one generally try to find normal forms which are polynomial in many variables, say $\underline{y} = (y_1, \ldots, y_q)$, and analytic in the other ones $\underline{x} = (x_1, \ldots, x_p)$, $p+q = n$. After complexification $(\underline{x}, \underline{y}) = (\underline{z}, \underline{w})$, this means that the object (into normal form) admits an analytic continuation along $\{\underline{z} = \underline{0}\}$ until infinity, i.e. on a full tubular neighborhood of $\{\underline{0}\} \times \mathbb{P}^q_{\mathbb{C}}$ in $\mathbb{C}^p \times \mathbb{P}^q_{\mathbb{C}}$. A classical idea which goes back to the works of G. D. Birkhoff is to do the converse. Given an analytic object defined on a neighborhood Ω of the origin of $\mathbb{C}^n$, extend it analytically along a trivial $\mathbb{P}^q_{\mathbb{C}}$-bundle M over $(\mathbb{C}^p, \underline{0})$. In trivializing coordinates $(\underline{z}, \underline{w}) \in M$, Chow's Theorem forces the object to be polynomial or rational in the $\underline{w}$-variable. The total manifold M is obtained after gluing Ω with other open sets equipped with similar object in such a way that those objects glue as well. In this paper, we provide such a construction for codimension one singular foliations with $q = 1$.

In Section 2, we prove that any germ $\mathcal{F}$ of singular codimension 1 foliation in $\mathbb{C}^{n+1}$ extends on a germ of trivial $\overline{\mathbb{C}}$-bundle M over $(\mathbb{C}^n, \underline{0})$, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, without additional singular point. In convenient trivializing coordinates $(\underline{z}, w)$ on M, $\underline{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^n, \underline{0})$ and $w \in \overline{\mathbb{C}}$, the extended foliation $\mathcal{F}$ is given by an integrable 1-form Ω, $\Omega \wedge d\Omega = 0$, which is polynomial in variable w of the form

$$\Omega = \omega_0 + \omega_1 w + \cdots + \omega_k w^k + (f_0 + f_1 w + \cdots + f_k w^k) dw$$

for germs of holomorphic 1-forms ω_i and functions $f_i(\underline{z})$, $i = 0, 1, \ldots, k$, depending only on the variable $\underline{z}$, and moreover $f_k \equiv 1$ and $f_{k-1} \equiv 0$. The degree k is bounded by the contact order between $\mathcal{F}$ and a generic line passing through the origin. For $k = 1$, we derive

Theorem 1.3. *Let $\mathcal{F}$ be the germ of singular foliation defined at the origin of $\mathbb{C}^{n+1}$ by an integrable holomorphic 1-form ω whose its linear part is not tangent to the radial vector field. Then, up to an analytic change of coordinates, there exist functions $f, g \in \mathbb{C}\{z\}$ (of a single variable z) and $\phi \in \mathbb{C}\{\underline{z}\}$ such that the foliation is defined by*

$$dF + wdG + wdw, \quad \textit{with} \quad F = f \circ \phi \quad \textit{and} \quad G = g \circ \phi.$$

In particular, the foliation is the pull-back by the map $\Phi(\underline{z}, w) = (\phi(\underline{z}), w)$ of the foliation defined by $df + wdg + wdw$ in the plane.

In dimension 2, this already provides the orbital part of Strozyna-Zoladek's Theorem. In dimension 3, this normal form was obtained after a formal change of coordinates by J.-P. Dufour and M. Zhitomirski in [7] but the convergence was not proved.

The result of [7] is in $\mathbb{R}^3$ but if one start with a (complex) foliation $\mathcal{F}$ commuting with the anti-holomorphic conjugation $(\underline{z}, w) \mapsto (\overline{\underline{z}}, \overline{w})$, then our construction can be carried out preserving this involution so that all our normal forms agree with the real setting.

The huge degree of freedom encountered during our construction can be used to preserve additional structure equipping the foliation. For instance, in Section 3, we extend closed meromorphic 1-forms and meromorphic functions. In particular, we derive a sort of Preparation Theorem for functions which is slightly different to the usual Weierstrass one: after a change of the w-coordinate, $w := \phi(\underline{z}, w)$, the given function really becomes a w-polynomial (no factor term).

In all above results, we construct by hands the $\overline{\mathbb{C}}$-fibration on M simultaneously with the extension of the foliation $\mathcal{F}$ by gluing bifoliated manifolds. Here, we use the local triviality of a pair of transversal foliations.

In dimension 2, when $\mathcal{F}$ is given by a vector field X, it is still possible to extend X on a 2-dimensional tubular neighborhood M of an embedded sphere $\overline{\mathbb{C}}$ but it is no more possible to extend a given local disc-bundle into a $\overline{\mathbb{C}}$-bundle at the same time. Here, we need the Rigidity Theorem of V. I. Savelev (see [19]): if the embedded sphere has zero self-intersection, then maybe replacing M by a sharper neighborhood of this sphere, this manifold is again a trivial $\overline{\mathbb{C}}$-bundle. In Section 4, we do this construction for non degenerate vector fields and derive Theorem 1.1 as well as Corollary 1.2 in a more general form including nilpotent singular points. From this, Écalle's orbital normal forms for saddle-nodes are shortly derived in Section 5 by further change of coordinates. In Section 6, we provide alternate orbital normal forms for saddles and saddle-nodes that extend analytically along an invariant curve. This is done by gluing a saddle and a saddle-node singularities along an invariant curve; the $\overline{\mathbb{C}}$-bundle is derived by Savelev's Theorem.

2. Preparation Theorem for codimension 1 foliations

We denote by $(\underline{z}, w)$ the variable of $\mathbb{C}^{n+1}$, $\underline{z} = (z_1, \ldots, z_n)$, for $n \geq 1$. Recall that a differential 1-form ω on an open set $\Omega \subset \mathbb{C}^{n+1}$ defines a codimension 1 singular foliation $\mathcal{F}$ (regular outside the zero-set of ω) if, and only if, it satisfies the Frobenius integrability condition $\omega \wedge d\omega = 0$. After dividing coefficients of ω by a common factor, the zero-set of ω has codimension 2 and the foliation $\mathcal{F}$ extends as a regular foliation outside this singular set. Therefore, this singular set is sharp for $\mathcal{F}$ and will be denoted by $\mathrm{Sing}(\mathcal{F})$. We begin with the main construction of this section

Lemma 2.1. *Let ω be a germ of integrable holomorphic 1-form at $(\underline{0}, 0) \in \mathbb{C}^{n+1}$ having codimension ≥ 2 zero set*

$$\omega = f_1(\underline{z}, w)dz_1 + \cdots + f_n(\underline{z}, w)dz_n + g(\underline{z}, w)dw, \quad \omega \wedge d\omega = 0$$

$f_1, \ldots, f_n, g \in \mathbb{C}\{\underline{z}, w\}$ and denote by $\mathcal{F}$ the induced singular foliation. Assume that $g(\underline{0}, w) \not\equiv 0$ vanishes at the order $k \in \mathbb{N}^*$ at 0

$$g(\underline{0}, w) = w^k \cdot \tilde{g}(w), \quad \tilde{g} \in \mathbb{C}\{w\}, \quad \tilde{g}(0) \neq 0.$$

Then, up to analytic change of the w-coordinate $w := \phi(\underline{z}, w)$, the foliation $\mathcal{F}$ is given by

$$P_1(\underline{z}, w)dz_1 + \cdots + P_n(\underline{z}, w)dz_n + Q(\underline{z}, w)dw = 0$$

for w-polynomials $P_1, \ldots, P_n, Q \in \mathbb{C}\{\underline{z}\}[w]$ of degree $\leq k$, Q unitary.

