Gorenstein graded rings associated to ideals

Shiro Goto and Shin-ichiro Iai*

1 Introduction.

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim A = d$. Let $I (\neq A)$ be an ideal in A and $s = \mathrm{ht}_A I$. This article studies the question of when the associated graded ring $\mathrm{G}(I) := \bigoplus_{i \geq 0} I^i/I^{i+1}$ is Gorenstein. To state our result, we set up some notation. Let ℓ be an integer such that $\ell \leq d$ and let J be a reduction of I generated by elements $a_1, a_2, \dots, a_\ell$. We denote by $\mathrm{r}_J(I)$ the reduction number of I with respect to J. The analytic spread of I is $\lambda(I) := \dim A/\mathfrak{m} \otimes_A \mathrm{G}(I)$. Then $s \leq \lambda(I) \leq \ell$. We always assume that the generating set $\{a_1, a_2, \dots, a_\ell\}$ of J is a *basic* generating set for J in the sense of Aberbach, Huneke, and Trung [AHT], which means that $J_i A_\mathfrak{q}$ is a reduction of $IA_\mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{V}(I)$ with $i = \mathrm{ht}_A \mathfrak{q} < \ell$. Here let $\mathrm{V}(I)$ be a set of prime ideals in A containing I and $J_i := (a_1, a_2, \dots, a_i)$ for $0 \leq i \leq \ell$. By [AHT], 7.2, there always exists a basic generating set for J if the field $A/\mathfrak{m}$ is infinite. Let

$$n := \max\{\mathrm{r}_{J_{s\mathfrak{q}}}(I_\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(I) \text{ and } \mathrm{ht}_A \mathfrak{q} = s\}$$

that is a generic reduction number. When the ring A is Cohen-Macaulay, the ideal J_s is a complete intersection (see 3.2), and if $n = 0$, we say that I is generically a complete intersection. Let $\mathrm{a}(\mathrm{G}(I))$ stand for the a-invariant of $\mathrm{G}(I)$ (cf. [GW], 3.1.4). Our ideal I is said to be height unmixed if all associated prime ideals of I have same codimension. With this notation the main result of this article can be stated as follows.

Theorem 1.1. *Let A be a Gorenstein local ring. Assume that I is a height unmixed ideal with $s > 0$ and* $\operatorname{depth} A/I^i + J_s \geq \min\{d-s, d-s+n-i\}$ *for all $1 \leq i \leq n-s+\ell$. Then the following two conditions are equivalent.*

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(1) $\mathrm{G}(I)$ *is a Gorenstein ring.*

(2) (i) $\mathrm{r}_J(I) \leq n - s + \ell$,

(ii) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \leq i \leq n$,

(iii) $(J_i : a_{i+1}) \cap I^{n-s+i+1} = J_i I^{n-s+i}$ *for all* $s \leq i < \ell$, *and*

(iv) $J_s^n : I^n \subseteq I^n$.

When this is the case, we have $\mathrm{a}(\mathrm{G}(I)) = n - s$.

The analytic deviation $\mathrm{ad}(I) := \lambda(I) - \mathrm{ht}_A(I)$ was firstly introduced by Huckaba and Huneke in their papers [HH1] and [HH2], which contain important results on the Cohen-Macaulay and/or Gorenstein property of $\mathrm{G}(I)$ in the case where $\mathrm{ad}(I) \leq 2$. The notion of analytic deviation has provided the impetus for large amount of research. And it has led to several general results on Cohen-Macaulayness and/or Gorensteinness in the associated graded rings $\mathrm{G}(I)$ of ideals having arbitrary analytic deviation. But almost all authors we recall assumed the number n to be at most one when $\mathrm{ad}(I)$ is positive, and we lack satisfactory references analyzing ideals for which n is large. Theorem 1.1 enables us to handle the case where the number n is arbitrary. There are, of course, examples in that case. Let us now give simple examples. Let n and s be positive integers, let $R = [[X_1, X_2, \ldots, X_{s+3}]]$ be a formal power series ring in $s+3$ variables over a field k, and let $L = (X_{s+1}^n X_{s+3}, X_{s+2}^2)$ be an ideal in R. We set $A = R/L$ that is a Gorenstein local ring of dimension $s+1$. For each $1 \leq i \leq s+2$, we denote by a_i a reduction of $X_i \mod L$. Put $I = (a_1, a_2, \ldots, a_{s+2})$ and $J = (a_1, a_2, \ldots, a_{s+1})$. Then $\mathrm{ht}_A I = s$. We obtain that J is a reduction of I with $\mathrm{r}_J(I) = 1$ and that the set $\{a_1, a_2, \ldots, a_s, a_{s+1}\}$ is a basic generating set for J. Then $n = \max\{\mathrm{r}_{J_{s\mathfrak{q}}}(I_\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(I) \text{ and } \mathrm{ht}_A \mathfrak{q} = s\}$, and furthermore, $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = n - s$ by 1.1.

We remark that J is not necessarily a minimal reduction. But the number ℓ being small is better for the conditions in Theorem 1.1. The conditions (i), (ii), and (iii) are necessary and sufficient conditions for $\mathrm{G}(I)$ to be a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{G}(I)) = n - s$ under the assumption of depths of $A/I^i + J_s$ (see 3.17). Hence the condition (iv) is the key to the ring $\mathrm{G}(I)$ being Gorenstein in this situation. We note that the condition of depths of $A/I^i + J_s$ is a necessary condition for Cohen-Macaulayness of $\mathrm{G}(I)$ in the case where $\ell = s$ (which implies $\mathrm{ad}(I) = 0$) or in the case where $\ell = s + 1$ and $n = 0$. When $\ell = s$, the equality $\mathrm{r}_J(I) = n$ follows from the condition (i), and hence the above theorem covers [GI1], 1.4 (i.e. $\mathrm{G}(I)$ is a Gorenstein ring if and only if A is a Gorenstein ring, $\mathrm{G}(I)$ is a Cohen-Macaulay ring, and $J^n : I^n = I^n$).

When I is generically a complete intersection, the Gorenstein property of $\mathrm{G}(I)$ has been studied closely in even the case where I has higher analytic deviation (cf. [GNN1], [JU], and [T]). However, the class of generically complete intersection ideals with which most of authors dealt remains within Cohen-Macaulay ideals. In Theorem 1.1, we have assumed that I is height unmixed, but the assumption that I is a Cohen-Macaulay ideal is removed in the case where $n = 0$. We can find references analyzing mixed ideals, whereas its analytic deviation is at most 2 (cf. [GNa], [GI1], and [GI2]). For the case where $n = 1$, the first author, Nakamura, and Nishida [GNN3] gave a characterization on the Gorensteinness of $\mathrm{G}(I)$ in the case where $\mathrm{ad}(I) = 1$. Our theorem is a generalization of their result.

We now mention what is in each section. In Section 2 we will prove our theorem in the case where I is generically a complete intersection. Recall that if $n = 0$, then the condition of depths of $A/I^i + J_s$ does not demand that the ring A/I is Cohen-Macaulay. When I is generically a complete intersection, there already exists a criterion on the Cohen-Macaulay property of $\mathrm{G}(I)$ in [GNN2], 6.5, which does not require the Cohen-Macaulayness of A/I. It seems then natural to ask when under the same hypothesis in their result the ring $\mathrm{G}(I)$ is also Gorenstein. Section 2 contains an answer of this question. Namely, under the hypotheses given by [GNN2], 6.5, $\mathrm{G}(I)$ is a Gorenstein ring if I is a height unmixed ideal in a Gorenstein local ring A (see 2.9).

In Section 3 we shall discuss the Cohen-Macaulay property of the associated graded ring. There are quite general conditions under which $\mathrm{G}(I)$ is a Cohen-Macaulay ring in [GNN2]. But the number n we can handle in their theorem is bounded above by one. Recently, Nishida [N2] showed criteria on the Cohen-Macaulayness in associated graded rings of filtrations having small analytic deviation in the case where the number n is arbitrary. The aim of Section 3 is to extend their theorems to the case where the numbers $\mathrm{ad}(I)$ and n are arbitrary (see 3.1, 3.19, 3.20, and 3.21).

Section 4 contains a proof of Theorem 1.1. This is based on reduction to the case where $s = 0$. But our result in such a case is entirely different from the case where s is positive. Namely, we have the following theorem in the case where $s = 0$.

Theorem 1.2. *Let A be a Gorenstein local ring. Assume that I is a height unmixed ideal with $s = 0$ and* $\operatorname{depth} A/I^i \geq \min\{d, d + n - i\}$ *for all* $1 \leq i \leq n + \ell$*. Then the following two conditions are equivalent.*

(1) $\mathrm{G}(I)$ *is a Gorenstein ring.*

(2) (i) $\mathrm{r}_J(I) \leq n + \ell$,

(ii) $(J_i : a_{i+1}) \cap I^{n+i+1} = J_i I^{n+i}$ *for all* $0 \leq i < \ell$, *and*

(iii) $(0) : I^i \subseteq I^{n-i+1}$ *for all* $1 \leq i \leq n$.

When this is the case, we have $\mathrm{a}(\mathrm{G}(I)) = n$.

To prove the above theorem, in Section 4 we will discuss a question of when the canonical module $\mathrm{K}_{\mathrm{G}(I)}$ of $\mathrm{G}(I)$ has the expected form in the sense of Herzog, Simis, and Vasconcelos [HSV] (i.e. $\mathrm{K}_{\mathrm{G}(I)}$ is the graded $\mathrm{G}(I)$-module associated to an ideal I with respect to the canonical module K_A of A). In their paper, they investigated originally the canonical modules of blow-up algebras and showed that $\mathrm{K}_{\mathrm{G}(I)}$ has the expected form if and only if so does the canonical module of the (extended) Rees algebra. Later, Zarzuela [Z] generalized their result to the case where the base ring A is not necessarily Cohen-Macaulay, and moreover, it led to a result give by Trung, Viêt, and Zarzuela [TVZ] concerning descriptions of the canonical modules of blow-up algebras in the term of a filtration of K_A. In our paper, we will utilize repeatedly such a filtration, which we call the canonical I-filtration of K_A (see [GI1], 1.1 and notice that it permits us to handle even the case where $s = 0$). For an answer of the above question, we have remarkable references [GNN1] and [JU]. Their results require that I is generically a complete intersection. We will generalize their result to the case where the number n is arbitrary (see 4.7).

Actually, in Section 4 we will see that the assumption of the depths yields the sharper estimation $\mathrm{r}_J(I) \leq \max\{n,\ \ell - s\}$ (see 4.19). Ultimately, we will give a criterion on the Gorensteinness of $\mathrm{G}(I)$ whose condition of the depths is weaker than that of Theorem 1.1 and Theorem 1.2 (see 4.23).

Throughout this paper let A be a Noetherian local ring, $I (\neq A)$ an ideal of A, and J a reduction of I. We always suppose that $\dim A/I = d - s$. Let ℓ be an integer with $\ell \leq d$ and assume that J contains a basic generating set $\{a_1, a_2, \ldots, a_\ell\}$. For $s \leq i \leq \ell$, let

$$\mathrm{r}_i(I) := \max\{\mathrm{r}_{J_{i\mathfrak{p}}}(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{V}(I) \text{ and } \mathrm{ht}_A \mathfrak{p} = i\}.$$

We put $\mathrm{r}_i = \mathrm{r}_i(I)$ for short. Then $n = \mathrm{r}_s$. When the ring $\mathrm{G}(I)$ is Cohen-Macaulay, we will use repeatedly the a-invariant formula: $\mathrm{a}(\mathrm{G}(I)) = \max\{\mathrm{r}_i - i \mid s \leq i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$ (cf. [U], 1.4). Our basic generating set $\{a_1, a_2, \ldots, a_\ell\}$ is said to be a *good* generating set in the sense of Aberbach [A] if the sequence $a_1, a_2, \ldots, a_\ell$ is I-filter regular, which means $a_i \notin \mathfrak{p}$ for any $1 \leq i \leq \ell$ and for any $\mathfrak{p} \in \mathrm{Ass}_A A/J_{i-1} \setminus \mathrm{V}(I)$. If the field $A/\mathfrak{m}$ is infinite, there exists a good generating set for J by [A], 2.3. Let t be an indeterminate over A. We define $\mathrm{R}(I) := A[It] \subseteq A[t]$ and $\mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}]$, which we call the Rees algebra and the extended Rees algebra. Then $\mathrm{G}(I) \cong \mathrm{R}(I)/I\mathrm{R}(I) \cong \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$ as graded rings. Let K_A, $\mathrm{K}_{\mathrm{R}'(I)}$, and $\mathrm{K}_{\mathrm{G}(I)}$ denote the graded canonical modules of A, $\mathrm{R}'(I)$, and $\mathrm{G}(I)$, respectively. We shall freely refer to [BH], [GN], [HK], and [HIO] for details of the theory on the canonical modules. We denote $\mathrm{G}(I)$ simply by G. Let $\mathfrak{M} = \mathfrak{m}G + G_+$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(\quad)$ ($i \in \mathbb{Z}$) the graded $i^{\underline{\mathrm{th}}}$ local cohomology functor of G with respect to $\mathfrak{M}$. For each graded G-module E, let E_j stand for the homogeneous component of E of degree j and let $\mathrm{a}_i(E) = \max\{j \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathfrak{M}}(E)]_j \neq (0)\}$ ($i \in \mathbb{Z}$). We put $E_{\geq m} = \oplus_{i \geq m} E_i$ for each $m \in \mathbb{Z}$.

2 The case of generic complete intersections.

In this section we always assume that A is a Gorenstein ring. The main purpose of this section is to prove the following proposition.

Proposition 2.1. *Assume G is a Cohen-Macaulay ring with* $\mathrm{a}(G) = -s$. *Then G is a Gorenstein ring if I is height unmixed and* $\operatorname{depth} A/I^i \geq d-s-i$ *for all* $1 \leq i \leq \max\{1, \ell - s - 1\}$.

To prove the proposition above, we may assume $\ell > s$. In fact, by the a-invariant formula we have $\mathrm{a}(G) \geq n - s$, and hence I is generically a complete intersection. Therefore if $\ell = s$, then we get $I = J_s$ because J_s is a complete intersection in a Cohen-Macaulay ring (see, e.g., 3.2). So we have nothing to prove.

The Cohen-Macaulayness of G induces the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular (see, e.g., 3.3), and hence we get the isomorphism $G/(a_1t, a_2t, \ldots, a_st)G \cong \mathrm{G}(I/J_s)$ of graded G-modules by [VV]. Therefore, passing to the ring A/J_s, we may also assume $s = 0$ (cf. [GNN2], 3.4).

In what follows, until 2.9 we assume that our height unmixed ideal $I (\neq (0))$ is generically a complete intersection with $s = 0$ and that G is a Cohen-Macaulay ring with $\mathrm{a}(G) = 0$. Furthermore, we suppose that $\operatorname{depth} A/I \geq d - 1$. Let $\mathfrak{a} = (0) : I$ and let $\overline{A} = A/\mathfrak{a}$. We have $I \cap \mathfrak{a} = (0)$ and hence a_1 is $\overline{A}$-regular element (see, e.g., [GNa], 2.1). Since I is height unmixed, $\mathrm{K}_{\overline{A}} = (0) :_A \mathfrak{a} \cong I$. Let $\mathrm{Q}\left(\overline{A}\right)$ denote the total quotient ring of $\overline{A}$. We consider a commutative A-algebra

$$B := I\overline{A} :_{\mathrm{Q}\left(\overline{A}\right)} I\overline{A}$$

that is finite as A-module. We have $\operatorname{depth} I\overline{A} = d$ because $\operatorname{depth} A/I \geq d-1$ and $I \cong I\overline{A}$. Therefore B is a maximal Cohen-Macaulay A-module of dimension d (see [AG], 2.2).

We put $T = \mathrm{G}(I\overline{A})$ and $S = \mathrm{G}(IB)$ for short. Look at the natural exact sequence $0 \to \overline{A} \to B \to B/\overline{A} \to 0$. Let $C = B/\overline{A}$. Then $\dim C \leq d - 2$.

Since $I\overline{A} = IB$, we get the exact sequence

$$0 \to T \xrightarrow{\varphi} S \to C \to 0 \qquad (\sharp)$$

of graded G-modules. Moreover we have the exact sequence

$$0 \to \mathfrak{a} \to G \xrightarrow{\varepsilon} T \to 0 \qquad (\sharp\sharp)$$

of graded G-modules (see, e.g., [GNa], 2.3). We note $\mathrm{K}_B \cong IB$ and hence $\mathrm{ht}_{B_\mathfrak{n}} IB_\mathfrak{n} = 1$ for all maximal ideal $\mathfrak{n}$ in B. Let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A. Notice that $\widehat{A} \otimes_A S \cong \prod_{j=1}^m \mathrm{G}(IB_j)$ is the direct product of associated graded rings $S_j := \mathrm{G}(IB_j)$ of ideals IB_j (with positive analytic spread) in Cohen-Macaulay local rings B_j, which are finite as $\widehat{A}$-modules.

Lemma 2.2. *S is a maximal Cohen-Macaulay G-module.*

Proof. We apply the local cohomology functors $\mathrm{H}^i_\mathfrak{M}(*)$ ($i \in \mathbb{Z}$) to the exact sequences ($\sharp$) and ($\sharp\sharp$). Then we have the resulting exact sequences

$$0 \to \mathrm{H}^{d-1}_\mathfrak{M}(T) \to \mathrm{H}^d_\mathfrak{M}(\mathfrak{a}) \to \mathrm{H}^d_\mathfrak{M}(G) \to \mathrm{H}^d_\mathfrak{M}(T) \to 0$$

and

$$\mathrm{H}^{i-1}_\mathfrak{M}(T) \cong \mathrm{H}^i_\mathfrak{M}(\mathfrak{a}) \qquad (i \le d-1)$$

of graded local cohomology modules from ($\sharp\sharp$). Therefore we get $\mathrm{H}^{i-1}_\mathfrak{M}(T) = [\mathrm{H}^{i-1}_\mathfrak{M}(T)]_0$ for all integers $i \le d-1$ and $\mathrm{a}(T) \le 0$ because $\mathrm{H}^i_\mathfrak{M}(\mathfrak{a}) = [\mathrm{H}^i_\mathfrak{M}(\mathfrak{a})]_0$ (see [GH], 2.2) and $\mathrm{a}(G) = 0$. And moreover, by the resulting exact sequences

$$\mathrm{H}^i_\mathfrak{M}(T) \to \mathrm{H}^i_\mathfrak{M}(S) \to \mathrm{H}^i_\mathfrak{M}(C) \qquad (i \le d-1)$$

and

$$\mathrm{H}^d_\mathfrak{M}(T) \cong \mathrm{H}^d_\mathfrak{M}(S)$$

of graded local cohomology modules from ($\sharp$), we get $\mathrm{H}^i_\mathfrak{M}(S) = [\mathrm{H}^i_\mathfrak{M}(S)]_0$ for all integers $i \le d-1$ and $\mathrm{a}_d(S) \le 0$ because $\mathrm{H}^i_\mathfrak{M}(T)$ and $\mathrm{H}^i_\mathfrak{M}(C)$ are concentrated in degree 0 for all $i \le d-1$ and $\mathrm{a}(T) \le 0$.

