Algebraic and geometric solutions of hyperbolic Dehn filling equations

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Abstract

We study the difference between algebraic and geometric solutions of the hyperbolic Dehn filling equations for a 3-manifold equipped with an ideal triangulation. We show that any geometric solution is an algebraic one, and we prove uniqueness of the geometric solutions.

We do explicit calculations for three interesting examples. With the first two examples we see that not all algebraic solutions are geometric and that algebraic solutions are not unique. The third example describes a non-hyperbolic manifold that admits a positive, partially flat solution of the compatibility and completeness (but not the angle) equations.

1 Introduction

One of the most useful tools for studying the hyperbolic structures on 3-manifolds is the technique of ideal triangulations, introduced by Thurston in [T] to study the hyperbolic structure of the complement of the figure-eight knot. An ideal triangulation of an open 3-manifold M is a description of M as a disjoint union of copies of the standard tetrahedron with vertices removed (ideal tetrahedron), glued together by a given set of face-pairing maps.

Given an ideal triangulation on an open 3-manifold M with toroidal ends (this is known to be necessary for hyperbolicity) the idea is to construct a hyperbolic structure on M by defining it on each tetrahedron and then by requiring that such structures are compatible with a global one on M. See [T], [NZ], [BP] for more details. A complete finite-volume hyperbolic structure with totally geodesic faces on an oriented ideal tetrahedron is described by a complex number with positive imaginary part, called modulus (see Section 2). The compatibility conditions on the hyperbolic structures of the tetrahedra translate in to algebraic equations on the moduli. These equations depend on the combinatorics of the triangulation and are called *compatibility equations* (see Section 2). When the moduli induce a structure on M, one can ask about completeness of such a structure, and this

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translates in to other algebraic equations on the moduli, called *completeness equations*. Once a complete structure on M is found, one can study small perturbations by taking moduli which are near the complete solution and which satisfy the compatibility equations. By studying the completions of the structures obtained in this way, one can see that in certain cases one obtains a Dehn filling of M for suitable filling-parameters. Conversely, given a Dehn filling N of M with assigned filling-parameters, there exists a system of equations on the moduli, which expresses the fact that the completion of the hyperbolic structure induced on M by the moduli is exactly N. We call such equations *hyperbolic Dehn filling equations*. The completeness equations can be viewed as a particular case of the hyperbolic Dehn filling equations, relative to the empty filling.

As the geometric conditions translate in to algebraic equations, the problem can be studied from an algebraic point of view. The question that naturally arises is:

Question 1 *Does every solution of the algebraic equations have a geometric meaning?*

It is well-known that when all the moduli have a positive imaginary part, the solutions of the compatibility equations have a natural geometric interpretation as a decomposition of M into geodesic ideal tetrahedra. In general it is not clear when an algebraic solution of the equations has a geometric interpretation leading to define a finite-volume hyperbolic structure on M. Moreover, it is still unknown whether every hyperbolic manifolds admits a decomposition into "positive" geodesic ideal tetrahedra.

Epstein and Penner [EP] have shown that a decomposition into convex geodesic ideal polyhedra always exists. By subdividing such a decomposition flat tetrahedra can appear. This translates in to the fact that some moduli become real. Petronio and Weeks [PW] have shown that a partially flat solution of the compatibility and completeness equations leads to a complete hyperbolic structure on M, while a solution of the compatibility equations alone does not in general lead to an (even incomplete) hyperbolic structure (we notice that in [PW] the system of compatibility equations contains additional equations on the angles of the moduli). By perturbing a partially flat decomposition of M, negatively oriented tetrahedra appear. This translates in to the fact that the imaginary part of some moduli become negative. The geometric meaning of a solution of the equations that involves positive, negative, and flat tetrahedra is not clear. Petronio and Porti [PP] have shown that the results known in the case in which all the tetrahedra are positive extend near a partially flat triangulation obtained as a subdivision of an Epstein-Penner decomposition. This leads to the following problem.

Question 2 *What is the geometric meaning of a mixed solution?*

Finally one can ask about uniqueness:

Question 3 *Are the algebraic/geometric solutions unique?*

In this paper we concentrate on the differences between algebraic and geometric solutions of the hyperbolic Dehn filling equations. Our interpretation of mixed solutions will be in terms of developing maps and holonomy. Roughly speaking, we will call *geometric solution* a choice of moduli whose holonomy is well-defined discrete and faithful (see Definition 3.6 for the exact definition).

In this setting, we show that *algebraic is strictly bigger than geometric*,in the sense that any geometric solution is algebraic but non all algebraic solutions are geometric, answering to Question 1 and partially to Question 2. We remark that the difference between algebraic and geometric solutions appears only with the presence of negative tetrahedra. Then we show that the geometric solutions are unique while the algebraic ones are not, answering to Question 3.

The paper is structured as follows: In Section 2 we define the notions of developing map and holonomy for a triangulation with moduli. In Section 3 we describe the system of hyperbolic Dehn filling equations, we give the definition of geometric solution and we prove the following fact:

Theorem 3.8 *Each geometric solution of the hyperbolic Dehn filling equations is also an algebraic one.*

In Section 4 we prove the uniqueness of the geometric solutions:

Theorem 4.3 *There exists at most one geometric solution of the hyperbolic Dehn filling equations.*

In Section 5 we do explicit calculations for some interesting examples:

- We study two one-cusped manifolds, namely two bundles over S^1 called LR^3 and L^2R^3 with as a fiber a punctured torus. These manifolds admit non-unique algebraic solutions and a (unique) geometric one.

- We study a manifold with non-trivial JSJ decomposition, obtained by gluing a Seifert manifold to the complement of the figure-eight knot. This manifold is not hyperbolic but it admits a partially flat solution of the compatibility and completeness equations.

The manifolds LR^3 and L^2R^3 are interesting because on one hand they show that the algebraic solutions are not unique, and on the other hand they provides examples of algebraic solutions which are not geometric (Proposition 5.1). We notice that these "bad" solutions do not involve flat tetrahedra and has a good behavior on the boundary. Namely, the boundary torus inherits an intrinsic Euclidean structure (up to scaling). This fact is surprising because the geometry of a finite-volume hyperbolic 3-manifold is strictly related to the geometry of its boundary. In fact, the equations on the moduli have an interpretation as conditions on the geometry of the boundary. More precisely, any ideal triangulation of M induces a triangulation of the boundary tori, by considering the manifold with boundary obtained by chopping off an open regular neighborhood of the ideal vertices. A modulus for the hyperbolic structure of an ideal tetrahedron determines a modulus for the similarity structure of the triangles obtained as horospherical sections near the vertices. So an ideal triangulation with moduli of M induces a triangulation with moduli of the boundary tori. The compatibility equations express the fact that the moduli for the triangles lead to similarity structures on the tori. The completeness equations express the fact that the structures of the boundary tori are Euclidean. Moreover, when the imaginary part of the moduli is not negative, the control of the geometry of the boundary implies a control of the one of the whole M. For example, in order to have a complete finite-volume hyperbolic structure on M, it suffices to check that the boundary tori have Euclidean structures.

In [F1] it is shown that any algebraic solution of the compatibility and completeness equations for the similarity structure of a triangulated torus leads to a Euclidean structure, even if there are negative triangles, provided that the algebraic sum of the areas of the triangles is not zero. So the example of LR^3 shows that the Euclidean situation in dimension 2 and the hyperbolic one in dimension 3 become quite different when we allow the moduli to have negative imaginary part.

The manifold with non-trivial JSJ decomposition that we study in the last example is a manifold that admits a positive, partially flat triangulation of the compatibility and completeness equations. Such a solution cannot be geometric as the manifold is not hyperbolic. This seems to contradicts [PW]. Actually there is no contradiction because in our example the conditions on the angles are not satisfied. This example shows that such conditions play a central role for a solution to be geometric.

2 Ideal triangulations with moduli, developing maps and holonomy

Let M be the interior of a compact 3-manifold $\overline{M}$ with boundary and let $\widehat{M}$ be the space obtained from $\overline{M}$ by collapsing each component of its boundary to a point. The space $\widehat{M}$ is homeomorphic to the space obtained from $\overline{M}$ by gluing to each boundary component C the cone over C. In the sequel we will often identify M with its image under the projection $\overline{M} \to \widehat{M}$.

Let Δ be the standard 3-simplex and let Δ^* be the standard ideal 3-simplex, i.e. Δ with vertices removed. We define now what we mean by ideal triangulation. Roughly speaking, an ideal triangulation of M is a presentation of M as a union of ideal tetrahedra.

Definition 2.1 *Let F_1 and F_2 be two-dimensional faces of two tetrahedra (possibly the same tetrahedron). A* face-pairing rule *$r : F_1 \dashrightarrow F_2$ is a bijective correspondence r from the vertices of F_1 to those of F_2. A* realization *of a face-pairing rule is a homeomorphism that extends the rule.*

Definition 2.2 *Let $\{\Delta_i,\ i \in I\}$ be a finite set of copies of Δ and let $\{r_j : F_{j1} \dashrightarrow F_{j2},\ j \in J\}$ be a set of face-pairing rules between two-dimensional faces of the Δ_i's. We say that $\tau = (\{\Delta_i\}, \{r_j\})$ is an* ideal triangulation *of M if there exists a set $\{f_j : F_{j1} \to F_{j2},\ j \in J\}$ of simplicial maps such that each f_j is an extension of the rule r_j and such that there exists a homeomorphism $\varphi : (\sqcup_i \Delta_i^*)/\{f_j\} \to M$. If M is oriented, we fix an orientation for Δ and we require the r_j's to be orientation-reversing and φ to be orientation-preserving. We say that $\mathcal{R} = (\{f_j\}, \varphi)$ is a* realization *of τ.*

Given a realization of τ, for each $i \in I$ we define φ_{Δ_i} to be the composition of φ with the projection $\Delta_i^* \to \sqcup_i(\Delta_i^*)/\{f_j\}$.

