A BORDISM APPROACH TO STRING TOPOLOGY

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Abstract. Using intersection theory in the context of Hilbert manifolds and geometric homology we show how to recover the main operations of string topology constructed by M. Chas and D. Sullivan, V. Godin and R. Cohen. We generalize some of these operations to spaces of maps from a sphere to a compact manifold.

1. Introduction

We call n-spheres spaces topological spaces of unbased maps from a n-sphere into a manifold. In fact, we work with Sobolev classes of maps rather than continuous or smooth maps. The aim of this paper is to study the algebraic structure of the homology of these spaces. We begin by giving some motivations for the study of such spaces by focusing on free loop spaces.

Free loop spaces in algebraic topology. The case $n = 1$ i.e. the case of free loop spaces is of particular interest. The study of free loop spaces over a compact oriented manifold plays a central role in algebraic topology, it is a place where different parts of mathematics interact. We review some of the appearances of free loop spaces in algebraic topology (all choices in this list are completely arbitrary):

- Free loop spaces are one of the main tool in order to study closed geodesics on Riemannian manifolds. Let M be Riemannian, compact, connected, simply connected, of dimension greater than one. D. Gromoll and W. Meyer proved that there exist infinitely many (geometrically distinct) periodic geodesics on an arbitrary Riemannian manifold if the Betti numbers of the free loop space of M are unbounded [24]. And using methods of rational homotopy theory M. Vigué and D. Sullivan showed that these rational Betti

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numbers are unbounded if and only if the rational cohomology has at least two generators [44].

- Following the works of D. Burghelea and Z. Fiedorowicz [6] of T. Goodwillie [23] and of J.D.S. Jones [29], we know that the cohomology of free loop spaces is strongly related to Hochschild homology and that the S^1-equivariant cohomology of free loop spaces is also related to cyclic homology and over a field of characteristic zero to Waldhausen algebraic K-theory. Hence rationally, the homotopy of free loop spaces has something to do with the space of automorphisms of manifolds (via Waldhausen's theory). To go further in this direction there are also some relations between the suspension spectrum of the free loop spaces and topological cyclic homology [4].

- Analysis on such spaces has always been a source of constant inspiration, for example one can cite the work of E. Witten on Index of Dirac operators on Free loop spaces [47], this was the beginning of the fascinating theory of elliptic genera and elliptic cohomology [41]. This shed new lights on the periodicity phenomena in stable homotopy and the analysis underlying it [1].

- One can also cite the work of K. T. Chen on a chain complex based on iterated integrals that computes the cohomology of free loop spaces over a manifold [9]. Iterated integrals give a De-Rham theory for Path spaces. In particular, this theory has found some applications in the algebraic interpretation of index theory on free loop spaces by means of cyclic homology [19] and [21]. Let us notice that iterated iterated have also to do with algebraic geometry (see [25] for a survey).

String topology. More recently the discovery by M. Chas and D. Sullivan of a Batalin Vilkovisky structure on the singular homology of these spaces [7] has had a deep impact on the subject and has revealed a part of a very rich algebraic structure [8], [11]. The BV-structure consists of

- A loop product $- \bullet -$ which is commutative and associative, it can be understood as an attempt to do intersection theory of families of closed curves,

- A loop bracket $\{-,-\}$ which controls the commutativity up to homotopy at the chain level of the loop product,

- An operator Δ coming from the action of S^1 on the free loop space (S^1 acts by reparametrization of the loops).

M. Chas and D. Sullivan use in [7] "classical intersection theory of chains in a manifold". This structure has also been defined in a purely homotopical way by R. Cohen and J. Jones using a ring spectrum structure on a Thom spectrum of a virtual bundle over the free loops space [12]. As discovered by S. Voronov [46], it comes in fact from a geometric operadic action of the cacti operad.

Plan of the paper and results. In this paper we adopt a quite different approach to string topology, namely we use a geometric version of singular homology introduced by M. Jakob [26]. And we show how it is possible to define Gysin morphisms, exterior products and intersection type products (such as the loop product of M. Chas and D. Sullivan) in the setting of Hilbert manifolds. Let us point out that three different types of free loop spaces are used in the mathematical literature:

- Spaces of continuous loops ([7] for example),

- Spaces of smooth loops, they are Fréchet manifolds but not Hilbert manifolds ([5] for some details),

- Spaces of Sobolev class of loops [31] or [24], as we need Hilbert manifolds in order to have a nice theory of transversality, we use this last version of free loop spaces. Of course these three spaces are very different from an analytical point of view, but they are homotopicaly equivalent.

In order to perform such intersection theory we recall in section 2 what is known about tranversality in the context of Hilbert manifolds. We also describe the manifold structure of free loop spaces used by W. Klingenberg [31] in order to study closed geodesics on Riemannian manifolds. The cornerstone of all the constructions of the next sections will be the "string pull-back", also used by R. Cohen and J. Jones [12, diagram 1.1]. In section 2.4 we extend these techniques to the n-spheres case and we show how to intersect geometrically families of n-spheres in M.

Section 3 is devoted to the introduction and main properties of geometric homology. This theory is based upon bordism classes of singular manifolds. In this setting families of n-spheres in M, which are families parametrized by smooth oriented compact manifolds, have a clear homological meaning. Of particular interest and crucial importance for applications to sphere topology is the construction of an explicit Gysin morphism for Hilbert manifolds in the context of geometric homology (section 3.3). This construction does not use any Thom spaces and is based on the construction of pull-backs for Hilbert manifolds. Such approach seems completely new in this context.

In section 4 we give the definitions and some properties of the operator Δ, the loop product, the loop bracket, the intersection morphism and the string bracket using geometric homology. We also deal with string topology operations, these operations are parametrized by the topological space of Sullivan's Chord diagrams $\mathcal{CF}^{\mu}_{p,q}(g)$, which is closely related to the combinatorics of Riemann surfaces of genus g, with p-incoming boundary components and q-outgoing. Pushing the work of R. Cohen and V. Godin further we prove:

Theorem: *For $q > 0$ we have morphisms:*

$$\mu_{n,p,q}(g) : H_n(C\mathcal{F}^{\mu}_{p,q}(g)) \to Hom(H_*(\mathcal{L}M^{\times p}), H_{*+\chi(\Sigma).d+n}(\mathcal{L}M^{\times q})).$$

where $\chi(\Sigma) = 2 - 2g - p - q$.

As a corollary we get the structure of Frobenius algebra on $H_{*+d}(\mathcal{L}M)$, build in [11]. And we also recover the operator Δ and the loop product of M. Chas and D. Sullivan [7]. We clearly obtain a lot of new operations acting on the homology of free loop spaces.

Finally, in section 5 we show how results of section 4 can be generalized to n-sphere spaces. In particular, we show that there exists a commutative and associative product on the homology of these spaces. We also focus on 3-sphere spaces.

Remark: In section 4 we use the language of operads and algebras over an operad (in order to state some results in a nice and appropriate framework). For definitions and examples of operads and algebras over an operad we refer to [20], [22], [35], [37], [38] and [46].

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2. Infinite dimensional manifolds

2.1. **Recollections on Hilbert manifolds.** This section is expository, we review the basic facts about Hilbert manifolds, we refer to [36] (see also [35] for a general introduction to infinite dimensional manifolds). Moreover all the manifolds we consider in this paper are hausdorff and second countable (we need these conditions in order to consider partitions of unity).

2.1.1. *Differential calculus.* Let E and F be two topological vector spaces, there is no difficulty to extend the notion of differentiability of a continuous map between E and F. Hence, let $f : E \to F$ be a continuous map we say that f is differentiable at $x \in E$ if for any $v \in E$ the limit:

$$\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists. One can define differentials, C^∞ morphisms, diffeomorphisms and so on.