Now assume that S is not a Cohen-Macaulay G-module. We put $t = \operatorname{depth} S (< d)$. Because $\widehat{A} \otimes_A \mathrm{H}^t_\mathfrak{M}(S) \cong \bigoplus_{j=1}^m \mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j)$ as graded $\widehat{A} \otimes_A G$-modules, we can find $1 \le j \le m$ such that $(0) \ne \mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j) = [\mathrm{H}^t_{\widehat{A} \otimes_A \mathfrak{M}}(S_j)]_0$. From [KN], 3.1 we obtain that $\mathrm{a}_t(S_j) < \mathrm{a}_{t+1}(S_j)$. However this is impossible because $\mathrm{a}_t(S_j) = 0$ and $\mathrm{a}_{t+1}(S_j) \le \mathrm{a}_{t+1}(S) \le 0$. $\square$

Applying the functor $\mathrm{Hom}_G(\quad, \mathrm{K}_G)$ to the exact sequences ($\sharp$) and ($\sharp\sharp$), we get the following commutative and exact diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{K}_T & \xrightarrow{\varepsilon^*} & \mathrm{K}_G & \longrightarrow & \mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G) & \longrightarrow & \mathrm{Ext}^1_G(T, \mathrm{K}_G) & \longrightarrow & 0 \\
 & & {\scriptstyle \varphi^*}\uparrow\wr & & \| & & \uparrow & & & & \\
0 & \longrightarrow & \mathrm{K}_S & \longrightarrow & \mathrm{K}_G & \longrightarrow & \mathrm{Coker}\, \varepsilon^* \circ \varphi^* & \longrightarrow & 0 & & \\
 & & & & & & \uparrow & & & & \\
 & & & & & & 0 & & & &
\end{array}
$$

of graded G-modules where K_S formally denotes $\mathrm{Hom}_G(S, \mathrm{K}_G)$. We put $X = \mathrm{Coker}\, \varepsilon^* \circ \varphi^*$. Since $\mathfrak{a}$ is concentrated in degree 0, so is $\mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G)$ by the local duality theorem together with [GH], 2.2. Therefore X is concentrated in degree 0 by the diagram above. Now let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ stand for the canonical I-filtration of A (see [GI1], 1.1). Then we have $I^{i+1} \subseteq \omega_i$ for all $i \in \mathbb{Z}$ and $\mathrm{K}_G \cong \bigoplus_{i \geq 0} \omega_{i-1}/\omega_i$ as graded G-modules. Because $A/\omega_0 \cong [\mathrm{K}_G]_0 \twoheadrightarrow X$ there exists an ideal F in A such that $X \cong A/F$.

Lemma 2.3. $I = F$.

Proof. $I \cdot X = (0)$ because $I \cdot \mathrm{Hom}_G(\mathfrak{a}, \mathrm{K}_G) = (0)$, so that $I \subseteq F$. Assume $I \subsetneq F$ and choose $\mathfrak{p} \in \mathrm{Ass}_A\, A/I$ such that $I_\mathfrak{p} \subsetneq F_\mathfrak{p}$. We have $\mathrm{ht}_A\, \mathfrak{p} = 0$, as I is height unmixed. Since $I_\mathfrak{p} = (0)$, we get $\mathfrak{p} \not\supseteq \mathfrak{a}$, and hence $S_\mathfrak{p} = (0)$. Therefore $[\mathrm{K}_G]_\mathfrak{p} \cong [A/F]_\mathfrak{p}$ by $[\mathrm{K}_S]_\mathfrak{p} \cong \mathrm{K}_{S_\mathfrak{p}} = (0)$. We have $\mathrm{K}_{G_\mathfrak{p}} \cong A_\mathfrak{p}$, as $I_\mathfrak{p} = (0)$. Thus $A_\mathfrak{p} \cong [A/F]_\mathfrak{p}$. Therefore $I_\mathfrak{p} = F_\mathfrak{p}$, which is impossible. $\square$

Therefore we get the exact sequence

$$0 \to \mathrm{K}_S \to \mathrm{K}_G \to A/I \to 0$$

of graded G-modules. Then we get the following.

Lemma 2.4. $\mathrm{a}(S_\mathfrak{n}) = -1$ *for all maximal ideals* $\mathfrak{n}$ *in* B.

Proof. Since $A/\omega_0 \cong [\mathrm{K}_G]_0 \twoheadrightarrow A/I$ and $\omega_0 \supseteq I$, we get $\omega_0 = I$, so that $[\mathrm{K}_S]_0 = (0)$. Thus $\mathrm{a}_d(S) \leq -1$ by the local duality theorem. Let $\mathfrak{n}$ be a maximal ideal in B. Then $S_\mathfrak{n} = \mathrm{G}(IB_\mathfrak{n})$ is a Cohen-Macaulay ring with $\mathrm{a}(S_\mathfrak{n}) \leq -1$ because $\mathrm{a}_d(S) \geq \mathrm{a}(S_j) = \mathrm{a}(S_\mathfrak{n})$ for some $j = 1, 2, \ldots, m$ (recall that $\widehat{A} \otimes_A S \cong \prod_{j=1}^m S_j$ as graded rings). The converse inequality $\mathrm{a}(\mathrm{G}(IB_\mathfrak{n})) \geq -1$ follows from [U], 1.4, as $\mathrm{ht}_{B_\mathfrak{n}}\, IB_\mathfrak{n} = 1$. $\square$

Let $\mathfrak{n}$ be a maximal ideal in B. Since $S_\mathfrak{n} = \mathrm{G}(IB_\mathfrak{n})$ is a Cohen-Macaulay ring, we have $\mathrm{r}_{JB_\mathfrak{n}}(IB_\mathfrak{n}) \leq \mathrm{a}(S_\mathfrak{n}) + \lambda(IB_\mathfrak{n})$ by [U], 1.4 (recall that

$\mathrm{r}_{JB_{\mathfrak{n}}}(IB_{\mathfrak{n}}) \le \mathrm{r}_{\mathfrak{J}}(IB_{\mathfrak{n}})$ for any minimal reduction $\mathfrak{J} \subseteq JB_{\mathfrak{n}}$ of $IB_{\mathfrak{n}}$). Hence by 2.4 we get $\mathrm{r}_{JB_{\mathfrak{n}}}(IB_{\mathfrak{n}}) \le \ell - 1$ because $\lambda(IB_{\mathfrak{n}}) \le \ell$. Thus $I^{\ell}B_{\mathfrak{n}} = JI^{\ell-1}B_{\mathfrak{n}}$, so that $I^{\ell}B = JI^{\ell-1}B$. Therefore we get the following two lemmas.

Lemma 2.5. *If $\ell \ge 2$, then $\mathrm{r}_J(I) \le \ell - 1$.*

Proof. Because $\ell \ge 2$, $I^{\ell}\overline{A} = JI^{\ell-1}\overline{A}$, as $IB = I\overline{A}$. Then $I^{\ell} \subseteq JI^{\ell-1} + \mathfrak{a}$, so that we have $I^{\ell} = JI^{\ell-1}$ because $I \cap \mathfrak{a} = (0)$. □

Lemma 2.6. *Assume that $\ell = 1$. Then $\mathrm{r}_J(I) = 0$ if A/I is Cohen-Macaulay.*

Proof. We have $IB = JB$, as $\ell = 1$. Because A/I is Cohen-Macaulay, so is $\overline{A}$. Then $B = \overline{A}$, and hence $I = J$. □

We now come to prove Proposition 2.1.

Proof of Proposition 2.1. We need the following claims.

Claim 2.7. $\operatorname{depth} B/I^kB \ge d - k$ *for all* $1 \le k \le \ell - 1$.

Proof. We will prove the claim by induction on k. Let $k = 1$. By the depth lemma applied to the exact sequence $0 \to IB \to B \to B/IB \to 0$ of A-modules, we get $\operatorname{depth} B/IB \ge d - 1$, as $IB = I\overline{A} \cong I$. Let $k > 1$ and assume that $\operatorname{depth} B/I^{k-1}B \ge d - k + 1$. By our standard assumption in 2.1 we see $\operatorname{depth} I^{k-1}/I^k \ge d - k$ because of applying the depth lemma to the exact sequence $0 \to I^{k-1}/I^k \to A/I^k \to A/I^{k-1} \to 0$ of A-modules. We have $I^{k-1}/I^k \cong I^{k-1}\overline{A}/I^k\overline{A} \cong I^{k-1}B/I^kB$ as A-modules because $I \cap \mathfrak{a} = (0)$ and $I\overline{A} = IB$. Therefore we get the exact sequence $0 \to I^{k-1}/I^k \to B/I^kB \to B/I^{k-1}B \to 0$ of A-modules. Hence the assertion follows by the depth lemma. □

Claim 2.8. K_S *is generated by homogeneous elements of degree* 1.

Proof. Let $\mathfrak{n}$ be a maximal ideal in B. The set $\{a_1, a_2, \ldots, a_{\ell}\}$ is a basic generating set for $JB_{\mathfrak{n}}$. In fact, since $\dim A = \dim B_{\mathfrak{n}}$, we have $\ell \le \dim B_{\mathfrak{n}}$. Let $1 \le i < \ell$. Take any $Q \in \mathrm{V}(IB)$ such that $Q \subseteq \mathfrak{n}$ and $\mathrm{ht}_B\, Q = i \le \ell$. Let $\mathfrak{q} = Q \cap A$. Then $\mathrm{ht}_A\, \mathfrak{q} = \mathrm{ht}_B\, Q$ because $\dim A/\mathfrak{q} = \dim B/Q$. Hence $[J_i]_{\mathfrak{q}}$ is a reduction of $I_{\mathfrak{q}}$, so that J_iB_Q is a reduction of IB_Q.

Because of 2.7, $\operatorname{depth} B_{\mathfrak{n}}/I^kB_{\mathfrak{n}} \ge (d-1) - k + 1$ for all $1 \le k \le \ell - 1$. Then the fact that $\mathrm{G}(IB_{\mathfrak{n}})$ is a Cohen-Macaulay ring with $\mathrm{a}(\mathrm{G}(IB_{\mathfrak{n}})) = -1$ implies that $\mathrm{K}_{S_{\mathfrak{n}}}$ is generated by homogeneous elements of degree 1 (see, e.g., 4.1), and hence so is K_S. □

Recall that $K_B = I$. From [HSV], 2.4 together with 2.8 above, we obtain $K_S = G_+$ because S is a Cohen-Macaulay ring. Therefore we have the exact sequence

$$0 \to G_+ \to K_G \to A/I \to 0$$

of graded G-modules. Look at the homogeneous components

$$\begin{array}{ccccccc} & 0 & \to & A/\omega_0 & \to & A/I & \to 0 \\ 0 \to & I/I^2 & \to & \omega_0/\omega_1 & \to & 0 & \\ 0 \to & I^2/I^3 & \to & \omega_1/\omega_2 & \to & 0 & \\ & & & \vdots & & & \end{array}$$

of the exact sequence above, where $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ is the canonical I-filtration of A (see [GI1], 1.1). By induction on i, we will see that $\omega_i = I^{i+1}$ for all integers $i \geq 0$. In fact, we get $\omega_0 = I$, as $\omega_0 \supseteq I$. Let $i > 0$ and assume $\omega_{i-1} = I^i$. We note that $\omega_i \supseteq I^{i+1}$. From bijections above we obtain that $I^i/I^{i+1} \cong \omega_{i-1}/\omega_i = I^i/\omega_i$, and hence the natural surjective map $I^i/I^{i+1} \to I^i/\omega_i$ is bijective. Thus we get $\omega_i = I^{i+1}$ for all $i \geq 0$. This means that G is a Gorenstein ring. $\square$

In the rest of this section, we assume that our basic generating set $\{a_1, a_2, \ldots, a_\ell\}$ is a good generating set. Let us give the main result in this section.

Theorem 2.9. *Assume that I is height unmixed and* $\operatorname{depth} A/I^i \geq d-s-i$ *for all $1 \leq i \leq \ell - s$. Then the following three conditions are equivalent.*

(1) *G is a Gorenstein ring with $\mathrm{a}(G) = -s$.*

(2) *$\mathrm{r}_i \leq i - s$ for all $s \leq i < \ell$ and $\mathrm{r}_J(I) \leq \ell - s$.*

If $\ell - s \geq 2$, we may add the following.

(3) *$\mathrm{r}_s = 0$, $\mathrm{r}_i \leq i - s - 1$ for all $s < i < \ell$, and $\mathrm{r}_J(I) \leq \ell - s - 1$.*

Proof. By [GNN2], 6.5, We may assume G is a Cohen-Macaulay ring (see also 3.23). Then the condition (2) is equivalent to saying that $\mathrm{a}(G) = -s$ because of the a-invariant formula: $\mathrm{a}(G) = \max\{\mathrm{r}_i - i \mid 0 \leq i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$ (cf. [U], 1.4). Therefore the implications (1) $\Leftrightarrow$ (2) follow from Proposition 2.1. We shall prove the implication (1) $\Rightarrow$ (3). Let $\ell - s \geq 2$. The sequence $a_1 t, a_2 t, \ldots, a_s t$ is G-regular, and hence we may assume $s = 0$ (cf. [GNN2], 3.4). Then we have $\mathrm{r}_J(I) \leq \ell - 1$ by 2.5. Let $1 \leq i < \ell$. Take any $\mathfrak{q} \in \mathrm{V}(I)$ such that $\mathrm{ht}_A \mathfrak{q} = i$. We must show $\mathrm{r}_{J_{i\,\mathfrak{q}}}(I_\mathfrak{q}) \leq i - 1$. We may assume $I_\mathfrak{q} \neq (0)$. Since I is height unmixed, so is $I_\mathfrak{q}$, and moreover $I_\mathfrak{q}$ is generically a complete intersection with $\mathrm{ht}_{A_\mathfrak{q}} I_\mathfrak{q} = 0$. Since G is a Gorenstein ring, $G_\mathfrak{q}$ is a Cohen-Macaulay ring with $\mathrm{a}(G_\mathfrak{q}) = 0$ (see, e.g., 4.6). If $i \geq 2$, then $\mathrm{r}_{J_{i\,\mathfrak{q}}}(I_\mathfrak{q}) \leq i-1$ by 2.5, so that $\mathrm{r}_i \leq i-1$. Suppose $i=1$. Then by $I_\mathfrak{q}$ is unmixed, $A_\mathfrak{q}/I_\mathfrak{q}$ is Cohen-Macaulay, and hence $\mathrm{r}_{J_{i\,\mathfrak{q}}}(I_\mathfrak{q}) = 0$ by 2.6. Thus we get $\mathrm{r}_1 = 0$. The implication (3) $\Rightarrow$ (2) is obvious. $\square$

The assumption that $\ell - s \geq 2$ can not be removed in the theorem above. There exist examples of Gorenstein associated graded rings of ideals in the case where $\ell - s = 1$, $n = 0$, and $\mathrm{r}_J(I) \neq 0$. For example, let $k[X_1, X_2, \ldots, X_5]$ and $k[s,t]$ be the polynomial rings over a field k. Let m be an integer with $m \geq 2$ and $\varphi : k[X_1, X_2, \ldots, X_5] \to k[s,t]$ be the homomorphism of k-algebras defined by $\varphi(X_1) = s^2$, $\varphi(X_2) = s^3$, $\varphi(X_3) = st$, $\varphi(X_4) = st^m$, $\varphi(X_5) = t^m$. We put $R = k[X_1, X_2, \ldots, X_5]$ and $A = R_M$, where $M = (X_1, X_2, \ldots, X_5)$. Let $P = \ker\varphi$ and $I = PA$. Then I is generically a complete intersection with $\mathrm{ht}_A\, I = 3$ and $\operatorname{depth} A/I = 1 (= \dim A - \mathrm{ht}_A\, I - 1)$. The ideal P is generated by the following five elements: $f_1 = X_1^3 - X_2^2$, $f_2 = X_1X_4 - X_2X_5$, $f_3 = X_4^2 - X_1X_5$, $f_4 = X_2X_4 - X_1^2X_5$, and

$$f_5 = \begin{cases} X_3^m - X_1X_2^{i-1}X_5 & (m = 3i-1) \\ X_3^m - X_2^iX_5 & (m = 3i) \\ X_3^m - X_1^2X_2^{i-1}X_5 & (m = 3i+1). \end{cases}$$

We have the relation $f_1f_3 = X_1f_2^2 - f_4^2$, so that $J = (f_1 + f_3, f_2, f_4, f_5)A$ is a reduction of I with $\mathrm{r}_J(I) = 1$. Hence $\mathrm{G}(I)$ is a Gorenstein ring by 2.9.

We close this section with the next corollary, which follows from [I], 3.1 together with 2.9.

Corollary 2.10. *Assume that I is a height unmixed ideal with $s = 2$ and $\operatorname{depth} A/I^i \geq d - i - 2$ for all $1 \leq i \leq \ell - 2$. Then the following three conditions are equivalent.*

(1) $\mathrm{R}(I)$ *is a Gorenstein ring.*

(2) $\mathrm{r}_i \leq i-2$ *for all* $2 \leq i < \ell$ *and* $\mathrm{r}_J(I) \leq \ell - 2$.