To require φ to be a homeomorphism from $(\sqcup_i \Delta_i^*)/\{f_j\}$ to M is equivalent to require φ to extend to a homeomorphism from $(\sqcup_i \Delta_i)/\{f_j\}$ to $\widehat{M}$ such that it is one-to-one from $(\sqcup_i \Delta_i/\{f_j\}) \setminus (\sqcup_i \Delta_i^*/\{f_j\})$ to $\widehat{M} \setminus M$. Hence the vertices of $\sqcup_i \Delta_i/\{f_j\}$ are in one-to-one correspondence with the connected components of $\partial\overline{M}$.

If U is an open regular neighborhood of the 0-skeleton of a tetrahedron Δ, we call Δ^- the truncated tetrahedron $\Delta \setminus U$ and we denote ∂U by $\partial^-\Delta$. Given an ideal triangulation $\tau = (\{\Delta_i\}, \{r_j\})$ it is possible to truncate each Δ_i in such a way that $(\sqcup_i \Delta_i^-/\{f_j\}, \sqcup_i \partial^-\Delta_i/\{f_j\})$ is homeomorphic to $(\overline{M}, \partial\overline{M})$. In other words, each ideal triangulation induces a triangulation of $\partial\overline{M}$.

We introduce now the notion of modulus of a hyperbolic ideal tetrahedron. Let A be a straight ideal tetrahedron in $\mathbb{H}^3$, i.e. a geodesic tetrahedron in $\overline{\mathbb{H}^3}$ such that $\partial\mathbb{H}^3 \cap A$ is the 0-skeleton of A. Since such a tetrahedron is the convex hull of its vertices, it is completely determined by them.

The hyperbolic ideal tetrahedra that we consider in the sequel can be flat, but not degenerate. This means that a tetrahedron can be completely contained in a 2-plane, but we always require that its vertices are four distinct points.

An orientation of an abstract tetrahedron is an ordering of its vertices up to even permutations. When A is fat (i.e. it is not contained in a 2-plane), the orientations of A as a subset of $\mathbb{H}^3$ are in correspondence with its orientations as an abstract tetrahedron.

Consider now the half-space model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$, let A be an oriented ideal tetrahedron in $\mathbb{H}^3$, and let (v_1, v_2, v_3, v_4) be a positive ordering of the vertices of A. Then the $\mathrm{Isom}^+(\mathbb{H}^3)$-class of A is determined by a complex number z by mapping (v_1, v_2, v_3, v_4) to $(0, 1, \infty, z)$ via an isometry. Since $\mathrm{Isom}^+(\mathbb{H}^3)$ acts transitively on the set of triples of points at infinity, it is always possible to map (v_1, v_2, v_3) to $(0, 1, \infty)$ via an element of $\mathrm{Isom}^+(\mathbb{H}^3)$. Moreover such an isometry is unique and the number z is the cross-ratio $[v_1 : v_2 : v_3 : v_4]$.

Since the vertices of A are four distinct points in $\mathbb{C} \cup \{\infty\}$, it follows that $z \notin \{0, 1, \infty\}$. By slicing A with a horosphere $\mathbb{C} \times \{t\}$, for a sufficiently large t we obtain a triangle with a Euclidean structure. Up to scaling, this structure is exactly that of the triangle in $\mathbb{C}$ with vertices in $\{0, 1, z\}$. It follows that the hyperbolic structure of A is determined by the similarity structure of a horospherical triangle at a vertex of A.

It is easily checked that as the ordering of the vertices varies in the same orientation class, then z varies on the set $\{z, \frac{1}{1-z}, 1 - \frac{1}{z}\}$. This ambiguity can be avoided by fixing a preferred edge e of A and arranging the vertices (v_1, v_2, v_3, v_4) in such a way that e joins v_1 and v_3 (note that $[v_1 : v_2 : v_3 : v_4] = [v_3 : v_4 : v_1 : v_2]$). The number z is called *modulus* of A associated to e. It is easy to see that the same modulus is associated to opposite edges.

Remark 2.3 In the sequel we tacitly assume that an orientation and an edge for each tetrahedron have been fixed.

From now on we assume M to be oriented.

Definition 2.4 *Let τ be an ideal triangulation of M. A choice of moduli $\mathbf{z} = \{z_i,\ i \in I\}$ for τ is a choice of a complex number $z_i \neq 0, 1$ for each tetrahedron Δ_i of τ. We write $(\tau, \mathbf{z})$ to mean an ideal triangulation τ with a choice of moduli $\mathbf{z}$ for τ.*

Definition 2.5 *Let Δ be the standard tetrahedron. We say that a continuous map $f : \Delta \to \overline{\mathbb{H}}^3$ is* straight *if*

- *f maps the vertices of Δ to $\partial\overline{\mathbb{H}}^3$;*
- *for each sub-simplex F of Δ, the image of F is contained in the convex hull of the image of its vertices;*
- *for each sub-simplex F of Δ, the map $f|_F$ is a homeomorphism whenever $f|_{\partial F}$ is a homeomorphism.*

Definition 2.6 *Let z be a a modulus for a tetrahedron Δ and $\xi : \Delta \to \overline{\mathbb{H}^3}$ be a continuous map. We say that ξ is* compatible *with z if it is straight and its image is a geodesic ideal tetrahedron of modulus z.*

Definition 2.7 *Let $\mathcal{R} = (\{f_j\}, \varphi)$ be a realization of an ideal triangulation with moduli $(\tau, \mathbf{z})$ of M. Let $\{g_i : \Delta_i \to \overline{\mathbb{H}^3}\}$ be a set of maps, each compatible with the corresponding z_i. For each $j \in J$ let ψ_j be the unique orientation-preserving isometry which realizes the face-pairing rule r_j between the corresponding faces of the hyperbolic tetrahedra $g_i(\Delta_i)$.*

We say that $\{g_i\}$ is compatible with $\mathcal{R}$ if for each j, called Δ_{i_1} and Δ_{i_2} the tetrahedra glued by the rule r_j, we have that

$$f_j = g_{i_2}^{-1} \circ \psi_j \circ g_{i_1}. \tag{1}$$

Remark 2.8 Each time we have some covering and some object o which can be lifted in some natural way, as usual we call $\widetilde{o}$ a lift of o. When it is clear, we leave to the reader to prove that what we do is independent of the chosen lift. In the following $\widetilde{M}$ will be the universal covering of M and $\widetilde{\tau}$ the lift of τ to $\widetilde{M}$.

Given $(\tau, \mathbf{z})$, the idea is to define a hyperbolic structure on M by taking, for each i, a straight ideal hyperbolic tetrahedron of modulus z_i and then by gluing the straight tetrahedra realizing the rules r_j via isometries. In order to succeed in this construction it is easy to see that a necessary condition is that for each edge e of τ the product of moduli around e is 1. As mentioned above, if z_i is the modulus of Δ_i associated to an edge, then changing edge, the modulus changes in the set $\{z_i, \frac{1}{1-z_i}, 1 - \frac{1}{z_i}\}$. It follows that the condition of the product of the moduli around the edges can be written as a system $\mathcal{C}$ of algebraic equations on the moduli, having the form

$$\pm \prod_i z_i^{\alpha_i}(1 - z_i)^{\beta_i} = 1$$

where $\alpha_i, \beta_i \in \mathbb{Z}$ depend on the triangulation and on the chosen preferred edges of the tetrahedra. These equations are called the *compatibility equations.*

Lemma 2.9 *Let $(\tau, \mathbf{z})$ be an ideal triangulation with moduli on M such that equations $\mathcal{C}$ are satisfied. Then for each realization $\mathcal{R}= (\{f_j\}, \varphi)$ of τ there exists a set of maps $\{g_i : \Delta_i \to \overline{\mathbb{H}}^3\}$ compatible with $\mathcal{R}$.*

Proof. We define the g_i's recursively on the $n-$skeleta of τ. On the 0-skeleton we define the maps simply looking at the compatibility with the moduli. Then a set of maps $\psi_j \in \mathrm{Isom}^+(\mathbb{H}^3)$ as in Definition 2.7 is well-defined.

Let e be an edge of a tetrahedron Δ_{i_0} with vertices e_0 and e_1. We define g_{i_0} on e to be a homeomorphism onto the geodesic between $g_{i_0}(e_0)$ and $g_{i_0}(e_1)$. Now we define the g_i's on the edges glued to e by the maps f_j using the formula (1) of Definition 2.7. Note that since $\mathcal{C}$ holds this is an unambiguous definition. We define the g_i's on the others edges in a similar way.

Once the g_i's are defined on the 1-skeleton there are no problems to use again formula (1) to define them on the 2-skeleton and there are no obstructions to extend such maps to the 3-cells.

□

Remark 2.10 From now on when we speak of an ideal triangulation, we suppose that a realization $\mathcal{R}$ has been fixed and that each set of maps compatible with the moduli is also compatible with $\mathcal{R}$.

