UNIFORM APPROXIMATION ON CLOSED SUBSETS OF $\mathbb{C}$ BY POLYANALYTIC FUNCTIONS

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ABSTRACT. We give necessary and/or sufficient conditions on a closed set F in $\mathbb{C}$ in order that any function f, continuous on F and n-analytic in F^0, be the uniform limit on F of a sequence of n-analytic entire or n-analytic meromorphic functions.

1. INTRODUCTION

Let $n \in \mathbb{N}$ be a fixed natural number, and let $\overline{\partial} = \partial/\partial\overline{z}$ denote the Cauchy-Riemann operator. A function f is said to be *n-analytic* (or *polyanalytic of order n*) in an open set $G \subset \mathbb{C}$ if f is n-times continuously differentiable and satisfies $\overline{\partial}^n f = 0$ in G.

It follows easily that f can always be written as

$$f(z) = f_0(z) + \overline{z} f_1(z) + \ldots + \overline{z}^{n-1} f_{n-1}(z), \tag{1}$$

where $f_0, \ldots, f_{n-1}$ are holomorphic functions in G, and that this representation is unique.

When the functions f_j in (1) are either all (analytic) polynomials, or all rational functions, or all meromorphic functions, we obtain the class of n-analytic polynomials, n-analytic rational functions and n-analytic meromorphic functions, respectively. Note that n-analytic rational functions are not defined as the quotient of n-analytic polynomials when $n \geq 2$, and that n-analytic meromorphic functions may be continuous at a singularity, for

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example the 3-analytic function

$$f(z) = \begin{cases} \overline{z}^2/z & z \neq 0; \\ 0 & z = 0; \end{cases}$$

at $z = 0$.

Let F be a closed subset of $\mathbb{C}$ and denote by F^0 its interior. Let $C(F)$ be the space of all continuous functions on F and for $f \in C(F)$, let $\|f\|_F = \sup_{z\in F} |f(z)|$. Set

$$\mathcal{A}_n(F) = \{f \in C(F) \mid \overline{\partial}^n f = 0 \text{ on } F^\circ\},$$

$$\mathcal{E}_n(F) = \{f \in C(F) \mid \forall\varepsilon > 0, \exists g \in \mathcal{A}_n(\mathbb{C}) \text{ with } \|f - g\|_F < \varepsilon\},$$

and

$$\mathcal{M}_n(F) = \{f \in C(F) \mid \forall\varepsilon > 0, \text{ there exists a } n\text{-analytic meromorphic function } h \text{ in } \mathbb{C} \text{ having no singularities in } F \text{ with } \|f - h\|_F < \varepsilon\}.$$

That is $\mathcal{E}_n(F)$ (respectively $\mathcal{M}_n(F)$) is the set of functions in $C(F)$ that can be approximated uniformly on F by *n-analytic entire* functions (respectively by *n-analytic meromorphic* functions on $\mathbb{C}$ having no singularities on F).

For a compact subset X of $\mathbb{C}$, set

$$\mathcal{P}_n(X) = \{f \in C(X) \mid \forall\varepsilon > 0, \text{ there exists a } n\text{-analytic polynomial } p \text{ with } \|f - p\|_X < \varepsilon\}.$$

For X and Y compact sets in $\mathbb{C}$ with $X \subseteq Y$, let $\mathcal{R}_n^Y$ denote the set of n-analytic rational functions without singularities on Y and set

$$\mathcal{R}_n^Y(X) = \{f \in C(X) \mid \forall\varepsilon > 0,\ \exists r \in \mathcal{R}_n^Y \text{ with } \|f - r\|_X < \varepsilon\}.$$

Note that $\mathcal{P}_n(X) \subseteq \mathcal{R}_n^X(X) \subseteq \mathcal{A}_n(X)$, that $\mathcal{E}_n(F) \subseteq \mathcal{M}_n(F) \subseteq \mathcal{A}_n(F)$ and that $\mathcal{E}_n(X) = \mathcal{P}_n(X)$.

In this paper, we give, for $n \geq 2$, several necessary or sufficient conditions on X or F, (apparently independent of n) in order that equality occurs in some of the inclusions above. In particular, for compact sets X, our results considerably generalize the main results of [4] (see Section 2).

We continue this section with a few more definitions and concepts that will be required to state our results. For any proper subset $S \subset \mathbb{C}$, we denote by S^0 its interior, ∂S its boundary and $\overline{S}$ its closure. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ and the open unit disk $\{|z| < 1\}$ will be denoted by $\overline{\mathbb{C}}$ and Δ respectively. For $X \subset \mathbb{C}$, X compact, $\widehat{X}$ will stand for the topological hull of X, that is, the union of X with all the bounded components of $\mathbb{C}\setminus X$ (holes of X).

Let f be a bounded holomorphic function on Δ. By Fatou's theorem the *nontangential limit* $f(e^{it})$ exists dt-almost everywhere (a.e.) on $\partial\Delta$. The following concept was introduced in [4].

Definition 1. A bounded simply connected domain $\Omega \subset \mathbb{C}$ is called a *Nevanlinna domain* if there exists a conformal mapping $k : \Delta \to \Omega$ of Δ onto Ω and two bounded holomorphic functions u and v in Ω (with $v \not\equiv 0$) such that dt-a.e. on $\partial\Delta$ one has

$$\overline{k(e^{it})} = \frac{(u \circ k)(e^{it})}{(v \circ k)(e^{it})}. \tag{2}$$

As pointed out in [4], this definition is independent of the choice of k, and (for Nevanlinna domains) the quotient u/v is uniquely defined in Ω. Indeed, if k_1, u_1, v_1 and k_2, u_2, v_2 satisfy (2), then k_1, u_2, v_2 also satisfy (2). By the Luzin-Privalov theorem (see [9, Chapter 3, §D3]), $(u_1/v_1)\circ k_1 = (u_2/v_2)\circ k_1$ and hence $u_1/v_1 = u_2/v_2$.