If $\ell \geq 4$, we may add the following.

(3) $\mathrm{r}_2 = 0$, $\mathrm{r}_i \leq i-3$ *for all* $2 < i < \ell$, *and* $\mathrm{r}_J(I) \leq \ell - 3$.

The defining ideals of projective monomial varieties of codimension 2 always satisfy the conditions stated in 2.10 (see [BM]). Hence their Rees algebras are necessarily Gorenstein rings.

3 On Cohen-Macaulay associated graded rings.

The main purpose of this section is to extend a theorem given by [GNN2] in the case where the number n is arbitrary (see Proposition 3.19). First of all, let us state the following.

Proposition 3.1. *For each a basic generating set* $\{a_1, a_2, \ldots, a_\ell\}$ *for* J, *the following two assertions hold true.*

1. *If* G *is a Cohen-Macaulay ring, then there exists an integer* $a \geq -s$ *such that*
   (1) $\mathrm{r}_J(I) \leq a + \ell$,
   (2) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \leq i \leq a + s$, *and*
   (3) $(J_i : a_{i+1}) \cap I^{a+i+1} = J_i I^{a+i}$ *for all* $s \leq i < \ell$.

2. *Conversely, assume that there exist an integer* $a \geq -s$, *which satisfies the conditions* (1), (2), *and* (3) *above. Let* J_s *be a complete intersection ideal and let* δ *be an integer with* $\operatorname{depth} A \geq \delta \geq s$. *Then*

$$\operatorname{depth} G \geq \delta$$

if $\operatorname{depth} A/I^i + J_s \geq \min\{\delta - s, \delta + a - i\}$ *for all* $1 \leq i \leq a + \ell$.

Remark. By [Tr], 3.6 together with 3.9, the least number a that satisfies the above conditions (1), (2), and (3) (in case it exists) becomes the a^*-invariant of G: $\mathrm{a}^*(G) := \max\{\mathrm{a}_i(G) \mid i \in \mathbb{Z}\}$ if J_s is a complete intersection. We note $\mathrm{a}^*(G) \geq -s$ because $\mathrm{a}^*(G) \geq -\operatorname{grade} G_+$ by [Tr], 2.3, and $-\operatorname{ht}_G G_+ \geq \dim G/G_+ - \dim G = \dim A/I - d = -s$ by our standard assumption that $\dim A/I = d - s$. The ideal J_s is a complete intersection if the ring A is Cohen-Macaulay (see 3.2).

We will prepare some results for a proof of 3.1. Take an integer i with $s \leq i < \ell$. Then we have $\operatorname{ht}_A I + [J_i I^{\mathrm{r}_i} : I^{\mathrm{r}_i + 1}] \geq i + 1$ because $I^{\mathrm{r}_i + 1} A_\mathfrak{p} = J_i I^{\mathrm{r}_i} A_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \in \mathrm{V}(I)$ with $\operatorname{ht}_A \mathfrak{p} \leq i$. Therefore we can choose a system of parameters $x_{s+1}, x_{s+2}, \ldots, x_d$ for the ring A/I such that $x_{i+1} \in J_i I^{\mathrm{r}_i} : I^{\mathrm{r}_i + 1}$ for all $s \leq i < \ell$. We put $\mathfrak{q} = (a_1, a_2, \ldots, a_s, x_{s+1} - a_{s+1}, x_{s+2} - a_{s+2}, \ldots, x_\ell - a_\ell, x_{\ell+1}, x_{\ell+2}, \ldots, x_d)$. Then we get the following.

Lemma 3.2. $\sqrt{\mathfrak{q}} = \mathfrak{m}$

Proof. Let $\mathfrak{p} \in \mathrm{V}(\mathfrak{q})$. We shall prove that $a_1, a_2, \ldots, a_i \in \mathfrak{p}$ for all $0 \leq i \leq \ell$ by induction on i. We may assume $s + 1 \leq i \leq \ell$ and $a_1, a_2, \ldots, a_{i-1} \in \mathfrak{p}$. We have $x_i^{\mathrm{r}_{i-1}+2} \equiv x_i a_i^{\mathrm{r}_{i-1}+1} \mod \mathfrak{p}$, as $x_i \equiv a_i$. By the choice of x_i

and by the inductive hypothesis on i, we get $x_i a_i^{\mathrm{r}_{i-1}+1} \in J_{i-1} I^{\mathrm{r}_{i-1}} \subseteq \mathfrak{p}$. And hence $x_i^{\mathrm{r}_{i-1}+2} \in \mathfrak{p}$. Therefore $x_i \in \mathfrak{p}$, so that $a_i \in \mathfrak{p}$. Thus we see J and $(x_{s+1}, x_{s+2}, \dots, x_d)$ are contained in $\mathfrak{p}$, and hence $\mathfrak{p} = \mathfrak{m}$ (recall that $I^{r_J(I)+1} \subseteq J$ and $x_{s+1}, x_{s+2}, \dots, x_d$ is a system of parameters for A/I). $\square$

For $0 \le i \le d$, we put

$$A_i := \begin{cases} (a_1t, a_2t, \dots, a_st)G & (0 \le i \le s) \\ A_s + (x_{s+1} - a_{s+1}t, x_{s+2} - a_{s+2}t, \dots, x_i - a_it)G & (s+1 \le i \le \ell) \\ A_\ell + (x_{\ell+1}, x_{\ell+2}, \dots, x_i)G & (\ell+1 \le i \le d) \end{cases}$$

and

$$B_i := \begin{cases} (a_1t, a_2t, \dots, a_st)G & (0 \le i \le s) \\ B_s + (x_{s+1}, x_{s+2}, \dots, x_i, a_{s+1}t, a_{s+2}t, \dots, a_it)G & (s+1 \le i \le \ell) \\ B_\ell + (x_{\ell+1}, x_{\ell+2}, \dots, x_i)G & (\ell+1 \le i \le d). \end{cases}$$

Then we get $\sqrt{A_j} = \sqrt{B_j}$ for all $0 \le j \le d$. In fact, let $0 \le j \le d$. We may assume $j > s$. The inclusion $A_j \subseteq B_j$ is trivial. Similarly as in the proof above, we take $Q \in \mathrm{V}(A_j)$. It is enough to show $a_1t, a_2t, \dots, a_it \in Q$ for all $0 \le i \le j$. Let us use induction on i. We may assume that $s+1 \le i \le j$ and that it holds true for $i-1$. Since $x_i \equiv a_it \mod Q$, we have $x_i^{\mathrm{r}_{i-1}+2} \equiv x_i(a_it)^{\mathrm{r}_{i-1}+1}$ because $(x_i a_i^{\mathrm{r}_{i-1}+1})t^{\mathrm{r}_{i-1}+1} \in (J_{i-1}I^{\mathrm{r}_{i-1}}t^{\mathrm{r}_{i-1}+1})G \subseteq Q$ by the choice of x_i and by the inductive hypothesis on i. Hence $x_i \in Q$. Therefore $a_it \in Q$.

In the case where J is a *special* reduction of I in the sense of Aberbach and Huneke, the following lemmas have been proved in their paper [AH].

Lemma 3.3. *G is a Cohen-Macaulay ring if and only if the sequence*

$$a_1t, a_2t, \dots, a_st, x_{s+1} - a_{s+1}t, x_{s+2} - a_{s+2}t, \dots, x_\ell - a_\ell t, x_{\ell+1}, x_{\ell+2}, \dots, x_d$$

is G-regular.

Proof. Let $0 \le j \le d$. The equality $\sqrt{A_j} = \sqrt{B_j}$ implies that all minimal prime ideals of A_j are graded because B_j is a graded ideal, and moreover $\sqrt{A_d} = \mathfrak{M}$ because $\sqrt{B_d} = \mathfrak{M}$ (recall that $x_{s+1}, x_{s+2}, \dots, x_d$ is a system of parameters for A/I). Therefore, if G is a Cohen-Macaulay ring, then that sequence is G-regular because all associated prime ideals of A_j are graded. The converse implication is trivial. $\square$

Lemma 3.4. *Let i be an integer with $0 \le i < \ell$. If G is a Cohen-Macaulay ring, then*

$$[(a_1t, a_2t, \dots, a_it)G :_G a_{i+1}t]_{\ge \mathrm{a}(G)+i+1} \subseteq (a_1t, a_2t, \dots, a_it)G.$$

Proof. Thanks to the a-invariant formula: $\mathrm{a}(G) = \max\{\mathrm{r}_i - j \mid s \leq j < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$ (cf. [U], 1.4), we have $\mathrm{r}_j \leq \mathrm{a}(G) + j$ for all $s \leq j < \ell$. Since the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by 3.3, we may assume $s = 0$ by [VV]. We need the following.

Lemma 3.5. *Let $s = 0$ and $0 \leq i \leq \ell$. Assume $\mathrm{r}_j \leq \mathrm{a}(G) + j$ for all $0 \leq j < \ell$. Then*

$$(x_1 - a_1t, x_2 - a_2t, \ldots, x_i - a_it)G \cap [G]_{\geq \mathrm{a}(G)+i+1} =$$
$$= [(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1}.$$

Proof. We have

$$[(x_1, x_2, \ldots, x_{i+1})G]_{\geq \mathrm{a}(G)+i+1} \subseteq [(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1}$$

for all $0 \leq i < \ell$ because $x_{i+1}I^{m+1} \subseteq J_iI^m$ for all $m \geq \mathrm{r}_i + 1$ by the choice of x_{i+1} (recall that $\mathrm{a}(G) + i + 1 \geq \mathrm{r}_i + 1$). Hence the left side of the required equality is contained in the right side. To prove the converse inclusion, we will use induction on i. We may assume $i > 0$ and it is true for $i - 1$. Let $m \geq \mathrm{a}(G) + i + 1$ and take any element $\eta \in [(a_1t, a_2t, \ldots, a_it)G]_m$. Write $\eta = \sum_{\alpha=1}^{i}(a_\alpha t)\eta_\alpha$ for some $\eta_\alpha \in [G]_{m-1}$. Then $\sum_{\alpha=1}^{i} x_\alpha\eta_\alpha \in [(x_1, x_2, \ldots, x_i)G]_{\geq a+i} \subseteq [(a_1t, a_2t, \ldots, a_{i-1}t)G]_{\geq \mathrm{a}(G)+i}$ (recall that the above inclusion). So $\sum_{\alpha=1}^{i} x_\alpha\eta_\alpha \in (x_1 - a_1t, x_2 - a_2t, \ldots, x_{i-1} - a_{i-1}t)G$ by the inductive hypothesis on i. Therefore $\eta = \sum_{\alpha=1}^{i} x_\alpha\eta_\alpha - \sum_{\alpha=1}^{i}(x_\alpha - a_\alpha t)\eta_\alpha \in (x_1 - a_1t, x_2 - a_2t, \ldots, x_i - a_it)G$. □

Lemma 3.5 leads to an injective homomorphism

$$[G/(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1} \hookrightarrow G/(x_1 - a_1t, x_2 - a_2t, \ldots, x_i - a_it)G$$

of G-modules for all $0 \leq i \leq \ell$. Let $0 \leq i < \ell$. Then since the element $x_{i+1} - a_{i+1}t$ is $G/(x_1 - a_1t, x_2 - a_2t, \ldots, x_i - a_it)G$-regular by 3.3, it is also $[G/(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1}$-regular by the injective map above. Hence $a_{i+1}t$ is $[G/(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1}$-regular (recall that $x_{i+1}[G]_{\geq \mathrm{a}(G)+i+1} \subseteq [(a_1t, a_2t, \ldots, a_it)G]_{\geq \mathrm{a}(G)+i+1}$). This completes the proof of 3.4 □

Corollary 3.6. *If G is a Cohen-Macaulay ring, then the conditions* (1), (2)*, and* (3) *in 3.1 are fulfilled for the integer* $\mathrm{a}(G)$.

Proof. We have $\mathrm{a}(G) \geq -s$ (see Remark in this section). Since G is a Cohen-Macaulay ring, the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by 3.3, so that $J_s \cap I^i = J_sI^{i-1}$ for all $i \in \mathbb{Z}$ by [VV]. Then Lemma 3.4 means that the sequence $a_1t, a_2t, \ldots, a_\ell t$ is G_+-filter regular, and hence the assertion follows from [Tr], 3.6. □

Let a be an integer with $a \geq -s$ and δ an integer with $s \leq \delta \leq \operatorname{depth} A$. For each integer $m \geq s$, we consider the following two conditions.

(A_m) $\operatorname{depth} A/J_j I^{a+i} \geq \delta - i$ whenever $s \leq i \leq m$ and $0 \leq j \leq i$.

(B_m) $(J_i : a_{i+1}) \cap I^{a+i+1} = J_i I^{a+i}$ whenever $s \leq i \leq m < \ell$.

Recall that ($\mathrm{B}_{\ell-1}$) is our condition (3) in 3.1. Now let us note the following lemmas due to [GNN2], which are delicately used in this section.

Lemma 3.7. *Let $s \leq m < \ell$ and assume the condition* (B_m) *is satisfied. Then $J_{\alpha-1} \cap I^{a+i} = J_{\alpha-1} I^{a+i-1}$ if $s+1 \leq \alpha \leq i \leq m+1$.*

Proof. See [GNN2], Proof of Claim 3. □

Lemma 3.8. *Let $s \leq i \leq \ell$. Assume $\mathrm{r}_J(I) \leq a+\ell$ and the condition* ($\mathrm{B}_{\ell-1}$) *is satisfied. Then $J_i \cap I^j = J_i I^{j-1}$ for all $j \geq a+i+1$, and hence we have*

$$[(a_1t, a_2t, \ldots, a_it)G :_G a_{i+1}t]_{\geq a+i+1} \subseteq (a_1t, a_2t, \ldots, a_it)G$$

if $s \leq i < \ell$.

Proof. Using the same arguments as in the proof of in [GNN2], 3.1, we get $J_i \cap I^j = J_i I^{j-1}$ for all $j \geq a+i+1$. For the last assertion, see in [GNN2], 4.1. □

As a direct consequence of 3.8, we note the next claim.

Corollary 3.9. *Assume that the conditions* (1)*,* (2)*, and* (3) *in 3.1 are fulfilled for an integer a. Then $J_s \cap I^j = J_s I^{j-1}$ for all $j \in \mathbb{Z}$.*

Proof. Lemma 3.8 implies $J_s \cap I^j = J_s I^{j-1}$ for all $i \geq a+s+1$. Then we get the required equality by condition (2). □

In the rest of this section we always assume J_s is a complete intersection ideal. We will use repeatedly the result of [VV] that, for each integer h, if $J_s \cap I^i = J_s I^{i-1}$ for all $1 \leq i \leq h$, then $J_j \cap I^i = J_j I^{i-1}$ whenever $1 \leq i \leq h$ and $0 \leq j \leq s$. We also use the isomorphism $M/N \tilde{\to} \alpha M/\alpha N$ form $x \mod N$ to $\alpha x \mod \alpha N$, where M, N, and α denote an A-module, an A-submodule of M, and an M-regular element, respectively.

Lemma 3.10. *Let h be an integer and let $0 \leq j \leq s$. Assume that $J_s \cap I^i = J_s I^{i-1}$ for all $1 \leq i \leq h$. Then $\operatorname{depth} A/I^i + J_j \geq \min\{\delta - s, \delta + a - i\}$ for all $1 \leq i \leq h$ if so is $\operatorname{depth} A/I^i + J_s$.*

Proof. Descending induction on j. The assertion being trivial for $j = s$, we may assume that $j < s$ and that it is true for $j + 1$. We need to show that the assertion holds true for j. Suppose that it is not true and take an integer $1 \leq i \leq h$ as small as possible, so that we have $i > 1$. We set $\overline{A} = A/J_j$. Then $\operatorname{depth} \overline{A}/I^i\overline{A} + a_{j+1}\overline{A} \geq \min\{\delta - s, \delta + a - i\}$ by the inductive hypothesis on j. We consider the canonical exact sequence

$$0 \to \frac{I^i\overline{A} + a_{j+1}\overline{A}}{I^i\overline{A}} \to \frac{\overline{A}}{I^i\overline{A}} \to \frac{\overline{A}}{I^i\overline{A} + a_{j+1}\overline{A}} \to 0$$

of A-modules. Since $J_{j+1} \cap I^i = J_{j+1}I^{i-1}$ and since a_{j+1} is an $\overline{A}$-regular element, we get the isomorphisms

$$\frac{I^i\overline{A} + a_{j+1}\overline{A}}{I^i\overline{A}} \cong \frac{a_{j+1}\overline{A}}{a_{j+1}I^{i-1}\overline{A}} \cong \frac{\overline{A}}{I^{i-1}\overline{A}}$$

as A-modules. We have that $\operatorname{depth} \overline{A}/I^{i-1}\overline{A} \geq \min\{\delta - s, \delta + a - i + 1\}$ by the minimality of i, and hence, applying the depth lemma to the short exact sequence above, we get $\operatorname{depth} \overline{A}/I^i\overline{A} \geq \min\{\delta - s, \delta + a - i\}$. This is a contradiction, which completes the proof. ☐

Corollary 3.11. *Let $s \leq m < \ell$. Assume that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+s$. Then* $\operatorname{depth} A/I^i \geq \min\{\delta - s, \delta + a - i\}$ *for all* $1 \leq i \leq a+m+1$ *if* (B_m) *is satisfied.*

Proof. By 3.7 together with the assumption that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+s$, we get $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+m+1$, so that the required inequality follows from 3.10. ☐

Lemma 3.12. *Let h be an integer and let $0 \leq j \leq s$. Assume that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq h$. Then* $\operatorname{depth} A/J_jI^i \geq \delta - s$ *for all $1 \leq i \leq h$ if so is* $\operatorname{depth} A/I^i + J_s$.