2.1.2. *Hilbert manifolds.* A topological space X is a manifold modelled on a separable Hilbert space E if there exists an atlas $\{U_i, \phi_i\}_{i\in I}$ such that:
i) each U_i is an open set of X and $X = \bigcup_{i\in I} U_i$,
ii) $\phi_i : U_i \to E$ is an homeomorphism,
iii) $\phi_i \phi_j^{-1}$ is a diffeomorphism whenever $U_i \bigcap U_j$ is not empty.

2.1.3. *Fredholm maps.* A smooth map $f : X \to Y$ between two Hilbert manifolds is a Fredholm map if for each $x \in X$, the linear map

$$df_x : T_x X \longrightarrow T_{f(x)} Y$$

is a Fredholm operator, that is to say:

$$index(f_x) = dim(ker df_x) - dim(coker df_x)$$

is finite. Recall that the index of a Fredholm map

$$index : X \to \mathbb{Z}$$

is a continuous map.

2.1.4. *Orientable morphisms.* A smooth map $f : X \to Y$ between two Hilbert manifolds is an orientable morphism if it is a proper map (the preimage of a compact set is compact) which is also Fredholm and such that the normal bundle $\nu(f)$ is orientable (for convenience we consider the notion of orientability with respect to singular homology but we could have chosen to work in a much more general setting).
Let us remark that a closed embedding is an orientable morphism if and only if $\nu(f)$ is finite dimensional and orientable. A closed Fredholm map is proper by a result of S. Smale [42].

2.1.5. *Partitions of unity.* A very nice feature of Hilbert manifolds with respect to Banach manifolds and other type of infinite dimensional manifolds is the existence of partitions of unity ([36, chapter II,3] for a proof). As a consequence mimicking techniques used in the finite dimensional case, one can prove that any continuous map

$$f : P \to X$$

from a finite dimensional manifold P to an Hilbert manifold X is homotopic to a C^∞ one. And we can also smooth homotopies.

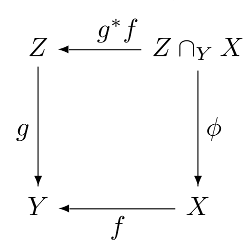
2.2. **Transversality.** We follow the techniques developed by A. Baker and C. Özel in [2] in order to deal with transversality in an infinite dimensional context.

2.2.1. *Transversal maps.* Let $f : X \to Y$ and $g : Z \to Y$ be smooth maps between two Hilbert manifolds. Then they are transverse at $y \in Y$ if

$$df(T_x X) + dg(T_z Z) = T_y Y$$

with $f(x) = g(z) = y$. The maps are transverse if they are tranverse at any point $y \in Imf \cap Img$.

2.2.2. *Pull-backs.* Let us recall the main results about pull-backs of Hilbert manifolds. We consider the following diagram:



where Z is a finite dimensional manifold and $f : X \to Y$ is an admissible orientable map.

Using an infinite dimensional version of the implicit function theorem [36, Chapter I,5], one can prove the following result:

2.2.3. **Proposition.** [2, prop. 1.17] *If the map*

$$f : X \to Y$$

is an orientable morphism and

$$g : Z \to Y$$

is a smooth map transverse to f, then the pull-back map:

$$g^*f : Z \cap_Y X \longrightarrow Z$$

is an orientable morphism.

Moreover the finite dimensional hypothesis on Z enables to prove:

2.2.4. **Theorem.** [2, Th. 2.1, 2.4] *Let*

$$f : X \to Y$$

be an orientable morphism and let

$$g : Z \to Y$$

be a smooth map from a finite dimensional manifold Z. Then g can be deformed by a smooth homotopy until it is transverse to f.

2.3. **Free loop spaces.** If we want to do intersection theory with spaces of closed curves, we need to consider them as smooth manifolds. Following [5, Chapter 3], one can consider the space $C^\infty(S^1, M)$ of all smooth curves as an Inverse Limit Hilbert manifold. But we prefer to enlarge this space and to consider $\mathbf{H}^1(S^1, M)$ the space of $\mathbf{H}^1$ curves. This space has the advantage to be an Hilbert manifold. With this choice we can apply all the techniques described in the sections 2.1 and 2.2.
In fact, we could also consider $\mathbf{H}^n(S^n, M)$ the space of $\mathbf{H}^n$-curves, all these spaces are Hilbert manifolds, and they are also all homotopy equivalent to the ILH manifold $C^\infty(S^1, M)$. And all these manifolds are homotopy equivalent to the space of continuous maps $C^0(S^1, M)$ equipped with the compact open topology.

2.3.1. *Manifold structure.* In order to define an Hilbert manifold structure on free loop spaces we follow W. Klingenberg's approach [31].
Let M be a simply-connected Riemannian manifold of dimension d. We set

$$\mathcal{L}M = \mathbf{H}^1(S^1, M).$$

The manifold $\mathcal{L}M$ is formed by the continuous curves $\gamma : S^1 \to M$ of class $\mathbf{H}^1$, it is modelled on the Hilbert space $\mathcal{L}\mathbb{R}^d = \mathbf{H}^1(S^1, \mathbb{R}^d)$. The space $\mathcal{L}\mathbb{R}^d$ can be viewed as the completion of the space $C_p^\infty(S^1, \mathbb{R}^d)$ of piecewise differentiable curves with respect to the norm $\| - \|_1$. This norm is defined via the scalar product:

$$< \gamma, \gamma' >_1 = \int \gamma(t) \diamond \gamma'(t) dt + \int \delta\gamma(t) \diamond \delta\gamma'(t) dt,$$

where $\diamond$ is the canonical scalar product of $\mathbb{R}^d$. As S^1 is 1-dimensional, we notice that by Sobolev's embedding theorem elements of $\mathcal{L}\mathbb{R}^d$ can be represented by continuous curves.

Let us describe an atlas $\{U_\gamma, \phi_\gamma\}$ of $\mathcal{L}M$. We take $\gamma \in C_p^\infty(S^1, M)$ a piecewise differentiable curve in M (notice that $C_p^\infty(S^1, M) \subset \mathbf{H}^1(S^1, M)$) and consider the pullback:

$$\begin{array}{ccc} \gamma^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow[\gamma]{} & M. \end{array}$$

Now let $T_\gamma^\epsilon \subset \gamma^*TM$ be the set of vectors of norm less than ϵ. The exponential map

$$exp : T_\gamma^\epsilon \longrightarrow M$$

identifies T_γ^ϵ with an open set of M and induces a map:

$$\mathcal{L}T_\gamma^\epsilon \longrightarrow \mathcal{L}M.$$

moreover as γ^*TM is a trivial vector bundle (because M is 1-connected), we fix a trivialization:

$$\varphi : \gamma^*TM \longrightarrow S^1 \times \mathbb{R}^d$$

this gives a chart:

$$\phi_\gamma : \mathcal{L}T_\gamma^\epsilon \longrightarrow \mathcal{L}\mathbb{R}^d.$$

2.3.2. **Remark.** In fact, the manifold structure on $\mathcal{L}M$ does not depend on a choice of a particular Riemannian metric on M.