The class of all Nevanlinna domains will be denoted by $\mathcal{N}$. For examples of domains in this class, see [4, §3] and the references therein.

Definition 2. A domain $\Omega \subset \mathbb{C}$ is called a *Carathéodory domain* if Ω is bounded and $\partial\Omega = \partial\Omega_\infty$, where Ω_∞ is the unbounded component of $\mathbb{C}\setminus\overline{\Omega}$.

Definition 3. Let A be a subset of $\mathbb{C}$. A non-empty subset $I \subset A$ is a *border* of A (or an *A-border*) if each $a \in I$ has a neighbourhood V in $\mathbb{C}$ such that there is a homeomorphism h of $V \cap A$ onto a relatively open subset of the closed upper half-plane and $V \cap I = \{z \in V \mid \operatorname{Im} h(z) = 0\}$. Thus, a border is locally a Jordan arc. We further require that *a border be locally rectifiable.*

Note that if Ω is an open set in $\mathbb{C}$, I is an $\overline{\Omega}$-border and f is a bounded holomorphic function in Ω, then the nontangential limit $f(\zeta)$ exists for almost every ζ on I with respect to length.

For a bounded domain Ω, a relatively open subset I of $\partial\Omega$ and $\varepsilon > 0$, set

$$\Omega_\varepsilon = \{z \in \Omega \mid \operatorname{dist}(z, \partial\Omega \setminus I) > \varepsilon\}$$

and

$$I_\varepsilon = \{z \in I \mid \operatorname{dist}(z, \partial\Omega \setminus I) > \varepsilon\}.$$

If $I = \partial\Omega$, we set $\Omega_\varepsilon = \Omega$ and $I_\varepsilon = \partial\Omega$.

Definition 4. Let Ω be a bounded domain in $\mathbb{C}$ and let I be a $\overline{\Omega}$-border. We say that Ω is *I-Nevanlinna* (denoted $\Omega \in \mathcal{N}(I)$) if for each sufficiently small $\varepsilon > 0$, there exist two bounded holomorphic functions u and v in Ω_ε (with $v \not\equiv 0$) such that a.e. on I_ε one has

$$\overline{\zeta} = \frac{u(\zeta)}{v(\zeta)}.$$

It follows from the Luzin-Privalov theorem that u/v is uniquely defined in the sense that if u_1, v_1 and u_2, v_2 satisfy the condition of the definition, then u_1/v_1 and u_2/v_2 coincide in Ω_ε, for sufficiently small ε. Letting ε go to zero, we have that u/v is uniquely defined in Ω. It is not obvious whether u/v can be written as the quotient of bounded holomorphic functions in Ω (see [8]).

Note also that if Ω is a Nevanlinna domain such that $\partial\Omega$ is a border of $\overline{\Omega}$, then (Ω is a Jordan domain and) $\Omega \in \mathcal{N}(\partial\Omega)$.

Definition 5. Let Ω be a bounded domain in $\mathbb{C}$ and let γ_1 and γ_2 be two $\overline{\Omega}$-border Jordan arcs on the boundary of Ω. Suppose that for $\ell = 1, 2$, there exist open neighbourhoods U_ℓ of γ_ℓ in $\mathbb{C}$ such that $\Omega_\ell = \Omega \cap U_\ell \in \mathcal{N}(\gamma_\ell)$. Let u_ℓ/v_ℓ be the meromorphic function appearing in the definition of γ_ℓ-Nevanlinna ($\ell = 1, 2$).

γ_1 and γ_2 are said to be *Nevanlinna Ω-independent* if there exists no path in Ω along which the two meromorphic elements $(u_\ell/v_\ell, \Omega_\ell)$, $\ell = 1, 2$, are meromorphic continuations of each other.

Remark 1. It follows from the Luzin-Privalov theorem that if γ_1 and γ_2 are Nevanlinna Ω-independent, then $\Omega \notin \mathcal{N}(I)$ for each $\overline{\Omega}$-border set I containing $\gamma_1 \cup \gamma_2$.

Notice also that if γ_1 and γ_2 are analytically independent $\overline{\Omega}$-border arcs on $\partial\Omega$ (see [4, §3]), then γ_1 and γ_2 are Nevanlinna Ω-independent. In particular, every polygon Ω has two Nevanlinna Ω-independent Jordan arcs on its boundary.

2. Main theorems

Our first two theorems give rather strong necessary conditions for the equalities $\mathcal{A}_n(F) = \mathcal{E}_n(F)$ and $\mathcal{A}_n(F) = \mathcal{M}_n(F)$ to hold, reducing the (necessity part of the) problem of approximating on closed sets by n-analytic entire functions and n-meromorphic functions to an approximation problem on compact sets. Though both Theorem 1 and Theorem 2 below are true for $n = 1$, much better results are known in this case. In fact, when $n = 1$, a complete characterization of the closed sets F for which either $\mathcal{A}_1(F) = \mathcal{E}_1(F)$ or $\mathcal{A}_1(F) = \mathcal{M}_1(F)$ holds has been obtained by N.U. Arakelian and A.H. Nersessian respectively. See the discussion following Problem 1.

