Bohr-angelicity and related properties for topological Abelian groups*

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Abstract

Applying the method proposed in [28], we study two independent classes of topological Abelian groups which are strictly angelic when endowed with their Bohr topology. In this way we get some extensions of Eberlein-Šmulyan type theorems for certain classes of topological Abelian groups. Finally, we show that under the presence of Bohr angelicity, to respect compactness (in the sense of [29]) and the Schur property are equivalent.

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Contents

Introduction

In many branches of mathematics the notion of compactness appears as a fundamental tool, and the optimal situation is when it can be used in its sequential version. Unfortunately this is not always the case, and there is a strong need to look for classes of topological spaces where compactness is equivalent to sequential or countable compactness. It was known from the early developments of general topology that metrizable spaces have this property. An important class of nonmetrizable spaces for which the equivalence also holds is provided by infinite-dimensional Banach spaces endowed with their weak topology. In fact, Eberlein-Šmulian theorem essentially states that a subset K of a Banach space is weakly compact if and only if it is sequentially weakly compact. Grothendieck was able to extend a part of this statement to the larger class of locally convex spaces which are quasi-complete in their Mackey topology. He proved that within the mentioned class, countable weak compactness is equivalent to relative weak compactness.

For an Abelian topological group G, there is a topology which can be considered analogous to the weak topology in the context of topological vector spaces. Namely, the topology generated by the set ΓG of all continuous characters of G, which we will denote by $\sigma(G, \Gamma G)$. The topology $\sigma(G, \Gamma G)$ is called the Bohr topology of G, because it coincides with the topology induced in G from the so called Bohr compactification of G. The existence of the Bohr compactification allows to define and study the Bohr topology in any topological group (i.e., not necessarily Abelian; see, e.g. [12]). In the present paper we deal only with Abelian groups, therefore we use the above given, direct definition of the Bohr topology.

Since the study of topological vector spaces through properties of their weak topologies is very important, we think that the investigation of similar questions in the larger class of topological Abelian groups endowed with their Bohr topology may be also interesting.

In this paper we will prove Eberlein-Šmulian type theorems for all metrizable and also for certain submetrizable dually separated groups. The obtained statements generalize some results in this direction contained in [8]. We also prove a Grothendieck type theorem for some sense complete (but not necessarily metrizable or submetrizable) Abelian topological groups.

Our approach is based upon general topological results of the following sort: for certain pairs of topological spaces X, Y the space of continuous functions $C(X, Y)$ with the pointwise convergence topology is angelic. The first significant statement of this style was obtained and then applied to topological vector spaces in [28]. We apply similar "angelicity" results to topological Abelian groups.

1 Angelic spaces. Definition and properties

We collect in this section the notions and results convenient for our subsequent developments.

In what follows $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will stand for the sets of natural, real and complex numbers with their ordinary topological and algebraic structures. Also $\aleph_0 := card(\mathbb{N})$ and $\mathfrak{c} := card(\mathbb{R})$.

As usual, $C(X, Y)$ will stand for the set of all continuous mappings between the topological spaces X and Y and $C_p(X, Y)$ will be the same set endowed with the pointwise convergence topology.

A subset A of a topological space X is:

- *relatively compact* if its closure $\overline{A}$ is compact;
- *relatively countably compact* if each sequence in A has a cluster point in X;
- *relatively sequentially compact* if each sequence in A has a subsequence converging to a point of X;
- *countably compact*, *sequentially compact* if in the above two definitions the cluster or the limit point is required to be in A;
- *sigma-compact* if it is the union of countably many compact subsets.

Clearly, every compact or sequentially compact subset is countably compact. Similar assertion holds for the corresponding relative properties, and it is well-known that the reverse implications hold in a metrizable space. In general, a sequentially compact subset of a completely regular Hausdorff space may be nonclosed, and its closure may be not countably compact (see Example 4).

A subset of $B \subset X$ of a completely regular space X is called *bounding* if $f(B)$ is a bounded subset of $\mathbb{R}$ for every $f \in C(X, \mathbb{R})$.

A completely regular space X is called *pseudocompact* if it is a bounding subset of itself. A subset of $B \subset X$ of a completely regular space X is called *pseudocompact* if it is pseudocompact space with respect to the topology induced from that of X. Any relatively countably compact subset is pseudocompct and any pseudocompact subset is bounding. The converse statements are not true in general.