2.3.3. *The tangent bundle.* Let $TM \to M$ be the tangent bundle of M. The tangent bundle of $\mathcal{L}M$ can be identified with $\mathcal{L}TM$, this is an infinite dimensional vector bundle where each fiber is isomorphic to the Hilbert space $\mathcal{L}\mathbb{R}^d$. Let $\gamma \in \mathcal{L}M$ we have:

$$\mathcal{L}TM_\gamma = \Gamma(\gamma^*TM),$$

where $\Gamma(\gamma^*TM)$ is the space of sections of the pullback of the tangent bundle of M along γ (this is the space of $\mathbf{H}^1$ vector fields along the curve γ). A trivialization φ of γ^*TM induces an isomorphism:

$$\mathcal{L}TM_\gamma \cong \mathcal{L}\mathbb{R}^d.$$

The tangent bundle of $\mathcal{L}M$ has been studied in [13] and [40].

2.3.4. *Riemannian structure.* The manifold $\mathcal{L}M$ has a natural Riemannian metric, the scalar product on $\mathcal{L}TM_\gamma \cong \mathcal{L}\mathbb{R}^d$ comes from $< -, - >_1$.

2.3.5. *The S^1-action.* The circle acts on $\mathcal{L}M$:

$$\Theta : S^1 \times \mathcal{L}M \longrightarrow \mathcal{L}M$$

by reparametrization:

$$\Theta(\theta, \gamma) : t \mapsto \gamma(t + \theta).$$

Of course this action is not free, it is continuous but not differentiable.
Let $\gamma \in \mathcal{L}M$, then the isotropy subgroup $Is(\gamma)$ of γ is S^1 if and only if γ is a constant map, in this case we say that γ is of multiplicity 0. Otherwise it is isomorphic to a finite cyclic group and the multiplicity of the curve is equal to the order of the isotropy subgroup. Let $\mathcal{L}M^{(m)}$ be the space of curves of multiplicity equal to m. This gives an S^1-equivariant partition of $\mathcal{L}M$:

$$\mathcal{L}M = \bigcup_m \mathcal{L}M^{(m)}.$$

The space $\mathcal{L}M^{(0)}$ can be identified with M, and $\mathcal{L}M^{(1)}$ is called the space of prime curves.

2.3.6. *The string pullback.* Let us consider the evaluation map

$$ev_0 : \mathcal{L}M \to M$$

$$\gamma \mapsto \gamma(0),$$

this is a submersion of Hilbert manifolds (this follows immediately from the definition of $\mathcal{L}TM$). As the map $ev_0 \times ev_0$ is transverse to the diagonal map Δ (because $ev_0 \times ev_0$ is a submersion), we can form the *string pull-back* [12, (1.1)]:

$$\begin{array}{ccc} \mathcal{L}M \times \mathcal{L}M & \xleftarrow{\widetilde{\Delta}} & \mathcal{L}M \cap_M \mathcal{L}M \\ \Big\downarrow {\scriptstyle ev_0 \times ev_0} & & \Big\downarrow {\scriptstyle ev} \\ M \times M & \xleftarrow{\Delta} & M, \end{array}$$

by transversality this is a diagram of Hilbert manifolds. We have:

$$\mathcal{L}M \cap_M \mathcal{L}M = \{(\alpha, \beta) \in \mathcal{L}M \times \mathcal{L}M / \alpha(0) = \beta(0)\}.$$

The map

$$\widetilde{\Delta} : \mathcal{L}M \cap_M \mathcal{L}M \to \mathcal{L}M \times \mathcal{L}M$$

is a closed embedding of codimension d.
As the normal bundle $\nu_{\tilde{\Delta}}$ is the pull-back of ν_{Δ} and as this last one is isomorphic to TM , we deduce that $\tilde{\Delta}$ is an orientable morphism.

2.3.7. *Families of closed strings.* A family of closed strings in M is a smooth map

$$f : P \to \mathcal{L}M$$

from a compact orientable manifold P.

The proposition below gives a necessary but not sufficient condition in order to do intersection of families of closed strings.

2.3.8. **Proposition.** *If $P \times Q \stackrel{f\times g}{\to} \mathcal{L}M \times \mathcal{L}M$ is transverse to $\widetilde{\Delta}$ then $ev_0 f$ and $ev_0 g$ are transverse in M.*

Now we suppose that (P, f) and (Q, g) are two orientable compact manifolds of dimensions p and q respectively. Moreover we suppose that they are such that $f \times g$ is transverse to $\widetilde{\Delta}$. We denote by $P * Q$ the pullback:

$$\begin{array}{ccc} P \times Q & \longleftarrow & P * Q \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{L}M \times \mathcal{L}M & \stackrel{\widetilde{\Delta}}{\longleftarrow} & \mathcal{L}M \cap_M \mathcal{L}M. \end{array}$$

Then $P * Q$ is a compact orientable submanifold of $P \times Q$ of dimension $p + q - d$.

2.3.9. *Composing loops.* Let us define the map:

$$\Upsilon : \mathcal{L}M \cap_M \mathcal{L}M \longrightarrow \mathcal{L}M.$$

Let (α, β) be an element of $\mathcal{L}M \cap_M \mathcal{L}M$ then $\Upsilon(\alpha, \beta)$ is the curve defined by:
$\Upsilon(\alpha, \beta)(t) = \alpha(2t)$ if $t \in [0, 1/2]$
$\Upsilon(\alpha, \beta)(t) = \beta(2t - 1)$ if $t \in [1/2, 1]$.
We notice that this map is well defined because we compose piecewise differential curves, hence no "dampening" constructions are needed as in [12, remark about construction (1.2)].
The construction of Υ comes from the co-H-space structure of S^1 i.e. the pinching map:

$$S^1 \longrightarrow S^1 \vee S^1.$$

2.3.10. *Intersection of families of closed strings.* Now consider two families of closed strings (P, f) and (Q, g), by deforming $f \times g$ one can produce a new family of closed strings $(P * Q, \Upsilon\psi)$ in M. We also notice that the image of $\Upsilon\psi$ lies in $\mathcal{L}M^{(0)} \bigcup \mathcal{L}M^{(1)}$.

2.3.11. **Remark.** All we have done with free loop spaces can be performed for manifolds of maps from a space which is a co-H-space and a compact orientable manifold to a compact Riemannian manifold.

2.4. **n-sphere spaces.** Let M be a n-connected d-dimensional compact oriented smooth manifold.

2.4.1. **Definition.** *We call the n-sphere space of M and we denote it by $\mathcal{S}_nM$ the space of $\mathbf{H}^n$-maps from S^n to M.*

2.4.2. **Remark.** By Sobolev's embedding theorem we know that

$$\mathbf{H}^n(S^n, M) \subset C^0(S^n, M).$$

2.4.3. **Theorem.** *$\mathcal{S}_nM$ is an Hilbert manifold.*

Proof As for free loop spaces an atlas of $\mathcal{S}_nM$ can be given by considering bfH^n vector fields along all maps

$$b : S^n \to M.$$

Then using a trivialization of b^*TM we deduce that $\mathcal{S}_nM$ is modelled on the separable Hilbert space $\mathcal{S}_n\mathbb{R}^d$. □

The tangent bundle of $\mathcal{S}_nM$ has the same description as $\mathcal{L}TM$. It can be identified with $\mathcal{S}_nTM$ and we have:

$$\mathcal{S}_nM_b = \Gamma(b^*TM).$$

Moreover, $\mathcal{S}_nM$ is a Riemannian manifold.

2.4.4. *The n-sphere pull-back.* Let fix a base point 0 in S^n, the evaluation map:

$$ev_0 : \mathcal{S}_nM \to M$$

is clearly a submersion of Hilbert manifolds, then we can form the pull-back of Hilbert manifolds:

$$\begin{array}{ccc} \mathcal{S}_nM \times \mathcal{S}_nM & \xleftarrow{\widetilde{\Delta}} & \mathcal{S}_nM \cap_M \mathcal{S}_nM \\ {\scriptstyle ev_0 \times ev_0}\downarrow & & \downarrow{\scriptstyle ev} \\ M \times M & \xleftarrow{\Delta} & M, \end{array}$$

and the map $\tilde{\Delta}$ is an orientable morphism.

