Supersequences and bounded sets in topological groups*

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Abstract

We show that the existence of a non-metrizable compact subspace of a topological group G often implies that G contains an uncountable *supersequence* (a copy of the one-point compactification of an uncountable discrete space). The existence of uncountable supersequences in a topological group has a non-trivial impact on functionally bounded subsets of the group. For example, if a topological group G contains an uncountable supersequence and K is a closed functionally bounded subset of G which does not contain uncountable supersequences, then any subset A of K is functionally bounded in $G \setminus (K \setminus A)$.

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1 Introduction

In the article, we study the problems of when a topological group contains an uncountable supersequence and what the impact of the presence of such supersequences on bounded subsets of the group is. In Section 2 we show that if a topological group G contains a duplicate of uncountable compact space, or the two arrows space, or a copy of the ordinal space ω_1 with the order topology, then G contains an uncountable supersequence (that is, an uncountable compact set with a single non-isolated point). These facts follow from a more general assertion based on the concept of a *fully closed mapping* (see Theorem 2.1).

It is proved in Section 3 that if a paratopological group G contains a copy of the ordinal space $\omega_1 + 1$ or an uncountable supersequence, then, for every sequence $S = \{y_n : n \in \omega\}$ converging to an element $g \in G$, the set S is bounded in $G \setminus \{g\}$. This result can be considerably strengthened in the case of topological groups: if a topological group G contains an uncountable supersequence, then for every first countable (or metrizable) compact subset K of G and for every $S \subseteq K$, the set S is bounded in $G \setminus (K \setminus S)$ (see Corollary 3.4).

1.1 Notation and terminology

All topological groups we consider are assumed to be Hausdorff. A *supersequence* in a space is a compact subset with a single non-isolated point. Clearly, every supersequence is a one-point compactification of an infinite discrete set. A sequence $\{x_n : n \in \omega\}$ in a space X is said to be *trivial* if it is eventually constant.

A subset B of a space X is called *bounded* (or *functionally bounded*) in X if the image $f(B)$ is bounded in the real line $\mathbb{R}$, for every continuous function $f\colon X \to \mathbb{R}$. It is clear that pseudocompact subspaces of X are bounded, but not vice versa [7].

The character of a point x in a space X is denoted by $\chi(x, X)$ and, similarly, $\chi(F, X)$ is the character of a subset F of X, that is, the minimal cardinality of a base for X at F.

A topological group G is *precompact* if G can be covered by finitely many translates of an arbitrary neighbourhood of the identity. Every pseudocompact group is precompact by a result in [1].

A *paratopological group* is an abstract group with topology in which the multiplication is (jointly) continuous. Clearly, every topological group is paratopological, but not vice versa.

2 Supersequences in topological groups

The existence of certain type compact sets in a topological group often implies that the group contains a long supersequence. Here we present a general result in this direction, Theorem 2.1, and deduce several corollaries. Our argument makes use of *fully closed mappings.*

Let $f\colon X \to Y$ be a continuous onto mapping. For every $U \subseteq X$, we put $f^{\#}(U) = Y \setminus f(X \setminus U)$. It is clear that if f is a closed mapping, then $f^{\#}(U)$ is open in Y for each open subset U of X. In addition, $f^{\#}(U)$ is the biggest subset of Y satisfying $f^{-1}f^{\#}(U) \subseteq U$.

Following [4], we say that a continuous closed mapping $f\colon X \to Y$ is *fully closed* if, for every finite open cover $U_1, \ldots, U_n$ of the space X, the complement $Y \setminus \bigcup_{i=1}^{n} f^{\#}(U_i)$ is finite. Fully closed mappings were defined and then applied by Fedorchuk to construct many highly non-trivial examples of compact spaces with certain combinations of properties (see [3, 5]).

Theorem 2.1 *Let $f\colon X \to Y$ be a fully closed mapping of a compact space X onto Y such that the set $T = \{y \in Y : |f^{-1}(y)| > 1\}$ has an infinite cardinality τ. Then every topological group containing X also contains a supersequence of the length τ.*

Proof. Suppose that a topological group G with identity e contains X as a subspace. For every $y \in T$, choose distinct points $a_y, b_y \in f^y$. Then

$$S = \{e\} \cup \{a_y^{-1} b_y : y \in T\}$$

is a supersequence in G with the single non-isolated point e. In fact, we claim even more: for every neighbourhood O of e in G, the set $\{y \in T : a_y^{-1} b_y \notin O\}$ is finite.