Theorem 1. *Let F be a closed subset of $\mathbb{C}$ and let $n \in \mathbb{N}$ be fixed. If $\mathcal{A}_n(F) = \mathcal{E}_n(F)$, then for each Carathéodory domain Ω, one has*

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega}) \tag{3}$$

and

$$\mathcal{A}_n(\overline{F \cap \Omega}) = \mathcal{R}_n^{\overline{\Omega}}(\overline{F \cap \Omega}). \tag{3'}$$

In particular, if in addition $\mathbb{C} \setminus \overline{\Omega}$ is connected, then

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{P}_n(F \cap \overline{\Omega}) \tag{4}$$

and

$$\mathcal{A}_n(\overline{F \cap \Omega}) = \mathcal{P}_n(\overline{F \cap \Omega}). \tag{4'}$$

Theorem 2. *Let F be a closed subset of $\mathbb{C}$ and let $n \in \mathbb{N}$ be fixed. If $\mathcal{A}_n(F) = \mathcal{M}_n(F)$, then for each Carathéodory domain Ω, one has*

$$\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{F \cap \overline{\Omega}}(F \cap \overline{\Omega}) \tag{5}$$

Problem 1. Is it true that, when $n \geq 2$, these necessary conditions for approximation are also sufficient?

When $n = 1$, it follows immediately from Arakelian's theorem (see [5, Chapter IV, §2]) that these conditions are not sufficient to have $\mathcal{A}_1(F) = \mathcal{E}_1(F)$. They are not sufficient either for the analogous result to hold when approximating by entire harmonic functions (see [7, Chapter 3, §10]). But condition (5) is both necessary and sufficient for $\mathcal{A}_1(F) = \mathcal{M}_1(F)$ to hold (see [5, Chapter IV, §1D]; note that the statement there must be corrected to say "for each *Carathéodory* domain g", otherwise it is incorrect).

In Section 4, we shall give examples to illustrate our theorems and to show that, in some circumstances, approximation by entire polyanalytic ($n \geq 2$) functions seems easier than by entire analytic ($n = 1$) or entire harmonic functions. For instance, it is not necessary for the set $\overline{\mathbb{C}} \setminus F$ to be locally connected at ∞ (compare with Arakelian's theorem [5, Chapter IV, §2C] and Gardiner's theorem [7, Corollary 3.21]).

Our next theorem is closely related to the question asked in Problem 1. It generalizes [4, Theorems 2.2].

Theorem 3. *Let $n \in \mathbb{N}$ be fixed. Let X be a compact subset of $\mathbb{C}$, and $\{\Omega_\ell\}_{\ell \in L}$ be the (connected) components of $(\widehat{X})^0$ that are not contained in X. Then*

$$\mathcal{A}_n(X) = \mathcal{P}_n(X). \tag{6}$$

if and only if

$$\mathcal{A}_n(X \cap \overline{\Omega}_\ell) = \mathcal{R}_n^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell) \tag{7}$$

for each $\ell \in L$.

Remark 2. When $n = 1$, it follows easily from Mergelyan's theorem (see [5, Chapter III, §2]) that (6) $\Longleftrightarrow$ $(X = \widehat{X})$ $\Longleftrightarrow$ (7), and thus that Theorem 3 holds in this case.

Notice also that when X is a *Carathéodory compact set* (i.e. when $\partial X = \partial \widehat{X}$), $\{\Omega_\ell\}_{\ell \in L}$ are precisely the bounded components of $\mathbb{C} \setminus X$, and thus Theorem 3 can be used, with the help of Theorem 1, to deduce Theorem 2.2 in [4].

In order to apply Theorem 3, it is important to be able to answer the following question:

Problem 2. Let Ω be a Carathéodory domain and X be a compact subset of $\overline{\Omega}$ such that $\mathcal{A}_n(X) = \mathcal{R}_n^{\overline{\Omega}}(X)$. When is it true then that $\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega)$?

In [4], Problem 2 was studied when Ω is a Jordan domain and X is strictly contained in Ω (see [4, Theorem 4.3]). We have obtained the following stronger result.

Theorem 4. *Let $n \in \mathbb{N}$, $n \geq 2$ be fixed, let Ω be a Carathéodory domain and let X be a compact set in $\overline{\Omega}$ such that*

$$\mathcal{A}_n(X) = \mathcal{R}_n^{\overline{\Omega}}(X) \tag{8}$$

and such that the set $E := \partial\Omega \setminus X$ is a $\overline{\Omega}$-border. Let $\{D_\ell\}_{\ell \in L}$ be all the components of $\Omega \setminus X$ for which $E_\ell := \partial D_\ell \setminus X \neq \emptyset$. If $D_\ell \notin \mathcal{N}(E_\ell)$ for each $\ell \in L$, then

$$\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega). \tag{9}$$

Corollary 1. *Let Ω be a Carathéodory domain with connected complement of it's closure, and such that $\partial\Omega$ contains two Nevanlinna Ω-independent arcs γ_1 and γ_2. Let X be a compact subset of Ω with $\mathcal{A}_n(X) = \mathcal{P}_n(X)$. Then $\mathcal{A}_n(K) = \mathcal{P}_n(K)$, where $K = X \cup \partial\Omega$.*

In [4, Theorem 4.3, Proposition 4.4] the previous result was obtained for Jordan domains Ω and analytically independent γ_1 and γ_2. Corollary 1, and a fortiori Theorem 4, fails when $n = 1$.

3. Proofs

3.1. **Proof of Theorem 1.** We will only establish (3), the proof of (3′) being essentially the same. Moreover (4) and (4′) follow immediately from (3) and (3′) respectively, from [3, Theorem 2] and Runge's Theorem.

Assume that $n \in \mathbb{N}$ is fixed, that $\mathcal{A}_n(F) = \mathcal{E}_n(F)$ and that $f \in \mathcal{A}_n(F \cap \overline{\Omega})$. Using Tietze's extension theorem, we first extend f to a function, still denoted by f, which is continuous on $\mathbb{C}$ and has compact support. We now apply the approximation scheme of A.G. Vitushkin [13], appropriately adapted to our situation. We will only outline the method. The details can be found in [4, Proof of Proposition 2.5].