2.4.5. *Composing n-spheres.* Thanks to the pinching map:

$$S^n \longrightarrow S^n \vee S^n$$

one can define:

$$\Upsilon : S_n M \cap_M \mathcal{S}_n M \longrightarrow \mathcal{S}_n M.$$

2.4.6. *Intersection of families of n-spheres.* As for families of closed strings, we consider two families of n-spheres in M denoted by (P, f) and (Q, g), by deforming $f \times g$ and taking the pullback $P * Q$ along $\tilde{\Delta}$ one can produce a new family of n-spheres $(P * Q, \Upsilon\psi)$ in M.

3. Geometric homology theories

As R. Thom proved it is not possible in general to represent singular homology classes of a topological space X by singular maps i.e maps:

$$f : P \longrightarrow X$$

from an oriented manifold to X. But, M. Jakob in [26], [27] proves that if we add a cohomological information to the map f (a singular cohomological class of P), then Steenrod's realizability problem with this additional cohomological data has an affirmative answer. In these two papers he develops a geometric version of homology. This geometric version seems to be very nice to deal with Gysin morphisms, intersection products and so on.
All the constructions we give below and also their applications to string topology work out for more general homology theories: bordism, topological K-theory for example. We refer the reader to [26], [27] and [28] for the definitions of these geometric theories.

3.1. **An alternative description of singular homology.**

3.1.1. *Geometric cycles.* Let X be a topological space, a geometric cycle is a triple (P, a, f) where:

$$f : P \longrightarrow X$$

is a continuous map from a compact connected orientable manifold P to X (i.e a singular manifold over X), and $a \in H^*(P, \mathbb{Z})$. If P is of dimension p and $a \in H^m(P, \mathbb{Z})$ then (P, a, f) is a geometric cycle of degree $p - m$. Take the free abelian group generated by all the geometric cycles and impose the following relation:

$$(P, \lambda.a + \mu.b, f) = \lambda.(P, a, f) + \mu.(P, b, f).$$

Thus we get a graded abelian group.

3.1.2. *Relations.* In order to recover singular homology we must impose the two following relations on geometric cycles:

i) (**Bordism relation**) If we have a map $h : W \to X$ where W is an orientable bordism between (P, f) and (Q, g) i.e.

$$\partial W = P \bigcup Q^-.$$

Let $i_1 : P \hookrightarrow W$ and $i_2 : Q \hookrightarrow W$ be the canonical inclusions, then for any $c \in H^*(W, \mathbb{Z})$ we impose:

$$(P, i_1^*(c), f) = (Q, i_2^*(c), g).$$

ii) (**Vector bundle modification**) Let (P, a, f) be a geometric cycle and consider a smooth orientable vector bundle $E \xrightarrow{\pi} M$, take the unit sphere bundle $S(E \oplus 1)$ of the Whitney sum of E with a copy of the trivial bundle over M. The bundle $S(E \oplus 1)$ admits a section σ, by $\sigma_!$ we denote the Gysin morphism in cohomology associated to this section. Then we impose:

$$(P, a, f) = (S(E \oplus 1), \sigma_!(a), f\pi).$$

An equivalence class of geometric cycle is denoted by $[P, a, f]$, let call it a geometric class. And $H'_q(X)$ is the abelian group of geometric classes of degree q.

3.1.3. **Theorem.** [26, Cor. 2.36] *The morphism:*

$$H'_q(X) \longrightarrow H_q(X, \mathbb{Z})$$

$$[P, a, f] \mapsto f_*(a \cap [P])$$

where $[P]$ *is the fundamental class of* P *is an isomorphism of abelian groups.*

3.2. **Cap product and Poincaré duality** [27, 3.2]**.** The cap product between $H^*(X, \mathbb{Z})$ and $H'_*(X)$ is given by the following formula:

$$\cap : H^p(X, \mathbb{Z}) \otimes H'_q(X) \longrightarrow H'_{q-p}(X)$$

$$u \cap [P, a, f] = [P, f^*(u) \cup a, f].$$

Let M be a d-dimensional smooth compact orientable manifold without boundary then the morphism:

$$H^p(M, \mathbb{Z}) \longrightarrow H'_{d-p}(M)$$

$$x \mapsto [M, x, Id_M]$$

is an isomorphism.

3.3. **Gysin morphisms.** ([28] for a finite dimensional version) We want to consider Gysin morphisms in the context of infinite dimensional manifolds. Let us recall two possible definitions for Gysin morphisms in the finite dimensional context. The following one is only relevant to the final dimensional case. Let us take a morphism:

$$f : M^m \longrightarrow N^n$$

of Poincaré duality spaces. Then we define:

$$f_! : H_*(M^m) \stackrel{D}{\to} H^{m-*}(M^m) \stackrel{f}{\to} H^{m-*}(N^n) \stackrel{D^{-1}}{\to} H_{*+n-m}(M^m),$$

where D is the Poncaré duality isomorphism.
For the second construction, if f is an embedding of smooth oriented manifold then one can apply the Pontryagin-Thom collapse c to the Thom space of the normal bundle of f and then apply the Thom isomorphism th:

$$f_! : H_*(M^m) \stackrel{c}{\to} H_*(Th(\nu(f))) \stackrel{th}{\to} H_{*+n-m}(M^m)$$

In the infinite dimensional context, we can not use Poincaré duality, and to be more explicit as possible we do not want to use the Pontryagin-Thom collapse and the Thom isomorphism. We prefer to use a very geometrical interpretation of the Gysin morphism which is to take pull backs of cycles along the map f.

So, we take $i : X \to Y$ an orientable morphism of Hilbert manifolds and we suppose that $\nu(i)$ is d-dimensional. Let us define:

$$i^! : H'_p(Y) \longrightarrow H'_{p-d}(X).$$

Let $[P, a, f]$ be a geometric class in $H'_p(Y)$, we can choose a representing cycle (P, a, f). If f is not smooth, we know that it is homotopic to a smooth map by the existence of partitions of unity on Y, moreover we can choose it transverse to i, by the bordism relation all these cycles represent the same class. Now we can form the pull-back:

$$\begin{array}{ccc} P & \xleftarrow{f^*i} & P \cap_Y X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y & \xleftarrow[i]{} & X \end{array}$$

we set:

$$i^!([P, a, f]) = (-1)^{d.|a|}[P \cap_Y X, (f^*i)^*(a), \phi].$$

The sign is taken from [28, 3.2c)], the Gysin morphism can be viewed as a product for bivariant theories [16].