A (completely regular) Hausdorff topological space space is called:

- a *g-space* if its relatively countably compact subsets are relatively compact (see [2]; in [20, Ch.5, Part 3, §2, Exercise 1 (p.209)] the term "(E)-space" is used for the same notion).
- a g_w-*space* if its countably compact subsets are relatively compact.
- a *IN-space* if its countably compact subsets are closed (in [24] the term "C-space" is used for the same notion).
- a *Š-space* if its compact subsets are sequentially compact.

Remark.

1. Clearly, if X is a g-space, then it is a g_w-space. Observe also that for a Hausdorff g_w+IN-space space X, the countable compact subsets of X are compact.

2. A compact (or a countably compact) Hausdorff space is trivially a g-space, however it may not be neither a IN-space nor a Š-space.

3. If X is a (completely regular Hausdorff) space, for which pseudocompact closed subspaces are compact, then X is a g-space (because a relatively countably compact $A \subset X$ is pseudocompact).

A Hausdorff topological space X is said to be an *angelic space*, if for every relatively countably compact subset A of X the following two claims hold:

(i) A is relatively compact.

(ii) If $b \in \overline{A}$, then there is a sequence in A that converges to b.

It can be said that a Hausdorff topological space X is angelic if and only if X is a g-space for which any compact subspace is a Frechet-Urysohn space. As proved in [2], an angelic space is exactly a hereditary g-space.

Proposition 1.1. *[28, Lemma 0.3]*
If X is an angelic space, and $A \subset X$, the following assertions are equivalent:

1. *A is countably compact,*
2. *A is sequentially compact,*
3. *A is compact.*

Using this statement it is straightforward to check the following proposition.

Proposition 1.2. *If X is an angelic space, then it is a g+IN+Š-space.*

We will see below that angelic spaces have quite good stability properties; nevertheless the product of two angelic spaces in general is not angelic. This fact motivated Govaerts [18] to define the strictly angelic spaces, a subclass of angelic spaces closed under countable products.

A Hausdorff topological space space X is called *strictly angelic* if it is angelic and every separable compact subset of X is first countable.

The properties of angelic and of strictly angelic spaces listed jointly in the next lemma are proved in [28] and [18] respectively.

Lemma 1.3. *(a) A homeomorph of a (strictly) angelic space is (strictly) angelic.*

(b) Any subspace of a (strictly) angelic space is (strictly) angelic.

(c) If there is a continuous one-to-one map f from a regular space E into a (strictly) angelic space X, then E is also (strictly) angelic.

(d) Let τ_1 and τ_2 be regular topologies in X such that $\tau_1 \leq \tau_2$. If (X, τ_1) is (strictly) angelic, (X, τ_2) is also (strictly) angelic.

(e) If X is a completely regular Hausdorff space such that $C_p(X, \mathbb{R})$ is (strictly) angelic and Y is a metrizable space, then $C_p(X, Y)$ is also (strictly) angelic.

Remark.[1] Let τ_1 and τ_2 be completely regular Hausdorff topologies in an (infinite) set X such that $\tau_1 \leq \tau_2$, $\tau_1 \neq \tau_2$. In connection with Proposition 1.3 (d) it is worthwhile to note that

[1] *Added in March, 26. 03.2003, pointed out to us by Professor M. Tkachenko.*

(1) (X, τ_1) is a IN-space $\Longrightarrow$ (X, τ_2) is a IN-space.

(2) (X, τ_1) is a Š-space $\Longrightarrow$ (X, τ_2) is a Š-space.

However, in general,

(3) (X, τ_1) is a g-space $\not\Longrightarrow$ (X, τ_2) is a g-space.

(4) (X, τ_1) is a g_w-space $\not\Longrightarrow$ (X, τ_2) is a g_w-space.

The items (1) and (2) are easy to verify.
In order to prove (3) and (4) take into account that (X, τ_1) may be compact, and (X, τ_2) may be a countably compact, non-compact space. Assuming the existence of a Ulam-measurable cardinal $\mathfrak{m}$ and taking $card(X) = \mathfrak{m}$, it can be achieved that (X, τ_1) be a compact topological Abelian group, while (X, τ_2) is a countably compact not compact topological group [13] (however, if a group X is such that $card(X)$ is not Ulam-measurable then X does not admit a pair of topologies with such a property [1]).