Definition 2.11 *Let $(\tau, \mathbf{z})$ be an ideal triangulation with moduli of M. A* developing map *for $(\tau, \mathbf{z})$ is a continuous map $D : \widetilde{M} \to \mathbb{H}^3$ such that, for each i, the map φ_{Δ_i} lifts to a map $\widetilde{\varphi}_{\Delta_i}$ and $D \circ \widetilde{\varphi}_{\Delta_i}$ extends to a map from Δ_i to $\overline{\mathbb{H}^3}$ which is compatible with z_i. Moreover, if $\widetilde{\varphi}^1_{\Delta_i}$ and $\widetilde{\varphi}^2_{\Delta_i}$ are two lifts of the same map, then we require that there exists an element Φ of* $\mathrm{Isom}^+(\mathbb{H}^3)$ *such that $D \circ \widetilde{\varphi}^1_{\Delta_i} = \Phi \circ D \circ \widetilde{\varphi}^2_{\Delta_i}$.*

Let $(\tau, \mathbf{z})$ be a triangulation with moduli of M and suppose $\mathcal{C}$ holds. Then there exists a representation $h : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ which is well-defined up to conjugation, defined as follows:

Let Δ_{i_1} be a base-tetrahedron. A path of tetrahedra is a sequence $\gamma = (\Delta_{i_1}, r_{j_1}, \Delta_{i_2}, r_{j_2}, \cdots, \Delta_{i_s}, r_{j_s}, \Delta_{i_{s+1}})$ such that the Δ_{i_k}'s are tetrahedra of τ and r_{j_k} is a face pairing rule of τ from a face of Δ_{i_k} to one of $\Delta_{i_{k+1}}$. A loop is a path in which $i_{s+1} = i_1$. For each i let $g_i : \Delta_i \to \overline{\mathbb{H}^3}$

be a map compatible with z_i and for each j let ψ_j the only orientation-preserving isometry which realizes the rule r_j between the corresponding faces of the geodesic tetrahedra $g_i(\Delta_i)$. We define h first on the set of paths of tetrahedra by setting

$$h(\gamma) = \psi_{j_1} \circ \cdots \circ \psi_{j_s}.$$

Let P be the space of loops of tetrahedra. It is a standard fact that P project to $\pi_1(M, x_0)$, where x_0 is any point of the image of $\varphi_{\Delta_{i_1}}$. The fact that $\mathcal{C}$ holds imply that $h(\gamma) = h(\gamma')$ if $[\gamma] = [\gamma']$ in $\pi_1(M, x_0)$. So h induces a map $h : \pi_1(M, x_0) \to \mathrm{Isom}^+(\mathbb{H}^3)$. It is easily checked that h is a representation and that its conjugacy class does not depend on the choices of the maps g_i and of the base-tetrahedron.

The representation h is called the *holonomy* of $\mathbf{z}$.

Proposition 2.12 *Let $(\tau, \mathbf{z})$ be a triangulation with moduli of M such that $\mathbf{z}$ is a solution of equations $\mathcal{C}$. Then there exist a developing map D and a representative h of the holonomy such that D is $\pi_1(M)$-equivariant, where $\pi_1(M)$ acts on $\widetilde{M}$ by deck transformations and on $\mathbb{H}^3$ via h.*

Proof. Let $\widetilde{\tau}$ be the triangulation of $\widetilde{M}$ and let Δ_{i_1} be a base tetrahedron of $\widetilde{\tau}$. We do the same construction used to define the holonomy. As above we fix maps g_i (the same g_i for all lifts of the same tetrahedron of τ) and the isometries ψ_j, and as above we define a map $\widetilde{h}$ from the space of paths of tetrahedra of $\widetilde{\tau}$ to $\mathrm{Isom}^+(\mathbb{H}^3)$.

Now let Δ_k be a tetrahedron of $\widetilde{\tau}$ and let γ be a path from Δ_{i_1} to Δ_k. We define $D_k : \Delta_k^* \subset \Delta_k \to \mathbb{H}^3$ by $D_k = \widetilde{h}(\gamma) \circ g_k$. Since $\widetilde{M}$ is simply connected, the definition of D_k does not depend on the choice of γ. Moreover, since the g_i's are compatible with the moduli, the union of the D_k's projects to a developing map $D : \widetilde{M} \to \mathbb{H}^3$.

If we fix a base point in $\widetilde{M}$ in the base-tetrahedron, then a representative h of the holonomy is fixed; and the $\pi_1(M)$-equivariance of D follows by construction.

□

Remark 2.13 It is easy to see that if a developing map exists, then $\mathcal{C}$ holds and so the holonomy is defined. Moreover, every developing map can be obtained via the above construction. It follows that for each developing map D the choices of a base point and its lift determine a representative h of the holonomy such that D is $\pi_1(M)$-equivariant.

Definition 2.14 *Let $(\tau, \mathbf{z})$ be a triangulation with moduli of M such that $\mathbf{z}$ is a solution of equations $\mathcal{C}$. Let N be an oriented, complete hyperbolic 3-manifold (hence $\widetilde{N} = \mathbb{H}^3$). We call a map $f : M \to N$* hyperbolic *if its lift $\widetilde{f} : \widetilde{M} \to \mathbb{H}^3$ is a developing map for $(\tau, \mathbf{z})$.*

Remark 2.15 Let $M, \tau, \mathbf{z}, N, f$ be as in Definition 2.14. Let $V(\Delta_i)$ denote the 0-skeleton of $\Delta_i \in \tau$. The maps φ_{Δ_i}, and so also $f \circ \varphi_{\Delta_i}$, are defined only on Δ_i^*. Nevertheless, since $\widetilde{f}$ is a developing map, then the maps $\widetilde{f} \circ \widetilde{\varphi}_{\Delta_i}$ extend to the whole Δ_i. Thus $(\widetilde{f} \circ \widetilde{\varphi}_{\Delta_i})(V(\Delta_i))$ is well-defined.

Definition 2.16 *Let $M, \tau, \mathbf{z}, N, f$ be as in Definition 2.14. Let γ be an oriented geodesic in N. Let v be a vertex of τ. We say that f* spirals around *γ near v if, in a suitable half-space model of $\mathbb{H}^3$ in which $\widetilde{\gamma}$ is the oriented line $(0, \infty)$, for each tetrahedron Δ_i having v as a vertex there exists a lift $\widetilde{f} \circ \widetilde{\varphi}_{\Delta_i}$ which maps v to ∞.*

Proposition 2.17 *Let $M, \tau, \mathbf{z}, N, f$ be as in definition 2.14. Let Γ be the subgroup of Isom$(\mathbb{H}^3)$ such that $N = \mathbb{H}^3/\Gamma$. Then the image of a holonomy representation relative to $\widetilde{f}$ consists of elements of Γ. Moreover, called h_N the holonomy of N, i.e. the isomorphism $h_N : \pi_1(N) \to \Gamma$, we have that $h = h_N \circ f_*$.*

Proof. The group $\pi_1(M)$ acts on $\widetilde{M}$ via deck transformations and $\pi_1(N)$ acts on $\widetilde{N} = \mathbb{H}^3$ via h_N. It is possible to chose the base-points in such a way that for all $\alpha \in \pi_1(M)$ and for all $x \in \widetilde{M}$ we have $\widetilde{f}(\alpha(x)) = h_N f_*(\alpha)(\widetilde{f}(x))$. Since $\widetilde{f}$ is a developing map, then $\widetilde{f}(\alpha(x)) = h(\alpha)(\widetilde{f}(x))$. It follows that $h_N f_*(\alpha)$ and $h(\alpha)$ coincide on the image of $\widetilde{f}$. Since the dimension of $\mathrm{Im}(\widetilde{f})$ is at least two and since both $h(\alpha)$ and $f_*(\alpha)$ are orientation-preserving isometries, then they coincide on the whole $\mathbb{H}^3$.

□

3 Hyperbolic Dehn filling equations

First of all, we recall the definition of Dehn filling of a manifold.

Definition 3.1 *Let M be the interior of a compact oriented 3-manifold $\overline{M}$ such that $\partial\overline{M}$ is a union of tori, $\partial\overline{M} = \sqcup_n T_n$. For each T_n let (μ_n, λ_n) be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n)\}$ where (p_n, q_n) is either a pair of coprime integers or the symbol ∞. For each n such that $(p_n, q_n) \neq \infty$, let L_n be an oriented solid torus, m_n be a meridian of $T_n' = \partial L_n$, l_n be a* *loop in T_n such that $[l_n] = p_n\mu_n + q_n\lambda_n$ and $\varphi_n : T_n \to T'_n$ be an orientation reversing homeomorphism such that $\varphi_n(l_n) = m_i$. The Dehn filling of M with parameters (p, q) is the manifold*

$$M_{(p,q)} = \mathrm{int}\Big(\overline{M} \sqcup \{L_n\}/\{\varphi_n\}\Big)$$

The tori L_n are called filling tori.

Remark 3.2 The resulting manifold $M_{(p,q)}$ actually depends only on the coefficients (p, q) and not on the maps φ_n's.

Remark 3.3 Not all the boundary tori are filled in $M_{(p,q)}$. Namely, a torus T_n is filled if and only if $(p_n, q_n) \neq \infty$. If $(p_n, q_n) = \infty$ for all n , then $M_{(p,q)} = M$.

Suppose that we have an ideal triangulations of a manifold M with a choice of moduli $\mathbf{z}$ that satisfy $\mathcal{C}$. We recall that if each z_n has positive imaginary part then the moduli define an (incomplete) hyperbolic structure on M. In this section we introduce some equations on the moduli, which we call *hyperbolic Dehn filling equations*. When the moduli have positive imaginary part, such equations imply that the completion of the hyperbolic structure defined by the moduli on M is a fixed Dehn filling of M. The principal condition expressed by the equations is that if m is a loop in a boundary torus killed in homology by the filling, then $h(m) = 1$.