Remark 2.2. *Equivalently, the foliation is defined by*

$$\omega_0 + \omega_1 w + \cdots + \omega_k w^k + Q(\underline{z}, w)dw = 0$$

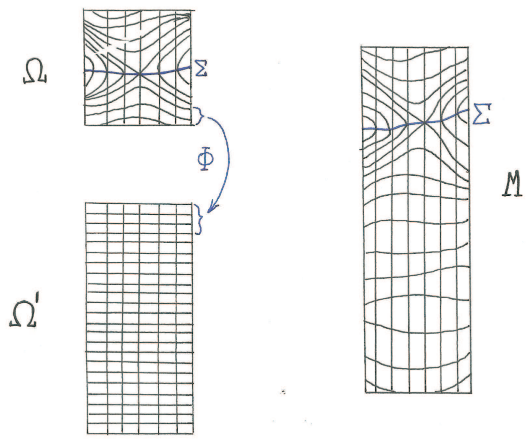
for germs $\omega_0, \omega_1, \ldots, \omega_k$ of holomorphic 1-forms at $\underline{0} \in \mathbb{C}^n$ (depending only on the $\underline{z}$-variable), and Q a unitary w-polynomial of degree k.

Proof. For $r > 0$, denote by $\mathbb{B}_r := \{\|\underline{z}\| < r\}$ and $\Delta_r := \{|w| < r\}$ the r-ball and r-disc in respective variables $\underline{z}$ and w. The assumption $g(\underline{0}, w) \not\equiv 0$ implies that there exist $\varepsilon > 0$ and $R > r > 0$ such that the singular foliation $\mathcal{F}$ defined by the kernel of ω is well-defined on the domain $\mathbb{B}_\varepsilon \times \Delta_R$ and is transversal to all vertical lines $\{\underline{z} = \text{constant}\}$ (in particular regular) in restriction to the corona $\mathbb{B}_\varepsilon \times (\Delta_R \setminus \Delta_r)$.

Maybe replacing ε, R, r by sharper constants $\varepsilon > \varepsilon' > 0$ and $R > R' > r' > r > 0$, there exists a unique holomorphic diffeomorphism (onto its image)

$$\Phi : \mathbb{B}_\varepsilon \times (\Delta_R \setminus \Delta_r) \to \mathbb{B}_\varepsilon \times \mathbb{C}; \ \ (\underline{z}, w) \mapsto (\underline{z}, \phi(\underline{z}, w))$$

preserving the vertical lines $\{\underline{z} = \text{constant}\}$, inducing the identity $\phi(\underline{0}, w) = w$ along the line $\{\underline{z} = \underline{0}\}$ and straightening the foliation $\mathcal{F}$ to the horizontal one $\mathcal{F}' : \{w = \text{constant}\}$. Indeed, just set for $\phi(\underline{z}, w)$ the w-coordinate of the unique intersection point of the fiber $\{\underline{z} = \underline{0}\}$ with the leaf of $\mathcal{F}$ passing through the point $(\underline{z}, w)$.



After gluing the foliated domain $\Omega := \mathbb{B}_\varepsilon \times \Delta_R$ together with the product $\Omega' := \mathbb{B}_\varepsilon \times (\overline{\mathbb{C}} \setminus \Delta_r)$ by means of Φ, the glued manifold we obtain $M := \Omega \cup_\Phi \Omega'$ is a $\overline{\mathbb{C}}$-bundle over $\mathbb{B}_\varepsilon$. By construction, the singular foliation $\mathcal{F}$, which was defined on the first chart Ω, extends to the whole of M and is transversal to the fibres in the second chart Ω'. After trivialisation, this manifold M identifies with the product $M \simeq \mathbb{B}_\varepsilon \times \overline{\mathbb{C}}$. In these new coordinates $(\underline{z}, w)$, the foliation $\mathcal{F}$ is defined by a global meromorphic 1-form $\omega' = R_1(\underline{z}, w)dz_1 + \cdots + R_n(\underline{z}, w)dz_n + dw$ where $R_1, \ldots, R_n \in \mathbb{C}\{\underline{z}\}(w)$ are w-rational functions.

Each pole of ω' or $\frac{\omega}{g(\underline{0},w)}$ corresponds to a tangency of the foliation $\mathcal{F}$ with fibration (counted with multiplicity). It follows from the assumption $g(\underline{0},w) = w^k \cdot \tilde{g}(w)$, $\tilde{g}(0) \neq 0$, that the total number of tangencies of a fibre with $\mathcal{F}$ in the first chart is k. Since $\mathcal{F}$ is transversal to the fibre in the second chart, we deduce that the w-rational coefficients $R_1, \ldots, R_n$ have exactly k poles (counted with multiplicity) in restriction to each fiber. Therefore, if Q denotes the unitary w-polynomial of degree k vanishing along these poles and if one write $R_i = \frac{P_i}{Q}$ for w-polynomials P_i, the transversality of $\mathcal{F}$ with the fibration at $\{w = \infty\}$ yields $deg_w(P_i) \leq k+2$. Notice that trivializing coordinates are unique up to permissible change

$$(\underline{\tilde{z}}, \tilde{w}) \mapsto (\varphi(\underline{\tilde{z}}), \frac{a(\underline{\tilde{z}})\tilde{w} + b(\underline{\tilde{z}})}{c(\underline{\tilde{z}})\tilde{w} + d(\underline{\tilde{z}})}).$$

After a global change of coordinates of the form $\Phi(\underline{z}, w)) = (\underline{z}, w/1 + f(\underline{z})w)$ on M, $f \in \mathbb{C}\{\underline{z}\}$, one may assume that the line $\{w = \infty\}$ at infinity is a leaf of the foliation (just straighten one leaf). In these new coordinates, w-polynomials P_i become of degree $k+1$. After a further global change of coordinates of the form $\Phi(\underline{z}, w)) = (\underline{z}, f(\underline{z})w)$ on M, $f \in \mathbb{C}\{\underline{z}\}$ non vanishing, $f(\underline{0}) = 0$, one may assume furthermore that the contact between $\mathcal{F}$ and the horizontal fibration $\{w = \text{constant}\}$ along the line $\{w = \infty\}$ has multiplicity 2 (no linear holonomy along this leaf in the w-coordinate). Finally, w-polynomials P_i become of degree k. □