Put $\mathcal{L} = \overline{\partial}^n$ and let $\Phi(z) = \bar{z}^{n-1}((n-1)!\,\pi z)^{-1}$ be the fundamental solution for $\mathcal{L}$, let $\partial := \partial/\partial z$ and let $B(a,r)$ denote the open disk with centre a and radius $r > 0$.

Now fix an arbitrary $\delta \in (0,1)$ and construct a δ-partition of unity as follows. For each $j = (j_1, j_2) \in \mathbb{Z}^2$, set $a_j = j_1\delta + ij_2\delta$, $B_j = B(a_j, \delta)$ and choose $\varphi_j \in C_0^\infty(B(a_j,\delta))$ such that $0 \le \varphi_j \le 1$, $\sum_{j\in\mathbb{Z}^2} \varphi_j \equiv 1$ and $\|\overline{\partial}^r \partial^s \varphi_j\| \le A\delta^{-(r+s)}$, $1 \le r+s \le n$, where A depends only on n.

Let $f_j = \Phi * (\varphi_j \mathcal{L} f)$. Set $J = \{j \in \mathbb{Z}^2 \mid B(a_j,\delta) \cap \operatorname{supp} f \neq \emptyset\}$, and note that J is a finite set. We have $\mathcal{L} f_j = \varphi_j \mathcal{L} f$ in the distributional sense, and $f = \sum_{j\in J} f_j$. Let $J_1 = \{j \in J \mid B_j \subset \Omega\}$, $J_2 = \{j \in J \mid B_j \cap \overline{\Omega} = \emptyset\}$ and $J_3 = \{j \in J \mid B_j \cap \partial\Omega \neq \emptyset\}$. Note that $J = J_1 \cup J_2 \cup J_3$.

If $j \in J_1$, then $f_j \in \mathcal{A}_n(F) = \mathcal{E}_n(F) \subset \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$. If $j \in J_2$, then $f_j \in \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ by Runge's Theorem.

It remains to consider the case $j \in J_3$. For each $j \in J_3$, one can find $a_j^* \in B(a_j,\delta) \setminus \overline{\Omega}$ and a Jordan curve γ_j in $\overline{B(a_j^*,\delta)} \setminus \overline{\Omega}$ with starting point a_j^* and with ending point in $\partial B(a_j^*,\delta)$. It follows (see [4, Proof of Proposition 2.5]) that we can construct functions g_j, n-analytic outside γ_j, and thus in $\mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$, such that

$$\|\sum_{j\in J_3}(f_j - g_j)\|_{F\cap\overline{\Omega}} \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

The desired approximant is $g = \sum_{j\in J_1\cup J_2} f_j + \sum_{j\in J_3} g_j$.

3.2. **Proof of Theorem 2.** Assuming that $\mathcal{A}_n(F) = \mathcal{M}_n(F)$ and that $f \in \mathcal{A}_n(F \cap \overline{\Omega})$, the proof of Theorem 2 is mutatis mutandis the proof of Theorem 1. The details are left to the reader.

3.3. **Proof of Theorem 3.** Notice that all Ω_ℓ, $\ell \in L$, are Carathéodory domains and so, by Theorem 1, (6) implies (7).

To show the sufficiency of (7), let μ be a finite complex-valued Borel measure supported on X such that $\mu \perp \mathcal{P}_n(X)$. We must prove that $\mu \perp \mathcal{A}_n(X)$ holds whenever (7) holds.

We start by noticing that if the index set L is empty, that is if X is simply connected, then (7) is an empty condition and (6) follows from [3, Theorem 2] and Runge's theorem. So we assume that $L \neq \emptyset$, and we fix $\ell \in L$. By [4, Lemma 2.4], one can find a sequence of complex analytic polynomials $\{q_j\}_{j=1}^\infty$ such that $\|q_j\|_{\widehat{X}} \leq C$, where C is an absolute constant, and, as $j \to \infty$, such that $q_j \to 1$ uniformly on compact subsets of Ω_ℓ, $q_j \to 0$ on compact subsets of $(\widehat{X})^0 \setminus \Omega_\ell$, and, in addition, such that the sequence of measures $\{q_j\mu\}_{j=1}^\infty$ converges weak-$*$ to some measure μ_{Ω_ℓ}. It follows that $\operatorname{supp} \mu_{\Omega_\ell} \subset (X \cap \overline{\Omega}_\ell) \cup \partial\widehat{X}$ and that $\mu_{\Omega_\ell} \perp \mathcal{P}_n(X)$.

Recall that

$$\widehat{\nu}(\zeta) = \frac{1}{2\pi i} \int \frac{d\nu(z)}{z - \zeta}$$

defines the Cauchy transform of a (finite) measure ν. We claim that

$$\widehat{\overline{z}^s\mu}(z_0) = 0, \quad \forall z_0 \in \Omega_\ell \setminus X, \quad s = 0, \dots, n-1. \tag{10}$$

Indeed, since $\widehat{\overline{z}^s\mu}(z_0) = \widehat{\overline{z}^s\mu_{\Omega_\ell}}(z_0)$ whenever $z_0 \in \Omega_\ell \setminus X$ (see [4, proof of Theorem 2.2 (2)]), we can assume without loss of generality that

$$\operatorname{supp} \mu \subset (X \cap \overline{\Omega}_\ell) \cup \partial\widehat{X}. \tag{11}$$

But if (11) is satisfied for μ, we then must have $\operatorname{supp} \mu_{\Omega_\ell} \subset X \cap \overline{\Omega}_\ell$. In fact, since $q_j \to 0$ on $(\widehat{X})^0 \setminus \Omega_\ell$,

$$\widehat{\overline{z}^s\mu_{\Omega_\ell}}(w) = 0, \quad \forall w \in \left[(\widehat{X})^0 \setminus \Omega_\ell\right] \cup \left[\mathbb{C} \setminus \widehat{X}\right], \quad s = 0, \dots, n-1. \tag{12}$$