Lemma 1.4. *[18] The product of countably many strictly angelic spaces is strictly angelic.*

In [7] it is proved (under the continuum hypothesis) that the product of two compact Hausdorff angelic spaces in general is not angelic (see also [31], where the the same is done in ZFC only).

We establish now a result important for our further work.

Proposition 1.5. *Let X be a Hausdorff topological space and Y be a metrizable topological space.*

(a) [21, Proposition 2.7] If $X = \overline{\cup_{n=1}^{\infty} K_n}$, where each K_n is a bounding (or in particular, a pseudocompact) subset of X, then $C_p(X, Y)$ is a an angelic space.

(b) [18, 28] If $X = \overline{\cup_{n=1}^{\infty} K_n}$, where each K_n is a relatively countably compact subsets of X, then $C_p(X, Y)$ is a strictly angelic space.

Proof. (b)[2]

Following [18], let us call a space X *sa-producing* if $C_p(X, \mathbb{R})$ is strictly angelic.

[2] Theorem 2.5 in [28] states that if X contains a dense sigma-compact subset, then $C_p(X, Y)$ is an angelic space. Theorem 2 in [18] asserts that if X contains a dense countably compact subset, then $C_p(X, Y)$ is a strictly angelic space. Since (b) is not directly formulated in [18] we are including an easy derivation of this statement from the other results of [18].

Put $X_n := \overline{K_n},\ n = 1, 2, \ldots$. Clearly, K_n is dense, relatively countably compact in X_n. By [18, Theorem 2], X_n is sa-producing. Consider $D := \cup_{n=1}^{\infty} X_n$ (equipped with the topology induced from that of X). Then, by [18, Theorem 3(b)] D is sa-producing. The set D is dense in X as shown by the inclusion $\cup_{n=1}^{\infty} K_n \subset \cup_{n=1}^{\infty} X_n = D$. Since X contains a dense sa-producing subspace, by [18, Theorem 3(a)], X itself is sa-producing. Finally, take into account Lemma 1.3 (e). □

The fact that $C_p(X, \mathbb{R})$ is a g-space for any countably compact X was already established in [19, Theorem 1].

We shall say that a completely regular Hausdorff space X is *angelic-producing* (or *a Grothendieck space* [2, §2]) if $C_p(X, \mathbb{R})$ is angelic. It seems to be unknown whether any angelic-producing space is sa-producing too.

Further study of angelic-producing spaces is done in [2, 10, 27] and some other papers. A general description of those spaces is posed as a problem in [2, 2.22].

2 The Bohr topology for a topological group

Let G be a topological Abelian group, and let $\mathbb{T}$ be the unit circle of the complex plane with its multiplicative structure and the topology induced by the usual of $\mathbb{C}$. The group of all continuous homomorphisms from G into $\mathbb{T}$ with pointwise operation will be denoted by ΓG. Any member of ΓG is called a *continuous character* (of G). If ΓG separates the points of G, then we will say that G is *dually separated* or G is a *DS-group* or G *has sufficiently many continuous characters*. A dually separated topological Abelian group is called also *a maximally almost periodic group, shortly MAP-group.* Since in our text a notion of almost periodic function will not appear at all, we prefer to use the term "dually separated group" which is defined directly by means of characters.

Any Hausdorff locally compact topological Abelian group is dually separated. This is well-known and a basic result. Evidently, any topological subgroup of a dually separated group is dually separated. It follows that, in particular, any Hausdorff totally bounded topological Abelian group is dually separated. This fact will be used in the sequel. Note however that in general neither a Hausdorff quotient, nor the completion of a dually separated topological Abelian group is dually separated (see [5] and [17] respectively).

The symbol $G^\wedge$ will denote ΓG endowed with the compact-open topology (*dual group*). We put $G^{\wedge\wedge} := (G^\wedge)^\wedge$ and thus obtained topological Abelian group is called the bidual of G.

For a fixed $g \in G$, let $g^\natural : \Gamma G \to \mathbb{T}$ be the evaluation mapping $\varphi \to \varphi(g)$ and $G^\natural := \{g^\natural : g \in G\}$. It is easy to see that $G^\natural \subset \Gamma G^\wedge$ and therefore the equality $\alpha_G(g) = g^\natural$ defines a natural mapping from G into $\Gamma G^\wedge$ which is a group homomorphism. If α_G is a topological isomorphism between G and the bidual $G^{\wedge\wedge}$, the group G is called *reflexive*.