Proof of Theorem 1.3. Since ω_1 not tangent to the radial vector field, up to a linear change of coordinates, one may assume that the tangency set $\Sigma = \{\omega(\partial_w) = 0\}$ between the foliation $\mathcal{F}$ defined by ω and the vertical fibration $\{\underline{z} = \text{constant}\}$ is smooth and transversal to the fibration. We are in the assumption of Lemma 2.1 with $k = 1$: up to a change of the w-coordinate, one may assume that $\mathcal{F}$ is defined by $\omega_0 + w\omega_1 + (w + F(\underline{z}))dw$ where ω_0 and ω_1 are holomorphic 1-form depending only on the $\underline{z}$-variable and $F \in \mathbb{C}\{\underline{z}\}$. After translation $w := w + F(\underline{z})$ (notice that $F(\underline{0}) = 0$), one may assume furthermore $F \equiv 0$ and integrability condition $\omega \wedge d\omega = 0$ yields

$$\omega_0 \wedge \omega_1 = 0, \quad d\omega_0 = 0 \quad \text{and} \quad d\omega_1 = 0.$$

After integration, we obtain

$$\omega_0 = dF \quad \text{and} \quad \omega_1 = dG$$

for functions $F, G \in \mathbb{C}\{\underline{z}\}$ with tangency condition $dF \wedge dG = 0$. By [14], there exists a primitive function $\phi \in \mathbb{C}\{\underline{z}\}$ (with connected fibres) through which F and G factorize. □

3. Preparation Theorem for closed meromorphic 1-forms

For simplicity, we start with the case of functions

Corollary 3.1. *Let $f(\underline{z},w)$ be a germ of holomorphic function at $(\underline{0},0)$ in $\mathbb{C}^{n+1}$ and assume that $f(\underline{0},w)$ vanishes at the order $k\in\mathbb{N}^*$ at* 0

$$f(\underline{0},w)=w^k\cdot\tilde{f}(w),\quad \tilde{f}\in\mathbb{C}\{w\},\quad \tilde{f}(0)\neq 0.$$

Then, up to analytic change of the w-coordinate $w:=\phi(\underline{z},w)$, the function f becomes a w-polynomial of degree k

$$f(\underline{z},w)=f_0(\underline{z})+f_1(\underline{z})w+\cdots+f_k(\underline{z})w^k$$

where $f_0,\ldots,f_n\in\mathbb{C}\{\underline{z}\}$ and $f_k(\underline{0})\neq 0$.

In Weierstrass Preparation Theorem, it is not necessary to change the coordinates but we inherit in general a non polynomial factor $f(\underline{z},w)=\tilde{f}(\underline{z},w)\cdot P(\underline{z},w)$, where $\tilde{f}\in\mathbb{C}\{\underline{z},w\}$ is non vanishing, $\tilde{f}(\underline{0},0)\neq 0$, and P is a w-polynomial.

Proof. After a preliminary change of the w-coordinates of the form $w:=\varphi(w)$, one may assume that $f(\underline{0},w)=w^k$. Now, if we proceed as in the proof of Lemma 2.1, the function f glue automatically with the function w^k defining the trivial horizontal foliation in the second chart of the manifold M. Then, choose global coordinates $(\underline{z},w)$ on M such that the pole of f coincide with $\{w=\infty\}$. In restriction to each fiber $\{\underline{z}=\text{constant}\}$, f is a degree k polynomial in w. □

By the same way, one can prove

Corollary 3.2. *Let f be a germ of meromorphic function at $(\underline{0},0)$ in $\mathbb{C}^{n+1}$ and assume that $f(\underline{0},w)$ is a well-defined and non constant germ of meromorphic function. Then, up to analytic change of the w-coordinate $w:=\phi(\underline{z},w)$, the function f becomes a w-rational function*

$$f(\underline{z},w)=\frac{f_0(\underline{z})+f_1(\underline{z})w+\cdots+f_k(\underline{z})w^k}{g_0(\underline{z})+g_1(\underline{z})w+\cdots+g_k(\underline{z})w^k}$$

where $f_0,\ldots,f_n,g_0,\ldots,g_n\in\mathbb{C}\{\underline{z}\}$.

Nevertheless, as one can see on the example $f=\frac{z+w+w^k}{z-w}$, the degree k of the w-polynomials occurring in the w-rational function are no more determined by the local divisor of the function.

Lemma 3.3. *Let ω be a germ of closed meromorphic 1-form at $(\underline{0},0)\in\mathbb{C}^{n+1}$ and assume that the vertical line $\{\underline{z}=\underline{0}\}$ is not invariant by the induced foliation. Then, up to analytic change of the w-coordinate $w:=\phi(\underline{z},w)$, the closed form ω takes the form*

$$\omega=\frac{P_1(\underline{z},w)dz_1+\cdots+P_n(\underline{z},w)dz_n+P(\underline{z},w)dw}{Q(\underline{z},w)}$$

for w-polynomials $P,Q,P_1,\ldots,P_n\in\mathbb{C}\{\underline{z}\}[w]$. More precisely, if k_0 and k_∞ denote the respective number of zeroes and poles of ω in restriction to a fiber $\{\underline{z}=\text{constant}\}$, then the numerator and denominator have respective degree k_0 and k_∞ if $k_0-k_\infty\geq-1$ and k_0 and $k_\infty+1$ if $k_0-k_\infty<-1$.

Proof. Denote by $\mathcal{F}$ the induced foliation and by Σ the tangency divisor between $\mathcal{F}$ and the vertical fibration $\{\underline{z}=\text{constant}\}$. Denote also by D_0 and D_∞ the zero and polar divisors of ω. Finally, denote by δ, d_0 and d_∞ the respective intersection number between a fiber $\{\underline{z}=\text{constant}\}$ and Σ, D_0 and D_∞: the restriction of ω to a generic fiber $\{\underline{z}=\text{constant}\}$ (not intersecting the indeterminacy set $D_0\cap D_\infty$) has exactly $k_0:=\delta+d_0$ zeroes and $k_\infty:=d_\infty$ poles counted with multiplicity. The restriction of ω to the fiber $L_0:=\{\underline{z}=\underline{0}\}$ has order $k=k_0-k_\infty$ at $w=0$. After a preliminary change of the w-coordinates of the form $w:=\varphi(w)$, one may put the restriction $\omega|_{L_0}$ into normal form

$$\begin{aligned}\omega|_{L_0}&=w^kdw\text{ if }k\geq0\\ \omega|_{L_0}&=\lambda\frac{dw}{w}\text{ if }k=-1\\ \omega|_{L_0}&=\lambda\frac{dw}{w^k(1-w)}\text{ if }k<-1\end{aligned}$$

where $\lambda\in\mathbb{C}$ denote the residue when $k\geq-1$. Now, if we proceed as in the proof of 2.1, the closed form ω glue automatically with the respective closed form w^kdw, $\lambda\frac{dw}{w}$ or $\lambda\frac{dw}{w^k(1-w)}$ defining the trivial horizontal foliation in the second chart. Choose trivializing coordinates in which $\omega|_{L_0}$ is still in the respective above normal form and $\{w=\infty\}$ is invariant. Then, one can write

$$\omega=\frac{P(\underline{z},w)}{Q(\underline{z},w)}\cdot\tilde{\omega}\quad\text{with}\tilde{\omega}=P_1(\underline{z},w)dz_1+\cdots+P_n(\underline{z},w)dz_n+P_0(\underline{z},w)dw$$

for w-polynomials $P,Q,P_0,\ldots,P_n$ and $\tilde{\omega}$ having codimension 2 singular set. The maximal degrees of P, Q and P_i are respectively given by δ, d_0 and d_∞ if $k\geq-1$ and by δ, d_0 and $d_\infty+1$ if $k<-1$ (due to the residue, we have to add an additional simple pole somewhere). $\square$