Since μ has no atoms on $\partial\widehat{X}$ (see [1, Lemma 9]), the function

$$f_{\Omega_\ell}(w) = \frac{1}{\pi} \int \frac{\overline{z} - \overline{w}}{z - w} d\mu_{\Omega_\ell}(z)$$

is continuous on $\mathbb{C} \setminus \overline{\Omega}_\ell$ and, by (12), it vanishes outside $\overline{\Omega}_\ell$. It thus follows that $\mu_{\Omega_\ell}|_{\mathbb{C}\setminus\overline{\Omega_\ell}} = 0$ since $\overline{\partial}^2 f_{\Omega_\ell} = \mu_{\Omega_\ell}$ in the sense of distributions. Therefore $\mu_{\Omega_\ell} \perp \mathcal{R}_n^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell) = \mathcal{A}_n(X \cap \overline{\Omega}_\ell)$ and (10) holds.

Now since ℓ was arbitrary in (10), we have that

$$\widehat{\overline{z}^s\mu}(z_0) = 0, \quad \forall z_0 \in \mathbb{C} \setminus X, \quad s = 0, \dots, n-1,$$

and thus $\mu \perp \mathcal{R}_n^X(X)$. We will complete the proof by showing that $\mathcal{R}_n^X(X) = \mathcal{A}_n(X)$. For this, we again apply Vitushkin's technique, using the notation from the proof of Theorem 1 above.

Take any $f \in \mathcal{A}_n(X)$ and extend it to a function (still denoted by f) continuous on all of $\mathbb{C}$ and with compact support. Let $\{f_j\}$ be the localized functions corresponding to the covering $\{B_j\}$. If $B_j \cap \widehat{X} = \emptyset$ or $B_j \cap \partial\widehat{X} \neq \emptyset$, we deal with the corresponding f_j as in the proof of Theorem 1. It remains to consider the case when $B_j \subset (\widehat{X})^0$. In this case, B_j belongs to some component Ω of $(\widehat{X})^0$. If $\Omega \subset X^0$, then $f_j \equiv 0$ and there is nothing to be done. If $\Omega \not\subset X^0$, then $\Omega = \Omega_\ell$ for some $\ell \in L$. Since $f \in \mathcal{A}_n(X \cap \overline{\Omega}_\ell) = \mathcal{R}_n{}^{\overline{\Omega}_\ell}(X \cap \overline{\Omega}_\ell)$, one can find $\{g_m\}_{m=1}^\infty$, g_m being n-analytic in some neighbourhood U_m of $X\cap\overline{\Omega}_\ell$ such that, after an appropriate extension of g_m (still denoted g_m) to a continuous function on all of $\overline{\Omega}_\ell$, one has $\|f - g_m\|_{\overline{\Omega}_\ell} \to 0$ as $m \to \infty$. Using well-known properties of the Vitushkin operator, we obtain that

$$\|\Phi * (\phi_j \mathcal{L} g_m) - f_j\|_X = \|\Phi * (\phi_j(\mathcal{L} g_m - \mathcal{L} f))\|_X \leq C\|g_m - f\|_{B_j} \to 0$$

as $m \to \infty$. Clearly $\Phi * (\phi_j \mathcal{L} g_m) \in \mathcal{R}_n^X(X)$. This completes the proof.

3.4. **Proof of Theorem 4.** We will proceed by contradiction. Let $K = X \cup \partial\Omega$ and suppose that

$$\mathcal{A}_n(K) \neq \mathcal{R}_n^{\overline{\Omega}}(K). \tag{13}$$

We will show that then there must exist an $\ell \in L$ such that $D_\ell \in \mathcal{N}(E_\ell)$.

Assuming (13), it follows that there exists a measure σ supported on K with $\sigma \perp \mathcal{R}_n^{\overline{\Omega}}(K)$ but $\sigma \not\perp \mathcal{A}_n(K)$. We notice that $\sigma|_E$ is a nonzero measure, since otherwise, we would have $\operatorname{supp}\sigma \subset X$, contradicting (8). Since $E = \cup_{\ell\in L} E_\ell$ and L is at most countable, we can choose an ℓ such that $\sigma|_{E_\ell}$ is also a nonzero measure. We shall prove that $D_\ell \in \mathcal{N}(E_\ell)$. Let $X_\ell = \overline{\Omega} \setminus (D_\ell \cup E_\ell)$, so that $\partial X_\ell \subset \partial\Omega \cup \partial D_\ell$. Fix $\varepsilon > 0$ so small that $E_\ell^\varepsilon := \{z \in E_\ell \mid \operatorname{dist}(z, X_\ell) \geq \varepsilon\} \neq \emptyset$ and $\sigma|_{E_\ell^\varepsilon} \neq 0$. By definition, E_ℓ is a $\overline{D}_\ell$-border, and so there exists an at most countable family of disjoint (open) Jordan arcs $\{\gamma_j\}_{j\in J_\ell}$ such that $E_\ell = \cup_{j\in J_\ell}\gamma_j$. Moreover, since E_ℓ^ε is compact in ∂D_ℓ and each γ_j is open in ∂D_ℓ, there exists a finite subset $J \subset J_\ell$ such that $E_\ell^\varepsilon \subset \cup_{j\in J}\gamma_j$.