Our equations can be written down even without restrictions on the imaginary part of the moduli. In this case, in general, there is not an obvious geometric interpretation of the solutions of the equations. For this reason we distinguish between algebraic and geometric solutions of the hyperbolic Dehn filling equations.

From now on, let M be the interior of an oriented compact 3-manifold $\overline{M}$ such that $\partial\overline{M}$ is the disjoint union of k tori, let τ be an ideal triangulation of M and let $\mathbf{z}$ be a choice of moduli for τ satisfying the compatibility equations $\mathcal{C}$.

Let $T \subset \partial\overline{M}$ be a boundary torus. We push T a little bit inside M and we consider the image of the natural map $\pi_1(T) \to \pi_1(M)$. We consider the half-space model $\mathbb{C}\times\mathbb{R}^+$ of $\mathbb{H}^3$ and a developing map D such that the vertex relative to T is lifted to a vertex mapped to ∞ by D. Then there exists a choice of the base-points such that the image of the holonomy $h(\pi_1(T))$ consists of maps which fix ∞. It follows that by considering the restriction to $\partial\mathbb{H}^3 \equiv \mathbb{CP}^1$ of the maps in $h(\pi_1(T))$, we obtain a representation h_T of $\pi_1(T)$ in the automorphism of $\mathbb{C}$. Moreover, since the restriction to $\partial\mathbb{H}^3$ of a positive isometry is a Möbius transformation, then h_T actually is a representation $h_T : \pi_1(T) \to \mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(\mathbb{C})$. Since h is well-defined up to conjugation, then the dilation component of h_T is well-defined, and it is a representation $\rho_T : \pi_1(T) \to \mathbb{C}^*$.

Since $\pi_1(T)$ is Abelian, it follows that $h_T(\pi_1(T))$ consists of maps which commute with each other. Then it is easy to see that either they are all translations, or they have a common fixed point. In the former case we have $\rho_T \equiv 1$. In the latter case, up to conjugation, we can suppose that the fixed point is 0. Thus we get $h_T = \rho_T$, in the sense that for all $\alpha \in \pi_1(T)$ and $\zeta \in \mathbb{C}$, $h_T(\alpha)(\zeta) = \rho_T(\alpha) \cdot \zeta$.

Remark 3.4 In the following, if there are no ambiguities, by writing $\rho_T \equiv 1$ we mean that $h_T(\pi_1(T))$ consists of translations and by $h_T = \rho_T$ we mean that $h_T(\pi_1(T))$ consists of maps which fix 0. We notice that h, h_T and ρ_T depend on $\mathbf{z}$. When we need to emphasize this, we write $\rho(\mathbf{z})$ and so on.

To write the equations, we need to work with $\log(\rho_T)$. In the following definition we fix a suitable determination of the logarithm of ρ_T. Formally, a boundary torus T of $\overline{M}$ is not contained in M, so a regular neighborhood U of T is not a subset of M. If there are no ambiguities, we do not distinguish between U and $U \cap M$.

Definition 3.5 *Let $M, \tau, \mathbf{z}$ be as above and let D be a developing map. Let T be a boundary component of $\overline{M}$ and $\widetilde{T}$ be a lift of T. Consider the model $\mathbb{C} \times \mathbb{R}^+$ of $\mathbb{H}^3$ such that the vertex relative to $\widetilde{T}$ is mapped to ∞. Suppose that $h_T = \rho_T$ and suppose that the following condition holds:*

> *There exists regular neighborhood $\widetilde{U} \subset \widetilde{M}$ of $\widetilde{T}$ such that the developed image of $\widetilde{U}$ does not intersect the line $(0, \infty)$.*

Then we choose a determination of $\log(\rho_T)$ as follows: let H be the universal covering of $\mathbb{H}^3 \setminus (0, \infty)$ made by using the covering $\exp : \mathbb{C} \to \mathbb{C}^*$. Let x_0 and $\widetilde{x}_0$ be base-points in T and $\widetilde{T}$. Let $\gamma : [0, 1] \to T$ be a loop at x_0 and $\widetilde{\gamma}$ be its lift starting from $\widetilde{x}_0$. After pushing a little T inside M, let $\widetilde{\alpha} : [0, 1] \to \mathbb{C}^*$ be the horizontal component of $D \circ \widetilde{\gamma}$. As $D \circ \widetilde{\gamma}$ lifts to H, the path $\widetilde{\alpha}$ lifts to a path $\overline{\alpha} : [0, 1] \to \mathbb{C}$. Since $h_T = \rho_T$, then $\widetilde{\alpha}(1) = \rho_T([\gamma]) \cdot \widetilde{\alpha}(0)$, and then $\overline{\alpha}(1) = \log(\rho_T([\gamma])) + \overline{\alpha}(0)$.

The points $\overline{\alpha}(0)$ and $\overline{\alpha}(1)$ depend only on the homotopy class of γ and on the choice of the base-points. If we change the base-points, then the determination of $\log(\rho_T([\gamma]))$ changes by a conjugation by translations and so it is well-defined.

Let us now fix a basis (μ, λ) for $H_1(T, \mathbb{Z})$ and let (a, b) be a pair of coprime integers. Consider the Dehn filling of M with parameters (a, b), i.e.

the filling in which an oriented loop homotopic to $a\mu + b\lambda$ is mapped to the meridian m of the solid torus. So the coefficient (a, b) induces an orientation of m. Since the gluing map inverts the orientations of the boundary tori, then the core γ of the filling tours is canonically oriented by requiring that m turns around γ by following the right-hand rule in the oriented solid tours.

We are now ready to give the hyperbolic Dehn filling equations.

Definition 3.6 *Let M be the interior of an oriented compact 3-manifold $\overline{M}$ such that $\partial\overline{M}$ is the disjoint union of k tori, let τ be an ideal triangulation of M and let $\mathbf{z}$ be a choice of moduli for τ satisfying the compatibility equations $\mathcal{C}$. For each boundary torus T_n let (μ_n, λ_n) be a basis for $H_1(T_n, \mathbb{Z})$. Let $(p, q) = \{(p_n, q_n),\ n = 1, \dots, k\}$ be such that (p_n, q_n) is either a pair of coprime integers or the symbol ∞. Let $\rho_n(\mathbf{z})$ be the dilation component of the holonomy of T_n when $\mathbf{z}$ varies on the space of solutions of the compatibility equations. We say that $\mathbf{z}$ is an* algebraic *solution of the (p, q)-equations if for each $n = 1, \dots, k$ we have:*

- *If $(p_n, q_n) = \infty$ then $\rho_n(\mathbf{z}) \equiv 1$.*
- *If $(p_n, q_n) \neq \infty$ then $h_{T_n}(\mathbf{z}) = \rho_n(\mathbf{z})$, the condition of Definition 3.5 holds, and*

$$p_n \log(\rho_n(\mathbf{z})[\mu_n]) + q_n \log(\rho_n(\mathbf{z})[\lambda_n]) = 2\pi i.$$

We say that z is a geometric *solution of the (p, q)-equations if, called $N = M_{(p,q)}$ the Dehn filling of M with parameters (p_n, q_n), we have:*

a) N is complete hyperbolic and the cores of the filling tori are disjoint geodesics $\{\gamma_n\}$.

b) There exists a proper map $f : M \to N \setminus \{\gamma_n\} \subset N$ of degree 1, which is hyperbolic w.r.t. $\mathbf{z}$.

c) For each boundary torus T_n with $(p_n, q_n) \neq \infty$, called v_n the vertex correspondent to T_n, f spirals around the relative γ_n near v_n, where γ_n has the orientation induced by the Dehn filling coefficient (p_n, q_n).

Remark 3.7 When all the coefficients (p_n, q_n) are ∞, then the system of the (p, q)-equations is exactly the classical system $\mathcal{M}$ of the so-called *completeness equations*. When the moduli have positive imaginary part the equations $\mathcal{M}$ imply that the hyperbolic structure defined by the moduli on M is complete (of finite volume).

Theorem 3.8 *Let $M, \tau, \{(\mu_n, \lambda_n)\}, (p, q)$ be as in Definition 3.6. Then each geometric solution of the (p, q)-equations is also algebraic.*

Proof. Let $\mathbf{z}$ be a geometric solution of the (p, q)-equation.

Let Γ be the subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ such that $N = \mathbb{H}^3/\Gamma$. So the holonomy of N, as a hyperbolic manifold, is an isomorphism $h_N : \pi_1(N) \to \Gamma$. By Proposition 2.17 the holonomy of M is obtained by composing the homomorphism $f_* : \pi_1(M) \to \pi_1(N)$ with h_N.

Each geodesic γ_n (the core of a filling torus) can be viewed as an element of $\pi_1(N)$. For each n, let $\Gamma_n \subset \Gamma$ be the set of all conjugates of $h_N(\gamma_n)$ and let P be the set of all parabolic elements of Γ. Note that P is exactly the image of all boundary elements of $\pi_1(N)$. Since f is proper, $\widetilde{f}$ maps each vertex of $\widetilde{\tau}$ to a fixed point of an element either of $\cup_n \Gamma_n$ or of P.

Moreover f is surjective on $N \setminus \{\gamma_n\}$ because it has degree 1. Since f spirals around γ_n near v_n, it maps the unfilled cusps of M to the cusps of N. This implies that for $(p_n, q_n) = \infty$ the holonomy of T_n consists of parabolic elements and so $\rho_n \equiv 1$.