Proof. We use induction on j. If $j = 0$, our claim is clear since $\operatorname{depth} A \geq \delta$. Assume that $j > 0$ and that $\operatorname{depth} A/J_{j-1}I^i \geq \delta - s$. Let $\overline{A} = A/J_{j-1}$. Recall that $\operatorname{depth} \overline{A}/I^i\overline{A} \geq \delta - s$ by 3.10. Looking at the following exact sequence and isomorphism:

$$0 \to \frac{a_j\overline{A}}{a_jI^i\overline{A}} \to \frac{\overline{A}}{a_jI^i\overline{A}} \to \frac{\overline{A}}{a_j\overline{A}} \to 0 \qquad \text{and} \qquad \frac{a_j\overline{A}}{a_jI^i\overline{A}} \cong \frac{\overline{A}}{I^i\overline{A},}$$

we see that $\operatorname{depth} \overline{A}/a_jI^i\overline{A} \geq \delta - s$ by the depth lemma. Hence $\operatorname{depth} A/J_jI^i + J_{j-1} \geq \delta - s$. We consider the two natural exact sequences

$$0 \to \frac{J_{j-1}}{J_jI^i \cap J_{j-1}} \to \frac{A}{J_jI^i} \to \frac{A}{J_jI^i + J_{j-1}} \to 0 \quad \text{and}$$

$$0 \to \frac{J_{j-1}}{J_{j-1}I^i} \to \frac{A}{J_{j-1}I^i} \to \frac{A}{J_{j-1}} \to 0$$

of A-modules. Here we have $J_{j-1}/J_jI^i \cap J_{j-1} = J_{j-1}/J_{j-1}I^i$ because

$$\begin{aligned} J_{j-1} \cap J_jI^i &= J_{j-1} \cap (J_{j-1}I^i + a_jI^i) \\ &= J_{j-1}I^i + (J_{j-1} \cap a_jI^i) \\ &= J_{j-1}I^i + a_j((J_{j-1} : a_j) \cap I^i) \\ &= J_{j-1}I^i + a_j(J_{j-1} \cap I^i) \\ &= J_{j-1}I^i + a_jJ_{j-1}I^{i-1} \\ &= J_{j-1}I^i. \end{aligned}$$

Therefore we get $\operatorname{depth} A/J_jI^i \geq \delta - s$ by the depth lemma. □

Corollary 3.13. *Assume that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+s$. Then the condition* (A_s) *is satisfied if* $\operatorname{depth} A/I^i \geq \delta - s$ *for all* $1 \leq i \leq a+s$.

Furthermore, we arrange the following three Lemmas.

Lemma 3.14. *Let $s \leq m < \ell$. Assume that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+s$ and the condition* (B_m) *is satisfied. Then $I^{a+i}/J_{\alpha-1}I^{a+i-1} \cong J_\alpha I^{a+i}/J_{\alpha-1}I^{a+i}$ as A-modules for all $1 \leq \alpha \leq i \leq m+1$.*

Proof. Let $I^{a+i}/J_{\alpha-1}I^{a+i-1} \to J_\alpha I^{a+i}/J_{\alpha-1}I^{a+i}$ ($x \mod J_{\alpha-1}I^{a+i-1} \mapsto a_\alpha x \mod J_\alpha I^{a+i}$). It suffices to show this is injective. Let $x \in I^{a+i}$ and suppose $a_\alpha x \in J_{\alpha-1}I^{a+i}$. Then we have $x \in (J_{\alpha-1} : a_\alpha) \cap I^{a+i} = [(J_{\alpha-1} : a_\alpha) \cap I^{a+\alpha}] \cap I^{a+i} \subseteq J_{\alpha-1} \cap I^{a+i}$ (use the fact that a_α is $A/J_{\alpha-1}$-regular if $\alpha \leq s$ and use the condition (B_m) if $\alpha > s$). We shall show $J_{\alpha-1} \cap I^{a+i} = J_{\alpha-1}I^{a+i-1}$. In fact, by 3.7, we obtain that $J_{\alpha-1} \cap I^{a+i} = J_{\alpha-1}I^{a+i-1}$ if $s+1 \leq \alpha \leq j \leq m+1$. And moreover, by 3.7 together with our standard assumption, we see $J_s \cap I^j = J_sI^{j-1}$ for all $1 \leq j \leq a+m+1$, so that $J_k \cap I^j = J_kI^{j-1}$ if $1 \leq k \leq s$ and $1 \leq j \leq a+m+1$ by [VV]. Thus the required equality follows. Then $x \in J_{\alpha-1}I^{a+i-1}$ and hence it is injective. □

Lemma 3.15. *Let $s \leq m < \ell$. Assume that $J_s \cap I^i = J_sI^{i-1}$ for all $1 \leq i \leq a+s$. Then the implication* $(\mathrm{B}_m) \Rightarrow (\mathrm{A}_{m+1})$ *holds true if* $\operatorname{depth} A/I^i + J_s \geq \min\{\delta - s, \delta + a - i\}$ *for all* $1 \leq i \leq a+m+1$.

Proof. We must prove that $\operatorname{depth} A/J_\alpha I^{a+i} \geq \delta - i$ whenever $s \leq i \leq m+1$ and $0 \leq \alpha \leq i$. Let us use induction on α. When $i = s$, our assertion follows form 3.13. Let $i > s$. If $\alpha = 0$, it is clear. We may assume $\alpha > 0$ and it holds true for $\alpha - 1$. Then, applying the depth lemma to the natural exact sequences

$$0 \to J_\alpha I^{a+i}/J_{\alpha-1}I^{a+i} \to A/J_{\alpha-1}I^{a+i} \to A/J_\alpha I^{a+i} \to 0$$

and

$$0 \to I^{a+i}/J_{\alpha-1}I^{a+i-1} \to A/J_{\alpha-1}I^{a+i-1} \to A/I^{a+i} \to 0$$

of A-modules, we get the assertion by 3.11 and 3.14. □

Lemma 3.16. *Assume that the conditions* (1), (2)*, and* (3) *in 3.1 are satisfied. Then*

$$\operatorname{depth} I^{a+i+1}/J_iI^{a+i} + I^{a+i+2} \geq \delta - i - 2$$

for all $s \leq i < \ell$ *if* $\operatorname{depth} A/I^i + J_s \geq \min\{\delta - s, \delta + a - i\}$ *for all* $1 \leq i \leq a + \ell$.

Proof. The condition (A_ℓ) is satisfied by 3.15. Looking at the natural exact sequences

$$0 \to I^{a+i+1}/J_iI^{a+i} \to A/J_iI^{a+i} \to A/I^{a+i+1} \to 0$$

and

$$0 \to I^{a+i+2}/J_iI^{a+i+1} \to A/J_iI^{a+i+1} \to A/I^{a+i+2} \to 0$$

of A-modules, we get $\operatorname{depth} I^{a+i+1}/J_iI^{a+i} \geq \delta - i$ and $\operatorname{depth} I^{a+i+2}/J_iI^{a+i+1} \geq \delta - i - 1$ by the depth lemma together with 3.11 and (A_ℓ). We consider the canonical exact sequence

$$0 \to I^{a+i+2}/J_iI^{a+i} \cap I^{a+i+2} \to I^{a+i+1}/J_iI^{a+i} \to I^{a+i+1}/J_iI^{a+i} + I^{a+i+2} \to 0$$

of A-modules. $J_iI^{a+i} \cap I^{a+i+2} \subseteq J_i \cap I^{a+i+2} = J_iI^{a+i+1}$ by 3.8. Hence we get the equality $I^{a+i+2}/J_iI^{a+i} \cap I^{a+i+2} = I^{a+i+2}/J_iI^{a+i+1}$, so that the assertion follows from applying the depth lemma to the exact sequence above. □

We are now ready to prove 3.1.

Proof of Proposition 3.1. The assertion 1 directly follows from 3.6. Let us prove the assertion 2. Since J_s is a complete intersection, the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by [VV] together with 3.9. Hence, passing to the ring A/J_s, we may assume $s = 0$ (cf. [GNN2], 3.4). For each $0 \leq i \leq \ell$, we put $\mathrm{U}^{(i)} = [G/J_itG]_{\geq a+i+1}$. Then $a_{i+1}t$ is an $\mathrm{U}^{(i)}$-regular element for

$0 \leq i < \ell$ by 3.8. Let $\mathrm{V}^{(i)} = \mathrm{U}^{(i)}/a_{i+1}t\,\mathrm{U}^{(i)}$. We have the natural exact sequence

$$0 \to \mathrm{U}^{(i+1)} \to \mathrm{V}^{(i)} \to \mathrm{W}^{(i)} \to 0 \qquad (\sharp)$$

of graded G-modules. Here $\mathrm{W}^{(i)} = [\mathrm{W}^{(i)}]_{a+i+1} \cong [\mathrm{U}^{(i)}]_{a+i+1} \cong I^{a+i+1}/J_iI^{a+i} + I^{a+i+2}$. Then $\operatorname{depth} \mathrm{W}^{(i)} \geq \delta - i - 2$ for all $0 \leq i < \ell$ by 3.16. We have the claim that

$$\mathrm{a}_j(\mathrm{U}^{(i)}) \leq a + i \text{ whenever } 0 \leq i \leq \ell \text{ and } j \in \mathbb{Z}.$$

In fact, We will prove it by descending induction on i. The assertion holds true if $i = \ell$ because $\mathrm{U}^{(\ell)} = (0)$ (recall that $\mathrm{r}_J(I) \leq a + \ell$). Let $i < \ell$ and assume that $\mathrm{a}_j(\mathrm{U}^{(i+1)}) \leq a + i + 1$ for all $j \in \mathbb{Z}$. Applying the local cohomology functors $\mathrm{H}^j_{\mathfrak{M}}(*)$ $(j \in \mathbb{Z})$ to the exact sequence $(\sharp)$, we get the resulting exact sequence

$$\cdots \to \mathrm{H}^j_{\mathfrak{M}}(\mathrm{U}^{(i+1)}) \to \mathrm{H}^j_{\mathfrak{M}}(\mathrm{V}^{(i)}) \to \mathrm{H}^j_{\mathfrak{M}}(\mathrm{W}^{(i+1)}) \to \cdots$$

of graded local cohomology modules. we have $\mathrm{a}_j(\mathrm{V}^{(i)}) \leq a + i + 1$ by the inductive hypothesis on i (recall that $\mathrm{W}^{(i)}$ is concentrated in degree $a+i+1$). Thus $\mathrm{a}_j(\mathrm{U}^{(i)}) \leq a + i$, as $a_{i+1}t$ is an $\mathrm{U}^{(i)}$-regular element.

Hence we have in particular that $\mathrm{a}_j(\mathrm{U}^{(0)}) \leq a$ for all $j \in \mathbb{Z}$. We consider the canonical exact sequence

$$0 \to \mathrm{U}^{(0)} \to G \to C \to 0 \qquad (\sharp\sharp)$$

of graded G-modules. Then $C = C_0 \oplus C_1 \oplus \cdots \oplus C_a$, so that $\mathrm{a}_j(C) \leq a$ (see [GH], 2.2). Hence we get $\mathrm{a}_j(G) \leq a$ for all $j \in \mathbb{Z}$.

Now let us prove $\operatorname{depth} G \geq \delta$. Firstly, we shall consider the case where $\ell = 0$. Then $G = G_0 \oplus G_1 \oplus \cdots \oplus G_{\mathrm{r}_J(I)}$. Let $0 \leq j \leq \mathrm{r}_J(I)$. By our standard assumption of the depths, we have $\operatorname{depth} A/I^j \geq \delta$, as $\mathrm{r}_J(I) \leq a$. Then applying the depth lemma to the exact sequence $0 \to I^j/I^{j+1} \to A/I^{j+1} \to A/I^j \to 0$ of A-modules, we get $\operatorname{depth} I^j/I^{j+1} \geq \delta$. Then $\operatorname{depth} G_j \geq \delta$, so that $\operatorname{depth} G \geq \delta$ by [GH], 2.2.

Let $\ell > 0$. Hence $\mathrm{U}^{(0)} \neq (0)$. Suppose that $\operatorname{depth} G < \delta$. We put $t = \operatorname{depth} G$ and $\alpha = \mathrm{a}_t(G)$. Then by [KN], 3.1, we have $\alpha < a$ because $\mathrm{a}_j(G) \leq a$ for all $j \in \mathbb{Z}$. We have $\operatorname{depth} C \geq t$ and $\mathrm{H}^t_{\mathfrak{M}}(C) = [\mathrm{H}^t_{\mathfrak{M}}(C)]_a$ by our standard assumption (recall that $\operatorname{depth} C_j = \operatorname{depth} I^j/I^{j+1} \geq \delta$ for all $0 \leq j \leq a-1$ and that $\operatorname{depth} C_a = \operatorname{depth} I^a/I^{a+1} \geq \delta - 1$). Therefore, from the resulting exact sequence

$$\cdots \to 0 \to 0 \to \mathrm{H}^t_{\mathfrak{M}}(\mathrm{U}^{(0)}) \to \mathrm{H}^t_{\mathfrak{M}}(G) \to \mathrm{H}^t_{\mathfrak{M}}(C) \to \cdots$$

of graded local cohomology modules from $(\sharp\sharp)$ we obtain $\operatorname{depth} \mathrm{U}^{(0)} = t$ and $\mathrm{a}_t(\mathrm{U}^{(0)}) = \alpha$ because $\alpha < a$. Then by [GNN2], *Claim*, we have the fact that

$$\mathrm{U}^{(i)} \neq (0),\ \operatorname{depth} \mathrm{U}^{(i)} = t - i, \text{ and } \mathrm{a}_{t-i}(\mathrm{U}^{(i)}) = \alpha + i \text{ for any } 0 \leq i \leq \ell.$$

But let us give a sketch of proof for the sake of completeness. In deed, assume that the assertion is not true and take i as small as possible. Then $i > 0$. Since $a_{i-1}t$ is $\mathrm{U}^{(i-1)}$-regular, we have $\operatorname{depth} \mathrm{V}^{(i-1)} = t - i$ and $\mathrm{a}_{t-i}(\mathrm{V}^{(i-1)}) = \alpha + i$ by minimality of i. By $\operatorname{depth} \mathrm{W}^{(i-1)} \geq \delta - i - 1$ we obtain the resulting exact sequence

$$\cdots \to 0 \to 0 \to \mathrm{H}^{t-i}_{\mathfrak{M}}(\mathrm{U}^{(i)}) \to \mathrm{H}^{t-i}_{\mathfrak{M}}(\mathrm{V}^{(i-1)}) \to \mathrm{H}^{t-i}_{\mathfrak{M}}(\mathrm{W}^{(i-1)}) \to \cdots$$

of graded local cohomology modules from ($\sharp$). Hence, because $\mathrm{W}^{(i-1)} = [\mathrm{W}^{(i-1)}]_{a+i}$ and because $\mathrm{a}_{t-i}(\mathrm{V}^{(i-1)}) = \alpha + i < a + i$, we get $\mathrm{U}^{(i)} \neq (0)$, $\operatorname{depth} \mathrm{U}^{(i)} = t - i$ and $\mathrm{a}_{t-i}(\mathrm{U}^{(i)}) = \alpha + i$, which is contradiction to the choice of i.

Thus we get $\mathrm{U}^{(i)} \neq (0)$ for any $0 \leq i \leq \ell$. However, this contradicts $\mathrm{U}^{(\ell)} = (0)$. Therefore $\operatorname{depth} G \geq \delta$. □

We note a direct consequence of 3.1.

Corollary 3.17. *Let A be a Cohen-Macaulay ring and assume that $\operatorname{depth} A/I^i + J_s \geq \min\{d - s, d - s + n - i\}$ for all $1 \leq i \leq n - s + \ell$. Then the following two conditions are equivalent.*

1. *G is a Cohen-Macaulay ring with $\mathrm{a}(G) = n - s$.*
2. (1) *$\mathrm{r}_J(I) \leq n - s + \ell$,*
   (2) *$J_s \cap I^i = J_s I^{i-1}$ for all $1 \leq i \leq n$, and*
   (3) *$(J_i : a_{i+1}) \cap I^{n-s+i+1} = J_i I^{n-s+i}$ for all $s \leq i < \ell$.*

Proof. Assume the condition 2 is fulfilled. Then the Cohen-Macaulayness of G follows from the assertion 2 in 3.1. We shall prove $\mathrm{a}(G) = n - s$. Since the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular, we may assume $s = 0$ (cf. [GNN2], 3.4). Recall that the exact sequence

$$0 \to \mathrm{U}^{(0)} \to G \to C \to 0 \tag{$\sharp\sharp$}$$

of G-modules in the proof of 3.1. Since $\mathrm{a}(G) \leq n$, it suffices to show $\dim C_n = d$. Suppose $[C_n]_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} A$ with $\dim A/\mathfrak{p} = d$. Then, since the ring A is Cohen-Macaulay and since $C_n \cong I^n/I^{n+1}$ as A-modules, $I^n_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \mathrm{V}(I)$ with $\operatorname{ht}_A \mathfrak{p} = 0$ by Nakayama's Lemma. This contradicts to the choice of n. The converse implication follows from 3.6. □

In the rest of this section we always assume our basic generating set $\{a_1, a_2, \ldots, a_\ell\}$ is a good generating set and consider the case where $\delta = d$. Then we get

Lemma 3.18. *Let $s \le m < \ell$. Then the implication* $(\mathrm{A}_m) \Rightarrow (\mathrm{B}_m)$ *holds true if* $\mathrm{r}_i \le a + i$ *for all* $s \le i \le m$.

Proof. Let $s \le i \le m$ and $L = (J_i : a_{i+1}) \cap I^{a+i+1}$. Then $L \supseteq J_i I^{a+i}$. Take any $\mathfrak{p} \in \mathrm{Ass}_A A/J_i I^{a+i}$. It suffices to show that $LA_\mathfrak{p} = J_i I^{a+i} A_\mathfrak{p}$. If $\mathfrak{p} \notin \mathrm{V}(I)$, then $a_{i+1} \notin \mathfrak{p}$ because $\{a_1, a_2, \cdots, a_\ell\}$ is a good generating set, so that we have nothing to prove. We may assume $\mathfrak{p} \in \mathrm{V}(I)$. The condition (A_m) implies $\mathrm{ht}_A \mathfrak{p} \le i$, as $\delta = d$. Then $LA_\mathfrak{p} = J_i I^{a+i} A_\mathfrak{p}$ since $\mathrm{r}_i \le a+i$. □

Let us note the conditions (1), (2), and (3) below are necessary conditions of the Cohen-Macaulayness of the ring G with $a = \mathrm{a}(G)$ (recall that the a-invariant formula).