Indeed, choose an open symmetric neighbourhood U of e such that $U^2 \subseteq O$. Since X is compact, we can find points $x_1, \ldots, x_n$ in X such that $X \subseteq \bigcup_{i=1}^{n} x_i U$. For every $i \leq n$, put $U_i = X \cap x_i U$. Then the sets $U_1, \ldots, U_n$ form an open cover of X, so the complement $F = X \setminus \bigcup_{i=1}^{n} f^{\#}(U_i)$ is finite. Take any point $y \in T \setminus F$. Then $y \in f^{\#}(U_i)$ for some $i \leq n$, so the points a_y and b_y lie in U_i. Hence $a_y^{-1} b_y \in (x_i U)^{-1} x_i U = U^2 \subseteq O$. This proves that $\{y \in T : a_y^{-1} b_y \notin O\} \subseteq F$, and our claim is proved.

It remains to verify that $|S| = |T| = \tau$. Suppose to the contrary that $|S| < |T|$. Then there exists an infinite set $Z \subseteq T$ and an element $g \in G$ such that $a_y^{-1} \cdot b_y = g$ for each $y \in Z$. It is clear that $g \neq e$. Take a neighbourhood O of e in G such that $g \notin O$. Then the set $\{y \in T : a_y^{-1} b_y \notin O\}$ is infinite, which contradicts the above claim.

We conclude, therefore, that S is a supersequence of the length τ in G. This finishes the proof. $\square$

Our next step is to give two simple examples of fully closed mappings of compact spaces. Let X be an arbitrary compact space X. Denote by X' a discrete copy of X, with a corresponding bijection $\varphi\colon X \to X'$, $\varphi(x) = x'$ for each $x \in X$. The *Alexandroff duplicate* of X is the union $A(X) = X \cup X'$, where all points of X' are isolated in $A(X)$ and a base of a point $x \in X$ in $A(X)$ consists of the sets $U \cup (\varphi(U) \setminus \{x\})$, where U is an arbitrary open neighbourhood of x in X (see [2, 3.1.26 and 3.1.G]). The space $A(X)$ with this topology is again compact and Hausdorff [2]. Let $\psi\colon A(X) \to X$ be a mapping defined by $\psi(x) = x$ and $\psi(x') = x$, for each $x \in X$. It is clear that ψ is a continuous retraction of $A(X)$ onto its closed subspace X.

Fact 2.2 *The mapping $\psi\colon A(X) \to X$ is fully closed for every compact space X.*

Proof. Consider an arbitrary open cover $U_1, \dots, U_n$ of $A(X)$. Without loss of generality we can assume that each U_i is a basic open set, i.e., either $U_i = \{x_i'\}$ or $U_i = V_i \cup (\varphi(V_i) \setminus \{x_i\})$ for some $x_i \in X$, where V_i is an open neighbourhood of x_i in X. Hence $X \setminus \bigcup_{i=1}^{n} \psi^{\#}(U_i) \subseteq \{x_i : 1 \leq i \leq n\}$. This proves the claim. □

Let $X = C_0 \cup C_1$, where $C_0 = (0,1] \times \{0\}$ and $C_1 = [0,1) \times \{1\}$, be the *two arrows space* (see [2, 3.10.C]). It is well known that X is a perfectly normal compact Hausdorff space. Denote by f the mapping of X onto the closed unit interval $I = [0,1]$ defined by $f(x,i) = x$, for all $x \in I$ and $i = 0,1$. It is clear that f is continuous and, since X is compact, f is a closed mapping. This simple observation can be refined as follows:

Fact 2.3 *The mapping $f\colon X \to I$ is fully closed.*

Proof. Let $U_1, \dots, U_n$ be an open cover of X. As in the proof of Fact 2.2, we can assume that each U_i is a basic open set, that is, either $U_i = \{(x_i, 0)\} \cup (x_i - \varepsilon_i, x_i) \times \{0,1\}$ or $U_i = \{(x_i,1)\} \cup (x_i, x_i + \varepsilon_i) \times \{0,1\}$, where $\varepsilon_i > 0$ for each $i \leq n$. In the first case, we have $f^{\#}(U_i) = (x_i - \varepsilon_i, x_i)$ and, in the second, $f^{\#}(U_i) = (x_i, x_i + \varepsilon_i)$. Hence $I \setminus \bigcup_{i=1}^{n} f^{\#}(U_i) \subseteq \{x_1, \dots, x_n\}$. This implies the claim. □

Proposition 2.4 *Suppose that a topological group G contains one of the following sets:*

(a) *duplicate of an uncountable compact space;*
(b) *two arrows space;*
(c) *a copy of ω_1 with the order topology.*

Then G contains an uncountable supersequence. In fact, in case (b), G contains a supersequence of the length 2^ω.

Proof. Items (a) and (b) follow from Facts 2.2 and 2.3, respectively, combined with Theorem 2.1. Therefore, it remains to verify (c). Let G contain a topological copy of the space ω_1 which we identify with ω_1. Then the subspace

$$K = \{e\} \cup \{\alpha^{-1} \cdot (\alpha+1) : \ \alpha < \omega_1\}$$

of G is homeomorphic to a supersequence of the length ω_1 and e is the unique non-isolated point of K. Indeed, suppose that U is a neighbourhood of the neutral element e of G. Choose an open symmetric neighbourhood V of e in G such that $V^2 \subseteq U$. For every $\alpha < \omega_1$, let $O_\alpha = \omega_1 \cap \alpha \cdot V$. Then

$$O = \bigcup \{O_\alpha \times O_\alpha : \ \alpha < \omega_1\}$$

is an open neighbourhood of the diagonal $\Delta = \{(\gamma,\gamma) : \gamma < \omega_1\}$ in $\omega_1 \times \omega_1$. It is clear that all accumulations points of the discrete set $C = \{(\alpha, \alpha+1) : \alpha < \omega_1\}$ lie in Δ. Hence the complement $C \setminus O$ is a closed discrete subset of the countably compact space $\omega_1 \times \omega_1$. We conclude that the set $C \setminus O$ is finite. This immediately implies that $K \setminus U$ is also finite, as required. As in the proof of Theorem 2.1, one easily verifies that $|K| = \omega_1$. So, K is an uncountable supersequence with the single non-isolated point e. $\square$

Remark 2.5 One can deduce both (b) and (c) of Proposition 2.4 from a more general fact. First, the two arrows space and ω_1 are linearly ordered spaces with uncountably many *twins* (two points x, y of a linearly ordered set $(X, <)$ are said to be twins if $x < y$ and the order interval (x, y) in X is empty or, equivalently, no point $z \in X$ satisfies $x < z < y$). Then, apply [9, Section 6] to show that if a topological group G with identity e contains a copy of a linearly ordered space $(X, <)$, then the set

$$K = \{e\} \cup \{x^{-1}y : \ x \text{ and } y \text{ are twins in } X\}$$

is either finite or a supersequence in G converging to e.

3 Bounded subsets of topological groups

Let $\omega_1 + 1$ be the space of ordinals $\leq \omega_1$ with the order topology and $\omega + 1$ be the usual sequence converging to the point ω. We identify the limit points ω_1 and ω of the spaces $\omega_1 + 1$ and ω, respectively, thus obtaining a space X in which the two "sequences" ω_1 and ω converge to the same point $x_0 \in X$. Clearly, ω_1 is countably compact, so it remains bounded in $X \setminus \{x_0\}$. However, the copy of $\omega \subseteq \omega_1$ is a clopen discrete subspace of X, so it is not bounded in X. It turns out that, in topological groups, the situation changes substantially.

We recall that a *paratopological group* is a group with topology in which the multiplication is jointly continuous (but the inversion can fail to be continuous).