Any Hausdorff locally compact topological Abelian group is reflexive. This is the famous Pontryagin duality theorem. For other examples and permanence properties of the class of reflexive groups we refer [4] and [5].

Let $A \subset G$, $B \subset \Gamma G$ be nonempty subsets. The annihilator of A is defined as:

$$A^\perp := \{\phi \in \Gamma G : \phi(x) = 1, \forall x \in A\}.$$

The polars of A and B respectively are:

$$A^\triangleright := \{\phi \in \Gamma G : Re(\phi(x)) \geq 0, \forall x \in A\}$$

and

$$B^\triangleleft := \{x \in G : Re(\phi(x)) \geq 0, \forall \phi \in B\},$$

where Re stands for real part.

It is not hard to see that the collection $\{K^\triangleright : K \subset G,\ K \text{ compact},\ K \neq \emptyset\}$ is a fundamental system of neighborhoods of the neutral element in $G^\wedge$.

A subset A of G is called *quasi-convex* if for every $x \in G \setminus A$, there is some $\chi \in A^\triangleright$, such that $Re(\chi(x)) < 0$. The quasi-convex hull of any subset $H \subset G$ is defined as the intersection of all quasi-convex subsets of G containing H. It coincides with the set $Q(H) := (H^\triangleright)^\triangleleft$.

The group G is *locally quasi-convex* if it has a neighborhood basis of the neutral element 0, given by quasi-convex sets. Any Hausdorff locally quasi-convex group is dually separated. Observe also that the dual $G^\wedge$ of G is locally quasi-convex. This implies that any reflexive group is locally quasi-convex.

The Bohr topology in G is the coarsest topology of all those that make continuous the elements of ΓG. It is a group topology and it will be denoted by $\sigma(G, \Gamma G)$, while $G^+ := (G, \sigma(G, \Gamma G))$. It is known that G^+ is a totally bounded topological group (see [12] from where the notation G^+ is taken and where several delicate questions about topological properties of G^+ are answered).[3]

[3]For a real topological vector space E *the weak topology* is the topology in E generated by the set LE of all continuous linear functionals. It is denoted by $\sigma(E, LE)$. On the other hand, E is in particular a topological Abelian group and it may be endowed with the Bohr topology $\sigma(E, \Gamma E)$. From the equality $\Gamma E = \{\exp(if) : f \in LE\}$, it follows that $\sigma(E, \Gamma E) \subset \sigma(E, LE)$. This fact implies that, whenever E^+ is angelic, $(E, \sigma(E, \Gamma E))$ is also angelic, provided E is dually separated. Thus, our results obtained in the context of topological groups imply (and in some cases refine too) the analogous statements known for locally convex spaces equipped with the weak topology.

It is known that for a topological Abelian group G the equality $G = G^+$ holds if and only if G is a totally bounded group.

The symbol $\sigma(\Gamma G, G)$ will denote the topology on ΓG of pointwise convergence on the elements of G, while $\Gamma_\sigma G := (\Gamma G, \sigma(\Gamma G, G))$. Observe that $\Gamma_\sigma G$ is totally bounded as a topological subgroup (in general non-closed) of the compact group $\mathbb{T}^G$.

Clearly, $\sigma(\Gamma G, G) \subset \sigma(G^\wedge, \Gamma G^\wedge)$. The latter is the Bohr topology of the dual group $G^\wedge$.

The following known fact is a group analogue of Alaoglu-Bourbaki's theorem.

Proposition 2.1. *Let G be a topological Abelian group and let U be a neighborhood of the neutral element of G. Then $U^\triangleright$ is a compact subset of $\Gamma_\sigma G$. Moreover, $U^\triangleright$ is an equicontinuous set of characters and therefore it is also compact in $G^\wedge$.*

Proof. It can be easily seen that $U^\triangleright$ is closed in the compact space $\mathbb{T}^G$. The rest of the proof is standard. □

In the next proposition we collect several important facts.

Proposition 2.2. *Let G be a topological Abelian group.*

(a) $\Gamma(\Gamma_\sigma G) = G^\natural$.

(b) If $H \subset \Gamma G$ is a subgroup which separates points of G, then H is dense in $\Gamma_\sigma G$.

(c) If $S \subset G$ is a subgroup such that $S^\perp = \{1\}$, then S is dense in G^+.