Proposition 3.19. *Let the ring A be Cohen-Macaulay and assume that there exists an integer $a \ge -s$ such that*

(1) $\mathrm{r}_J(I) \le a + \ell$,

(2) $\mathrm{r}_i \le a + i$ *for all* $s \le i < \ell$,

(3) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \le i \le a + s$, *and*

(4) $\mathrm{depth}\, A/I^i + J_s \ge \min\{d - s, d + a - i\}$ *for all* $1 \le i \le a + \ell$.

Then G is a Cohen-Macaulay ring.

Proof. Since (A_s) is satisfied by 3.13, $(\mathrm{B}_{\ell-1})$ is satisfied by 3.15 and 3.18. Therefore the assertion follows from 3.1 □

Let us give some consequences of 3.19. The following corollary is a generalization of a result due to [VV].

Corollary 3.20. *Put* $a = \max\{\mathrm{r}_i - i \mid s \le i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$ *and assume that* $\mathrm{depth}\, A/I^i + J_s \ge \min\{d - s, d + a - i\}$ *for all* $1 \le i \le a + \ell$. *Then the following two conditions are equivalent.*

1. *G is a Cohen-Macaulay ring.*
2. *A is a Cohen-Macaulay ring and* $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \le i \le a + s$.

When this is the case, we have $a = \mathrm{a}(G)$.

The last assertion directly follows form a-invariant formula (cf. [U], 1.4). The next corollary covers a result given by [N2] in the case of ideal adic filtrations.

Corollary 3.21. *Let A be a Cohen-Macaulay ring and put* $a = \max\{\mathrm{r}_i - i \mid s \le i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$. *Assume that* $\mathrm{depth}\, A/I^i + J_s \ge \min\{d - s, d + a - i\}$ *for all* $1 \le i \le a + \ell$. *Then the following two conditions are equivalent.*

1. *G is a Cohen-Macaulay ring.*

2. $\mathrm{G}(I_{\mathfrak{p}})$ *is a Cohen-Macaulay ring for all* $\mathfrak{p} \in \mathrm{V}(I)$ *with* $\mathrm{ht}_A \mathfrak{p} = s$.

When this is the case, we have $a = \mathrm{a}(G)$.

Proof. It is enough to prove that the condition 2 implies $J_s \cap I^i = J_s I^{i-1}$ for all $1 \leq i \leq a+s$ by 3.20. We will use induction on i. If $i=1$, the assertion is clear. Let $i \geq 2$ and assume that it holds true for $i-1$. Take any $\mathfrak{p} \in \mathrm{Ass}_A A/J_s I^{i-1}$. It suffices to show that $(J_s \cap I^i)A_{\mathfrak{p}} = J_i I^{i-1} A_{\mathfrak{p}}$ because it is trivial that $J_s \cap I^i \supseteq J_s I^{i-1}$. We may assume $\mathfrak{p} \in \mathrm{V}(I)$. Thanks to 3.12, we get $\operatorname{depth} A/J_s I^{i-1} \geq d-s$ by the inductive hypothesis on i, and hence $\mathrm{ht}_A \mathfrak{p} = s$. Since $\mathrm{G}(I_{\mathfrak{p}})$ is a Cohen-Macaulay ring, the sequence $a_1t, a_2t, \ldots, a_st$, is $\mathrm{G}(I_{\mathfrak{p}})$-regular, so that $J_s A_{\mathfrak{p}} \cap I^j A_{\mathfrak{p}} = J_s I^{j-1} A_{\mathfrak{p}}$ for all $j \in \mathbb{Z}$ by [VV]. □

By [VV], when I is an $\mathfrak{m}$-primary ideal, G is a Cohen-Macaulay ring if so is A and $\mathrm{r}_J(I) \leq 1$. Hence the following result follows from 3.21.

Corollary 3.22. *Let A be a Cohen-Macaulay ring and put* $a = \max\{\mathrm{r}_i - i \mid s \leq i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$. *Assume that* $\operatorname{depth} A/I^i + J_s \geq \min\{d-s, d+a-i\}$ *for all* $1 \leq i \leq a+\ell$. *Then G is a Cohen-Macaulay ring if a generic reduction number n is at most* 1.

Let us close this section with the following result.

Theorem 3.23. *Let A be a Cohen-Macaulay ring and assume that* $\operatorname{depth} A/I^i + J_s \geq \min\{d-s, d-s+n-i\}$ *for all* $1 \leq i \leq n-s+\ell$. *Then the following three conditions are equivalent.*

(1) *G is a Cohen-Macaulay ring with* $\mathrm{a}(G) = n-s$.

(2) (i) $\mathrm{r}_J(I) \leq n-s+\ell$,
(ii) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \leq i \leq n$, *and*
(iii) $\mathrm{r}_i \leq n-s+i$ *for all* $s \leq i < \ell$.

(3) (i) $\mathrm{r}_J(I) \leq n-s+\ell$,
(ii) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \leq i \leq n$, *and*
(iii) $(J_i : a_{i+1}) \cap I^{n-s+i+1} = J_i I^{n-s+i}$ *for all* $s \leq i < \ell$.

Proof of Theorem 3.23. (1) ⇒ (2): Lemma 3.3 implies the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular, so that the condition (ii) holds true by [VV]. The conditions (i) and (iii) follow from the a-invariant formula: $\mathrm{a}(G) = \max\{\mathrm{r}_i - i \mid s \leq i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$ (cf. [U], 1.4). (2) ⇒ (3): Since the condition (A_s) is satisfied for $n-s$ by 3.13, we get $(\mathrm{B}_{\ell-1})$ is also satisfied for $n-s$ by 3.15 and 3.18. The implication (3) ⇒ (1) directly follow from 3.17. □

4 The case of arbitrary generic reduction numbers.

In this section we aim to prove Theorem 1.1 and Theorem 1.2, and discuss the question of when K_G has the expected form ([HSV]). Throughout this section, let A be a Cohen-Macaulay local ring with the canonical module K_A. Put $a = \mathrm{a}(G)$. We say that K_G has the expected form if $\mathrm{K}_G \cong \mathrm{gr}_I(\mathrm{K}_A)(a)$ as graded G-modules, where $\mathrm{gr}_I(\mathrm{K}_A) := \mathrm{K}_A/I\mathrm{K}_A \oplus I\mathrm{K}_A/I^2\mathrm{K}_A \oplus \cdots$. To begin with we state the following result due to [GNN1] and [JU].

Theorem 4.1 ([GNN1] and [JU]). *Assume* $\operatorname{depth} A/I^i \geq d-s-i+1$ *for all* $1 \leq i \leq \ell - s$. *Then* K_G *has the expected form if* G *is a Cohen-Macaulay ring with* $\mathrm{a}(G) = -s$.

The next lemma, which contains the above result, does not require that I is a generically a complete intersection. We get it using arguments from [GNN1] and [JU]. But let us now give a proof for the sake of completeness.

Lemma 4.2. *Assume that* $\operatorname{depth} A/I^i + J_s \geq \min\{d-s, d+a-i+1\}$ *for all* $1 \leq i \leq a+\ell$. *Then* K_G *is generated by homogeneous elements in degrees at least* $-a$ *and at most* s *if the ring* G *is Cohen-Macaulay.*

Before proving the lemma above, we note the special case of it.

Corollary 4.3. *Let* $\ell = s$. *Then* K_G *is generated by homogeneous of elements in degrees at least* $-a$ *and at most* s *if the ring* G *is Cohen-Macaulay.*

Proof. When $\ell = s$, we have $\mathrm{r}_J(I) = a + \ell$ by the a-invariant formula. Moreover, G is a Cohen-Macaulay ring if and only if $\operatorname{depth} A/I^i + J_s = d-s$ for all $1 \leq i \leq \mathrm{r}_J(I)$ (see, e.g., [GI1], 2.3). Hence the assertion follows from 4.2. ☐

Proof of Lemma 4.2. The Cohen-Macaulayness of G induces the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by 3.3. Hence $G/(a_1t, a_2t, \ldots, a_st)G \cong \mathrm{G}(I/J_s)$ as rings by [VV]. So we have $\mathrm{K}_G/\mathfrak{M}\mathrm{K}_G \cong [\mathrm{K}_{\mathrm{G}(I/J_s)}/\mathfrak{M}\mathrm{K}_{\mathrm{G}(I/J_s)}](-s)$ as graded G-modules and $\mathrm{a}(G) = \mathrm{a}(\mathrm{G}(I/J_s)) - s$. Thus, passing to the ring A/J_s, we may assume $s = 0$. Besides, we may assume $\ell > 0$ (see, e.g., [GI1], 2.3). Let $0 \leq i \leq \ell$. We put $\mathrm{U}^{(i)} = [G/J_itG]_{\geq a+i+1}$. Then $a_{i+1}t$ is an $\mathrm{U}^{(i)}$-regular element for $0 \leq i < \ell$ by 3.8. Let $\mathrm{V}^{(i)} = \mathrm{U}^{(i)}/a_{i+1}t\,\mathrm{U}^{(i)}$. We have the natural exact sequence

$$0 \to \mathrm{U}^{(i+1)} \to \mathrm{V}^{(i)} \to \mathrm{W}^{(i)} \to 0 \qquad (\sharp)$$

of graded G-modules. Here $\mathrm{W}^{(i)} = [\mathrm{W}^{(i)}]_{a+i+1} \cong [\mathrm{U}^{(i)}]_{a+i+1} \cong I^{a+i+1}/J_iI^{a+i} + I^{a+i+2}$. Let us prove the following claim.

Claim 4.4. depth $\mathrm{W}^{(i)} \geq d-i-1$ *for all* $0 \leq i < \ell$.

Proof. Firstly, we shall prove $A/J_\alpha I^{a+i} \geq d-i+1$ for all $0 \leq \alpha < i \leq \ell$ by induction on α. The assertion being trivial for $\alpha = 0$, we may assume $\alpha > 0$ and it holds true for $\alpha - 1$. Hence depth $A/J_{\alpha-1}I^{a+i} \geq d-i+1$ and $A/J_{\alpha-1}I^{a+i-1} \geq d-i+2$ by the inductive hypothesis on α. Then applying the depth lemma to the natural exact sequences

$$0 \to J_\alpha I^{a+i}/J_{\alpha-1}I^{a+i} \to A/J_{\alpha-1}I^{a+i} \to A/J_\alpha I^{a+i} \to 0$$

and

$$0 \to I^{a+i}/J_{\alpha-1}I^{a+i-1} \to A/J_{\alpha-1}I^{a+i-1} \to A/I^{a+i} \to 0$$

of A-modules, we get depth $A/J_\alpha I^{a+i} \geq d-i+1$ because we have the isomorphism $I^{a+i}/J_{\alpha-1}I^{a+i-1} \cong J_\alpha I^{a+i}/J_{\alpha-1}I^{a+i}$ of A-modules by 3.14 (recall that the Cohen-Macaulay property of G induces the condition $(\mathrm{B}_{\ell-1})$ by 3.1).

Let us now show that depth $I^{a+i+1}/J_iI^{a+i} + I^{a+i+2} \geq d-i-1$ for all $0 \leq i < \ell$. Thanks to the depth lemma, by the natural exact sequences

$$0 \to I^{a+i+1}/J_iI^{a+i} \to A/J_iI^{a+i} \to A/I^{a+i+1} \to 0$$

of A-modules, we get depth $I^{a+i+1}/J_iI^{a+i} \geq d-i$ because depth $A/J_iI^{a+i} \geq d-i$ by 3.15. Applying the depth lemma to the natural exact sequences

$$0 \to I^{a+i+2}/J_iI^{a+i+1} \to A/J_iI^{a+i+1} \to A/I^{a+i+2} \to 0$$

of A-modules, we get depth $I^{a+i+2}/J_iI^{a+i+1} \geq d-i$ because depth $A/J_iI^{a+i+1} \geq d-i$ as is showed above. Look at the canonical exact sequence

$$0 \to I^{a+i+2}/J_iI^{a+i} \cap I^{a+i+2} \to I^{a+i+1}/J_iI^{a+i} \to I^{a+i+1}/J_iI^{a+i} + I^{a+i+2} \to 0$$

of A-modules. We have $J_iI^{a+i} \cap I^{a+i+2} \subseteq J_i \cap I^{a+i+2} = J_iI^{a+i+1}$ by 3.8. Hence we have $I^{a+i+2}/J_iI^{a+i} \cap I^{a+i+2} = I^{a+i+2}/J_iI^{a+i+1}$, so the required inequality follows from applying the depth lemma to the exact sequence above. Hence we get depth $\mathrm{W}^{(i)} \geq d-i-1$ if $0 \leq i < \ell$. $\square$

We need to prove depth $\mathrm{U}^{(i)} \geq d-i$ for all $0 \leq i < \ell$. When $0 \leq i < \ell-1$, depth $\mathrm{U}^{(i)} \geq d-i$ if and only if depth $\mathrm{U}^{(i+1)} \geq d-i-1$. In fact, the later condition is equivalent to saying that depth $\mathrm{V}^{(i)} \geq d-i-1$ by 4.4 (apply the depth lemma to ($\sharp$)). Since $a_{i+1}t$ is $\mathrm{U}^{(i)}$-regular, depth $\mathrm{V}^{(i)} \geq d-i-1$ if and only if depth $\mathrm{U}^{(i)} \geq d-i$.

Hence it suffice to show that depth $\mathrm{V}^{(\ell-1)} \geq d-\ell$ by descending induction on i. Thanks to a-invariant formula, we have $\mathrm{r}_J(I) \leq a+\ell$, and hence

$\mathrm{U}^{(\ell)} = (0)$. Then by the exact sequence ($\sharp$), we see $\mathrm{V}^{(\ell-1)} \cong \mathrm{W}^{(\ell-1)}$ as G-modules, so that $\operatorname{depth} \mathrm{V}^{(\ell-1)} \geq d - \ell$ by 4.4.

Let $[\quad]_* := \mathrm{Hom}_G(G/\mathfrak{M}, \quad)$ that is a functor of graded G-modules. Then we have

Claim 4.5. *Let $0 \leq i \leq \ell$. Then $[\mathrm{H}^{d-i}_{\mathfrak{M}}(\mathrm{U}^{(i)})]_*$ is concentrated in degree $a + i$.*

Proof. Descending induction on i. Since $\mathrm{U}^{(\ell)} = (0)$, there is nothing to prove for $i = \ell$. Let $i < \ell$ and assume the assertion holds true for $i+1$. By 4.4 we have the resulting exact sequence $0 \to \mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{U}^{(i+1)}) \to \mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{V}^{(i)}) \to \mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{W}^{(i+1)})$ of graded local cohomology modules from ($\sharp$), so that we get the exact sequence

$$0 \to [\mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{U}^{(i+1)})]_* \to [\mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{V}^{(i)})]_* \to [\mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{W}^{(i)})]_*$$

of graded G-modules. Therefore $[\mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{V}^{(i)})]_*$ is concentrated in degree $a + i + 1$ by the inductive hypotheses on i (recall that $\mathrm{W}^{(i)} = [\mathrm{W}^{(i)}]_{a+i+1}$). Applying the local cohomology functor to the canonical exact sequence $0 \to \mathrm{U}^{(i)}(-1) \xrightarrow{a_{i+1}t} \mathrm{U}^{(i)} \to \mathrm{V}^{(i)} \to 0$ of graded G-modules, we obtain the resulting exact sequence $0 \to \mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{V}^{(i)}) \to \mathrm{H}^{d-i}_{\mathfrak{M}}(\mathrm{U}^{(i)})(-1) \xrightarrow{a_{i+1}t} \mathrm{H}^{d-i}_{\mathfrak{M}}(U^{(i)})$ of graded local cohomology modules (recall that $\operatorname{depth} \mathrm{U}^{(i)} \geq d - i$). Then we get $[\mathrm{H}^{d-i-1}_{\mathfrak{M}}(\mathrm{V}^{(i)})]_* \cong [\mathrm{H}^{d-i}_{\mathfrak{M}}(\mathrm{U}^{(i)})]_*(-1)$ as graded G-modules since $a_{i+1}t \in \mathfrak{M}$. Therefore $[\mathrm{H}^{d-i}_{\mathfrak{M}}(\mathrm{U}^{(i)})]_*$ is concentrated in degree $a + i$. $\square$

Hence we have in particular that $[\mathrm{H}^{d}_{\mathfrak{M}}(\mathrm{U}^{(0)})]_*$ is concentrated in degree a. We consider the canonical exact sequence

$$0 \to \mathrm{U}^{(0)} \to G \to C \to 0 \qquad (\sharp\sharp)$$

of graded G-modules. Then $C = C_0 \oplus C_1 \oplus \cdots \oplus C_a$ and $C_j \cong I^j/I^{j+1}$. And hence $\operatorname{depth} C \geq d$ because $\operatorname{depth} C_j \geq d$ for all $0 \leq j \leq a$ by our standard assumption that $\operatorname{depth} A/I^{j+1} \geq d$. Thus we obtain the resulting graded exact sequence $0 \to \mathrm{H}^{d}_{\mathfrak{M}}(\mathrm{U}^{(0)}) \to \mathrm{H}^{d}_{\mathfrak{M}}(G) \to \mathrm{H}^{d}_{\mathfrak{M}}(C) \to 0$ of graded local cohomology modules from ($\sharp\sharp$), so that we get the exact sequence

$$0 \to [\mathrm{H}^{d}_{\mathfrak{M}}(\mathrm{U}^{(0)})]_* \to [\mathrm{H}^{d}_{\mathfrak{M}}(G)]_* \to [\mathrm{H}^{d}_{\mathfrak{M}}(C)]_*$$

of G-modules. Therefore $[\mathrm{H}^{d}_{\mathfrak{M}}(G)]_*$ is concentrated in degrees between 0 and a because so is $[\mathrm{H}^{d}_{\mathfrak{M}}(C)]_*$. This completes the proof of 4.2 by the local duality theorem. $\square$

Now let us note the following lemma.