4. Non degenerate vector fields in the plane

Let X be a germ of analytic vector field at $(0,0)\in\mathbb{C}$

$$X = f(z,w)\partial_z + g(z,w)\partial_w$$

having an isolated singularity at the origin ($f(0,0)=g(0,0)=0$) and a non trivial linear part

$$X_1 := \begin{pmatrix} \frac{\partial f}{\partial z}(0,0) & \frac{\partial f}{\partial w}(0,0) \\ \frac{\partial g}{\partial z}(0,0) & \frac{\partial g}{\partial w}(0,0) \end{pmatrix} \neq 0.$$

Proof of Theorem 1.1. When the linear part X_1 of X is not radial, one can find linear coordinates in which

$$X_1 = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$$

where $-\alpha$ and β respectively stand for the product and the sum of the eigenvalues λ_1 and λ_2. The eigenvector corresponding to λ_i is $(1,\lambda_i)$; in the case $\lambda_1=\lambda_2$, the matrix X_1 is not the diagonal one. After a change of the w-coordinate of the form $w:=\varphi(w)$, we may assume that restriction of $f(z,w)$ to the vertical line $\{z=0\}$ takes the form $f(0,w)=w$. Similarly to the proof of Lemma 2.1, we consider a small polydisc $\Omega=\Delta_\varepsilon\times\Delta_R$ on which the vector field X has no other singularity and is transversal to the vertical fibration $\{z=\text{constant}\}$ in restriction to some corona $\Delta_\varepsilon\times(\Delta_R\setminus\Delta_r)$, $\epsilon>0$, $R>r>0$. Maybe replacing ε,R,r by sharper constants $\varepsilon>\varepsilon'>0$ and $R>R'>r'>r>0$, there exists a unique holomorphic diffeomorphism

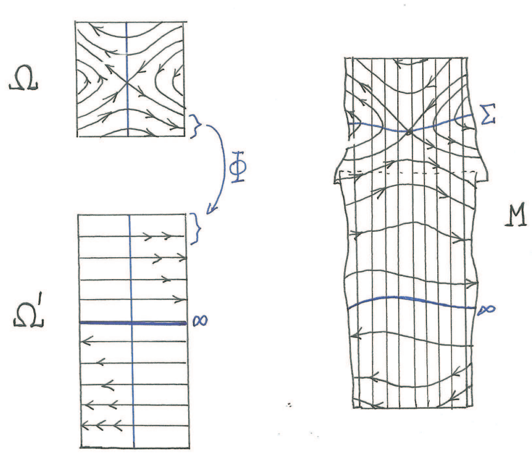
$$\Phi:\Delta_\varepsilon\times(\Delta_R\setminus\Delta_r)\to\mathbb{C}\times\mathbb{C};\ \ (z,w))\mapsto(\phi(z,w),w+z\psi(z,w))$$

conjugating the vector field X with $w\partial_z$ and inducing the identity $\phi(\underline{0},w)=w$ along the fiber $\{z=0\}$. Indeed, each point of the corona is the image of a point $(0,w)$ by time t map of the vector field X for unique $t,w\in\mathbb{C}$; just send it by Φ onto the image of $(0,w)$ by the time t map of $w\partial_z$. After gluing the domain $\Omega'=\Delta_\varepsilon\times\Delta_R$ together with the product $\mathbb{B}_\varepsilon\times(\overline{\mathbb{C}}\setminus\Delta_r)$ by mean of Φ, this latter map Φ becomes the transition map of a surface M along which the vector field X extends. By construction, the surface M is the neighborhood of an embedded sphere $\overline{\mathbb{C}}$ ($\{z=0\}$ in both charts). Since the ∂_z-component of X agree with $w\partial_z$ along $\{z=0\}$, it follows that the Jacobian of Φ takes the form

$$D_{(z,w)}\Phi = \begin{pmatrix} 1 & 0 \\ \tilde{\psi}(z,w) & 1 \end{pmatrix}$$

and the embedded sphere has zero self-intersection.

Following [19], maybe replacing M by a smaller tubular neighborhood of the embedded sphere, the surface M is biholomorphic to the trivial bundle $\Delta \times \overline{\mathbb{C}}$. Choose trivializing coordinates (z, w) on M.



The vector field X has exactly one isolated zero, say $(z, w) = (0, 0)$, and a simple pole along a trajectory (previously given by $\{w = \infty\}$ in the second chart where X wrote $w\partial_z$) that we may assume still given by $\{w = \infty\}$. The tangency divisor Σ between the induced foliation $\mathcal{F}$ and the fibration $\{z = \text{constant}\}$ still is a smooth curve intersecting the fiber $\{z = 0\}$ at the singular point $(z, w) = (0, 0)$ without multiplicity. Indeed, the Jacobian of the change of coordinates (from the first chart to the global coordinates) at the singular point is fixing the w-direction, so that the linear part of the vector field takes the form

$$X_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b \neq 0.$$

As in the proof of Lemma 2.1, one may choose the (global) w-coordinates so that the foliation has a contact of order 2 with the horizontal foliation $\{w = \text{constant}\}$ along the polar trajectory $\{w = \infty\}$ and the tangency set $\Sigma = \{w = 0\}$ is horizontal as well.

Therefore, the vector field X writes

$$X = g(z)w\partial_z + (f_0(z) + wf_1(z))\partial_w$$

for germs $f, g, h \in \mathbb{C}\{z\}$. Indeed, the coefficients of $X = P(z,w)\partial_z + Q(z,w)\partial_w$ become automatically rational in the w-variable. Since the unique pole of X is simple and located at $\{w = \infty\}$, P and Q are in fact polynomials of maximal degree 1 and 3 (notice that ∂_w has a double zero at $\{w = \infty\}$). Finally, conditions on tangency and polar sets imply the special form above. By a change of z-coordinate, we may furthermore assume $g(z) \equiv 1$ ($g(0) = b \neq 0$). Automatically, the linear part X_1 for the new coordinates writes

$$X_1 = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$$

i.e. $f_0(z) = \alpha z + \cdots$ and $f_1(z) = \beta + \cdots$ where dots mean higher order terms. Finally, the Bogdanov-Takens normal form $X = (w + f(z))\partial_z + f_0(z)\partial_w$ is derived after the change of coordinate $w := w - f$ with $f'(z) = f_1(z)$, $f(0) = 0$. □

Corollary 4.1. *Let X be a germ of analytic vector field as in Theorem 1.1. Then, by a further change of (complex or real) analytic coordinates, one may assume that we are in one of the following cases*

(1) *X has an invariant curve of the form $C : \{w^2 - z^k = 0\}$ and*

$$X = f(z)(2w\partial_z + kz^{k-1}\partial_w) + g(z)z^l(2z\partial_z + kw\partial_w), \quad l + 1 \geq \frac{k}{2} \geq 1$$

(2) *X has an invariant curve of the form $C : \{w = 0\}$ and*

$$X = f(z)(w + z^k)\partial_z + g(z)z^l w\partial_w, \quad l + 1 \geq k \geq 1$$

(3) *X is a real center or focus and*

$$X = f(z)(-w\partial_z + kz^{2k-1}\partial_w) + g(z)z^l(z\partial_z + kw\partial_w), \quad l + 1 \geq k \geq 1$$

where, in every case, $f(0) \neq 0$.