Proof. (a) Is a particular case of [11, Corollary 3.8].

(b) Assume otherwise that $H_1 := \overline{H} \neq \Gamma G$. Taking into account that $\Gamma_\sigma G$ is a totally bounded Abelian group, and that $\Gamma(\Gamma_\sigma G) = G^\natural$ (by (*a*)), there would exist $g \in G$ distinct from the neutral element of G such that ${g^\natural}_{|H_1} = 1$. In particular, $\varphi(g) = g^\natural(\varphi) = 1$ for all $\varphi \in H$, which contradicts the assumption that H separates points of G.

(c) Through item (a), item (c) is consequence of (b). We give also a proof which avoids a usage of (a).

Suppose by contradiction that $S_1 := \overline{S} \neq G$. Taking into account that G^+ is a totally bounded Abelian group, and that $\Gamma G^+ = \Gamma G$ (this is evident), there would exist $\varphi \in \Gamma G$ such that $\varphi \neq \mathbf{1}$ and $\varphi_{|S_1} = 1$. In particular, $\varphi(g) = 1$ for all $g \in S$, i.e. $\varphi \in S^\perp$. This contradicts the assumption that $S^\perp = \{1\}$.

□

The following easy observation will be useful in the sequel.

Proposition 2.3. *Let G be a dually separated group. Then, the natural mapping $\alpha := \alpha_G$ is a topological isomorphism between G^+ and $(G^\natural, \sigma(G^\natural, \Gamma_\sigma G))$.*

Proof.

- α is 1-1 due to the fact that G is a DS-group.
- α is continuous: If (x_β) is a $\sigma(G, \Gamma G)$-convergent net in G, say $x_\beta \to x$, then for every $\varphi \in \Gamma G$, $\alpha(x_\beta)(\varphi) = \varphi(x_\beta) \to \varphi(x) = \alpha(x)(\varphi)$. This implies that $\alpha(x_\beta) \to \alpha(x)$.
- $\alpha^{-1} \colon \alpha(G^+) \to G^+$ is also continuous. Let $\alpha(x_\beta)$ be a net convergent to $\alpha(x)$ in $\alpha(G^+) \subset C_p(\Gamma_\sigma G, \mathbb{T})$. For each $\varphi \in \Gamma G$, $\alpha(x_\beta)(\varphi) \to \alpha(x)(\varphi)$; therefore $\varphi(x_\beta) \to \varphi(x)$, $\forall \varphi \in \Gamma G$. Thus $x_\beta \to x$ in G^+.

□

3 Angelicity of the Bohr topology

Two classes of topological groups angelic with respect to their Bohr topology are provided by Theorems 3.3 and 3.5. They are distinct, as can be inferred from the Examples 1 and 2 in Section 7. The following propositions are crucial to obtain them.

Proposition 3.1. *Let X be a topological Abelian group.*

(a) If $X = \overline{\cup_{n=1}^{\infty} K_n}$, where $K_n,\ n = 1, 2, \dots$ are bounding subsets, then $\Gamma_\sigma X$ is angelic.
(b) If $X = \overline{\cup_{n=1}^{\infty} K_n}$, where $K_n,\ n = 1, 2, \dots$ are relatively countably compact subsets, then $\Gamma_\sigma X$ is strictly angelic.

Proof. (a) By Proposition 1.5(a), $C_p(X, \mathbb{T})$ is angelic. Since $\Gamma_\sigma X$ is a topological subspace of $C_p(X, \mathbb{T})$, by Lemma 1.3(b), it is angelic.

(b) By Proposition 1.5(b), $C_p(X, \mathbb{T})$ is strictly angelic. Since $\Gamma_\sigma X$ is a topological subspace of $C_p(X, \mathbb{T})$, by Lemma 1.3(b), it is strictly angelic.

□

Remark.

(1) Proposition 3.1 is not true in general: e.g., $\Gamma_\sigma X$ is a not a Š-space (hence, non-angelic) if X is a discrete group with $card(X) \geq \mathfrak{c}$. In fact, it is known that $\Gamma_\sigma X$ is a compact Hausdorff group having topological weight $\kappa = card(X) \geq \mathfrak{c}$. Therefore $\Gamma_\sigma X$ contains a homeomorphic copy of the compact space $\{-1, 1\}^\kappa$ (see [30]), which is not sequentially compact.