Lemma 4.6. *Assume G is a Cohen-Macaulay ring. Then* $\mathrm{a}(G) = n - s$ *if* K_G *has the expected form.*

Proof. We may assume $s = 0$ by [VV] together with 3.3. Since $[\mathrm{K}_G]_{-\mathrm{a}(G)} \cong \mathrm{K}_A/I\mathrm{K}_A$ as A-modules, $\mathrm{a}(G) = \mathrm{a}(G_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{V}(I)$ (recall that $\mathrm{a}(G) = -\min\{k \in \mathbb{Z} \mid [\mathrm{K}_G]_k \neq (0)\}$ and $[\mathrm{K}_G]_\mathfrak{p} \cong \mathrm{K}_{[G_\mathfrak{p}]}$ as graded G-modules). Take $\mathfrak{p} \in \mathrm{V}(I)$ such that $\mathrm{ht}_A \mathfrak{p} = 0$ and $n = \mathrm{r}_{(0)_\mathfrak{p}}(I_\mathfrak{p})$. Then $G_\mathfrak{p} = [G_0]_\mathfrak{p} \oplus [G_1]_\mathfrak{p} \oplus \cdots \oplus [G_n]_\mathfrak{p}$ and $[G_n]_\mathfrak{p} \cong I_\mathfrak{p}^n \neq (0)$, and hence $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = n$ by [GH], 2.2. Thus we get $\mathrm{a}(G) = n$. $\square$

In what follows, until 4.19, we maintain to assume that $s = 0$. Then it is a necessary condition of the Gorensteinness of the associated graded ring G that $I^{n-i+1} \subseteq (0) : I^i$ for all $1 \leq i \leq n$ (see [GI1], 4.4).

We extend Theorem 4.1 to the case where the number n is arbitrary as follows.

Lemma 4.7. *Let G be a Cohen-Macaulay ring with* $\mathrm{a}(G) = n$. *Assume* $\mathrm{K}_A/I^i\mathrm{K}_A$ *is Cohen-Macaulay and* $(0) : I^i \subseteq I^{n-i+1}$ *for all* $1 \leq i \leq n$. *Then the following two assertions hold true.*

1. $\mathrm{r}_J(I) \leq \max\{n, \ell\}$ *and* $\mathrm{r}_i \leq \max\{n, i\}$ *for all* $0 \leq i < \ell$.
2. K_G *has the expected form if* $\operatorname{depth} A/I^i \geq d - i + 1$ *for all* $1 \leq i \leq \ell$.

We will arrange some lemmas for proving 4.7. Let $[\quad]^* = \mathrm{Hom}_A(\quad, \mathrm{K}_A)$ and let $\mathfrak{a} = (0) : I$. Then $\mathfrak{a} \cong [\mathrm{K}_A/I\mathrm{K}_A]^*$ because $\mathfrak{a} \cong \mathrm{Hom}_A(A/I, A) \cong \mathrm{Hom}_A(A/I, \mathrm{Hom}_A(\mathrm{K}_A, \mathrm{K}_A))$ and $\mathrm{Hom}_A(A/I \otimes \mathrm{K}_A, \mathrm{K}_A) \cong \mathrm{Hom}_A(\mathrm{K}_A/I\mathrm{K}_A, \mathrm{K}_A)$ as A-modules. The following three lemmas are well-known in the case where $\mathrm{K}_A = A$ but let us note proofs for completeness.

Lemma 4.8. *The following two conditions are equivalent.*

(1) $\mathrm{K}_A/I\mathrm{K}_A$ *satisfies the Serre's condition* (S_1).

(2) $I\mathrm{K}_A = (0) :_{\mathrm{K}_A} \mathfrak{a}$.

Proof. Taking the K_A-dual of the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ of A-modules, we get a commutative and exact diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(A/\mathfrak{a}, \mathrm{K}_A) & \longrightarrow & \mathrm{Hom}_A(A, \mathrm{K}_A) & \longrightarrow & \mathrm{Hom}_A(\mathfrak{a}, \mathrm{K}_A) & & \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & I\mathrm{K}_A & \longrightarrow & \mathrm{K}_A & \longrightarrow & \mathrm{K}_A/I\mathrm{K}_A & \longrightarrow & 0 \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

of A-modules. Then because $\mathrm{Hom}_A(\mathfrak{a}, \mathrm{K}_A) \cong [\mathrm{K}_A/I\mathrm{K}_A]^{**}$ as A-modules, $\mathrm{K}_A/I\mathrm{K}_A$ satisfies the Serre's condition (S_1) if and only if the right-hand homomorphism is injective. The later condition is equivalent to saying that the left-hand homomorphism is bijective, which means $I\mathrm{K}_A = (0) :_{\mathrm{K}_A} \mathfrak{a}$. $\square$

Lemma 4.9. *Assume that* $\mathrm{K}_A/I\mathrm{K}_A$ *satisfies the Serre's condition* (S_1). *Then the following two conditions are equivalent.*

(1) $A/\mathfrak{a}$ *satisfies the Serre's condition* (S_2).

(2) $\mathrm{H}^{d-1}_{\mathfrak{m}}(\mathrm{K}_A/I\mathrm{K}_A) = (0)$.

Proof. We recall $[\mathrm{K}_A/I\mathrm{K}_A]^* \cong \mathfrak{a}$ as A-modules. Then applying the functor $[\quad]^*$ to the exact sequence $0 \to I\mathrm{K}_A \to \mathrm{K}_A \to \mathrm{K}_A/I\mathrm{K}_A \to 0$ of A-modules, we get a commutative and exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & [\mathrm{K}_A/I\mathrm{K}_A]^* & \longrightarrow & [\mathrm{K}_A]^* & \longrightarrow & [I\mathrm{K}_A]^* & \longrightarrow & \mathrm{Ext}^1_A(\mathrm{K}_A/I\mathrm{K}_A, \mathrm{K}_A) \longrightarrow 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathfrak{a} & \longrightarrow & A & \longrightarrow & [A/\mathfrak{a}]^{**} & &
\end{array}$$

of A-modules because $I\mathrm{K}_A \cong [A/\mathfrak{a}]^*$ (see 4.8). Therefore $A/\mathfrak{a}$ satisfies the Serre's condition (S_2) if and only if $\mathrm{Ext}^1_A(\mathrm{K}_A/I\mathrm{K}_A, \mathrm{K}_A) = (0)$. The later condition is equivalent to saying that $\mathrm{H}^{d-1}_A(\mathrm{K}_A/I\mathrm{K}_A) = (0)$ by the local duality theorem. $\square$

Lemma 4.10. *The following conditions are equivalent.*

(1) $\mathrm{K}_A/I\mathrm{K}_A$ *is a Cohen-Macaulay A-module.*

(2) $A/\mathfrak{a}$ *is a Cohen-Macaulay ring with* $(0) :_{\mathrm{K}_A} \mathfrak{a} = I\mathrm{K}_A$.

Proof. We may assume $\mathrm{K}_A/I\mathrm{K}_A$ satisfies the condition (S_1) by 4.8. Then it is enough to show $\mathrm{K}_A/I\mathrm{K}_A$ is Cohen-Macaulay if and only if so is $A/\mathfrak{a}$.

Assume $\mathrm{K}_A/I\mathrm{K}_A$ is a Cohen-Macaulay A-module. Therefore we have in particular $\mathrm{H}^{d-1}_{\mathfrak{m}}(\mathrm{K}_A/I\mathrm{K}_A) = (0)$, so that the ring $A/\mathfrak{a}$ satisfies the Serre's condition (S_2) by 4.9. Hence $A/\mathfrak{a} \cong [A/\mathfrak{a}]^{**} \cong [I\mathrm{K}_A]^*$. Therefore we get $A/\mathfrak{a}$ is a Cohen-Macaulay ring because $I\mathrm{K}_A$ is a maximal Cohen-Macaulay A-module (apply the depth lemma to the exact sequence $0 \to I\mathrm{K}_A \to \mathrm{K}_A \to \mathrm{K}_A/I\mathrm{K}_A \to 0$ of A-modules).

Conversely, suppose $A/\mathfrak{a}$ is a Cohen-Macaulay ring. Then $I\mathrm{K}_A$ $(\cong \mathrm{K}_{A/\mathfrak{a}})$ is a maximal Cohen-Macaulay A-module, so that $\operatorname{depth} \mathrm{K}_A/I\mathrm{K}_A \geq d-1$. Since $A/\mathfrak{a}$ satisfies the condition (S_2), we have $\mathrm{H}^{d-1}_{\mathfrak{m}}(\mathrm{K}_A/I\mathrm{K}_A) = (0)$ by 4.9. Therefore $\mathrm{K}_A/I\mathrm{K}_A$ is a Cohen-Macaulay A-module. $\square$

Now let $\{\omega_i\}_{i\in\mathbb{Z}}$ stand for the canonical I-filtration of K_A (see [GI1], 1.1). Hence $I^{i+1}\mathrm{K}_A \subseteq \omega_{-\mathrm{a}(G)+i}$ and, if G is a Cohen-Macaulay ring, we have $[\mathrm{K}_G]_i \cong \omega_{i-1}/\omega_i$ for all $i \in \mathbb{Z}$. Put $T = \mathrm{G}(IA/\mathfrak{a})$. Then we get the following lemma.

Lemma 4.11. *Assume that G is a Cohen-Macaulay ring with $\mathrm{a}(G) = n$ and $I^n \supseteq \mathfrak{a}$. Then the following two assertion holds true.*

1. *T is a Cohen-Macaulay ring if $A/\mathfrak{a}$ is a Cohen-Macaulay ring.*
2. *$\omega_{-n} = I\mathrm{K}_A$ and $\mathrm{a}(T) \leq n-1$ if $\mathrm{K}_A/I\mathrm{K}_A$ satisfies the Serre's condition (S_1).*

Proof. We consider the natural exact sequence

$$0 \to \ker\varepsilon \to G \xrightarrow{\varepsilon} T \to 0 \qquad (\sharp)$$

of graded G-modules where ε is the map of associated graded rings induced by the natural map $A \to A/\mathfrak{a}$. We put $L = \ker\varepsilon$. Then $L = L_n \cong \mathfrak{a}$ because $I^{n+1} \cap \mathfrak{a} = (0)$ and $I^n \supseteq \mathfrak{a}$.

The assertion 1: Assume that $A/\mathfrak{a}$ is a Cohen-Macaulay ring. Then $\operatorname{depth}\mathfrak{a} = d$, so that $\operatorname{depth} L = d$. We apply the local cohomology functors $\mathrm{H}^i_{\mathfrak{M}}(*)$ ($i \in \mathbb{Z}$) to the graded exact sequences $(\sharp)$. Then by the resulting graded exact sequence

$$0 \to \mathrm{H}^{d-1}_{\mathfrak{M}}(T) \to \mathrm{H}^d_{\mathfrak{M}}(L) \to \mathrm{H}^d_{\mathfrak{M}}(G) \to \mathrm{H}^d_{\mathfrak{M}}(T) \to 0$$

of graded local cohomology modules, we have $\mathrm{H}^{d-1}_{\mathfrak{M}}(T) = [\mathrm{H}^{d-1}_{\mathfrak{M}}(T)]_n$ and $\mathrm{a}(T) \leq n$ because $L = L_n$ and $\mathrm{a}(G) = n$. Then from [KN], 3.1 we obtain that T is a Cohen-Macaulay ring.

The assertion 2: Suppose that $\mathrm{K}_A/I\mathrm{K}_A$ satisfies the Serre's condition (S_1). Take the K_G-dual of the sequence $(\sharp)$, and we get the exact sequence

$$0 \to \mathrm{K}_T \to \mathrm{K}_G \to \mathrm{Hom}_G(L, \mathrm{K}_G) \to \mathrm{Ext}^1_G(T, \mathrm{K}_G) \to 0$$

of graded G-modules. We see

$$\mathrm{Hom}_G(L, \mathrm{K}_G) = [\mathrm{Hom}_G(L, \mathrm{K}_G)]_{-n} \cong [\mathrm{K}_A/I\mathrm{K}_A]^{**}$$

by the local duality theorem (recall that $L = L_n \cong \mathfrak{a} \cong [\mathrm{K}_A/I]^*$). From $[\mathrm{K}_G]_{-n} \cong \mathrm{K}_A/\omega_{-n}$ we obtain a homomorphism $\phi : \mathrm{K}_A/\omega_{-n} \to [\mathrm{K}_A/I\mathrm{K}_A]^{**}$ of A-modules. Since $I\mathrm{K}_A \subseteq \omega_{-n}$, we have the natural surjective map $\pi : \mathrm{K}_A/I\mathrm{K}_A \to \mathrm{K}_A/\omega_{-n}$. Let $\varphi = \phi \circ \pi$. We put $K = \ker\varphi$. We want to show $K = (0)$. Assume that $K \neq (0)$ and choose $\mathfrak{p} \in \mathrm{Ass}_A K$. Then $\mathrm{ht}_A \mathfrak{p} = 0$ because $\mathfrak{p} \in \mathrm{Ass}_A \mathrm{K}_A/I\mathrm{K}_A$. Hence we see a homomorphism

$\varphi_\mathfrak{p} : (\mathrm{K}_A/I\mathrm{K}_A)_\mathfrak{p} \to ([\mathrm{K}_A/I\mathrm{K}_A]^{**})_\mathfrak{p}$ is surjective, as $\mathrm{Ext}^1_G(T, \mathrm{K}_G)_\mathfrak{p} = (0)$. Moreover we have the canonical isomorphism $([\mathrm{K}_A/I\mathrm{K}_A]^{**})_\mathfrak{p} \cong (\mathrm{K}_A/I\mathrm{K}_A)_\mathfrak{p}$ because $\dim A_\mathfrak{p} = 0$, and hence the surjective homomorphism $\varphi_\mathfrak{p}$ must be bijective. This is a contradiction. Therefore $K = (0)$, so that φ, π, and ϕ are injective. Thus $\omega_{-n} = I\mathrm{K}_A$ and $\mathrm{a}(T) \le n-1$. □

We now come to prove 4.7.

Proof of Lemma 4.7. When $n = 0$, the assertion 1 follows from the a-invariant formula. Moreover, the assertion 2 directly follows from 4.1. Hence we may assume $n > 0$. Firstly, we consider the case where $\ell = 0$. Then the assertion 1 follows from the a-invariant formula again. We must show the assertion 2. Let $1 \le i \le n$. Then $(0) : I^i = I^{n-i+1}$ by our standard assumption. Since $\mathrm{K}_A/I^i\mathrm{K}_A$ satisfies the Serre's condition (S_1), we have $I^i\mathrm{K}_A = (0) :_{\mathrm{K}_A} [(0) : I^i]$ by 4.8, so that $I^i\mathrm{K}_A = (0) :_{\mathrm{K}_A} I^{n-i+1}$. This implies that K_G has the expected form by [GI1], 2.3.

We will consider the case where $\ell > 0$. Let $0 \le j \le n$ and let $A_j := A/(0) : I^j$.

Claim 4.12. *The ring A_j is Cohen-Macaulay and* $\mathrm{K}_{A_j} = I^j\mathrm{K}_A$

Proof. We are going to prove by induction on j. The assertion is trivial for $j = 0$. We may assume $j > 0$ and it holds true for $j-1$. We have $\mathrm{K}_{A_{j-1}} = I^{j-1}\mathrm{K}_A$ by the inductive hypothesis, and hence $\mathrm{K}_{A_{j-1}}/I\mathrm{K}_{A_{j-1}} = I^{j-1}\mathrm{K}_A/I^j\mathrm{K}_A$ that is a Cohen-Macaulay A-module. In fact, if $j = 1$, this is clear by our standard assumption that $\mathrm{K}_A/I\mathrm{K}_A$ is Cohen-Macaulay. If $j \ge 2$, this follows from applying the depth lemma to the exact sequence $0 \to I^j\mathrm{K}_A/I^{j-1}\mathrm{K}_A \to \mathrm{K}_A/I^{j-1}\mathrm{K}_A \to \mathrm{K}_A/I^j\mathrm{K}_A \to 0$ of A-modules. Therefore by 4.10, we get the assertion because $A_j \cong A_{j-1}/(0) : IA_{j-1}$ as rings. □

Let $T_j := \mathrm{G}(IA_j)$. We have $\mathrm{ht}_{A_j} IA_j = 0$, as $I \supseteq (0) : I^j$. We also have $\lambda(IA_j) \le \lambda(I) \le \ell$ because the kernel of the natural map $G \to T_j$ of associated graded rings is a finitely graded ideal in G (recall that $I^{n+1} \cap [(0) : I^j] = (0)$, as I^{n+1} is generically a complete intersection). Let $n_j := \mathrm{r}_0(IA_j)$. It is routine to check $n_j = n-j$ and $I^{n_j}A_j \supseteq (0) : IA_j$. Let $\{\omega_i^{(j)}\}_{i\in\mathbb{Z}}$ denote the canonical IA_j-filtration of $I^j\mathrm{K}_A$.

Claim 4.13. *T_j is a Cohen-Macaulay ring with* $\mathrm{a}(T_j) = n_j$.

Proof. We prove by induction on j. If $j = 0$, the assertion is trivial, as $G = T_0$ and $n = n_0$. Let $j > 0$ and assume that it holds true for $j-1$. We have $A_{j-1}/(0) : IA_{j-1} \cong A_j$ that is a Cohen-Macaulay ring.

$\mathrm{K}_{A_{j-1}}/I\mathrm{K}_{A_{j-1}} (= I^{j-1}\mathrm{K}_A/I^j\mathrm{K}_A)$ satisfies the Serre's condition (S_1) because $\mathrm{K}_A/I^j\mathrm{K}_A$ is Cohen-Macaulay. Then by 4.11 we get T_j is a Cohen-Macaulay ring with $\mathrm{a}(T_j) \leq n_{j-1}-1$. Hence $\mathrm{a}(T_j) = n_j$ because $n_{j-1}-1 = n-(j-1)-1 = n-j = n_j$ and because $n_j = \mathrm{r}_0(IA_j) = \max\{\mathrm{a}([T_j]_\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{V}(I),\ \mathrm{ht}_A\,\mathfrak{q} = 0\} \leq \mathrm{a}(T_j)$ (recall that $\mathrm{K}_{[T_j]_\mathfrak{q}} = [\mathrm{K}_{T_j}]_\mathfrak{q}$). □

Claim 4.14. *If* $0 \leq j < n$, *then* $\omega^{(j)}_{-n_j} = I^{j+1}\mathrm{K}_A$.