If $(p_n, q_n) \neq \infty$ then the image of h_{T_n} is contained in the subgroup of Γ generated by γ_n, so $h_{T_n} = \rho_n$. The fact that $\mathrm{Im}(f) = N \setminus \{\gamma_n\}$ implies the condition of Definition 3.5, and the fact that $N = M_{(p,q)}$ implies that $p_n \log(\rho_n(\mathbf{z})[\mu_n]) + q_n \log(\rho_n(\mathbf{z})[\lambda_n]) = 2\pi i$.

□

For each subgroup Γ of $\mathrm{Isom}(\mathbb{H}^3)$, let $\mathrm{Fix}(\Gamma)$ denote the set of fixed points of all the non-trivial elements of Γ.

Remark 3.9 In the proof of Theorem 3.8 we showed that for each Δ_n of τ, we have $(\widetilde{f} \circ \widetilde{\varphi}_{\Delta_n})(V(\Delta_n)) \subset \mathrm{Fix}(\Gamma)$.

Remark 3.10 It is well-known that if each z_n has positive imaginary part, then an algebraic solution is also a geometric solution.

In Section 5 we give examples of algebraic solutions that are not geometric.

4 Uniqueness

In this section we prove the uniqueness of the geometric solutions. We will need the following version of the rigidity theorem for complete hyperbolic 3-manifolds of finite volume, which can be found in [BCS] and in [F2].

Theorem 4.1 (Strong statement of Mostow's rigidity) *Let M_1 and M_2 be two complete connected hyperbolic 3-manifolds of finite volume. Let*

$f : M_1 \to M_2$ be a continuous proper map such that $\mathrm{vol}(M_1) = |\deg(f)|$ $\mathrm{vol}(M_2)$*. Then f is proper homotopic to a locally isometric covering of degree* $\deg(f)$ *of M_1 onto M_2.*

This theorem in particular implies that a 3-manifold carries at most one hyperbolic structure up to isometries. In the following when we speak about a hyperbolic 3-manifold M, we mean that M is equipped with its unique hyperbolic structure.

In the following we fix an oriented 3-manifold M with k toroidal cusps equipped with an ideal triangulation $\tau = (\{\Delta_i\}, \{r_j\})$ and a realization $\mathcal{R} = (\{f_j\}, \varphi)$. For each boundary torus T_n we fix a basis (μ_n, λ_n) for $H_1(T_n, \mathbb{Z})$. We fix the Dehn filling coefficients $(p, q) = \{(p_n, q_n)\}$ and a geometric solution $\mathbf{z} = \{z_i\}$ of the (p, q)-equation.

Let $N = M_{(p,q)}$ be equipped with its hyperbolic structure $N = \mathbb{H}^3/\Gamma$ with $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ and let $\{\gamma_n\}$ be the set of the geodesic cores of the filling tori. The following proposition shows that the set of the geodesics γ_n is uniquely determined by (p, q).

Proposition 4.2 *Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two finite-volume, complete hyperbolic structure on N such that each γ_n is geodesic for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Then there exists an isometry $\alpha : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ such that $\alpha(\gamma_n) = \gamma_n$ for all n.*

Proof. By rigidity, the identity $Id : (N, \mathfrak{S}_1) \to (N, \mathfrak{S}_2)$ is homotopic to an isometry α. Thus for each n the loop γ_n is homotopic to $\alpha(\gamma_n)$.

By hypothesis γ_n is geodesic for both $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Since α is an isometry it follows that $\alpha(\gamma_n)$ is a geodesics for $\mathfrak{S}_2$. Hence γ_n and $\alpha(\gamma_n)$ are geodesics for $\mathfrak{S}_2$ and they are homotopic, so they must coincide.

□

Theorem 4.3 *In the hypotheses fixed above, the moduli z_i's are uniquely determined by the coefficients (p, q).*

Proof. Let $f : M \to N$ be a hyperbolic map as in Definition 3.6. By definition of hyperbolic map, the lift $\widetilde{f}$ is a developing map. By Proposition 2.17 the holonomy of $\mathbf{z}$ is the composition $h = h_N \circ f_*$, where h_N is the holonomy of the hyperbolic manifold N.

Let v_n be the vertex of τ relative to the n-th cusp of M and let $\widetilde{v}_n$ be one of its lifts. Let $\mathrm{Stab}(\widetilde{v}_n)$ be the stabilizer of $\widetilde{v}_n$ in $\pi_1(M)$ which acts on $\widetilde{M}$ via deck transformations. Then $\widetilde{f}(\widetilde{v}_n)$ is fixed by $h(\mathrm{Stab}(\widetilde{v}_n))$. If T_n is the boundary torus relative to v_n, then $\mathrm{Stab}(\widetilde{v}_n)$ is conjugate to $\pi_1(T_n)$. It follows that $h(\mathrm{Stab}(\widetilde{v}_n))$ is either parabolic (if $(p_n, q_n) = \infty$) or generated by a conjugate of $h_N(\gamma_n)$. In the former case $h(\mathrm{Stab}(\widetilde{v}_n))$ consists of exactly one point so $\widetilde{f}(\widetilde{v}_n)$ is completely determined. In the latter case $h(\mathrm{Stab}(\widetilde{v}_n))$ consists of two points, but the condition (c) of Definition 3.6 allows us to determine $\widetilde{f}(\widetilde{v}_n)$.

Since $\widetilde{f}$ is a developing map, the modulus of a tetrahedron is completely determined by the $\widetilde{f}$-image of its vertices. It follows that the moduli z_i's depend only on $h_N \circ f_*$. We now prove that the moduli do not depend on f.

Let g be another hyperbolic map as in Definition 3.6. Let us call h_f and h_g the holonomy relative to f and g respectively. From the properties of the holonomy (see Section 2) it follows that there exists an element $\varphi \in \mathrm{PSL}(2, \mathbb{C})$ such that $h_g = \varphi h_f \varphi^{-1}$. Thus if a point p is fixed by $h_f(\mathrm{Stab}(\widetilde{v}_n))$, then the point $\varphi(p)$ is fixed by $\varphi h_f(\mathrm{Stab}(\widetilde{v}_n))\varphi^{-1} = h_g(\mathrm{Stab}(\widetilde{v}_n))$. Using again condition (c) of Definition 3.6, we see that $\widetilde{g}(v) = \varphi\widetilde{f}(v)$ for each vertex v of $\widetilde{\tau}$. Since φ is an isometry, the modulus of a tetrahedron of $\widetilde{\tau}$ depends only on h_N. In other words the moduli z_i's depend only on the hyperbolic structure of N, which is unique by Theorem 4.1 and the next Lemma.

□

Lemma 4.4 *The manifold N has finite volume.*

Proof. Let $\mathrm{vol}(\Delta_i)$ denote the hyperbolic volume of the ideal tetrahedron of modulus z_i, where $\mathrm{vol}(\Delta_i)$ is taken negative if $\mathrm{Im}(z_i) < 0$. Since f is a hyperbolic map, then the volume of its image satisfies $\mathrm{vol}(\mathrm{Im}(f)) \leq \sum |\mathrm{vol}(\Delta_i)| < \infty$. Moreover f is a degree-1 map from $N \setminus \{\gamma_n\}$ to $N \setminus \{\gamma_n\}$, hence $\mathrm{vol}(N) = \mathrm{vol}(N \setminus \{\gamma_n\}) = \mathrm{vol}(\mathrm{Im}(f)) < \infty$.

□

5 Examples

In this section we explicitly compute of the solutions of the compatibility and completeness equations for some particular one-cusped 3-manifold.

To begin we fix some notation. Let L and R be the following matrices of $\mathrm{SL}(2, \mathbb{Z})$:

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Each element A of $\mathrm{SL}(2,\mathbb{Z})$ can be written as a product $A = \prod_{i=1}^{n} A_i^{n_i}$, with $A_i \in \{L, R\}$ and $n_i \in \mathbb{N}$.

Let S be the punctured torus $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$. Then each element $A \in \mathrm{SL}(2,\mathbb{Z})$ induces a homeomorphism φ_A of S. Given $A = \prod A_i^{n_i}$, we call $\prod A_i^{n_i}$ the manifold obtained from $S \times [0,1]$ by gluing $(x,0)$ to $(\varphi_A(x),1)$. For such a manifold, using the algorithm described in [FH], one easily obtains an ideal triangulation with $\sum n_i$ tetrahedra.

We notice that the complement of the figure-eight knot is the manifold LR, and its standard ideal triangulation with two tetrahedra is the one obtained according to [FH].

We use the following notation to labeling simplices. For each vertex v of a tetrahedron X, we call X_v the triangle obtained by chopping off the vertex v from X and X^v the face of X opposite to v. Given a tetrahedron X and two vertices v, w of X, by abuse of notation, we use the label w also for the edge of the triangle X_v corresponding to the face X^w. A modulus for a tetrahedron X is named z_X and we will specify the edge to which it is referred.

5.1 The manifold LR^3

Let M be the manifold LR^3, i.e. the manifold obtained as described above by using the element $LR^3 = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$. Using the algorithm described in [FH], we get the ideal triangulation τ of M with four tetrahedra, labeled A, B, C, D and pictured in Figure 1.

We label the vertices of the tetrahedra as in Figure 1 (we use such labels because they are natural using the algorithm of [FH]). The moduli are referred to the edge $\overline{0\,\frac{1}{1}}$ (note that this edge is common to all the tetrahedra).