For each $j \in J$, we can choose points a_j and b_j on γ_j such that γ_j has tangents at a_j and b_j, the path $\widetilde{\gamma}_j$ which is the part of γ_j strictly between a_j and b_j has positive orientation with respect to Ω when "starting" at a_j and "ending" at b_j, and moreover such that $E_\ell^\varepsilon \subset \cup_{j\in J}\widetilde{\gamma}_j =: \Gamma_1$ and $\sigma|_{\Gamma_1} \neq 0$. Notice that

$$\operatorname{dist}(z, X_\ell) < \varepsilon \quad \text{if} \quad z \in E_\ell \setminus \Gamma_1.$$

Let Ω_∞ be the unbounded (connected) component of $\overline{\mathbb{C}} \setminus \overline{\Omega}$. Then Ω_∞ is simply connected since $\partial\Omega_\infty = \partial\Omega$ is connected, and each $\widetilde{\gamma}_j$ is a $\overline{\Omega}_\infty$-border (here we extend Definition 3 to include domains in $\overline{\mathbb{C}}$ which contain ∞). Choose an enumeration of J such that the $\widetilde{\gamma}_j$'s are in consecutive order on $\partial\Omega_\infty$. We may as well assume that this enumeration is $\{1, 2, \dots, N\}$, where $N = |J|$. Notice that the $\widetilde{\gamma}_j$'s have negative orientation with respect to Ω_∞. Therefore, for each $j \in J$, we can find a Jordan rectifiable arc β_j in Ω_∞ which starts at b_j, ends at a_{j+1}, is normal to γ_j at b_j and is normal to γ_{j+1} at a_{j+1} (where $a_{N+1} := a_1$ and $\gamma_{N+1} = \gamma_1$). In addition we can (and will) require that the curve $\Gamma = \cup_{j\in J}(\widetilde{\gamma}_j \cup \beta_j)$ is a Jordan rectifiable curve which surrounds and meets nontangentially the compact set

$$Y = X_\ell \cup (E_\ell \setminus (\cup_{j\in J} \widetilde{\gamma}_j))$$

and has positive orientation with respect to the domain G surrounded by Γ. We notice that $Y \subset G \cup (\cup_{j\in J}\{a_j, b_j\})$, $\operatorname{supp}\sigma \subset Y \cup \Gamma_1$, $\Gamma_1 \subset \Gamma$, and recall that $\sigma|_{\Gamma_1} \neq 0$. Moreover, since Γ meets Y nontangentially, for each $z \in Y$,

$$\operatorname{dist}(z, \Gamma) \asymp \operatorname{dist}(z, \cup_{j\in J}\{a_j, b_j\}). \tag{14}$$

Let $\sigma_0 = (\prod_{j\in J}(z - a_j)(z - b_j))\sigma$. Then $\sigma_0 \perp \mathcal{P}_n(Y \cup \Gamma)$, since this is also true for σ. Recall that $\widehat{\mu}(\zeta) = (2\pi i)^{-1} \int (z - \zeta)^{-1} d\mu(z)$ defines the Cauchy transform of a measure μ. We need the following Lemma.

Lemma 1. *Let μ be a measure such that $\left(\prod_{j\in J} |z - a_j||z - b_j|\right)^{-1} \mu$ is a finite measure on Y. Then the measure $\widehat{\mu}(\zeta)d\zeta|_\Gamma$ is finite and*

$$(\mu + \widehat{\mu}(\zeta)d\zeta|_\Gamma) \perp \mathcal{P}_1(Y \cup \Gamma). \tag{15}$$

Proof of Lemma 1. Let us first prove that $\widehat{\mu}(\zeta)d\zeta|_\Gamma$ is a finite measure:

$$\begin{aligned}
\int_\Gamma |\widehat{\mu}(\zeta)||d\zeta|| &= \frac{1}{2\pi} \int_\Gamma \left| \int_Y \frac{d\mu(z)}{z - \zeta} \right| |d\zeta| \\
&\leq \int_Y \left(\int_\Gamma \frac{|d\zeta|}{|z - \zeta|} \right) |d\mu(z)| \\
&\leq \text{Const} \int_Y \frac{\text{length}(\Gamma)}{\prod_{j\in J} |z - a_j||z - b_j|} |d\mu(z)| \\
&< \infty,
\end{aligned}$$

where the penultimate inequality follows from (14).

This last estimate allows us to use Fubini's theorem in order to prove (15).

Indeed, if p is a complex polynomial, then

$$\begin{aligned}
&\int_Y p(z)d\mu(z) + \int_\Gamma p(\zeta)\widehat{\mu}(z)d\zeta \\
&\quad = \int_Y p(z)d\mu(z) + \int_\Gamma p(\zeta)\left(\frac{1}{2\pi i}\int_Y \frac{d\mu(z)}{z-\zeta}\right)d\zeta \\
&\quad = \int_Y p(z)d\mu(z) + \int_Y \left(\frac{1}{2\pi i}\int_\Gamma \frac{p(\zeta)}{z-\zeta}d\zeta\right)d\mu(z) \\
&\quad = 0
\end{aligned}$$

by Cauchy's formula. □

Let us apply Lemma 1 to the measures $\mu_s := \overline{z}^s\sigma_0|_Y$, $s = 0,\dots,n-1$. We then have

$$(\overline{z}^s\sigma_0 - (\mu_s + \widehat{\mu}_s(\zeta)d\zeta|_\Gamma)) \perp \mathcal{P}_1(\Gamma),$$

so that, by an analog of the F. and M. Riesz theorem on rectifiable Jordan domains due to I.I. Privalov [11, Chapter 3, §7], for each $s \in \{0,\dots,n-1\}$ there exists a function h_s in the the Smirnov class $E_1(G)$ such that one has

$$\overline{z}^s\sigma_0 = \mu_s + \widehat{\mu}_s(\zeta)d\zeta|_\Gamma + h_s(\zeta)d\zeta|_\Gamma\,. \tag{16}$$