Proof. T_j is a Cohen-Macaulay ring with $\mathrm{a}(T_j) = n_j$ and $\mathrm{K}_{A_j}/I\mathrm{K}_{A_j} = I^j\mathrm{K}_A/I^{j+1}\mathrm{K}_A$ that satisfies the Serre's condition (S_1) because $\mathrm{K}_A/I^{j+1}\mathrm{K}_A$ is Cohen-Macaulay, as $j < n$. Hence the equality $\omega^{(j)}_{-n_j} = I^{j+1}\mathrm{K}_A$ follows from Lemma 4.11. □

We note

Claim 4.15. $(0) : I^n \cap I^i = (0) : I^{n-i+1}$ *for all* $i \geq 1$.

Proof. When $i \geq n+1$, I^i is generically a complete intersection, and hence $(0) : I^n \cap I^i = (0)$. Let $i \leq n$. The inclusion $(0) : I^n \cap I^i \supseteq (0) : I^{n-i+1}$ follows from our standard assumption. Assume that $(0) : I^n \cap I^i \supsetneq (0) : I^{n-i+1}$. Then there exists $\mathfrak{p} \in \mathrm{Ass}_A[(0) : I^n \cap I^i]/(0) : I^{n-i+1}$. Since $\mathrm{ht}_A\,\mathfrak{p} = 0$, $I^i_\mathfrak{p} = (0) : I^{n-i+1}_\mathfrak{p}$ by the definition of n, so that $(0) : I^n_\mathfrak{p} \cap I^i_\mathfrak{p} = (0) : I^{n-i+1}_\mathfrak{p}$. This is a contradiction. □

The assertion 1 directly follows from the next claim.

Claim 4.16. $\mathrm{r}_J(I) \leq \max\{n, \ell\}$ *and* $\mathrm{r}_i \leq \max\{n, i\}$ *for all* $0 \leq i < \ell$.

Proof. By [U], 1.4, we get $\mathrm{r}_{JA_n}(IA_n) \leq \ell$ because T_n is a Cohen-Macaulay ring with $\mathrm{a}(T_n) = 0$ (recall that $\lambda(IA_n) \leq \ell$ and that $\mathrm{r}_{JA_n}(IA_n) \leq \mathrm{r}_\mathfrak{J}(IA_n)$ for any minimal reduction $\mathfrak{J}(\subseteq JA_n)$ of IA_n). Then $I^{\ell+1} \subseteq JI^\ell + (0) : I^n$, so that $I^{\ell+1} \subseteq JI^\ell + [(0) : I^n \cap I^{\ell+1}]$. If $\ell \geq n$, then $(0) : I^n \cap I^{\ell+1} = (0)$ by 4.15 and hence $\mathrm{r}_J(I) \leq \ell$. If $\ell < n$, then $(0) : I^n \cap I^{\ell+1} = (0) : I^{n-\ell}$ by 4.15, so that $I^{\ell+1} \cdot I^{n-\ell} \subseteq [JI^\ell + (0) : I^{n-\ell}] \cdot I^{n-\ell}$ and hence $\mathrm{r}_J(I) \leq n$. Thus $\mathrm{r}_J(I) \leq \max\{n, \ell\}$.

Let $0 \leq i < \ell$. We must show $\mathrm{r}_i \leq \max\{n, i\}$. Take any $\mathfrak{q} \in \mathrm{V}(I)$ such that $\mathrm{ht}_A\,\mathfrak{q} = i$. By [U], 1.4, we have $\mathrm{r}_i(IA_n) \leq i$ because $[T_n]_\mathfrak{q}$ is a Cohen-Macaulay ring with $\mathrm{a}([T_n]_\mathfrak{q}) \leq \mathrm{a}(T_n) = 0$. So $I^{i+1}_\mathfrak{q} \subseteq J_{i\mathfrak{q}} I^i_\mathfrak{q} + [(0) : I^n_\mathfrak{q} \cap I^{i+1}_\mathfrak{q}]$. Then, using the same argument above, we get $\mathrm{r}_i \leq \max\{n, i\}$ for all $0 \leq i < \ell$. □

Notice that IA_n is generically a complete intersection since $I_\mathfrak{p} = (0) : I^n_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{V}(I)$ with $\mathrm{ht}_A\,\mathfrak{p} = 0$.

Claim 4.17. $\operatorname{depth} A_n/I^i A_n \geq d-i+1$ *for all* $1 \leq i \leq \ell$.

Proof. We may assume $i > 1$ because $I \supseteq (0) : I^n$. Look at the exact sequence

$$0 \to A/(0) : I^n \cap I^i \to A/(0) : I^n \oplus A/I^i \to A/(0) : I^n + I^i \to 0$$

of A-modules. By 4.15, $A/(0) : I^n \cap I^i = A/(0) : I^{n-i+1}$, which is a Cohen-Macaulay ring by 4.10. Then we get $\operatorname{depth} A_n/I^i A_n \geq d-i+1$ by applying the depth lemma to the short exact sequence above (recall that $A/(0) : I^n$ is a Cohen-Macaulay ring). $\square$

Hence by 4.1, K_{T_n} has the expected form, which means that $\mathrm{K}_{T_n} \cong \operatorname{gr}_I(\mathrm{K}_A)_{\geq n}(n)$ as graded G-modules since $\mathrm{K}_{A_n} = I^n \mathrm{K}_A$.

Claim 4.18. $\mathrm{K}_{T_j} \cong \operatorname{gr}_I(\mathrm{K}_A)_{\geq j}(n)$ *as graded* G*-modules.*

Proof. Descending induction on j. We may assume $j < n$ and it is true for $j+1$. Consider the natural exact sequence

$$0 \to L \to T_j \to T_{j+1} \to 0$$

of graded G-modules. We have $L = [L]_{n_j} = (0) : IA_j$. Taking the K_{T_j}-dual of the above exact sequence, we get the exact sequence

$$0 \to \mathrm{K}_{T_{j+1}} \to \mathrm{K}_{T_j} \to \operatorname{Hom}_{T_j}(L, \mathrm{K}_{T_j}) \to 0$$

of graded G-modules. By the inductive hypothesis on j, we have $\mathrm{K}_{T_{j+1}} \cong \operatorname{gr}_I(\mathrm{K}_A)_{\geq j+1}(n)$ and hence $\operatorname{gr}_I(\mathrm{K}_A)_{\geq j+1}(n) \cong [\mathrm{K}_{T_j}]_{\geq -n_j+1}$ as graded G-modules because $\operatorname{Hom}_{\mathrm{K}_{T_j}}(L, \mathrm{K}_{T_j})$ is concentrated in degree $-n_j$.

Let us prove $\omega^{(j)}_{-n_j+i} = I^{i+j+1}\mathrm{K}_A$ for all $i \geq 0$ by induction on i. By 4.14 we may assume that $i > 0$ and $\omega^{(j)}_{-n_j+i-1} = I^{i+j}\mathrm{K}_A$. Then by $\operatorname{gr}_I(\mathrm{K}_A)_{\geq j+1}(n) \cong [\mathrm{K}_{T_j}]_{\geq -n_j+1}$ we get

$$I^{i+j}\mathrm{K}_A/I^{i+j+1}\mathrm{K}_A \cong \omega^{(j)}_{-n_j+i-1}/\omega^{(j)}_{-n_j+i} = I^{i+j}\mathrm{K}_A/\omega^{(j)}_{-n_j+i},$$

and hence the natural surjective map $I^{i+j}\mathrm{K}_A/I^{i+j+1}\mathrm{K}_A \to I^{i+j}\mathrm{K}_A/\omega^{(j)}_{-n_j+i}$ is bijective (recall that $I^{i+j+1}\mathrm{K}_A \subseteq \omega^{(j)}_{-n_j+i}$). Therefore $\omega^{(j)}_{-n_j+i} = I^{i+j+1}\mathrm{K}_A$. $\square$

Thus we see $\mathrm{K}_G \cong \operatorname{gr}_I(\mathrm{K}_A)(n)$ as graded G-modules and hence K_G has expected form. This completes the proof of 4.7. $\square$

Let us state the following corollaries of 4.7.

Corollary 4.19. *Assume G is a Gorenstein ring. Then $\mathrm{r}_J(I) \le \max\{n, \ell - s\}$ and $\mathrm{r}_i \le \max\{n, i-s\}$ for all $s \le i < \ell$ if the ring $A/I^i + J_s$ is Cohen-Macaulay for all $1 \le i \le n$.*

Proof. Since $a_1t, a_2t, \ldots, a_st$ is G-regular, we may assume $s = 0$ (cf. [GNN2], 3.4). We have $\mathrm{a}(G) = n$ by 4.6. Because G is a Gorenstein ring, so is A and $I^{n-i+1} \supseteq (0) : I^i$ for all $1 \le i \le n$ by [GI1], 4.4. Then the assertion follows from 4.7. $\square$

Corollary 4.20. *Let A be a Gorenstein ring and let G be a Cohen-Macaulay ring with $\mathrm{a}(G) = n - s$. Assume $A/I^i + J_s$ is Cohen-Macaulay for all $1 \le i \le n$. Then*

$$\mathrm{G}(I) \text{ is a Gorenstein ring } \iff \begin{cases} J_s^n : I^n \subseteq I^n & (s > 0); \\ (0) : I^i \subseteq I^{n-i+1} \quad \text{for all } 1 \le i \le n & (s = 0) \end{cases}$$

if depth $A/I^i \ge d - s - i + 1$ *for all* $1 \le i \le \ell - s$.

Proof. When $n = 0$, the assertion holds true by 4.1 (cf., also, 2.9), so that we my assume $n > 0$. The assertion in the case where $s = 0$ directly follows form 4.7 together with [GI1], 4.4. We must prove the case where $s > 0$. Suppose G is a Gorenstein ring. Let $L = J_s^n : I^n + I^n/I^n$. We want to show $L = (0)$. Assume that $L \ne (0)$ and take $\mathfrak{p} \in \mathrm{Ass}_A L$. Then $\mathrm{ht}_A \mathfrak{p} = s$ since A/I^n is Cohen-Macaulay (see 3.11). By the a-invariant formula, we have $\mathrm{r}_{J_{s\mathfrak{p}}}(I_\mathfrak{p}) = \mathrm{a}(\mathrm{G}(I_\mathfrak{p})) - s$, so that $\mathrm{r}_{J_{s\mathfrak{p}}}(I_\mathfrak{p}) = n$ because $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = \mathrm{a}(G)$ (cf. the proof of 4.6). Then since $\mathrm{G}(I_\mathfrak{p})$ is a Gorenstein ring, we get ${J_{s\mathfrak{p}}}^n : I_\mathfrak{p}^n = I_\mathfrak{p}^n$ by [GI1], 1.4. This is a contradiction. Thus $L = (0)$ and hence we have $J_s^n : I^n \subseteq I^n$.

Conversely suppose $J_s^n : I^n \subseteq I^n$. Let $1 \le i \le n$. By the result in the case where $s = 0$, it is enough to show $J_s : I^i \subseteq I^{n-i+1} + J_s$ because the sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by 3.3. Let $L = J_s : I^i + I^{n-i+1}/I^{n-i+1} + J_s$ and assume that there exists $\mathfrak{p} \in \mathrm{Ass}_A L$. Then $\mathrm{ht}_A \mathfrak{p} = s$, as the ring $A/I^{n-i+1} + J_s$ is Cohen-Macaulay. Let $r = \mathrm{r}_{J_{s\mathfrak{p}}}(I_\mathfrak{p})$. Then $r \le n$. We have $I_\mathfrak{p}^r \subseteq {J_{s\mathfrak{p}}}^r : I_\mathfrak{p}^r \subseteq {J_{s\mathfrak{p}}}^n : I_\mathfrak{p}^n \subseteq I_\mathfrak{p}^n$ (cf. [GI1], Claim 2). Therefore $r = n$ and ${J_{s\mathfrak{p}}}^n : I_\mathfrak{p}^n = I_\mathfrak{p}^n$. Then $\mathrm{G}(I_\mathfrak{p})$ is a Gorenstein ring by [GI1], 1.4 and hence so is $\mathrm{G}(I(A/J_s)_\mathfrak{p})$ by [VV]. Therefore $(0) : I^i(A/J_s)_\mathfrak{p} = I^{n-i+1}(A/J_s)_\mathfrak{p}$ by [O]. Consequently $J_{s\mathfrak{p}} : I_\mathfrak{p}^i = I_\mathfrak{p}^{n-i+1} + J_{s\mathfrak{p}}$, which is impossible. Thus $L = (0)$ and hence we get $J_s : I^i \subseteq I^{n-i+1} + J_s$. $\square$

Corollary 4.21. *Let $\{a_1, a_2, \cdots, a_\ell\}$ be a good generating set and put $a = \max\{\mathrm{r}_i - i \mid s \le i < \ell\} \cup \{\mathrm{r}_J(I) - \ell\}$. Let A be a Gorenstein ring and I a height unmixed ideal. Assume that* depth $A/I^i + J_s \ge \min\{d - s, d + a - i\}$ *for all $1 \le i \le a + \ell$. Then the following two conditions are equivalent.*

(1) *G is a Gorenstein ring.*

(2) $\mathrm{G}(I_\mathfrak{p})$ *is a Gorenstein ring with* $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = a$ *for all* $\mathfrak{p} \in \mathrm{V}(I)$ *such that* $\mathrm{ht}_A \mathfrak{p} = s$.

When this is the case, we have $\mathrm{a}(G) = a = n - s$.

Proof. It suffices to show the condition (2) implies the condition (1) and the last assertion. By 3.21, we may assume that G is a Cohen-Macaulay ring with $\mathrm{a}(G) = a$. Let $\mathfrak{p} \in \mathrm{V}(I)$ such that $\mathrm{ht}_A \mathfrak{p} = s$. Then we obtain that $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = \mathrm{r}_{J_{s\mathfrak{p}}}(I_\mathfrak{p}) - s$ from the a-invariant formula. Therefore $\mathrm{a}(G) = n - s$. The sequence $a_1t, a_2t, \ldots, a_st$ is G-regular by 3.3, so that we may also assume $s = 0$ by [VV] (cf. [GNN2], 3.4). When $n = 0$, we have $\mathrm{a}(G) = 0$, and hence the assertion follows form 2.1. Let $n > 0$. Let $1 \le i \le n$ and let $L = (0) : I^i + I^{n-i+1}/I^{n-i+1}$. Assume that there exists $\mathfrak{q} \in \mathrm{Ass}_A L$. Then $\mathrm{ht}_A \mathfrak{q} = 0$ because the ring A/I^{n-i+1} is Cohen-Macaulay. Since $\mathrm{G}(I_\mathfrak{q})$ is a Gorenstein ring, we get $I_\mathfrak{q}^{n-i+1} = (0) : I_\mathfrak{q}^i$ by [O] (notice that $n = \mathrm{r}_{(0)}(I_\mathfrak{q})$). So $L_\mathfrak{q} = (0)$, which is a contradiction. Thus $L = (0)$ and hence $I^{n-i+1} \supseteq (0) : I^i$. This completes the proof by 4.20. □

We are now ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1 and Theorem 1.2. We may assume G is a Cohen-Macaulay ring with $\mathrm{a}(G) = n - s$ by 3.17. When $n = 0$, the assertions follows form 2.1 because the condition that $\operatorname{depth} A/I^i \ge d - s - i$ for all $1 \le i \le \ell - s$ is equivalent to saying that $\operatorname{depth} A/I^i + J_s \ge d - s - i$ for all $1 \le i \le \ell - s$ (cf. 3.10 and [GNN2], 3.4). Therefore we may assume the number n is positive. Then Theorem 1.1 and Theorem 1.2 directly follows from 4.20 together with 4.6. □

In the rest of this section we assume that $\mathrm{K}_A = A$. We put $R' = \mathrm{R}(I)$. Let $\omega = \{\omega_i\}_{i \in \mathbb{Z}}$ denote the canonical I-filtration of A (see [GI1], 1.1). Hence $\omega_{-\mathrm{a}(G)+i} \supseteq I^{i+1}$ for all $i \in \mathbb{Z}$ and $\mathrm{K}_{R'} \cong \oplus_{i \in \mathbb{Z}} \omega_i$ as graded R'-modules.

Lemma 4.22. *Assume that $G_\mathfrak{p}$ is a Gorenstein ring with* $\mathrm{a}(G_\mathfrak{p}) = \mathrm{a}(G)$ *for all* $\mathfrak{p} \in \cup_{i \in \mathbb{Z}} \mathrm{Ass}_A A/I^i$. *Then* $\mathrm{K}_{R'} \cong R'(\mathrm{a}(G) + 1)$ *as graded R'-modules.*

Proof. Take any $\mathfrak{p} \in \cup_{i \in \mathbb{Z}} \mathrm{Ass}_A A/I^i$. Since $G_\mathfrak{p}$ is a Gorenstein ring with $\mathrm{a}(G_\mathfrak{p}) = \mathrm{a}(G)$, we get $[\omega_{-\mathrm{a}(G)+i}]_\mathfrak{p} = I_\mathfrak{p}^{i+1}$ for all $i \in \mathbb{Z}$, so that $\omega_{-\mathrm{a}(G)+i} = I^{i+1}$ for all $i \in \mathbb{Z}$ (recall that $\omega_{-\mathrm{a}(G)+i} \supseteq I^{i+1}$ in general). Thus $\mathrm{K}_{R'} \cong R'(\mathrm{a}(G) + 1)$ as graded R'-modules. □

For each $\mathfrak{p} \in \mathrm{V}(I)$, we put

$$\delta_{\mathfrak{p}} := \min\left\{\dim A_{\mathfrak{p}},\ \frac{1}{2}(\dim A_{\mathfrak{p}} + \lambda(I_{\mathfrak{p}}) + 1)\right\}.$$

Then the main result in this section can be stated as follows.