Proposition 3.1 *Suppose that a paratopological group G contains a copy of the ordinal space $\omega_1 + 1$ or an uncountable supersequence. If a sequence $S = \{y_n : n \in \omega\}$ converges to an element $g \in G$, then S is bounded in $G \setminus \{g\}$.*

Proof. Let $K = \{e\} \cup \{x_\alpha : \alpha < \omega_1\}$ be a subspace of G homeomorphic to a supersequence of the length ω_1, with the unique non-isolated point e, the identity of G. We assume that $e \neq x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. Since translations in a paratopological group are homeomorphisms, we can assume that $g = e$, that is, the sequence S converges to the identity e. Clearly, we can also assume that $e \notin S$. Then $T = S \cup \{e\}$ is a one-point compactification of S.

For every $n \in \omega$, there can exist only one index $\alpha_n < \omega_1$ such that $x_{\alpha_n} \cdot y_n = e$. Take $\beta < \omega_1$ such that $\alpha_n < \beta$ for each $n \in \omega$. Then $L = \{e\} \cup \{x_\alpha : \beta < \alpha < \omega_1\}$ is a subspace of G homeomorphic to K. Let $P = L \times T$ be the product space. It is easy to verify that the subset $\{e\} \times S$ of the space $X = P \setminus \{(e, e)\}$ is bounded in X. Consider the multiplication mapping $f : G \times G \to G$, $f(x, y) = x \cdot y$ for $x, y \in G$. Since f is continuous, the image $S = f(\{e\} \times S)$ is bounded in $f(X)$. Our choice of the ordinal β implies that $e \notin f(X)$, so S is bounded in $G \setminus \{e\}$.

Similarly, suppose that $\{e\} \cup \{x_\alpha : \alpha < \omega\}$ is a subspace of G homeomorphic to $\omega_1 + 1$, where e plays the role of the point ω_1 in $\omega_1 + 1$. Choose an ordinal $\beta < \omega_1$ as above and put $L = \{e\} \cup \{x_\alpha : \beta < \alpha < \omega_1\}$. Note that the subspace $X = (L \times T) \setminus \{(e, e)\}$ of the compact space $L \times T$ is pseudocompact, so the the set $S = f(\{e\} \times S)$ is bounded in $f(X) \subseteq G \setminus \{e\}$. □

In the case of topological groups, the above result can be given a much sharper form (see Theorem 3.3). First, we need an auxiliary result.

Proposition 3.2 *Let K be a closed bounded subset of a topological group H, and let $A \subseteq K$. If K does not contain any non-empty G_δ-subset of the group H, then A is bounded in $H \setminus (K \setminus A)$.*

Proof. Suppose to the contrary that K does not contain any non-empty G_δ-subset of the group H, but A is unbounded in $X = H \setminus (K \setminus A)$. Take a continuous real-valued function f on X such that the image $f(A)$ is unbounded in $\mathbb{R}$. By induction, we can choose a sequence $\{x_n : n \in \omega\} \subseteq A$ such that $|f(x_n)| \geq n$ and $|f(x_{n+1})| \geq |f(x_n)| + 3$, for each $n \in \omega$. Since f is continuous, for every $n \in \omega$ there exists an open neighbourhood U_n of the identity e in H such that $|f(x) - f(x_n)| < 1$ for all $x \in X \cap x_n \cdot U_n$. Clearly, all accumulation points of the family $\{x_n \cdot U_n : n \in \omega\}$ lie in K. Choose a sequence $\{V_n : n \in \omega\}$ of open symmetric neighbourhoods of e in H satisfying $V_{n+1}^2 \subseteq U_n \cap V_n$, for each $n \in \omega$. Then $N = \bigcap_{n\in\omega} V_n$ is a closed subgroup of type G_δ in H. Let $\pi\colon G \to G/N$ be the quotient mapping of G onto the left coset space G/N. The space G/N is submetrizable by [8, Prop. 3]; in particular, G/N has countable pseudocharacter. Then $\pi(K)$ is a bounded subset of G/N. Since every submetrizable space is Dieudonné-complete by [2, 8.5.13 (g)], the closure of $\pi(K)$ in G/N, say, C is compact.