In Section 7, we give criteria to decide from the Bogdanov-Takens normal form of X in which case we are. For the moment, just notice that saddles, saddle-nodes respectively correspond to cases 1 and 2.

Proof of Corollaries 1.2 and 4.1. We go back to the preliminary normal forms

$$X = w\partial_z + (f(z) + g(z)w)\partial_w$$

proof of Theorem 1.1. Following Lemma 7.1, the foliation $\mathcal{F}$ either admits an invariant curve of the form $C : \{w^2 + a(x)w + b(x) = 0\}$, where $a(z)$ and $b(z)$ are (real or complex) analytic functions vanishing at 0, or admits a smooth (real or complex) analytic invariant curve transversal to the fibration $\{w = \text{constant}\}$. We want to simplify the invariant curve by a change of coordinates of the form $(z, w) := (\varphi(z), w + \phi(z))$. Notice that the vector field will therefore take the form $X = (f_1(z) + f_2(z)w)\partial_z + (g_1(z) + g_2(z)w)\partial_w$.

In the former case, the invariant curve is a 2-fold covering of the z-variable. One can use a vertical translation $w := w + \phi(z)$ so that C becomes invariant by the involution $i(z, w) = (z, -w)$, i.e. $C = \{w^2 = \tilde{b}(z)\}$. Then, by a change of the z-coordinate, one can normalize $\tilde{b}(z) = z^k$ (or $\tilde{b}(z) = -z^k$ when k is even in the real setting). In these new coordinates, the vector field becomes of the form

$$X = (f_1(z) + f_2(z)w)\partial_z + (g_1(z) + g_2(z)w)\partial_w;$$

therefore, writing that each of the vector fields $X \pm i_*X$ vanish identically along the curve $t \mapsto (t^k, t^2)$, we deduce that X is takes the form (1) (or (3) when k is even in the real setting) of Corollary 4.1.

In the saddle case, we have $k = 2$. We set $\tilde{f}(z) := \frac{f(z)}{f(0)}$ and $\tilde{g}(z) := g(z) - \frac{g(0)}{f(0)}f(z)$ so that $\tilde{f}(0) = 1$, $\tilde{g}(0) = 0$ and the vector field X writes

$$X = \tilde{f}(z)X_1 + \tilde{g}(z)(z\partial_z + w\partial_w) \quad \text{with} \quad X_1 = \begin{pmatrix} g(0) & f(0) \\ f(0) & g(0) \end{pmatrix}$$

($f(0) = \pm(\lambda_2 - \lambda_1) \neq 0$). Finally, after a rotation $(z, w) := (z - w, z + w)$, we obtain normal forms (1) of Corollary 1.2 for saddles.

In the case $\mathcal{F}$ admits a smooth analytic invariant curve transversal to the fibration $\{w = \text{constant}\}$, we first use a vertical translation $w := w + \phi(z)$ to straighten it onto the horizontal axis and then use change of z-coordinate to send the tangency set $\{\Sigma\}$ between the foliation $\mathcal{F}$ and the vertical fibration onto the line $\{w = z\}$. We immediately obtain normal form (2) of Corollary 4.1 (resp. of Corollary 1.2 in the saddle-node case $k = 1$). □

5. Écalle orbital normal forms for saddle-nodes

In the saddle-node case with formal invariants $(k, \mu) = (1, 0)$, we easily derive the alternate orbital normal forms given by Écalle in [8].

Theorem 5.1. *[Écalle]Assume that X is a saddle-node singularity with formal invariants $k = 1$ and $\mu = 0$ (for instance, a small enough perturbation of the formal model $X = z\partial_z + w^2\partial_w$). Then, up to analytic change of coordinates, the foliation $\mathcal{F}$ is defined by a vector field of the form*

$$X = z\partial_z + w^2\partial_w + wf(z)\partial_z$$

where $f \in \mathbb{C}\{z\}$ has no linear term $f'(0) = 0$ ($\mu = 0$).

Proof. We go back to preliminary normal forms given in the proof of Lemma 2.1

$$\omega = (f_0(z) + f_1(z)w + f_2(z)w^2 + f_3(z)w^3)dz + (w + g(z))dw.$$

We choose projective vertical coordinates $w := \{\frac{a(z)w+b(z)}{c(z)w+d(z)}\}$ on M (see proof of Lemma 2.1) so that $\{w = \infty\}$ is a leaf of the global foliation on M and $\{w = 0\}$ is the persistent invariant curve of the saddle-node singularity, having a contact of order 2 with the horizontal foliation $\{w = \text{constant}\}$. Therefore, the foliation writes

$$\omega = f(z)w^2dz + (w + g(z))dw.$$

Since formal invariant $k = 1$ and tangency set $\Sigma = \{\omega(\partial_w) = 0\}$ is transversal to the invariant curve, we have respectively $f(0) \neq 0$ and $g(z) = z\tilde{g}(z)$ with $\tilde{g}(0) \neq 0$ (formal computation). After change of z-coordinate, one may normalize the regular holomorphic 1-form $\frac{f(z)}{g(z)}dz = \lambda\frac{dz}{z}$ (λ is the residue), i.e. $f(z) = \lambda\tilde{g}(z)$, so that after division the 1-form ω writes

$$\omega = \lambda w^2dz + (u(z)w + z)dw$$

with $u(0) \neq 0$ and the vector field $X = -\lambda w^2\partial_w + (wf(z) + z)\partial_z$. After a linear change of w-coordinate, one linearize the coefficient $\lambda = -1$. The formal invariant μ is therefore given by the linear term $f'(0) = 0$ of f (formal computation). ☐

Remark 5.2. *Notice that the previous proof provides normal forms*

$$X = z\partial_z + z^kw^2\partial_w + wf(z)\partial_z, \quad k \in \mathbb{N}$$

for the general saddle-node singularity (other formal invariants $k > 1$) by writing $f(z) = z^k\tilde{f}(z)$, $\tilde{f}(0) \neq 0$ and normalizing $\frac{f(z)}{g(z)}dz = z^{k-1}dz$.