3.4. The cross product [27, 3.1]**.** The cross product is given by the pairing:

$$\times : H'_q(X) \otimes H'_p(Y) \longrightarrow H'_{p+q}(X \times Y)$$

$$[P, a, f] \times [Q, b, g] = (-1)^{dim(P).|b|}[P \times Q, a \times b, f \times g].$$

The sign makes the cross product commutative. let

$$\tau : X \times Y \to Y \times X$$

be the interchanging morphism then:

$$\tau_*(\alpha \times \beta) = (-1)^{|\alpha||\beta|}\beta \times \alpha.$$

3.5. The intersection product([28, sect.3]**).** Let us return to the finite dimensional case and consider M an orientable compact d-dimensional manifold. As for Gysin morphisms in order to be very explicit we avoid the classical constructions of the intersection product that use either Poincaré duality or the Thom isomorphism.
Let $[P, x, f] \in H'_{n_1}(M)$ and $[Q, y, g] \in H'_{n_2}(M)$, we suppose that f and g are transverse in M, then we form the pull back:

$$\begin{array}{ccc} P \times Q & \xleftarrow{\ j\ } & P \cap_M Q \\ {\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle \phi} \\ M \times M & \xleftarrow[\Delta]{} & M \end{array}$$

and define the pairing:

$$- \bullet - : H'_{n_1}(M) \otimes H'_{n_2}(M) \xrightarrow{\times} H'_{n_1+n_2}(M \times M) \xrightarrow{\Delta^!} H'_{n_1+n_2-d}(M).$$

Hence, we set:

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+dim(P).|b|}[P \cap_M Q, j^*(a \times b), \phi].$$

Let $l : P \cap_M Q \to P$ and $r : P \cap_M Q \to Q$ be the canonical maps, then we also have:

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+dim(P).|b|}[P \cap_M Q, l^*(a) \cup r^*(b), \phi].$$

With these signs conventions the intersection product $\bullet$ makes $H'_{*+d}(M)$ into a graded commutative algebra:

$$[P \cap_M Q, l^*(a) \cup r^*(b), \phi] = (-1)^{(d-dim(P)-|a|)(d-dim(Q)-|b|)} [Q \cap_M P, l^*(b) \cup r^*(a), \phi].$$

4. String topology

In this section, using the theory of geometric cycles we show how to recover the BV-structure on

$$\mathbb{H}_*(\mathcal{L}M) := H_{*+d}(\mathcal{L}M, \mathbb{Z})$$

introduced in [7] and studied from a homotopical point of view in [12]. We also define the intersection morphism, the string bracket of [7] and string topology operations (we extend the Frobenius structure given in [11] to a homological action of the space of Sullivan's chord diagrams).

4.1. **The operator Δ.** First we define the Δ-operator on $\mathbb{H}_*(\mathcal{L}M)$. Let us consider a geometric cycle $[P, a, f] \in H'_{n+d}(\mathcal{L}M)$, we have a map:

$$\Theta_f : S^1 \times P \stackrel{Id \times f}{\to} S^1 \times \mathcal{L}M \stackrel{\Theta}{\to} \mathcal{L}M.$$

4.1.1. **Definition.** *There is an operator*

$$\Delta : H'_{n+d}(\mathcal{L}M) \to H'_{n+d+1}(\mathcal{L}M)$$

given by the following formula:

$$\Delta([P, a, f]) = (-1)^{|a|}[S^1 \times P, 1 \times a, \Theta_f].$$

4.1.2. **Proposition** [7, prop. 5.1]**.** *The operator verifies:* $\Delta^2 = 0$.

Proof This follows from the associativity of the cross product and the nullity of $[S^1 \times S^1, 1 \times 1, \mu] \in h'_2(S^1)$ where μ is the product on S^1. $\square$

4.2. **Loop product.** Let us take $[P, a, f] \in H'_{n_1+d}(\mathcal{L}M)$ and $[Q, b, g] \in H'_{n_2+d}(\mathcal{L}M)$. We can smooth f and g and make them transverse to $\widetilde{\Delta}$, then we form the pull-back $P * Q$.

4.2.1. **Definition.** *Let $l : P * Q \to P$ and $r : P * Q \to Q$ be the canonical maps, then we have the pairing:*

$$- \bullet - : H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d}(\mathcal{L}M)$$

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+dim(P).|b|}[P * Q, l^*(a) \cup r^*(b), \Upsilon\psi],$$

let call it the loop product.

4.2.2. **Proposition** [7, Thm. 3.3]**.** *The loop product is associative and commutative.*

Proof The associativity of the loop product follows from the associativity of the intersection product, the cup product and the fact that Υ is also associative up to homotopy.
In order to prove the commutativity of $\bullet$ we follow the strategy of [7, Lemma 3.2].
There is a smooth interchanging map:

$$\tau : \mathcal{L}M \cap_M \mathcal{L}M \to \mathcal{L}M \cap_M \mathcal{L}M.$$

Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes the formula of [7, lemma 3.2] gives an homotopy H (so this is also a bordism) between:

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M.$$

This bordism identifies $[P * Q, l^*(a) \cup r^*(b), \Upsilon\psi]$ and

$$[P * Q, \tau^*(l^*(a) \cup r^*(b)), \Upsilon\tau\psi]$$

which is equal to:

$$(-1)^{(dim(P)-d-a)(dim(P)-d-b)}[Q * P, l^*(b) \cup r^*(a), \Upsilon\psi].$$

□

4.3. **Loop bracket.** Let $[P, a, f]$ and $[Q, b, g]$ be two geometric classes, in the preceding section we have defined a bordism between

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M.$$

Using the same homotopy one can define another bordism between

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau} \mathcal{L}M$$

and

$$P * Q \xrightarrow{\psi} \mathcal{L}M \cap_M \mathcal{L}M \xrightarrow{\Upsilon\tau^2} \mathcal{L}M.$$

Composing these two bordisms one obtains a geometric class:

$$(-1)^{|a|+|b|}[S^1 \times P * Q, 1 \times l^*(a) \cup r^*(b) \times 1, \tilde{H}].$$

4.3.1. **Definition.** *The loop bracket is the pairing:*

$$\{-,-\}: H'_{n_1+d}(\mathcal{L}M) \otimes H'_{n_2+d}(\mathcal{L}M) \longrightarrow H'_{n_1+n_2+d+1}(\mathcal{L}M)$$
$$\{[P,a,f],[Q,b,g]\} =$$
$$(-1)^{(d+1).(|a|+|b|)+dim(P).|b|}[S^1 \times P * Q, 1 \times l^*(a) \cup r^*(b), \tilde{H}].$$

There is another way to define the bracket by setting [7, Cor. 5.3]:

$$\{\alpha,\beta\} = (-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta).$$

Together with this bracket, $(\mathbb{H}_*(\mathcal{L}M), \bullet, \{-,-\})$ is a Gerstenhaber algebra.

4.3.2. **Theorem.** [7, Thm. 4.7] *The triple* $(\mathbb{H}_*(\mathcal{L}M), \bullet, \{-,-\})$ *is a gerstenhaber algebra:*
i) $(\mathbb{H}_*(\mathcal{L}M), \bullet)$ *is a graded associative and commutative algebra.*
ii) The loop bracket $\{-,-\}$ *is a Lie bracket of degree* $+1$*:*

$$\{\alpha,\beta\} = (-1)^{(|\alpha|+1)(|\beta|+1)}\{\beta,\alpha\},$$
$$\{\alpha,\{\beta,\gamma\}\} = \{\{\alpha,\beta\}.\gamma\} + (-1)^{(|\alpha|+1)(|\beta|+1)}\{\beta,\{\alpha,\gamma\}\},$$

iii) $\{\alpha, \beta \bullet \gamma\} = \{\alpha,\beta\} \bullet \gamma + (-1)^{|\beta|(|\alpha|+1)}\beta \bullet \{\alpha,\gamma\}$.