The face-pairing rules of τ are, according to the arrows in the picture:

$$\begin{array}{cccc} A^{\frac{0}{1}} \longleftrightarrow B^{\frac{2}{1}} & B^{\frac{1}{0}} \longleftrightarrow C^{\frac{3}{2}} & C^{\frac{2}{1}} \longleftrightarrow D^{\frac{4}{3}} & D^{\frac{3}{2}} \longleftrightarrow A^{\frac{1}{1}} \\ A^{\frac{1}{0}} \longleftrightarrow B^{0} & B^{\frac{1}{1}} \longleftrightarrow C^{0} & C^{\frac{1}{1}} \longleftrightarrow D^{0} & D^{\frac{1}{1}} \longleftrightarrow A^{0} \end{array}$$

The induced triangulation on the boundary torus is described in Figure 2.

We can now write down the compatibility and completeness equations. It is easy to check that $\mathcal{C} + \mathcal{M}$ is equivalent to the system (2).

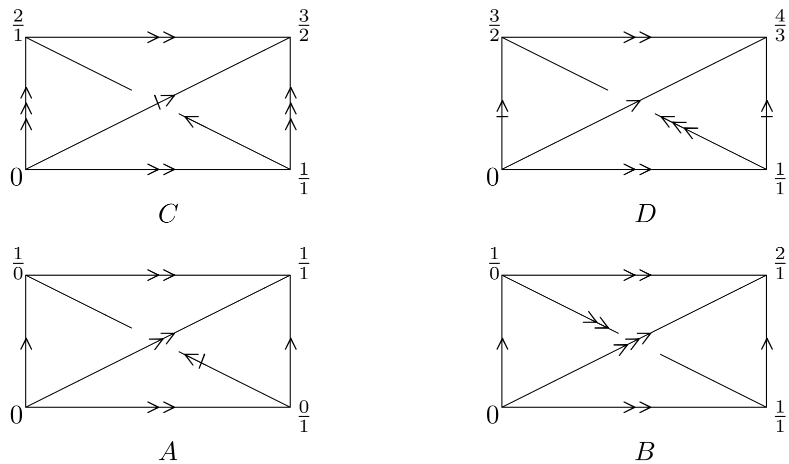

Figure 1: Ideal triangulation of M

$$
\mathcal{C}\begin{cases}
\mathcal{C}_1. & z_A(1-\frac{1}{z_A})^2 z_D^2 z_C^2 z_B^2 \frac{1}{1-z_B} = 1 \\[2ex]
\mathcal{C}_2. & (\frac{1}{1-z_A})^2 \frac{1}{1-z_D}(1-\frac{1}{z_B})^2 \frac{1}{1-z_C} = 1 \\[2ex]
\mathcal{C}_3. & (1-\frac{1}{z_D})^2 \frac{1}{1-z_C} z_A = 1 \\[2ex]
\mathcal{C}_4. & (1-\frac{1}{z_C})^2 \frac{1}{1-z_D}\frac{1}{1-z_B} = 1
\end{cases} \tag{2}
$$

$$
\mathcal{M}. \qquad z_D z_C z_B (1-z_A) = 1
$$

Moreover, the product of the four equations $\mathcal{C}$ is exactly the square of the product of all the moduli, and so it is 1. So if three equations are satisfied, then the remaining one must be. It follows that we can discard one of the $\mathcal{C}$'s.

We discard $\mathcal{C}_2$. Using $\mathcal{M}$ in $\mathcal{C}_1$ and then $\mathcal{C}_1$ in $\mathcal{C}_4$ and $\mathcal{M}$ we obtain the following system of equations, equivalent to $\mathcal{C}+\mathcal{M}$:

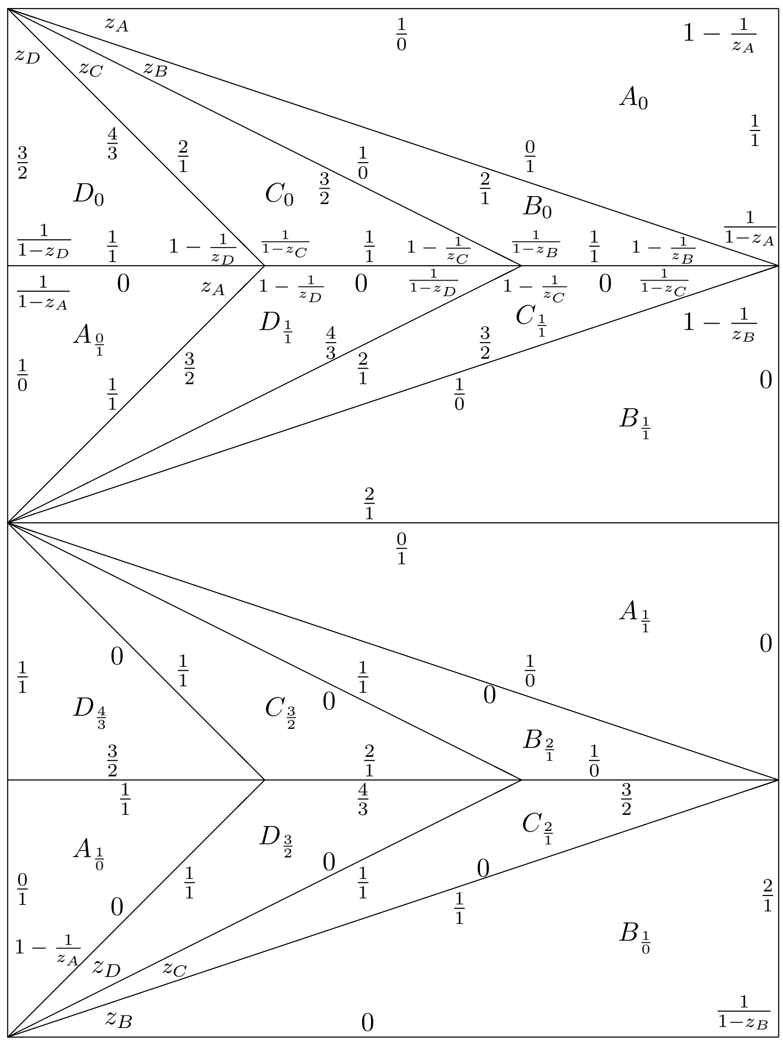

Figure 2: The triangulation of the boundary torus

$$
\begin{cases}
\mathcal{M}. & z_D z_C (1 - z_A)^2 = -z_A \\
\mathcal{C}_1. & z_A (1 - z_B) = 1 \\
\mathcal{C}_3. & (\frac{z_D - 1}{z_D})^2 \frac{z_A}{1 - z_C} = 1 \\
\mathcal{C}_4. & (\frac{z_C - 1}{z_C})^2 \frac{z_A}{1 - z_D} = 1
\end{cases}
\tag{3}
$$

Solving the system, one finds four non-degenerate solutions; one completely positive, giving the hyperbolic structure of M, and one with two negative tetrahedra, and their conjugates (i.e. the same situations but with inverted orientations). The following table contains numerical approximations of the solutions. Note that even if the modulus z_B is different from the modulus z_A, equation $\mathcal{C}_1$ implies that the geometric versions of A and B are isometric.

Solution 1		Volumes
z_A	$0.4275047 + i1.5755666$	0.9158907
z_B	$0.8395957 + i0.5911691$	0.9158907
z_C	$0.7271548 + i0.2284421$	0.5786694
z_D	$0.7271548 + i0.2284421$	0.5786694

Solution 2		Volumes
z_A	$1.0724942 + i0.5921114$	0.8144270
z_B	$0.2854042 + i0.3945194$	0.8144270
z_C	$-1.7271548 - i0.6779619$	-0.2398640
z_D	$-1.7271548 - i0.6779619$	-0.2398640

Note that for Solution 2, the total volume is particularly small, which imply that, even if the identification space is defined, it cannot be a hyperbolic manifold.

Figure 3: The triangles $D_0,\ C_0,\ B_0,\ A_0$ with the moduli of Solution 2.

In Figures 3 and 4, we describe what the triangulation of the boundary torus of M looks like when we choose the moduli of Solution 2. There are two types of triangles, the positive ones, relative to the tetrahedra A and B and the negative ones, relative to C and D. In Figure 3 the four triangles of the top quarter of the triangulation of Figure 2 are pictured. The two parts of Figure 4 are the top and bottom part of the triangulation of Figure 2.

Figure 4: Geometric triangulation of the boundary torus, Solution 2.

Now we look at the algebraic expression of the solutions. A simple calculation shows that the moduli can be expressed by equations (4):

$$\begin{cases} z_C = z_D = w \\ z_A = \dfrac{w^2}{1-w} \\ z_B = 1 - \dfrac{1}{z_A} = \dfrac{w^2+w-1}{w^2} \\ w^4 + 2w^3 - w^2 - 3w + 2 = 0 \end{cases} \tag{4}$$

The four solutions correspond to the four roots $w_1, \overline{w_1}, w_2, \overline{w_2}$ of the polynomial $P(w) = w^4 + 2w^3 - w^2 - 3w + 2$. Note that looking at the reduction (mod 2) of P, one can see that P is irreducible over $\mathbb{Z}$, and then also over $\mathbb{Q}$.