6. Further normal forms for saddles and saddle-nodes

In this section, we modify our construction in order to derive normal forms for saddle or saddle-node singular points of foliations in the real or complex plane admitting endless analytic continuation along one invariant curve.

Theorem 6.1. *Let $\mathcal{F}$ be a germ of analytic foliation having a saddle singularity at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$). Then, up to a real (resp. complex) analytic change of coordinate, the foliation $\mathcal{F}$ is defined by a vector field of the form*

$$X = (f(z) + w)z\partial_z + w\partial_w$$

where $f(0) \in \mathbb{R}_*^-$.

Proof. In convenient coordinates, $\mathcal{F}$ is defined by

$$X = g(z,w)z\partial_z + w\partial_w, \quad g(0,0) \in \mathbb{R}_*^-.$$

Here, we have just straightened the total invariant curve on $\{zw = 0\}$. Denote by $\varphi(z) = e^{2i\pi\lambda}z + \cdots$ the holonomy map of the vertical invariant curve $C : \{z = 0\}$ computed on some transversal $T : \{w := w_0\}$. Following [13], the inverse φ^{-1} is also the holonomy map of the unstable analytic curve of a saddle-node $\mathcal{F}'$ with invariants $(1, -\lambda)$, i.e. defined by a vector field of the form

$$X' = z\partial_z + w^2\partial_w + wg'(z,w)z\partial_z, \quad g'(0,0) = -\lambda.$$

Therefore, one can glue those two foliations along a small corona by a diffeomorphism of the form $\Phi(z,w) = (\phi(z,w), \varepsilon/w)$ and inherit a complex surface M with an embedded sphere L transversal to a fibration by discs (respectively given in charts by $L : \{z = 0\}$ and $\{w = \text{constant}\}$) and a foliation $\mathcal{F}$ having the following properties. The sphere L is a global invariant curve of $\mathcal{F}$ carrying two singular points of $\mathcal{F}$, namely a saddle and a saddle-node. The foliation is transversal to the fibration except along the invariant curves of the saddle and saddle-node singular points where they have a contact of respective order 1 and 2 (notice that $dw(X) = w$ while $dw(X') = w^2$). Moreover, Camacho-Sad formula (see [3]) shows that the embedded sphere L has self-intersection 0 so that, again, we can apply Savelef's Theorem [19] and assume without loss of generality that M is a trivial $\mathbb{P}^1$-bundle over a disc. One can choose global coordinates (z,w) so that the fibration by discs on M is given by $\{w = \text{constant}\}$ and the saddle and saddle-node singular points are respectively located at $w = 0$ and $w = \infty$.

Therefore, the foliation $\mathcal{F}$ is given by a vector field of the form

$$X = z\partial_z + \frac{f(z)w}{g(z)+w}\partial_w$$

where $f(0) \neq 0$. Equivalently, the foliation $\mathcal{F}$ is defined by $(\tilde{g}(z)+\tilde{f}(z)w)\partial_z + w\partial_w$ where $\tilde{f}(z) = 1/f(z)$ and $\tilde{g}(z) = g(z)/f(z)$. Finally, one can normalize $\tilde{f}(z) \equiv 1$ by a local change of the z-coordinate. □

The same construction provides

Theorem 6.2. *Let $\mathcal{F}$ be a germ of analytic foliation having a saddle-node singularity with formal invariants $(1, -\lambda)$ at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$). Assume that $\mathcal{F}$ has an analytic invariant curve in the direction of the vanishing eigenvalue and $\lambda \notin \mathbb{R}^+$. Then, up to a real (resp. complex) analytic change of coordinate, the foliation $\mathcal{F}$ is defined by a vector field of the form*

$$X = z\partial_z + w^2\partial_w + wf(z)z\partial_z$$

where $f(0) = -\lambda$.

Proof. We proceed as in the previous proof, starting with a saddle-node

$$X' = z\partial_z + w^2\partial_w + wg'(z,w)z\partial_z, \quad g'(0,0) = -\lambda.$$

Following Poincaré Linearization Theorem when $\lambda \notin \mathbb{R}$ or the result proved by R. Pérez-Marco and J.-C. Yoccoz in [18] when X' is a saddle, one can realize the holonomy map $\varphi(z)$ of the vertical invariant curve of X as the inverse of the holonomy map of a singularity of the form

$$X = g(z,w)z\partial_z + w\partial_w, \quad g(0,0) = \lambda$$

as soon as $\lambda \notin \mathbb{R}^+$. Then, after proceeding as in the proof of Theorem 6.1, one obtain a global foliation of the form $X = z\partial_z + \frac{w}{f(z)+w}\partial_w$ with $f(0) = \lambda$. After change of coordinate $w := -1/w$, one derives normal forms for saddle-node

$$X = z\partial_z + \frac{w^2}{1-f(z)w}\partial_w.$$

□

The normal forms of Theorem 6.2 cannot be generalized for all values of λ. Indeed, look at the analytic continuation of the corresponding foliation along the line $L_0 = \{z = 0\}$. In the finite part $(w \in \mathbb{C})$, $L_0 \setminus \{\underline{0}\}$ is a regular leaf of the foliation. At $w = \infty$, in coordinates $(z, \tilde{w})$, $\tilde{w} = 1/w$, the foliation $\mathcal{F}$ is given by

$$\tilde{w} \cdot X = (z\tilde{w} + zf(z))\partial_z - \tilde{w}\partial_{\tilde{w}}.$$

The singular point at $w = \infty$ is a node when $\lambda = -f(0) \in \mathbb{R}^+$ and a saddle-node when $\lambda = 0$ having L_0 as persistent invariant curve. It follows that the holonomy map φ of $L_0 \setminus \{0,\infty\}$ belongs to a restrictive conjugacy class. For instance, when $\lambda \in \mathbb{R}^+$ is irrational, the node singularity forces φ to be linearizable by Poincaré's Theorem. On the other hand, there exist saddle-nodes with invariants $(1,-\lambda)$ having non linearizable holonomy when λ does not satisfy Brjuno condition. Actually, our construction shows that this is the unique obstruction to put a saddle-node with irrational invariant $\lambda \in \mathbb{R}^+$ into the normal form of Theorem 6.2. By the same way, for $\lambda = 0$, we immediately derive the counterpart of Écalle's Theorem in [8]:

Theorem 6.3. *[Écalle]Let X be a saddle-node singularity with formal invariants $k = 1$ and $\mu = 0$ as in Theorem 5.1 and assume that X has an analytic invariant curve C in the direction of the vanishing eigenvalue. Then, up to analytic change of coordinates, the foliation $\mathcal{F}$ can be defined by a vector field of the form*

$$X = z\partial_z + w^2\partial_w + wf(z)z^2\partial_z,$$

$f \in \mathbb{C}\{z\}$, if, and only if analytic invariants $\mathbb{A}_\omega$ of the holonomy map φ of C (which is tangent to the identity) are all vanishing when ω runs over $2i\pi\mathbb{N}^*$.