(2) $\Gamma_\sigma X$ may not be an Š-space also in the case when X a non-separable Banach space (see [14], where a chapter is devoted to Banach spaces with w^*-sequentially compact dual unit ball).

A main consequence of Proposition 3.1 is the following statement.

Proposition 3.2. *Let G be a dually separated group.*

(a) If $\Gamma_\sigma G = \overline{\cup_{n=1}^{\infty} K_n}$, where each K_n is a bounding subset of $\Gamma_\sigma G$, then G^+ is angelic.

(b) If $\Gamma_\sigma G = \overline{\cup_{n=1}^{\infty} K_n}$, where each K_n is a relatively countably compact subset of $\Gamma_\sigma G$, then G^+ is strictly angelic.

Proof. By Proposition 2.2(a), $\Gamma(\Gamma_\sigma G) = G^\natural$. Hence, applying Proposition 3.1(a) (respectively, (b)) to $X := \Gamma_\sigma G$, we get that $(G^\natural, \sigma(G^\natural, \Gamma_\sigma G))$ is angelic (respectively, strictly angelic). The fact that G^+ is angelic (respectively, strictly angelic) follows now from Proposition 2.3 and Lemma 1.3(a). □

Now we derive from Proposition 3.2 some internal conditions of strict angelicity.

Theorem 3.3. *Let G be a metrizable dually separated group. Then G^+ is strictly angelic.*

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable neighborhood basis of the neutral element of G. Clearly, $\bigcup_{n\in\mathbb{N}} U_n^\triangleright = \Gamma G$. Since the sets $U_n^\triangleright$ are compact in the space $\Gamma_\sigma G$ (Proposition 2.1), $\Gamma_\sigma G$ is sigma-compact and Proposition 3.2(b) applies. □

Under the Continuum Hypothesis, Theorem 3.3 is the best possible in the class of Hausdorff compactly generated LCA groups, as we prove next.

Proposition 3.4. *(CH) For a compactly generated Hausdorff locally compact Abelian group G the following statements are equivalent:*

(i) G^+ is angelic.

(ii) G^+ is a Š-space.

(iii) G is metrizable.

(iv) G^+ is strictly angelic.

Proof. (*i*) $\Longrightarrow$ (*ii*) by Proposition 1.2

(*ii*) $\Longrightarrow$ (*iii*)

Step 1. Suppose that G is compact and therefore its Bohr topology coincides with the original. If G were non-metrizable, its topological weight κ would be strictly greater than $\aleph_0$. This and (CH) imply: $\kappa \geq \mathfrak{c}$. Then G would contain a homeomorphic copy of the compact space $\{-1, 1\}^\kappa$ (see [30]), which is not sequentially compact. This contradicts *(ii).*

Step 2. Let G be a compactly generated Hausdorff locally compact Abelian group. Then $G = \mathbb{R}^a \times \mathbb{Z}^b \times F$, for some nonnegative integers a and b and some compact Abelian group F [23, Theorem 9.8]. The group F is a Š-space, since it can be considered as a subgroup of G^+. By step 1, F is metrizable, and therefore G is also metrizable.

(*iii*) $\Longrightarrow$ (*iv*) by Theorem 3.3, while (*iv*) $\Longrightarrow$ (*i*) is evident. □

Theorem 3.5. *Let G be a topological Abelian group which contains a neighborhood U of the neutral element such that $U^\triangleright$ separates the points of G. Then G^+ is strictly angelic.*

Proof. Let U be as in the claim. By Propositions 2.2 (b) and 2.1, the group H generated by $U^\triangleright$ is dense and sigma-compact in $\Gamma_\sigma G$. It follows now, from Proposition 3.2(b), that G^+ is strictly angelic. □

In the framework of locally quasi-convex groups it is possible to obtain a unified version of Theorems 3.3 and 3.5, as we give next.

Theorem 3.6. *Let G be a topological Abelian group with neutral element* 0*.*

(a) If G locally quasi-convex and admits a coarser first countable at 0 *Hausdorff (not necessarily group) topology $\mathcal{T}_1$, then G^+ is strictly angelic.*

(b) If G admits a coarser metrizable dually separated group topology $\mathcal{T}_1$, then G^+ is strictly angelic.