The holonomy representation can be explicitly calculated as a function of w. Let us fix a fundamental domain F for M obtained by taking one copy of each tetrahedron and then performing the gluings:

$$A^{\frac{1}{0}} \longleftrightarrow B^0 \qquad B^{\frac{1}{1}} \longleftrightarrow C^0 \qquad C^{\frac{1}{1}} \longleftrightarrow D^0$$

Consider now the geometric version of F, i.e. a developed image of F. The holonomy is generated by the isometries corresponding to the remaining face-pairing rules. We consider the upper half-space model of $\mathbb{H}^3$ with coordinates in which the points $0, 1, \infty$ of $\partial\mathbb{H}^3$ are the vertices of D labeled respectively $\frac{3}{2}, 0, \frac{4}{3}$. Calculations show that in this model the holonomy is generated by the elements of $\mathrm{PSL}(2, \mathbb{C})$ represented by the matrices:

$$\begin{pmatrix} 1 & \frac{w^2}{w^2+w-1} \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -w \\ \frac{1}{w} & -w-1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -w^2 \\ -1 & w^2+w-1 \end{pmatrix}$$

that respectively correspond to the face-pairing rules

$$A^0 \longrightarrow D^{\frac{1}{1}} \qquad C^{\frac{2}{1}} \longrightarrow D^{\frac{4}{3}} \qquad B^{\frac{2}{1}} \longrightarrow A^{\frac{0}{1}}$$

What is important is that the entries of such matrices are numbers belonging to $\mathbb{Q}(w)$ (and this can be proved even without the explicit calculations).

Proposition 5.1 *The Solution 2 is not geometric.*

Proof. This obviously follows from the uniqueness of geometric solutions, but we also give an alternative proof. Let w_1 (resp. w_2) be the root of P relative to Solution 1 (resp. 2) of $\mathcal{C} + \mathcal{M}$. So w_1 gives the hyperbolic structure of M. Let $h_j : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ be the holonomy representation relative to w_j for $j = 1, 2$. Since P is irreducible and the entries of the holonomy-matrices are in $\mathbb{Q}(w)$, it follows that a relation between elements holds for h_1 if and only if it holds for h_2. Since h_1 is the holonomy of the complete hyperbolic structure of M, then it is faithful, and it follows that also h_2 is faithful.

The image of h_2 cannot be discrete because otherwise $\mathbb{H}^3/h_2$ will be a hyperbolic manifold M' with too small a volume. We notice that by the rigidity of representations (see [F2]) it follows that to obtain an absurd it suffices that $\mathrm{vol}(h_2) \neq \mathrm{vol}(h_1)$. By Proposition 2.17 the holonomy of any geometric solution is discrete, so Solution 2 cannot be geometric.

□

From the fact that h_2 is not discrete and Proposition 2.17 it follows that there is no map, which is hyperbolic w.r.t. Solution 2, from LR^3 to any hyperbolic manifold. Finally, we show that the image of h_2 is dense in $\mathrm{PSL}(2, \mathbb{C})$. We need the following standard fact about $\mathrm{PSL}(2, \mathbb{C})$ (see for example [K] or [G]).

Lemma 5.2 *Let G be a non-elementary subgroup of* $\mathrm{PSL}(2, \mathbb{C})$ *and suppose that G is not discrete. Then the closure of G is either* $\mathrm{PSL}(2, \mathbb{C})$ *or it is conjugate to* $\mathrm{PSL}(2, \mathbb{R})$ *or to a $\mathbb{Z}_2$-extension of* $\mathrm{PSL}(2, \mathbb{R})$.

Proposition 5.3 *The image of the holonomy relative to Solution* 2 *is dense in* $\mathrm{PSL}(2,\mathbb{C})$.

Proof. It is easy to check that the image of h_2 is a non-elementary subgroup of $PSL(2,\mathbb{C})$. Suppose that its closure is conjugate to $PSL(2,\mathbb{R})$ or to a $\mathbb{Z}_2$-extension of $PSL(2,\mathbb{R})$. Then there exist a line in $\mathbb{C}\cup\{\infty\} = \partial\mathbb{H}^3$ which is h_2-invariant. Looking at the parabolic elements of h_2, it is easy to see that such a line does not exist. The thesis follows by Lemma 5.2.

□

The example discussed so far is interesting for several reasons. On one hand it shows that an algebraic solution of $\mathcal{C}+\mathcal{M}$ can be non-geometric. On the other hand it shows that there is no uniqueness of the algebraic solutions.

Moreover this example does not involve flat tetrahedra, so it is quite "regular." Finally, the bad solution of $\mathcal{C}+\mathcal{M}$ of LR^3 has the property that "everything works OK at the boundary," namely, the triangulation with moduli induced on the boundary torus defines on it a Euclidean structure (up to scaling). Roughly speaking, this means that the cusp of LR^3 would like to have a complete hyperbolic structure of finite volume according to the bad solution of $\mathcal{C}+\mathcal{M}$, but the rest of the manifold does not agree.

5.2 The manifold L^2R^3

In this section we do calculations for the manifold L^2R^3.

$$\mathrm{L}^2\mathrm{R}^3 = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = \begin{pmatrix}7 & 2\\3 & 1\end{pmatrix}.$$

Using the algorithm described in [FH], we get the ideal triangulation τ of M with five tetrahedra, labeled A, B, C, D, E and pictured in Figure 5.

We label the vertices of the tetrahedra as in Figure 5. The moduli z_A and z_B are referred to the edge $\overline{0\,\frac{1}{0}}$ while z_C, z_D, z_E to the edge $\overline{0\,\frac{2}{1}}$. The induced triangulation on the boundary torus is that of Figure 5.2. It is easy to see that the system of compatibility and completeness equations $\mathcal{C}+\mathcal{M}$

Figure 5: Ideal triangulation of M

is equivalent to the following one:

$$\begin{cases} z_A z_B = z_C z_D z_E \\ z_C(1 - z_A) = 1 \\ (1 - z_D)^2 z_E^2 = (1 - z_E)^2 z_D^2 \\ (z_A - 1)^2 = z_A^2 (1 - z_B)^2 \\ (1 - \frac{1}{z_E})^2 \frac{1}{1 - z_D} (1 - \frac{1}{z_A}) = 1 \end{cases}$$

Solving this system, we have found eight solutions. The following tables contain numerical approximations of the solutions. Note that even if the modulus z_A is different from the modulus z_C, the second equation implies that the geometric versions of A and C are isometric.

	Solution 1	volume	Solution 2	volume
z_A	$0.75 + i0.6614378$	0.9626730	$0.75 - i0.6614378$	-0.9626730
z_B	$1.25 + i0.6614378$	0.7413987	$1.25 - i0.6614378$	-0.7413987
z_C	$0.5 + i1.3228756$	0.9626730	$0.5 - i1.3228756$	-0.9626730
z_D	1	$*$	1	$*$
z_E	1	$*$	1	$*$

Figure 6: Triangulation of the boundary torus

Solution 3		volume	Solution 4	volume
z_A	1.588633261	0	1.127804076	0
z_B	1.370528159	0	1.113321168	0
z_C	−1.69885025	0	−7.824476637	0
z_D	0.3783840018	0	0.2518509745	0
z_E	−3.387066549	0	−0.6371698130	0

Solution 5		volume	Solution 6	volume
z_A	$0.4950484 + i0.3298695$	0.7399514	$0.4950484 - i0.3298695$	−0.7399514
z_B	$0.6011109 + i0.9321327$	1.0089809	$0.6011109 - i0.9321327$	−1.0089809
z_C	$1.3880304 + i0.9067580$	0.7399514	$1.3880304 - i0.9067580$	−0.7399514
z_D	$0.5022247 + i0.2691269$	0.6433681	$0.5022247 - i0.2691269$	−0.6433681
z_E	$0.6077815 + i0.3441339$	0.7596486	$0.6077815 - i0.3441339$	−0.7596486

Solution 7		volume	Solution 8	volume
z_A	$0.1467328 + i1.2472524$	0.9386051	$0.1467328 - i1.2472524$	−0.9386051
z_B	$1.9069644 + i0.7908171$	0.4782906	$1.9069644 - i0.7908171$	−0.4782906
z_C	$0.3736330 + i0.5461534$	0.9386051	$0.3736330 - i0.5461534$	−0.9386051
z_D	$1.1826577 - i2.5849142$	−0.7155138	$1.1826577 + i2.5849142$	0.7155138
z_E	$-0.5956636 + i1.2429350$	0.7019645	$-0.5956636 - i1.2429350$	−0.7019645

Solutions 1 and 2 contain degenerated tetrahedra. We notice that the non-degenerate moduli of such solutions are exactly those that give the hyperbolic structure on the manifold obtained by removing the tetrahedra D and E and adding the gluing rules:

$$\begin{array}{lll} C^{\frac{1}{0}} \leftrightarrow A^{\frac{1}{1}} & \text{via} & (0, \frac{3}{1}, \frac{2}{1}) \leftrightarrow (0, \frac{1}{0}, \frac{0}{1}) \\ C^{\frac{2}{1}} \leftrightarrow A^{0} & \text{via} & (0, \frac{1}{0}, \frac{3}{1}) \leftrightarrow (\frac{0}{1}, \frac{1}{0}, \frac{1}{1}). \end{array}$$

Now we look at the algebraic expression of Solutions 3-8. Let $P(x) = x^6 + 4x^5 + 3x^4 + 3x^3 - 4x^2 + 2$. A simple calculation shows that the moduli can be expressed in terms of roots of P by the following expressions:

$$\begin{cases} z_A = \dfrac{1}{22}(5w^5 + 19w^4 + 9w^3 + 6w^2 - 8w + 17) \\ z_B = \dfrac{1}{44}(10w^5 + 49w^4 + 62w^3 + 34w^2 - 16w + 34) \\ z_C = \dfrac{1}{11}(-12w^5 - 39w^4 - 4w^3 - 10w^2 + 72w - 32) \\ z_D = \dfrac{1}{22}(-4w^5 - 13w^4 + 6w^3 + 15w^2 + 2w + 4) \\ z_E = w \\ P(w) = 0 \end{cases}$$

Solutions $3, 4, 7, 8$ are not geometric because of uniqueness of geometric solutions. Moreover, as in the case of LR^3, the polynomial P is irreducible, and the argument of Proposition 5.1 works in the present case.