We refer to [8] for a precise definition of the analytic invariants. Let us briefly recall that the complete list of invariants for k-tangent to the identity maps $\varphi(z) = z + z^{k+1} + \cdots$ up to analytic change of coordinates is given by a collection of (real or complex) scalars $\mathbb{A}_\omega$ where ω runs over a k-fold covering of the lattice $2i\pi\mathbb{Z}^* \subset \mathbb{C}^*$. Such a map is always the holonomy map of the unstable analytic curve of a saddle-node with invariants $(1,0)$. Nevertheless, it is the holonomy map of the persistent invariant curve of a saddle-node if, and only if, all $\mathbb{A}_\omega$ vanish when ω runs over $2i\pi n$ for $n \leq -2$. The k invariants $\mathbb{A}_\omega$ over $-2i\pi$ stand for the divergence of the formal invariant curve: this curve is convergent if, and only if, those k invariants are zero. Since invariants of the inverse φ^{-1} are given by $\mathbb{A}_{-\omega}$, it follows that necessary and sufficient condition of Theorem 6.3 reads:

φ^{-1} is the holonomy map of the persistent analytic invariant curve of a saddle-node whose formal invariant curve is convergent!

This condition is strictly what is imposed by the normal form of Theorem 6.3 (the singularity at $w = \infty$ is a saddle-node) and what we need to deduce such normal form from the construction above.

7. Zoology of non degenerate planar vector fields

We now discuss on the possible invariant curve of X and start with

Theorem (Poincaré-Dulac). *If the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ of the vector field X are both non zero and their ratio $\lambda_2/\lambda_1 \in \mathbb{C} \setminus \mathbb{R}^-$ is not real negative, then, up to (real or complex) analytic change of coordinates, the vector field becomes either linear diagonal*

$$X = \lambda_1 z\partial_z + \lambda_2 w\partial_w$$

or, in the resonant node case $\{\lambda_1, \lambda_2\} = \{\lambda, k\lambda\}$, $k \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}^*$, possibly of the form (including the non diagonal linear case for $n = 1$)

$$X = \lambda(kz + w^k)\partial_z + \lambda w\partial_w$$

or, in the real hyperbolic focus case $\{\lambda_1, \lambda_2\} = \{a \pm ib\}$, $a, b \in \mathbb{R}^*$

$$X = (az - bw)\partial_z + (bz + aw)\partial_w.$$

Those normal forms are unique (up to a permutation of λ_1 and λ_2) so that this statement cannot be improved. In the case $\alpha := \lambda_2/\lambda_1 \in \mathbb{R}^-$, we have the weaker statement (see [15] or [22] for formal models for the corresponding vector fields)

Theorem (Poincaré-Dulac). *In the case $\lambda_2/\lambda_1 \in \mathbb{R}^-$, then, up to a formal change of coordinates and multiplication by a non vanishing formal power series, the vector field becomes either linear*

$$X = \lambda_1 z\partial_z + \lambda_2 w\partial_w$$

or, in the resonant saddle case $\alpha = -\frac{p}{q} \in \mathbb{Q}^*_-$, possibly of the form

$$X = \lambda_1 z\partial_z + \left(\lambda_2 + (z^p w^q)^k + \mu(z^p w^q)^{2k}\right) w\partial_w$$

or, in the saddle-node case $\alpha = 0$, always of the form

$$X = z\partial_z + \left(w^k + \mu w^{2k}\right) w\partial_w$$

for unique positive integer $k \in \mathbb{N}^*$ and scalar $\mu \in \mathbb{C}$.

Following works of Briot-Bouquet and Dulac, both in the saddle and saddle-node case, there is exactly one smooth analytic curve at $(0,0)$ tangent to each eigenvector corresponding to a non zero eigenvalue which is identically tangent to the vector field X. Tangent to the zero eigenvector of the saddle-node, there exists exactly one smooth formal invariant curve, i.e. a formal parametrization $t \mapsto (\varphi(t), \phi(t))$ satisfying the corresponding differential equation, $\varphi, \phi \in \mathbb{C}[[t]]$, which can be convergent (exceptional) or divergent (generic). There is no other analytic or even formal, smooth or not, invariant curve at $(0,0)$ for a saddle or a saddle node vector field.

In the nilpotent case $\lambda_1 = \lambda_2 = 0$, the invariant curve is determined after reduction of the singularity by several blowings-up. One can find in [6] and [16] (see also [21] and [20]) the following complete description

Lemma 7.1. *Let X be a germ of complex (or real) singular analytic vector field of the form*

$$X = 2w\partial_z - (2\alpha r z^{r-1} f(z) w + \beta s z^{s-1} g(z))\partial_w, \quad f(0) = g(0) = 1,$$

for complex numbers $\alpha, \beta \in \mathbb{C}^*$ and integers $r > 0$ and $s > 1$. Then, we are in one of the following exclusive cases.

(1) *When $2r > s$, the invariant curve of X is analytic, given by*

$$C : \{w^2 + z^r a(z) w + z^s b(z) = 0\}, \quad \left\{ \begin{array}{rcl} a(0) & = & \frac{4r}{s+2r}\alpha, \\ b(0) & = & \beta \end{array} \right.$$

and writes $C : \{\tilde{w}^2 \pm \tilde{z}^s = 0\}$ after a change of coordinates. Here, "$\pm$" is relevant only in the real case and is the sign of β.

(2) *When $2r = s$, the invariant curve of X is analytic, given by*

$$C : \{w^2 + z^r a(z) w + z^s b(z) = 0\}, \quad \left\{ \begin{array}{rcl} a(0) & = & \alpha, \\ b(0) & = & \beta \end{array} \right.$$

and writes $C : \{\tilde{w}^2 \pm \tilde{z}^{2r} = 0\}$ after a change of coordinates **except** *when the roots the roots $t_1, t_2 \in \mathbb{C}$ of $t^2 + \alpha t + \beta = 0$ satisfy the one of the following two conditions*

(a) *when $r(\frac{t_1}{t_2} - 1) \in \mathbb{Q}^+ \setminus \mathbb{N}$ (up to a permutation of t_1 and t_2): in this case, there are infinitely many additional invariant curves which are analytic and singular;*

(b) *when $r(\frac{t_1}{t_2} - 1) \in \mathbb{N}$ (up to a permutation of t_1 and t_2): in this case, either there is only one irreducible invariant curve $C : \{w - t_1 z^r a(z) = 0\}$, $a(0) = 1$, or there are infinitely many additional invariant curves to this former one which are analytic of the form $w = t_2 z^r + \cdots$. The former case is generic; the latter one does not occur when $t_1 = t_2$.*

(3) *When $2r < s$, the invariant curve of X splits into an analytic one C and a generically divergent one C' given by*

$$\left\{ \begin{array}{rcl} C & : & \{w + z^r a(z) = 0\} \\ C' & : & \{w + z^{s-r} b(z) = 0\} \end{array} \right. \quad \left\{ \begin{array}{rcl} a(0) & = & \alpha, \\ b(0) & = & \frac{s}{2r}\frac{\beta}{\alpha} \end{array} \right.$$

In convenient coordinates, the analytic invariant curve is given either by $C : \{\tilde{w} = 0\}$, or by $C \cup C' : \{\tilde{w}^2 + \tilde{z}^{2r} = 0\}$.