First setting $s = 0$ in (16), then multiplying the resulting equation by $\overline{z}^s$, $s = 0,\dots,n-1$, and observing that $\mu_s := \overline{z}^s\mu_0$, we thus also have

$$\overline{z}^s\sigma_0 = \overline{z}^s\mu_0 + \overline{\zeta}^s\widehat{\mu}_0(\zeta)d\zeta\Big|_\Gamma + \overline{\zeta}^s h_0(\zeta)d\zeta\Big|_\Gamma\,.$$

Subtracting this last equality from (16), we get, for $s = 0,\dots,n-1$,

$$\overline{\zeta}^s(\widehat{\mu}_0(\zeta) + h_0(\zeta))\,d\zeta|_\Gamma = (\widehat{\mu}_s(\zeta) + h_s(\zeta))\,d\zeta|_\Gamma\,.$$

Since $\sigma_0|_{\Gamma_1} = (\widehat{\mu}_0(\zeta) + h_0(\zeta))\,d\zeta|_{\Gamma_1} \neq 0$, by the Luzin-Privalov theorem, we have that $\widehat{\mu}_0(z)+h_0(z) \not\equiv 0$ in D_ℓ, and so the nontangential limit $\widehat{\mu}_0(\zeta)+h_0(\zeta)$ is nonzero at almost all points $\zeta \in \Gamma_1$, which gives that

$$\overline{\zeta}^s = \frac{\widehat{\mu}_s(\zeta) + h_s(\zeta)}{\widehat{\mu}_0(\zeta) + h_0(\zeta)}$$

(nontangentially) a.e. on Γ_1. Since each $\widehat{\mu}_s$ is bounded outside an ε-neighbourhood of Y and each h_s is a quotient of bounded holomorphic functions in G, a simple calculation gives (taking just $s = 1$) that $\overline{\zeta} = u(\zeta)/v(\zeta)$ nontangentially $d\zeta$ - a.e. on Γ_1 for some holomorphic functions u and v bounded in $\{z \in D_\ell \mid \mathrm{dist}(z,K) > \varepsilon\}$, which precisely means that $D_\ell \in \mathcal{N}(E_\ell)$. Theorem 4 is proved.

3.5. **Proof of Corollary 1.** It follows from the hypotheses that $\gamma_1 \cap \gamma_2 = \emptyset$. Let $X_1 = X \cup (\partial\Omega \setminus (\gamma_1 \cup \gamma_2))$. Corollary 1 will follow immediately from Theorem 4 applied to this X_1. It thus only suffices to verify that the hypotheses of Theorem 4 are satisfied in this case.

Since $\mathcal{A}_n(X) = \mathcal{P}_n(X)$, $\widehat{X}$ is contained in Ω and $\partial\Omega \setminus (\gamma_1 \cup \gamma_2)$ has connected complement, it follows from Runge's theorem that

$$\mathcal{A}_n(X_1) = \mathcal{P}_n(X_1).$$

Moreover, in this case, $E := \partial\Omega \setminus X_1 = \gamma_1 \cup \gamma_2$, and the collection $\{D_\ell\}_{\ell \in L}$ consists of a single domain $D_1 = \Omega \setminus \widehat{X}$. But by Remark 1, $D_1 \notin \mathcal{N}(\gamma_1 \cup \gamma_2)$. Thus Theorem 4 applies and Corollary 1 is proved.

4. Examples

Example 1. We first give an example that shows that Theorem 1 would fail if we would replace either $\mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ or $\mathcal{R}_n^{\overline{\Omega}}(\overline{F \cap \Omega})$ by $\mathcal{P}_n(F \cap \overline{\Omega})$ or $\mathcal{P}_n(\overline{F \cap \Omega})$ in (3) or (3′) respectively.

Let U_1 be an open bounded ribbon which winds around the outside of the circle $\{|z| = 1\}$ and accumulates on that circle. One says that U_1 is a "cornucopia" (see [6, Chapter VI, §5, Figure 4] or [4, §3]). Note in passing that U_1 is a Carathéodory domain with the property that $\mathbb{C} \setminus \overline{U}_1$ is <u>not</u> connected.

Let $F = \{|z| \leq 1\} \cup \overline{U_1}$ and $\Omega = U_1$. We remark that F is a Carathédory compact set, i.e. that $\partial F = \partial \widehat{F}$, and that $F \cap \overline{\Omega} = \overline{F \cap \Omega} = \overline{\Omega}$. For all $n \in \mathbb{N}$, we have $\mathcal{A}_n(F) = \mathcal{P}_n(F)$ by [3, Theorem 2], and $\mathcal{A}_n(F \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(F \cap \overline{\Omega})$ by [4, Proposition 2.5], but $\mathcal{A}_n(F \cap \overline{\Omega}) \neq \mathcal{P}_n(F \cap \overline{\Omega})$ by [4, Theorem 2.2(2)].

Example 2. The following is an application of Theorem 3 which cannot be obtained by using [4, Theorem 2.2(2)] instead.

We again start with a cornucopia. This time the open ribbon U_2 winds around the outside of the ellipse $\{x^2 + 2y^2 = 1\}$, and accumulates on that ellipse. Let X be the union of $\overline{U}_2$ with a segment $[-a, a]$, where $a > 0$ and the segment does not contain the foci of the ellipse. Let $\Omega = \{x^2 + 2y^2 < 1\}$ be the interior of the ellipse. Notice that $\Omega \notin \mathcal{N}$ (see [4, §3]), that Ω is the only component of $(\widehat{X})^0$ which is not contained in X, that X is not a Carathéodory compact set (so [4, Theorem 2.2(2)] does not apply in this situation), and that $X \cap \overline{\Omega} = \partial\Omega \cup [-a, a]$. Fix $n \in \mathbb{N}$, $n \geq 2$. From [4, Theorem 4.3(2)], it follows that

$$\mathcal{A}_n(X \cap \overline{\Omega}) = \mathcal{P}_n(X \cap \overline{\Omega}) = \mathcal{R}_n^{\overline{\Omega}}(X \cap \overline{\Omega})$$

and consequently, from Theorem 3, that $\mathcal{A}_n(X) = \mathcal{P}_n(X)$.