5.3 A manifold with non-trivial JSJ decomposition

The manifold we consider in this section is obtained by gluing to the boundary torus of the complement of the figure-eight knot a Seifert manifold. The resulting manifold, which we call M, clearly is not hyperbolic because it contains an incompressible tours (the old boundary torus).

This example is interesting because the manifold M admits an ideal triangulation with four tetrahedra such that there exists a positive, partially flat solution of $\mathcal{C} + \mathcal{M}$. Obviously such a solution cannot be geometric, as M is not hyperbolic. We remark that in the present example the moduli do not satisfy the equations on the angles. Namely, when a modulus is positive then it is well defined the angle associated to a modulus, in such a way that the sum of angles of any horospherical triangle is always 2π. Then in addiction to equations $\mathcal{C}$ one can require that the sum of the angles around any edge is exactly 2π. Such equations are called $\mathcal{C}^*$. In [PW] is proved that every partially flat solution of $\mathcal{C}^* + \mathcal{M}$ is geometric. Here we show a non-geometric, partially flat solution of $\mathcal{C} + \mathcal{M}$ that does not satisfy $\mathcal{C}^*$. This shows that the equations $\mathcal{C}^*$ play a fundamental role in order to have hyperbolicity.

We describe now our manifold M. We use the techniques of standard spines to construct an ideal triangulation of M, referring to [M] for details on the theory of spines. Let A be the following subset of $\mathbb{C}$:

$$A = \{z \in \mathbb{C} : |z| \leq 4,\ |z-2| > 1, |z+2| > 1\}.$$

A is a disc with two holes. Let $I \subset A$ be the set of the point with zero real part. Let S be the space obtained from $A \times [0, 1]$ by gluing $(z, 0)$ to $(-z, 1)$ and let L be the Möbius strip coming from I. The manifold S is Seifert manifold we want to glue to complement of the figure-eight knot. We call C_e and C_i the external and internal components of ∂S. Note that $\partial L \subset C_e$.

We glue C_e to the boundary torus of the complement of the figure-height knot. To do this, we specify where we glue the boundary of the Möbius strip. We use the classical triangulation of the complement of the figure-eight knot. If one imagines to look from the cusp inside the complement of the figure-eight knot, one gets the following picture:

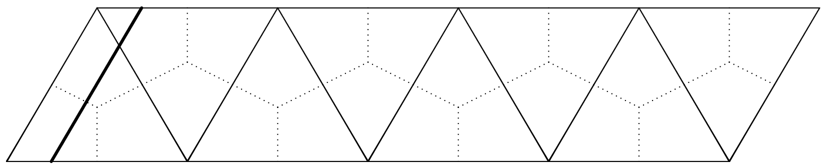

Figure 7: The boundary of the complement of the figure-eight knot

There the eight equilateral triangles of the boundary are pictured. The dashed lines represent the standard spine dual of the ideal triangulation, and the marked line is where we glue ∂L.

Since S retracts to $C_e \cup L$, a spine of M is obtained simply by gluing a Möbius strip to the spine of the complement of the figure-eight knot as in Figure 7. Such a spine has a vertex more than the old one, but is not standard. Performing a *lune* move along the Möbius strip we obtain a standard spine of M with five vertices. As the new spine is standard, its dual is an ideal triangulation with five tetrahedra. Such a triangulation can be simplified with an *MP*-move, replacing the three new tetrahedra with an equivalent pair of tetrahedra. At the end, we get the triangulation of M sketched in Figure 8.

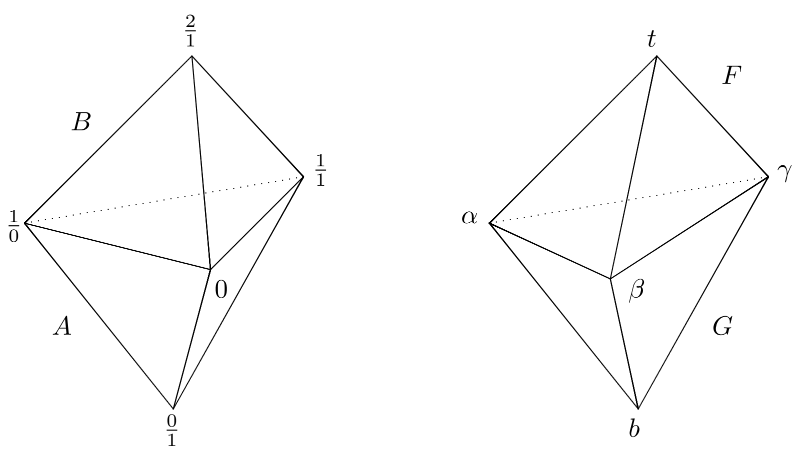

Figure 8: The ideal triangulation of M

The tetrahedra labeled A and B are the old ones (those of the complement of the figure-eight knot). The gluing rules are the following:

$$A^{\frac{0}{1}} \leftrightarrow B^{\frac{2}{1}} : \ (0, \tfrac{1}{0}, \tfrac{1}{1}) \leftrightarrow (0, \tfrac{1}{0}, \tfrac{1}{1}) \qquad A^{\frac{1}{0}} \leftrightarrow B^{0} : \ (0, \tfrac{0}{1}, \tfrac{1}{1}) \leftrightarrow (\tfrac{1}{0}, \tfrac{1}{1}, \tfrac{2}{1})$$
$$A^{\frac{1}{1}} \leftrightarrow B^{\frac{1}{0}} : \ (0, \tfrac{0}{1}, \tfrac{1}{0}) \leftrightarrow (0, \tfrac{1}{1}, \tfrac{2}{1}) \qquad A^{0} \leftrightarrow F^{\gamma} : \ (\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{1}{0}) \leftrightarrow (t, \alpha, \beta)$$
$$B^{\frac{1}{1}} \leftrightarrow G^{\gamma} : \ (0, \tfrac{1}{0}, \tfrac{2}{1}) \leftrightarrow (b, \beta, \alpha) \qquad F^{t} \leftrightarrow G^{b} : \ (\alpha, \beta, \gamma) \leftrightarrow (\alpha, \beta, \gamma)$$
$$F^{\alpha} \leftrightarrow G^{\beta} : \ (\beta, \gamma, t) \leftrightarrow (\gamma, \alpha, b) \qquad F^{\beta} \leftrightarrow G^{\alpha} : \ (\alpha, t, \gamma) \leftrightarrow (\gamma, b, \beta)$$

The moduli z_A and z_B are referred to the edge $\overline{0\frac{1}{1}}$ and z_F, z_G to $\overline{\alpha\beta}$. The triangulation of the boundary torus is that of Figure 9. It is readily checked

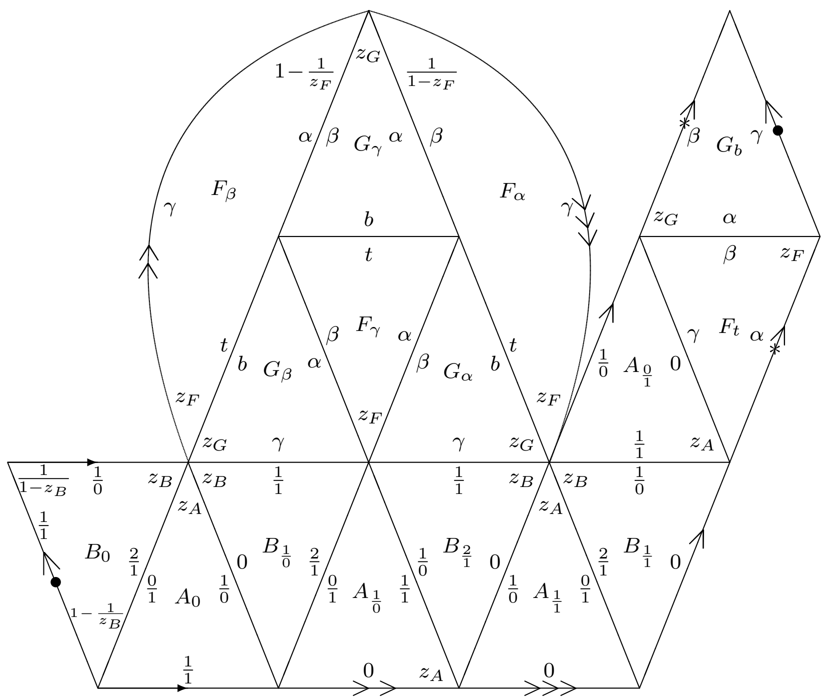

Figure 9: Triangulation with moduli of the boundary torus

that the system of compatibility and completeness equations is equivalent to the following one:

$$\begin{cases} \dfrac{1}{1-z_A} \cdot \dfrac{1}{z_B} \cdot \dfrac{z_F}{z_G} = 1 \\ z_G z_F = 1 \end{cases} \qquad \begin{cases} \dfrac{(1-Z_A)^2}{z_A} \cdot \dfrac{z_B^2}{1-z_B} = 1 \\ z_B(1-z_A) = 1 \end{cases}$$

From this we easily get $z_G = z_F$ and $z_F^2 = 1$. Since we are looking for non-degenerate solutions, we have $z_F = z_G = -1$. Using this we get $z_A = z_B$ and

$$z_A^2 - z_A + 1 = 0$$

and then $z_A = z_B = \frac{1 \pm i\sqrt{3}}{2}$. That is, the ideal tetrahedra F and G are flat but not degenerate, while A and B are regular, exactly as in the complement of the figure-eight knot. We notice that the space obtained by gluing together the geometric versions of the tetrahedra A, B, F, G is not a manifold.

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