Sketch of the proof. The singular foliation $\mathcal{F}$ induced by X is equivalently defined by the 1-form

$$\omega = 2wdw + (2\alpha r z^{r-1} f(z)w + \beta s z^{s-1} g(z))dz.$$

Notice that by a change of z-variable, one can normalize either $f(z) \equiv 1$ or $g(z) \equiv 1$.

When $s < 2r$, the total invariant curve of the foliation is given in convenient coordinates by $C : \{w^2 \pm z^s = 0\}$ (see [6] or [21]). In fact, from the formal point of view, the quasi-homogeneous graduation $\deg(x) = \deg(dx) = 2$ and $\deg(y) = \deg(dy) = s$ imposes a hierarchy on the coefficients (see [12]). The reduction of the singularity is therefore given by that of the lowest quasi-homogeneous part of ω, namely $d(w^2 + \beta z^s)$. After blowing-up, the foliation is tangent to the exceptional divisor D with only complex saddle singularities. Only those located along the smooth part of D have a local invariant curve outside D and give rise to an invariant irreducible analytic curve after blowing down. There are 1 or 2 such singularities depending whether when s is odd or even. After setting $g(z) \equiv 1$, a formal computation shows that the Puiseux expansions $w = \sum_{n>0} a_n z^{\frac{n}{k}}$, $k \in \mathbb{N}^*$, of those solutions to $\omega = 0$ take the form $w = \pm\sqrt{\beta} z^{s/2} - \frac{2r\alpha}{s+2r} z^r + \cdots$ which gives the desired equation for C

$$(w + \sqrt{\beta} z^{s/2} + \frac{2r\alpha}{s+2r} z^r + \cdots)(w - \sqrt{\beta} z^{s/2} + \frac{2r\alpha}{s+2r} z^r + \cdots)$$
$$= w^2 + z^r a(z) w + z^s b(z) = (w + z^r \tilde{a}(z))^2 \pm (z\tilde{b}(z))^s = \tilde{w}^2 \pm \tilde{z}^s$$

for analytic functions $\tilde{a}(z)$ and $\tilde{b}(z)$ satisfying $\tilde{a}(z) = \frac{a(z)}{2}$ and $\tilde{b}(0) = \sqrt[s]{b(0)}$ (or $i\sqrt[s]{b(0)}$ in the real center/focus case).

When $s \geq 2r$, the adapted quasi-homogeneous graduation is given by $\deg(x) = \deg(dx) = 1$ and $\deg(y) = \deg(dy) = r$ and the reduction of the singularity is dictated by $(w + \alpha z^r)dw + \beta dz^s$ (see [15] or [20]). The singularity of the foliation is reduced after r successive blowing-up. The exceptional divisor D is a global invariant curve of the lifted foliation $\tilde{\mathcal{F}}$. Each of the $r-1$ normal crossings of D is a resonant saddles singular point of $\tilde{\mathcal{F}}$, the invariant curves of which is contained in D and disappear after blowing-down. Beside, the foliation $\tilde{\mathcal{F}}$ has exactly 1 or 2 additional singularities along D, namely along the smooth part of the irreducible component of D appearing after the last blowing-up. After setting $f(z) \equiv 1$, in the chart $(z, t) = (z, w/z^r)$, the lifted foliation $\tilde{\mathcal{F}}$ is defined by

$$\frac{\omega}{2rz^{2r-1}} = tzdt + r(t^2 + \alpha t + \frac{s}{2r}\beta z^{s-2r} g(z))dz.$$

When $s = 2r$, the foliation $\tilde{\mathcal{F}}$ is singular at $(0, t_1)$ and $(0, t_2)$ (where t_1 and t_2 are the roots of $t^2 + \alpha t + \beta = 0$). If those points are distinct

($t_1 \neq t_2$, i.e. $\alpha^2 \neq 4\beta$), the corresponding eigenvalues have respective ratio $\frac{t_2}{r(t_1-t_2)}$ and $\frac{t_1}{r(t_2-t_1)}$: the foliation is defined near $\tilde{t} := t - t_1 = 0$ by a vector field whose linear part is $r(t_1-t_2)\tilde{t}\partial_{\tilde{t}} + t_2 w\partial_w$ (permute 1 and 2 for the other singularity). Since $\frac{t_2}{r(t_1-t_2)} + \frac{t_1}{r(t_2-t_1)} = -\frac{1}{r}$, we notice that the singular points cannot be nodes at the same time. At least one is neither a node, nor a saddle node and gives rise to an analytic invariant curve. We omit the obvious discussion when the other one is a resonant node. Finally, when $t_1 = t_2$ (i.e. $\alpha^2 = 4\beta$), the foliation is defined near this singular point by a vector field whose linear part is $\tilde{t}^2\partial_{\tilde{t}} + t_i w\partial_w$: it is a saddle node whose persistent invariant curve is transversal to D. When it does exist, an invariant curve through the singular point $(0, t_i)$ has a parametrization of the form $w = t_i z^r + \cdots$ which give the desired equation for the total invariant curve.

Finally, when $s > 2r$, the foliation $\tilde{\mathcal{F}}$ is singular at $(0,0)$ and $(0,-\alpha)$. The first singularity is a saddle node whose persistent invariant curve is contained in D and possibly gives rise only to a formal curve. The second singularity is a saddle whose eigenvalues have ratio $-\frac{1}{r}$. The invariant curves have respective formal parametrization $w = -\frac{s}{2r}\frac{\beta}{\alpha}z^{s-r} + \cdots$ and $w = -\alpha z^r + \cdots$. □

Remark 7.2. *In the nilpotent case, we have the following dictionary between normal forms of Corollary 4.1 and cases listed in Lemma 7.1. We can set $f(0) = 1$ in normal forms (linear change of w-coordinate) and we choose l sharp so that $\alpha = g(0) \neq 0$.*

(1) *Here, $k \geq 2$. Either $l + 1 > \frac{k}{2}$ and we are in case (1) $2r > s$ of Lemma 7.1 with $k = s$, or $l + 1 = \frac{k}{2}$ and we are in cases $2r \leq s$ with $k = 2r$. More precisely, the case (3) $2r < s$ of Lemma 7.1 corresponds to $\alpha = \pm 1$.*

(2) *Here, $k \geq 1$. Either $l + 1 > k$ and we are in case (3) $2r < s$ of Lemma 7.1 with $k = r$, or $l + 1 = k$ and and we are in cases $2r \leq s$ with s even and $k = s/2$. More precisely, case (1) $2r > s$ corresponds to $\alpha = -1$ and case (2) $2r = s$ with a single invariant curve, to $\alpha = \frac{k}{n+k}$, $n \in \mathbb{N}$ (including saddle-node case $\alpha = 1$).*

(3) *Here, $k \geq 1$. Either $l + 1 \geq k$ and we are in case (1) $2r > s$ of Lemma 7.1 with $k = s/2$, or $l + 1 = k$ and we are in cases $2r = s$ with $k = r$. The center case corresponds to the former one.*

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