Theorem 4.23. *Let A be a Gorenstein local ring and let I be a height unmixed ideal. Assume that* $\operatorname{depth}[A/I^i + J_s]_{\mathfrak{p}} \geq \min\{\delta_{\mathfrak{p}} - s, \delta_{\mathfrak{p}} - s + n - i\}$ *for all $1 \leq i \leq n - s + \min\{\operatorname{ht}_A \mathfrak{p}, \ell\}$ and for all $\mathfrak{p} \in \mathrm{V}(I)$. Then* $\mathrm{G}(I)$ *is a Gorenstein ring with* $\mathrm{a}(G) = n - s$ *if*

(1) $\mathrm{r}_J(I) \leq n - s + \ell$,

(2) $J_s \cap I^i = J_s I^{i-1}$ *for all* $1 \leq i \leq n$,

(3) $(J_i : a_{i+1}) \cap I^{n-s+i+1} = J_i I^{n-s+i}$ *for all* $s \leq i < \ell$, *and*

(4) $\begin{cases} J_s^n : I^n \subseteq I^n & (s > 0); \\ (0) : I^i \subseteq I^{n-i+1} \quad \textit{for all } 1 \leq i \leq n & (s = 0). \end{cases}$

Proof. Let $\mathrm{a}^*(G) := \max\{\mathrm{a}_i(G) \mid i \in \mathbb{Z}\}$. Furthermore, we set $\underline{\mathrm{a}}_i(G) := \max\{m \in \mathbb{Z} \mid [\mathrm{H}^i_{G_+}(G)]_m \neq (0)\}$ for each $i \in \mathbb{Z}$ and put $\underline{\mathrm{a}}^*(G) := \max\{\underline{\mathrm{a}}_i(G) \mid i \in \mathbb{Z}\}$. From the next claim we can see that the above conditions (1)-(4) commute with localization.

Claim 4.24. *Let $\mathfrak{p} \in \mathrm{V}(I)$ and $i = \min\{\operatorname{ht}_A \mathfrak{p}, \ell\}$. Then $\mathrm{r}_{J_{i\,\mathfrak{p}}}(I_{\mathfrak{p}}) \leq n - s + i$.*

Proof. Let $\mathfrak{p} \in \mathrm{V}(I)$. We may assume $i = \operatorname{ht}_A \mathfrak{p} < \ell$. We have $\mathrm{r}_s(I_{\mathfrak{p}}) = n$ and $\operatorname{ht}_{A_{\mathfrak{p}}} I_{\mathfrak{p}} = s$ because I is a height unmixed ideal. From [Tr], 2.7, we obtain $\mathrm{r}_{J_{i\,\mathfrak{p}}}(I_{\mathfrak{p}}) - i \leq \underline{\mathrm{a}}^*(\mathrm{G}(I_{\mathfrak{p}}))$. Since local cohomology commutes with localization, we get $\underline{\mathrm{a}}^*(\mathrm{G}(I_{\mathfrak{p}})) \leq \underline{\mathrm{a}}^*(G)$. Further, $\underline{\mathrm{a}}^*(G) = \mathrm{a}^*(G)$ (see [Hy], 2.3 or [Tr], 2.8). The inequality $\mathrm{a}^*(G) \leq n - s$ follows from [Tr], 3.6 together with 3.8 and 3.9 (recall that J_s is a complete intersection by 3.2). Thus $\mathrm{r}_{J_{i\,\mathfrak{p}}}(I_{\mathfrak{p}}) - i \leq n - s$. □

Let $\mathcal{A} := \{\mathfrak{p} \in \mathrm{V}(I) \mid \dim A_{\mathfrak{p}} = \lambda(I_{\mathfrak{p}})\}$. Recall that the rings R' and G are quasi-unmixed because so is A (see [HIO], 18.24). We need the next claim.

Claim 4.25. *The ring G fulfills Serre's condition (S_1).*

Proof. Take any $Q \in \operatorname{Ass} G$ and put $\mathfrak{q} = Q \cap A$. We have $\operatorname{depth} G_\mathfrak{q} \leq \dim G_\mathfrak{q}/QG_\mathfrak{q}$. Suppose $\dim A_\mathfrak{q} \geq (\dim A_\mathfrak{q} + \lambda(I_\mathfrak{q}) + 1)/2$. Then $\operatorname{depth} G_\mathfrak{q} \geq (\dim A_\mathfrak{q} + \lambda(I_\mathfrak{q}) + 1)/2$ by 3.1. Since $\dim G_\mathfrak{q}/QG_\mathfrak{q} \leq \lambda(I_\mathfrak{q})$, we get $\operatorname{depth} G_\mathfrak{q} \leq \lambda(I_\mathfrak{q})$. But $\operatorname{depth} G_\mathfrak{q} - \lambda(I_\mathfrak{q}) \geq (\dim A_\mathfrak{q} + \lambda(I_\mathfrak{q}) + 1)/2 - \lambda(I_\mathfrak{q}) = (\dim A_\mathfrak{p} - \lambda(I_\mathfrak{p}) + 1)/2 > 0$, as $\dim A_\mathfrak{p} \geq \lambda(I_\mathfrak{p})$. This is impossible. Hence $\dim A_\mathfrak{p} < (\dim A_\mathfrak{p} + \lambda(I_\mathfrak{p}) + 1)/2$. Then $\operatorname{depth} G_\mathfrak{q} \geq \dim A_\mathfrak{q}$ by 3.1. Then $\operatorname{ht}_G Q = \operatorname{ht}_{G_\mathfrak{q}} QG_\mathfrak{q} \leq \dim G_\mathfrak{q} - \operatorname{depth} G_\mathfrak{q} = 0$. Therefore $\operatorname{ht}_G Q = 0$ and hence the ring G fulfills Serre's condition (S_1). □

Hence G is unmixed (i.e. $\operatorname{Ass} G = \{Q \in \operatorname{Spec} G \mid \dim G = \dim G/Q\}$) because G is equi-dimensional. Then $\mathcal{A} = \operatorname{Ass}_A G$ (see [M], 4.1). Let us prove the following two claims.

Claim 4.26. $\mathrm{K}_{R'} \cong R'(n-s+1)$ *as graded R'-modules.*

Proof. Take any $\mathfrak{p} \in \mathcal{A}$. Then $\dim A_\mathfrak{p} < (\dim A_\mathfrak{p} + \lambda(I_\mathfrak{p}) + 1)/2$, so that the ring $G_\mathfrak{p}$ is a Gorenstein ring with $\mathrm{a}(G_\mathfrak{p}) = n - s$ by 1.1 and 1.2. We shall show $\mathrm{a}(G) = n - s$. In deed, the inequality $\mathrm{a}^*(G) \leq n - s$ follows from [Tr], 3.6 together with 3.8 and 3.9, and hence $\mathrm{a}(G) \leq n - s$. The converse inequality is clear because $n - s = \mathrm{a}(G_\mathfrak{p}) \leq \mathrm{a}(G)$ (recall that $\operatorname{Supp}_G \mathrm{K}_G = \operatorname{Spec} G$, as $\operatorname{Ass}_G \mathrm{K}_G = \{Q \in \operatorname{Spec} G \mid \dim G = \dim G/Q\} = \operatorname{Ass} G$). Since $\mathcal{A}(= \operatorname{Ass}_A G) = \cup_{i \in \mathbb{Z}} \operatorname{Ass}_A A/I^i$, we get $\mathrm{K}_{R'} \cong R'(n-s+1)$ as graded R'-modules because of 4.22. □

Claim 4.27. $\operatorname{depth} R'_P \geq \min\left\{\dim R'_P, \frac{1}{2}\dim R'_P + 1\right\}$ *for all* $P \in \operatorname{Spec} R'$.

Proof. Let $P \in \operatorname{Spec} R'$. We may assume $\operatorname{ht}_{R'} P \geq 3$ by the claim above. Therefore it is enough to show $\operatorname{depth} R'_P \geq \frac{1}{2}\dim R'_P + 1$. If $t^{-1} \notin P$, then $R'_P = A[t, t^{-1}]_P$, so that we have nothing to prove. Hence we may also assume $t^{-1} \in P$. Put $\mathfrak{p} = P \cap A$. Then

$$\begin{aligned}
\operatorname{depth} R'_P &\geq \operatorname{depth} R'_\mathfrak{p} - \dim R'_\mathfrak{p}/PR'_\mathfrak{p} = \operatorname{depth} R'_\mathfrak{p} - \dim R'_\mathfrak{p} + \dim R'_P \\
&= \operatorname{depth} G_\mathfrak{p} - \dim G_\mathfrak{p} + \dim R'_P \\
&= \left(\frac{1}{2}\dim R'_P + 1\right) + \frac{1}{2}\dim R'_P - 1 + \operatorname{depth} G_\mathfrak{p} - \dim G_\mathfrak{p} \\
&= \left(\frac{1}{2}\dim R'_P + 1\right) + \frac{1}{2}\dim G_P - \frac{1}{2} + \operatorname{depth} G_\mathfrak{p} - \dim G_\mathfrak{p} \\
&= \left(\frac{1}{2}\dim R'_P + 1\right) + \operatorname{depth} G_\mathfrak{p} - \\
&\qquad - \frac{1}{2}\dim G_\mathfrak{p} - \frac{1}{2}\left(\dim G_\mathfrak{p} - \dim G_P + 1\right).
\end{aligned}$$

Hence we must to show the inequality

$$\operatorname{depth} G_{\mathfrak{p}} \geq \frac{1}{2} \dim G_{\mathfrak{p}} + \frac{1}{2} \left(\dim G_{\mathfrak{p}} - \dim G_P + 1 \right).$$

If the ring $G_{\mathfrak{p}}$ is Cohen-Macaulay, then we have nothing to prove because $\operatorname{ht}_G P \geq 1$. Assume the ring $G_{\mathfrak{p}}$ is not Cohen-Macaulay. Then $\dim A_{\mathfrak{p}} > \frac{1}{2}(\dim A_{\mathfrak{p}} + \lambda(I_{\mathfrak{p}}) + 1)$ by 3.1. Since $\lambda(I_{\mathfrak{p}}) \geq \dim G_{\mathfrak{p}} - \dim G_P$, it suffices to prove $\operatorname{depth} G_{\mathfrak{p}} \geq \frac{1}{2} \dim G_{\mathfrak{p}} + \frac{1}{2} \left(\lambda(I_{\mathfrak{p}}) + 1 \right)$. This follows form 3.1, as $\delta_{\mathfrak{p}} = \frac{1}{2}(\dim A_{\mathfrak{p}} + \lambda(I_{\mathfrak{p}}) + 1)$. □

Thanks to [H], 5.8 together with two claims above, we get R' is a Gorenstein ring and hence so is G. This completes the proof of 4.23. □

Concluding this paper, let us give an example to illustrate Theorem 4.23. Let $A = k[[X_{ij} \mid i = 1, 2,\ 1 \leq j \leq 5]]$ be a formal power series ring in 10 variables over an infinite field k and let I be an ideal in A generated by the maximal minors of the 2 by 5 generic matrix $X = [X_{ij}]$. Then $\operatorname{ht}_A I = 4$ and $\lambda(I) = 7$. Let J be a minimal reduction of I and let $\{a_1, a_2, \cdots, a_7\}$ be a good generating set for J. Since A/I is an isolated singularity, J is a special reduction (cf. [AH], 6.4 or [N1], 2.5), which means $\mathrm{r}_i(I) = 0$ for all $4 \leq i < 7$. We have $\mathrm{r}_J(I) = 2$, $\operatorname{depth} A/I = 6$, $\operatorname{depth} A/I^2 = 3$, and $\operatorname{depth} A/I^3 = 3$. We will show that Theorem 4.23 is practical about this example. The conditions (1), (2), and (4) in 4.23 is naturally satisfied. We must show that the condition (3) is satisfied, which means the condition (B_6) is fulfilled for $a = -4$ (see Section 3). To begin with let us note the following lemma.

Lemma 4.28. *Let h be an integer and let α be an integer with $\alpha \leq \operatorname{depth} A/I + 1$. Assume that $J_s \cap I^i = J_s I^{i-1}$ for all $1 \leq i \leq h$. Then $\operatorname{depth} A/I^i + J_s \geq \alpha - i$ for all $1 \leq i \leq h$ if so is $\operatorname{depth} A/I^i$.*

Proof. See the proof of [GNN2], 3.4. □

We have $(J_4 : a_5) \cap I = J_4$. In fact, it is clear that $(J_4 : a_5) \cap I \supseteq J_4$. Take any $\mathfrak{p} \in \operatorname{Ass}_A A/J_4$ and it is enough to show $[(J_4 : a_5) \cap I]_{\mathfrak{p}} = [J_4]_{\mathfrak{p}}$. Then $\operatorname{ht}_A \mathfrak{p} = 4$ because J_4 is a complete intersection. We may assume $\mathfrak{p} \in \mathrm{V}(I)$ because $a_5 \notin \mathfrak{p}$ if $\mathfrak{p} \notin \mathrm{V}(I)$ (recall that $\{a_1, a_2, \cdots, a_7\}$ is a good generating set for J). Then $I_{\mathfrak{p}} = [J_4]_{\mathfrak{p}}$ because $\mathrm{r}_4 = 0$. This completes a proof of the required equality, which means the condition (B_4) is fulfilled for $a = -4$.

Claim 4.29. *The following two assertion holds true.*

1. $J_4 \cap I^i = J_4 I^{i-1}$ *for all* $1 \le i \le 3$.
2. *The condition* (B_6) *is fulfilled for* $a = -4$.

Proof. Since the condition (B_4) is fulfilled for $a = -4$, we obtain that the condition (A_5) is fulfilled for $\delta = 10 (= \dim A)$ and $a = -4$ from 3.15, namely $\operatorname{depth} A/J_j I \ge 5$ for $j = 4, 5$ because $\operatorname{depth} A/I = 6 \ge 10 - 4 - 1$. Then the condition (B_5) is fulfilled for $a = -4$ by 3.18. And moreover, $J_4 \cap I^2 = J_4 I$. In fact, assume that there exists $\mathfrak{p} \in \operatorname{Ass}_A A/J_4 I$ such that $[J_4 \cap I^2]_\mathfrak{p} \supsetneq [J_4 I]_\mathfrak{p}$. Then $\mathfrak{p} \ne \mathfrak{m}$, as $\operatorname{depth} A/J_4 I \ge 5$. Hence $\mathrm{G}(I_\mathfrak{p})$ is a Cohen-Macaulay ring because A/I is an isolated singularity, so that the sequence $a_1 t, \ldots, a_4 t$ is $\mathrm{G}(I_\mathfrak{p})$-regular by 3.3. Therefore $J_{4\mathfrak{p}} \cap I_\mathfrak{p}^i = J_{4\mathfrak{p}} I_\mathfrak{p}^{i-1}$ for all $i \in \mathbb{Z}$ by [VV], which is contradiction.

Hence by 4.28, $\operatorname{depth} A/I^2 + J_s = 3 (= 9 - 4 - 2)$ because $\operatorname{depth} A/I^2 = 3$. Then the condition (A_6) is fulfilled for $\delta = 9$ and $a = -4$ by 3.15, namely $\operatorname{depth} A/J_j I^2 \ge 3$ for $j = 4, 5, 6$. Then using the same argument above, we get $J_4 \cap I^3 = J_4 I^2$.

We must prove $(J_6 : a_7) \cap I^3 = J_6 I^2$. Take any $\mathfrak{p} \in \operatorname{Ass}_A A/J_6 I^2$ and it suffices to prove $[(J_6 : a_7) \cap I^3]_\mathfrak{p} = [J_6 I^2]_\mathfrak{p}$. We may assume $\mathfrak{p} \in \mathrm{V}(I)$ because $\{a_1, a_2, \cdots, a_7\}$ is a good generating set for J. We have $\operatorname{ht}_A \mathfrak{p} \le 7$ because $\operatorname{depth} A/J_6 I^2 \ge 3$. If $\operatorname{ht}_A \mathfrak{p} \le 6$, then $I_\mathfrak{p} = [J_6]_\mathfrak{p}$ because $\mathrm{r}_i = 0$ for all $4 \le i < 7$, so that we have nothing to prove. Let $\operatorname{ht}_A \mathfrak{p} = 7$. Then $G_\mathfrak{p}$ is Cohen-Macaulay, and hence $[(J_6 : a_7)]_\mathfrak{p} \cap I_\mathfrak{p}^3 = [J_6]_\mathfrak{p} I_\mathfrak{p}^2$ by 3.6 (recall that $\mathrm{a}(G_\mathfrak{p}) = -4$, as $I_\mathfrak{p}$ is a complete intersection of height 4). Thus we obtain that the condition (B_6) is fulfilled for $a = -4$. $\square$

Let $\mathfrak{p} \in \mathrm{V}(I)$. We want to prove the inequality $\operatorname{depth}[A/I^i + J_s]_\mathfrak{p} \ge \delta_\mathfrak{p} - 4 - i$ for all $1 \le i \le \min\{\operatorname{ht}_A \mathfrak{p}, 7\} - 4$. If $\mathfrak{p} \ne \mathfrak{m}$, then $G_\mathfrak{p}$ is Cohen-Macaulay and hence the sequence $a_1 t, \ldots, a_4 t$ is $G_\mathfrak{p}$-regular by 3.3, so that it is enough to show $\operatorname{depth}[A/I^i]_\mathfrak{p} \ge \delta_\mathfrak{p} - 4 - i$ for all $1 \le i \le \min\{\operatorname{ht}_A \mathfrak{p}, 7\} - 4$ by 4.28. Since the ideal $I_\mathfrak{p}$ is a complete intersection, $[A/I^i]_\mathfrak{p}$ is a Cohen-Macaulay ring of dimension $\dim A_\mathfrak{p} - 4$, so that we have nothing to prove. Let $\mathfrak{p} = \mathfrak{m}$. Then $\delta_\mathfrak{p} = 9$, and hence it suffice to show that $\operatorname{depth} A/I^i \ge 5 - i$ for all $1 \le i \le 3$ by 4.28 together with 4.29, 1. We have $\operatorname{depth} A/I = 6$, $\operatorname{depth} A/I^2 = 3$, and $\operatorname{depth} A/I^3 = 3$, which completes the proof of required inequality. Thus we get $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I)) = -4$ by Theorem 4.23.

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Department of Mathematics
School of Science and Technology
Meiji University
214-8571 JAPAN
e-mail: goto@math.meiji.ac.jp

Mathematics Laboratory
Hokkaido University of Education, Sapporo
002-8502 JAPAN
e-mail: iai@sap.hokkyodai.ac.jp