4.3.3. **Remark.** Let us recall that there are two important examples of Gerstenhaber algebras:
- The first one is the Hochschild cohomology of a differential graded associative algebra A:

$$HH^*(A,A),$$

this goes back to M. Gerstenhaber [17].
- The second example is the singular homology of a double loop space:

$$H_*(\Omega^2 X),$$

this is due to F. Cohen [10].
Both examples are related by the Deligne's conjecture proved in many different ways by C. Berger and B. Fresse [6], M. Kontsevich [32], M. Kontsevich and Y. Soibelman [33], J. McClure and J. Smith [39], D. Tamarkin [43] and S. Voronov [45]. This conjecture states that there is a natural action of an operad C_2 quasi-isomorphic to the chain operad of little 2-discs on the Hochschild cochain complex of an associative algebra.
Hochschild homology enters the theory by the following results of R. Cohen and J.D.S. Jones [12, Thm. 13]:
if $C^*(M)$ denotes the singular cochains of a manifold M, then there is an isomorphism of associative algebras:

$$HH^*(C^*(M), C^*(M)) \cong \mathbb{H}_*(\mathcal{L}M).$$

4.4. **The BV-structure.** In [12] and [7] it is proved that $\mathbb{H}_*(\mathcal{L}M)$ is a BV-algebra (we refer to [18] for BV-structures).

4.4.1. **Theorem** [7, Th. 5.4]**.** *The loop product $\bullet$ and the operator Δ makes $\mathbb{H}_*(\mathcal{L}M)$ into a Batalin Vilkovisky algebra, we have the following relations:*
i) $(\mathbb{H}_*(M^{S^1}), \bullet)$, *is a graded commutative associative algebra.*
ii) $\Delta^2 = 0$
iii) $(-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet \beta - \alpha \bullet \Delta(\beta)$ *is a derivation of each variable.*

The proof of is the theorem in the context of geometric homology is given by building explicit bordisms between geometric cycles. All these bordisms are described in [7].

4.4.2. **Remark.** E. Getzler introduced BV-algebras in the context of 2-dimensional topological field theories [18]. And he proved that $H_*(\Omega^2 M)$ is a BV-algebra if M has a S^1 action. Other examples are provided by the de Rham cohomology of manifolds with S^1-action.
The BV-structure on $\mathbb{H}_*(\mathcal{L}M)$ comes in fact form a geometric action of the *cacti* operad [12], [46] (normalized cacti with spines in the terminology of R. Kaufmann [31]). This cacti operad is homotopy equivalent to the little framed discs operad [46]. And we know since the work of E. Getzler that the homology of the little framed discs operad gives the BV operad [18].
Let us explain this geometric action. In fact we take the Gysin morphism along a map:

$$cacti(n) \times \mathcal{L}M^{\times n} \longleftarrow \mathcal{L}^{cacti(n)}M$$

and we compose with a map:

$$\mathcal{L}^{cacti(n)}M \longrightarrow \mathcal{L}M.$$

For $n = 1, 2$ we know from R. Kaufmann's description of *cacti* [30] that $cacti(n)$ is a smooth manifold. In that case the map defined above are maps of Hilbert manifolds and they give also a very nice description of the action of $H'_*(cacti)$ on $H'_{*+d}(\mathcal{L}M)$.
So, it is certainly worth building a smooth structure on *cacti* or on an operad homotopy equivalent that acts in the same way. This would give a more conceptual proof of the preceding theorem.

4.5. **Constant strings.** We have a canonical embedding:

$$c : M \hookrightarrow \mathcal{L}M$$

c induces a map:

$$c_* : H'_{n+d}(M) \to H'_{n+d}(\mathcal{L}M).$$

The morphism c_* is clearly a morphism of commutative algebras.

4.6. **Intersection morphism.** Let recall that the map

$$ev_0 : \mathcal{L}M \longrightarrow M$$

is a submersion (in fact this is a smooth fiber bundle of Hilbert manifolds). Hence if we choose a base point $m \in M$ the fiber of ev_0 in m is the Hilbert manifold ΩM of based loops in M. Consider the morphism:

$$i : \Omega M \hookrightarrow \mathcal{L}M$$

from the based loops in M to the free loops in M, this is an orientable morphism of codimension d.
Let us describe the intersection morphism:

$$I = i^{!} : \mathbb{H}_*(\mathcal{L}M) \to H_*(\Omega M).$$

Let $[P, a, f] \in H'_{n+d}(\mathcal{L}M)$ be a geometric class, one can define $I([P, a, f])$ in two ways:

i) using the Gysin morphism: $I([P, a, f]) = (-1)^{d.|a|}[P \cap_{\mathcal{L}M} \Omega M, (f^* i)^*(a), \phi]$.

A better way is certainly to notice that this is the same as doing the loop product with $[c_m, 1, c]$ where c_m is a point and $c : c_m \to \mathcal{L}M$ is the constant loop space at the point m, then we have:

ii) $I([P, a, f]) = (-1)^{d.|a|}[P * c_m, l^*(a), \psi]$.

We remark that $P * c_m$ is either empty or equal to m (depending on the dimension of P, for example when $dimP < d$). And if $|a| > 0$ we also have $I([P, a, f]) = 0$.

4.6.1. **Proposition** [7, Prop 3.4]**.** *The intersection morphism I is a morphism of associative algebras.*

Proof. The algebra structure on $H'_*(\Omega M)$ comes from the Pontryagin product which is the restriction of Υ to $\Omega M \times \Omega M$, we have the following diagram:

$$\begin{array}{ccc} \Omega M \times \Omega M & \xrightarrow{\Upsilon_{\Omega M \times \Omega M}} & \Omega M \\ \downarrow{\scriptstyle i \times i} & & \downarrow{\scriptstyle i} \\ \mathcal{L}M \cap_M \mathcal{L}M & \xrightarrow{\Upsilon} & \mathcal{L}M. \end{array}$$

The Pontryagin product is given by the formula:

$$[P, a, f].[Q, b, g] = (-1)^{dim(P).|b|}[P \times Q, a \times b, \Upsilon_{\Omega M \times \Omega M}(f \times g)].$$

This product is associative but not commutative. The intersection morphism is a morphism of algebras by commutativity of the diagram above. □

This morphism has been studied in details in [15], in particular it is proved that the kernel of I is nilpotent.

4.7. **Bordism and string topology.** Let $\Omega_*^{SO}(X)$ be the bordism group of a topological space X, we recall that it is isomorphic to the bordism classes of singular oriented manifolds over X (morphism $f : M \to X$).
We remark that $\Omega_*^{SO}(\mathcal{L}M)$ is also a BV-algebra (all the construction described above immediately adapt to Ω_*^{SO}).
Let call a geometric class $[P, a, f]$ realizable if it is equivalent to a class $[Q, 1, g]$ this is equivalent to condition of being in the image of the Steenrod-Thom morphism:

$$st : \Omega_{n+d}^{SO}(\mathcal{L}M) \to \mathbb{H}_n(M)$$
$$[M, f] \mapsto [M, 1, f].$$

This morphism is clearly a morphism of BV-algebras and we have:

4.7.1. **Proposition.** *If $c \notin Im(st)$ then $I(c) = 0$.*

Proof This follows from the fact that if a geometric class is not realizable it has the form $[P, a, f]$ with $|a| > 0$ and in this case $I([P, a, f]) = 0$. □

4.7.2. **Remark:** We recall that in general st is neither injective nor surjective. However it is surjective when $n + d < 6$ (using Atiyah-Hirzebruch spectral sequence one prove that it is an isomorphism for $n + d = 0, 1, 2$) and it is also sujective if we work over $\mathbb{F}_2$, over this field the orientability condition in the definition of geometric homology is unnecessary.