Note that $\pi(x_n) \in \pi(K) \subseteq C$ for each $n \in \omega$. In addition, $\pi(x_n) \neq \pi(x_k)$ if $n \neq k$. Indeed, otherwise $x_n^{-1}x_k \in N$ for some $n, k \in \omega$ with $n < k$. But then $x_k \in x_n \cdot N \subseteq x_n \cdot V_n$, whence it follows that $|f(x_k) - f(x_n)| < 1$. This contradicts our choice of the points x_n's. Therefore, the infinite subset $\{\pi(x_n) : n \in \omega\}$ of the compact set C has an accumulation point $p \in C$. Since the mapping π is open, the fiber $\pi^{-1}(p)$ is in the closure of the set $\bigcup_{n\in\omega} \pi^{-1}\pi(x_n) = \bigcup_{n\in\omega} x_n \cdot N$. Since K is closed in H and does not contains non-empty G_δ-subsets of H, the intersection $\pi^{-1}(y) \cap K$ is nowhere dense in $\pi^{-1}(y)$, for each $y \in G/N$. Hence $\pi^{-1}(p)$ is in the closure of the set $(\bigcup_{n\in\omega} x_n \cdot N) \setminus K \subseteq X$. Note that $x_n \cdot N \subseteq x_n \cdot V_n$ for each $n \in \omega$, so all accumulation points of the family $\{x_n \cdot N : n \in \omega\}$ lie in K. This proves that $\pi^{-1}(p) \subseteq K$, contradicting our assumption about K and finishing the proof. □

Theorem 3.3 *Suppose that a topological group H contains an uncountable supersequence. If K is a closed bounded subset of H which does not contain uncountable supersequences and $A \subseteq K$ is an arbitrary set, then A is bounded in $H \setminus (K \setminus A)$.*

Proof. Let $F \subseteq H$ be an uncountable supersequence. One can assume that F contains the identity e of H and e is the unique non-isolated point of F. By Proposition 3.2, it suffices to verify that if P is a non-empty G_δ-set in H, then $P \setminus K \neq \emptyset$. Clearly, P can be written as $P = \bigcap_{n\in\omega} U_n$, where each U_n is open in H. Suppose to the contrary that $P \subseteq K$ and pick a point $x \in P$. The complement $F_n = F \setminus x^{-1}U_n$ is finite for each $n \in \omega$, so the set $F \setminus x^{-1}P = \bigcup_{n\in\omega} F_n$ is countable. Hence $F' = x^{-1}P \cap F$ is an uncountable supersequence and $xF' \subseteq P \subseteq K$ is also an uncountable supersequence, thus contradicting our assumption about K. This proves the theorem. □

Corollary 3.4 *Suppose that a topological group G contains an uncountable supersequence. If K is a first countable (or metrizable) compact subset of G and $S \subseteq K$, then S is bounded in $G \setminus (K \setminus S)$.*

Remark 3.5 Suppose that a subgroup H of a topological group G contains an uncountable supersequence. Then H can contain a sequence S converging to an element of $G \setminus H$ such that S is unbounded in H. Indeed, let G be the Σ-product of ω_1 copies of the discrete group $\mathbb{Z}(2) = \{0, 1\}$, i.e., G is the subgroup of 2^{ω_1} which consists of all elements x with the property that $|\{\alpha < \omega_1 : x(\alpha) = 1\}| \leq \omega$. Let also H be the the corresponding σ-product in 2^{ω_1}, that is,

$$H = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x(\nu) = 1\}| < \omega\}.$$

Then H is a dense subgroup of the Fréchet–Urysohn group G, so every point of $G \setminus H$ is a limit of a convergent sequence lying in H. For every $\alpha < \omega_1$, denote by x_α the element of H defined by $x_\alpha(\nu) = 1$ iff $\nu = \alpha$. Then the subspace $\{\bar{0}\} \cup \{x_\alpha : \alpha < \omega_1\}$ of H is a supersequence of the length ω_1, where $\bar{0}$ is the neutral element of G. Note that the group H is σ-compact, hence normal. Therefore, every closed subset of H is C-embedded in H. Since every sequence S in H converging to a point of $G \setminus H$ is closed in H, S is unbounded in H.

Problem 3.6 *Suppose that a paratopological group G contains a copy of the ordinal space ω_1. Does then G contain a copy of $\omega_1 + 1$?*

Problem 3.7 *Does there exist a "complete" paratopological group which contains a closed copy of the ordinal space ω_1?*

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