Example 3. We now give an example of a nowhere dense closed set F in $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus F$ is connected, but *not* locally connected at ∞, and still F is a set of Carleman approximation (see [5, Chapter IV, §3]) by n-analytic functions when $n \geq 2$. In particular, $\mathcal{A}_n(F) = C(F) = \mathcal{E}_n(F)$. This is in sharp contrast with (the case $n = 1$) Arakelian's theorem [5, Chapter IV, §2] and with harmonic Carleman approximation [7, Theorem 4.6].

For $k = 2, 3, 4, \dots$, let

$$A_k = \left\{x = \tfrac{1}{k},\ 0 \leq y \leq k\right\}$$
$$B_k = \left\{x = \tfrac{1}{2}\left(\tfrac{1}{k} + \tfrac{1}{k-1}\right),\ 0 \leq y \leq k\right\}$$
$$C_k = \left\{\tfrac{1}{k} \leq x \leq \tfrac{1}{2}\left(\tfrac{1}{k} + \tfrac{1}{k-1}\right),\ y = k\right\}$$

and set

$$F = \left(\bigcup_{k=2}^{\infty} (A_k \cup B_k \cup C_k)\right) \cup \left\{x = 0,\ 0 \leq y < \infty\right\}.$$

The set F is often refer to as an Arakelian glove. Clearly, $\overline{\mathbb{C}} \setminus F$ is connected, but not locally connected at ∞. From [2, Proposition 5], to prove that F is a n-analytic Carleman set, it suffices to show the existence of a "compatible exhaustion" $\{K_j\}_{j=1}^{\infty}$. For this purpose, set $K_1 = \emptyset$,

$$K_j = \{|x| \leq j,\ |y| \leq j\}, \quad \text{when} \quad j \geq 2,$$

and let $X_j = K_j \cup (K_{j+2} \cap F)$. It remains to check (see [2, equation (10)]) that

$$\mathcal{A}_n(X_j) = \mathcal{P}_n(X_j). \tag{17}$$

When $j = 1$, this follows from [3, Theorem 2] and Runge's theorem. Now fix $j \in \mathbb{N}$, $j \geq 2$, and let Ω be the Jordan domain whose closure is

$$\overline{\Omega} = X_j \setminus \Bigg[\Bigg(\bigcup_{k=j+3}^{\infty} \Big(\left\{x = \tfrac{1}{k},\ k < y < \infty\right\} \cup \left\{x = \tfrac{1}{2}(\tfrac{1}{k} + \tfrac{1}{k-1}),\ k < y < \infty\right\}\Big)\Bigg) \cup \left\{x = 0,\ y > k\right\}\Bigg].$$

We now apply Theorem 4 with this Ω and $X = K_j$. Since, in this case, $\Omega \setminus X$ consists of two rectangles, which we denote $D_{j,1}$ and $D_{j,2}$, and since $D_{j,i} \notin \mathcal{N}(\partial D_{j,i} \setminus X)$ for $i = 1, 2$, one has $\mathcal{A}_n(X \cup \partial\Omega) = \mathcal{R}_n^{\overline{\Omega}}(X \cup \partial\Omega) = \mathcal{P}_n(X \cup \partial\Omega)$. Now we apply Theorem 3 to obtain (17), and this completes the example.

References

[1] E. Bishop, *Boundary measures of analytic differentials*, Duke Math. J. **27** (1960), 331–340.

[2] A. Boivin, P.M. Gauthier and P. Paramonov, *Approximation on closed sets by analytic or meromorphic solutions of elliptic equations and applications*, Canad. J. Math. **54**(5) (2002), 945–969.

[3] J.J. Carmona, *Mergelyan approximation theorem for rational modules*, J. Approx. Theory **44** (1985), 113–126.

[4] J.J. Carmona, P.V. Paramonov and K.Yu. Fedorovski, *On uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions* (Russian), Mat. Sb. **193**:10 (2002), 75–98; English translation in Sb. Math. **193**:10 (2002), 1469–1492.

[5] D. Gaier, "Vorlesungen über Approximation in Komplexen", Birhäuser Verlag, 1980; English translation: "Lectures on Complex Approximation", Birkäuser, 1987.

[6] T.W. Gamelin, "Uniform Algebras", 2nd Ed., Chelsea, 1984.

[7] S.J. Gardiner, "Harmonic Approximation", LMS Lecture Notes Series vol. 221, Cambridge University Press, 1995.

[8] P.M. Gauthier and W. Hengartner, *Local harmonic majorants of functions subharmonic in the unit disc*, J. Analyse Math. **26** (1973), 405–412.

[9] P. Koosis, "Introduction to H_p Spaces", 2nd ed., Cambridge Tracts in Mathematics vol. 115, Cambridge University Press, 1998.

[10] P.V. Paramonov and K.Yu. Fedorovski, *Uniform and C^1-approximability of functions on compact subsets of $\mathbb{R}^2$ by solutions of second order elliptic equations*, Mat. Sb. **190**:2 (1999), 123–144 (Russian); English translation in: Sb. Math. **190**:2 (1999), 285–307.

[11] I.I. Privalov, "Boundary Properties of Analytic Functions" (Russian), 2nd ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; (German translation) "Randeigenschaften Analytischer Funktionen", Deutscher V. der Wiss., Berlin, 1956.

[12] H.S. Shapiro, "The Schwarz Function and its Generalization to Higher Dimensions", Wiley, 1992.

[13] A.G. Vitushkin, *Analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk. **22**:6 (1967), 141–199 (Russian); English translation in Russian Math. Surveys **22**:6 (1967), 139–200.

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