4.8. **String bracket.**

4.8.1. *The string space.* Let us consider the fibration:

$$S^1 \to ES^1 \to BS^1.$$

There exists a smooth model for this fibration, for ES^1 we take S^∞ the inductive limit of S^n. By [35, chapter X] this is an Hilbert manifold modelled on $\mathbb{R}^{(\mathbb{N})}$. As S^1 acts freely and smoothly on S^∞ we have a S^1 fiber bundle of Hilbert manifolds:

$$S^1 \to S^\infty \xrightarrow{\pi} \mathbb{C}P^\infty.$$

We get the S^1-fibration:

$$S^1 \to \mathcal{L}M \times S^\infty \to \mathcal{L}M \times_{S^1} S^\infty.$$

The projection:

$$\mathcal{L}M \times S^\infty \to \mathcal{L}M$$

is a homotopy equivalence of Hilbert manifolds. As we know from [14] that an homotopy equivalence between two separable Hilbert manifolds is homotopic to a diffeomorphism, we deduce that they are diffeomorphic.
The space $\mathcal{L}M \times_{S^1} S^\infty$ is not an Hilbert manifold because the action of S^1 on $\mathcal{L}M$ is not smooth. Let call this space the string space of M.

4.8.2. *String homology.* Let $\mathcal{H}_i$ be the homology group $H'_{i+d}(\mathcal{L}M \times_{S^1} S^\infty)$, this is the string homology of M. In what follows we give explicit definitions of the morphism c, M, E of [7, 6].

The morphism c. Let $e \in H^2(\mathcal{L}M \times_{S^1} S^\infty)$ be the Euler class of the S^1-fibration defined above:

$$c : \mathcal{H}_i \to \mathcal{H}_{i-2}$$

$$c([P, a, f]) = [P, f^*(e) \cup a, f].$$

The morphism E. This morphism is $E = \pi_*$:

$$E : \mathbb{H}_i(\mathcal{L}M) \to \mathcal{H}_i$$

$$E([P, a, f]) = [P, a, \pi f].$$

The morphism M. Let $[P, a, f]$ a geometric class in $\mathcal{H}_i$. For a point $p \in P$, we can choose $(c_p, u) \in \mathcal{L}M \times S^\infty$ that represents a class $[c_p, u] \in \mathcal{L}M \times_{S^1} S^\infty$, as this choice is non-canonical we take all the orbit of (c_p, u) under the action of S^1. In this way we produce a map

$$\tilde{f} : S^1 \times P \to \mathcal{L}M \times S^\infty.$$

Identifying $\mathcal{L}M \times S^\infty$ with $\mathcal{L}M$ one get a map:

$$M : \mathcal{H}_i \to \mathbb{H}_{i+1}(\mathcal{L}M)$$

$$M([P, a, f]) = (-1)^{|a|}[S^1 \times P, 1 \times a, \tilde{f}].$$

We have the following exact sequence, which is the Gysin exact sequence associated to the S^1-fibration π:

$$\cdots \to \mathbb{H}_i(\mathcal{L}M) \xrightarrow{E} \mathcal{H}_i \xrightarrow{c} \mathcal{H}_{i-2} \xrightarrow{M} \mathbb{H}_{i-1}(\mathcal{L}M) \to \cdots$$

4.8.3. *The bracket.* The string bracket is given by the formula:

$$[\alpha, \beta] = (-1)^{|a|} E(M(a) \bullet M(b)).$$

Together with this bracket $(\mathcal{H}_*, [-,-])$ is a graded Lie algebra of degree $(2-d)$ [7, Th. 6.1].

4.9. **Riemann surfaces operations.** These operations are defined by R. Cohen and V. Godin in [11] by means of Thom spectra technology.
Let Σ be an oriented surface of genus g with $p+q$ boundary components, p incoming and q outgoing. We fix a parametrization of these components. Hence we have two maps:

$$\rho_{in} : \coprod_p S^1 \to \Sigma,$$

and

$$\rho_{out} : \coprod_q S^1 \to \Sigma.$$

If we consider the space of $\mathbf{H}^2$-maps $\mathbf{H}^2(\Sigma, M)$, we get a Hilbert manifold, we also look at $\mathcal{L}M$ as a Hilbert manifold modelled on $\mathbf{H}^2(S^1, \mathbb{R}^d)$, then we get the diagram of Hilbert manifolds:

$$\mathcal{L}M^{\times q} \xleftarrow{\rho_{out}} \mathbf{H}^2(\Sigma, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times p}.$$

Let $\chi(\Sigma)$ be the Euler characteristic of the surface. Using Sullivan's Chord diagrams it is proved in [11] that the morphism

$$\mathbf{H}^2(\Sigma, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times q}$$

has a homotopy model:

$$\mathbf{H}^2(c, M) \xrightarrow{\rho_{in}} \mathcal{L}M^{\times q}$$

where $c \in \mathcal{CF}^{\mu}_{p,q}(g)$ is the Sullivan's chord diagram associated to Σ. This is an embedding of Hilbert manifolds of codimension $-\chi(\Sigma).d$. Hence by using the Gysin morphism for Hilbert manifold one can define the operation:

$$\mu_\Sigma : H'_*(\mathcal{L}M^{\times p}) \xrightarrow{\rho^!_{in}} H'_{*+\chi(\Sigma).d}(\mathbf{H}^2(\Sigma, M)) \xrightarrow{\rho_{out*}} H'_{*+\chi(\Sigma).d}(\mathcal{L}M^{\times q}).$$

All these operations are parametrized by the topological space of marked, metric chord diagrams $C\mathcal{F}^{\mu}_{p,q}(g)$ [11, sect1]. Using geometric homology and part of the techniques of [11, sect2] one proves:

4.9.1. **Theorem.** *For $q > 0$ we have morphisms:*

$$\mu_{n,p,q}(g) : H'_n(C\mathcal{F}^{\mu}_{p,q}(g)) \to Hom(H'_*(\mathcal{L}M^{\times p}), H'_{*+\chi(\Sigma).d+n}(\mathcal{L}M^{\times q}))$$

Proof Let us give the construction of $\mu_{n,p,q}(g)$:
consider an element of $H'_n(C\mathcal{F}^{\mu}_{p,q}(g))$ and suppose that is represented by a geometric cycle (S, α, g) where $g : S \to \mathcal{CF}^{\mu}_{p,q}(g)$. And let define:

$$Map(g, M) = \{(s, f)/s \in S, f \in Map(g(s), M)\},$$

we also have maps:

$$\rho_{in} : Map(g, M) \to \mathcal{L}M^{\times p},$$
$$\rho_{out} : Map(g, M) \to \mathcal{L}M^{\times q}.$$

Using [11, lemma3] we get an embedding of codimension $-\chi(\Sigma).d$:

$$\rho_{in} \times p : Map(g, M) \to \mathcal{L}M^{\times p} \times S$$

where p is the canonical projection. If this map is an embedding of Hilbert manifolds (we actually don't know if $\mathcal{CF}^{\mu}_{p,q}(g)$ is a manifold), we can form the following diagram:

$$\begin{array}{ccccc}
N_1 \times \ldots \times N_p \times S & \xleftarrow{\ j\ } & N_1 * \ldots * N_p * S & & \\
\downarrow{\scriptstyle f_1 \times \ldots \times f_p \times Id_S} & & \downarrow{\scriptstyle \phi_g} & & \\
\mathcal{L}M^{\times p} \times S & \xleftarrow[\rho_{in} \times p]{} & Map(g, M) & \xrightarrow[\rho_{out}]{} & \mathcal{L}M^{\times q}.
\end{array}$$

We define $\mu_{n,p,q}$ in the following way:

□

Let us define the morphism in a less heuristic and more rigorous way. Rather than using an hypothetical embedding of Hilbert manifolds let use the Thom collapse map of [11, lemma5]:

$$\tau_g : \mathcal{L}M^{\times p} \times S \to (Map(g, M))^{\nu(g)}$$

where $\nu(g)$ is an open neighborhood of the embedding $\rho_{in} \times p$ and let th_g be the Thom isomorphism:

$$th_g : H'_*((Map(g, M))^{\nu(g)}) \to H'_{*+\chi(\Sigma).d}(Map(g, M)).$$

Then $\mu_{n,p,q}$ is defined by:

□

4.9.2. **Remark.** We obtain homological string topology operations which are 4-graded, 3 gradings being purely geometric (p, q and g) and one grading purely homological. Actually, we do not know if $H'_*(\mathcal{C}F^{\mu}_{p,q})$ is a PROP, but that seems a reasonable conjecture ([11, theorem6]).

4.9.3. **Corollary.** *If we fix $n = 0$, then the action of homological degree 0 string topology operations induces on $H'_{*+d}(\mathcal{L}M)$ a structure of Frobenius algebra without co-unit.*

4.9.4. *BV-structures.* Let us give some hints of how to recover the BV-structure on $H'_{*+d}(\mathcal{L}M)$. From E. Getzler's work [18], we know that topological conformal field theories when restricted to the genus 0 Riemann surfaces carry a BV-structure. Hence, let work with $g = 0$ and $q = 1$, we conjecture a map of operads:

$$H'_*(cacti) \to H'_*(\mathcal{CF}^{\mu}_{p,1}(g)).$$

$S^{1^{\times p+q}}$ acts on $\mathcal{CF}^{\mu}_{p,q}(g)$ by rotating the markings. In particular S^1 acts on $\mathcal{CF}^{\mu}_{1,1}(g)$ by rotating the marking of the outgoing boundary cycle, hence we get maps:

$$S^1 \to S^1 \times \mathcal{CF}^{\mu}_{1,1}(g) \to \mathcal{CF}^{\mu}_{1,1}(g)$$

which induce a morphism:

$$H'_n(S^1) \to Hom(H'_{*+d}(\mathcal{L}M), H'_{*+d+n}(\mathcal{L}M)).$$

The generator of $H'_0(S^1)$ gives the unit, and the fundamental class of S^1 gives the operator Δ.
The loop product comes from the Frobenius structure, it is induced by the pair of pant i.e. the surface of genus 0 with 2 incoming boundary components and one outgoing.

5. n-sphere topology.

Let fix an integer $n > 1$, and suppose that M is a n-connected compact oriented smooth manifold. Some results about the algebraic structure of the homology of n-sphere spaces were announced in [46, Th. 2.5]. We show how to recover a part of this structure.

5.1. **Sphere product.** Let us take $[P, a, f] \in H'_{n_1+d}(\mathcal{S}_n M)$ and $[Q, b, g] \in H'_{n_2+d}(\mathcal{S}_n M)$ two families of n-spheres. We can smooth f and g and make them transverse to $\widetilde{\Delta}$, then we form the pull-back $P * Q$.

5.1.1. **Definition.** *Let $l : P * Q \to P$ and $r : P * Q \to Q$ be the canonical maps, then we have the pairing:*

$$- \bullet - : H'_{n_1+d}(\mathcal{S}_n M) \otimes H'_{n_2+d}(\mathcal{S}_n M) \longrightarrow H'_{n_1+n_2+d}(\mathcal{S}_n M)$$

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d.(|a|+|b|)+dim(P).|b|}[P * Q, l^*(a) \cup r^*(b), \Upsilon\psi],$$

let call it the sphere product.

5.1.2. **Proposition.** *The sphere product is associative and commutative.*

Proof The associativity and commutativity of the sphere product follows from the associativity and commutativity of the intersection product, the cup product and the fact that Υ is also associative and commutative up to homotopy. $\square$

5.2. **Constant spheres.** We have a canonical embedding:

$$c : M \hookrightarrow \mathcal{S}_n M$$

c induces a map:

$$c_* : H'_{*+d}(M) \to H'_{*+d}(\mathcal{S}_n M).$$

The morphism c_* is clearly a morphism of commutative algebras.

5.3. **Intersection morphism.** Let us recall that the map

$$ev_0 : \mathcal{S}_n M \longrightarrow M$$

is a submersion (in fact this is a smooth fiber bundle of Hilbert manifolds). Hence if we choose a base point $m \in M$ the fiber of ev_0 in m is the Hilbert manifold $\Omega^n M$ of n-iterated based loops in M. Consider the morphism:

$$i : \Omega^n M \hookrightarrow \mathcal{S}_n M$$

this is an orientable morphism of codimension d.
Let us describe the intersection morphism:

$$I = i^! : H_{*+d}(\mathcal{S}_n M) \to H_*(\Omega M).$$

Let $[P, a, f] \in H'_{n+d}(\mathcal{S}_n M)$ be a geometric class, one can define $I([P, a, f])$ in the following way:

take the sphere product with $[c_m, 1, c]$ where c_m is a point and $c : c_m \to \mathcal{S}_n M$ is the constant sphere at the point m, then we set:

$$I([P, a, f]) = (-1)^{d.|a|}[P * c_m, l^*(a), \psi].$$

As in the case $n = 1$, we remark we have:

5.3.1. **Proposition.** *The intersection morphism I is a morphism of graded commutative and associative algebras.*

5.4. **3-sphere topology.** Let emphasis on the case $n = 3$, in this case we show the existence of an operator of degree 3 acting on the homology. And we also obtain some results about the S^3-equivariant homology of 3-sphere spaces.

5.4.1. *The operator Δ_3.* The sphere S^3 acts on $\mathcal{S}_3 M$, let denote this action by Θ. Hence, if we consider a family of 3-spheres in M

$$f : P \to \mathcal{S}_3 M$$

we can build a new familly:

$$\Theta_f : S^3 \times P \xrightarrow{id \times f} S^3 \times \mathcal{S}_3 \xrightarrow{\Theta} \mathcal{S}_3 M.$$

5.4.2. **Definition.** *The operator Δ_3 is given by the following formula:*

$$\Delta_3 : H'_*(\mathcal{S}_3 M) \to H'_{*+3}(\mathcal{S}_3 M)$$

$$[P, a, f] \mapsto (-1)^{|a|}[S^3 \times P, 1 \times a, \Theta_f].$$

5.4.3. **Proposition.** *The operator verifies:*

$$\Delta_3^2 = 0.$$

Proof The proof is exactly the same as in the case of the operator Δ and follows from the associativity of the cross product and the nullity of the geometric class:

$$[S^3 \times S^3, 1 \times 1, \mu].$$

□

5.4.4. 3-*sphere bracket.* As in the case $n = 1$, we have a smooth model of the S^3-fibration:

$$S^3 \to ES^3 \to BS^3$$

which is given by the S^3 fiber bundle of Hilbert manifolds:

$$S^3 \to S^\infty \to \mathbb{H}P^\infty.$$

We consider the S^3-fibration:

$$S^3 \to \mathcal{S}_3 M \times ES^3 \to \mathcal{S}_3 M \times_{S^3} ES^3.$$

Let $\mathcal{H}_*^3 = H_{*+d}(S_3 M \times ES^3)$ and consider the Gysin exact sequence of the fibration:

$$\cdots \to H'_{i+d}(\mathcal{S}_3 M) \xrightarrow{E} \mathcal{H}_i^3 \xrightarrow{c} \mathcal{H}_{i-4}^3 \xrightarrow{M} H'_{i+d-1}(\mathcal{S}_3 M) \to \cdots.$$

5.4.5. **Definifion.** *We define the* 3*-sphere bracket by the formula:*

$$[\alpha, \beta] = (-)^{|\alpha|} E(M(\alpha) \bullet M(\beta)).$$

5.4.6. **Remark.** This bracket is anti-commutative.

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