Proof. (a) Let $\{U_n : n \in \mathbb{N}\}$ be a countable $\mathcal{T}_1$-neighborhood basis of the neutral element 0. Since $\mathcal{T}_1$ is coarser than the original topology of G, for each natural n there exists a quasi-convex neighborhood V_n of the neutral element of G, such that $V_n \subset U_n$. Put $H_0 := \bigcup_{n\in\mathbb{N}} V_n^\triangleright$, and let us prove that H_0 separates the points of G. Take any $g \in G$ with $g \neq 0$. Since $\mathcal{T}_1$ is a Hausdorff topology, there exists a natural number n such that $g \notin U_n$, and consequently $g \notin V_n$. Being V_n quasi-convex, there exists $\varphi \in V_n^\triangleright \subset H_0$ with $Re(\varphi(g)) < 0$. Hence $\varphi(g) \neq 1$.

Denote by H the group generated by H_0. Taking into account that H is sigma-compact in $\Gamma_\sigma G$, and by Proposition 2.2(b) also is dense, Proposition 3.2 can be applied to obtain that G^+ is strictly angelic.

(b) Denote G_1 the abstract group G equipped with the topology $\mathcal{T}_1$. Since $\mathcal{T}_1$ is coarser than the original topology of G, we have $\Gamma G_1 \subset \Gamma G$. Hence, the Bohr topology of G_1 is coarser than the Bohr topology of G. Since G_1 is a dually separated metrizable group, by Theorem 3.3 $(G_1)^+$ is strictly angelic. Consequently, G^+ is strictly angelic too.

□

Note that Theorem 3.6(a) for the case when G is a Hausdorff locally convex space improves the Dieudonné-Schwartz's version of Šmulian theorem, as quoted in [25, Theorem 9.8.2].

4 Groups which are g-spaces with respect to their Bohr topologies

Since the property mentioned in the title for locally convex spaces and their weak topologies are established under some assumption on completeness [20], the same can be expected in the context of topological Abelian groups.

For a topological Abelian group G let us consider the following condition:

(*) *any character $\psi : \Gamma G \to \mathbb{T}$, whose restriction to any equicontinuous subset H of ΓG is $\sigma(\Gamma G, G)_{|H}$-continuous, is $\sigma(\Gamma G, G)$-continuous.*

We recall a criterion of completeness of Grothendieck: a locally convex space G is complete if and only if it satisfies (*).

In the previous work [9] (see Theorem 4.1), we obtained that (*) is *a sufficient condition* of completeness in the class of Hausdorff locally quasi-convex topological Abelian groups. In the same paper it is proved that (*) is not necessary: there exists a complete metrizable separable locally quasi-convex group which does not satisfy (*).

Based upon these considerations, we introduce a new notion. A topological Abelian group G is *demi-complete* if it satisfies the following condition:

(**) *any character $\psi : \Gamma G \to \mathbb{T}$, whose restriction to any angelic-producing subspace H of $\Gamma_\sigma G$ is $\sigma(\Gamma G, G)_{|H}$-continuous, is $\sigma(\Gamma G, G)$-continuous.*

Observe that any metrizable group G is demi-complete (the whole $\Gamma_\sigma G$ is then angelic-producing, since it is sigma-compact and Proposition 1.5 applies). Also the additive group G of any complete locally convex space is demi-complete (by Grothendieck's criterion, since every $\sigma(\Gamma G, G)$-closed equicontinuous subset H of ΓG is a compact subspace of $\Gamma_\sigma G$ (by Proposition 2.1) and Proposition 1.5 applies).

Theorem 4.1. *Let G be a demi-complete dually separated group, and $K \subset G$. If K is relatively countably compact in G^+, it is also relatively compact in G^+. In other words, G^+ is a g-space.*

Proof. Put $K^\natural := \{g^\natural : g \in K\}$ and let F be the closure of $K^\natural$ in $\mathbb{T}^{\Gamma G}$. Since F is compact in $\mathbb{T}^{\Gamma G}$, it is sufficient to show that $F \subset G^\natural$ (then, by Proposition 2.3, the set $\alpha_G^{-1}(F)$ will be a compact subset of G^+ containing K). Fix an arbitrary $\psi \in F$. Clearly ψ is a character on ΓG. So, by Proposition 2.2(a), we will have $\psi \in G^\natural$ if we can show that ψ is continuous on $\Gamma_\sigma G$. To this end, take an arbitrary $H \subset \Gamma G$ which is angelic-producing as a subspace of $\Gamma_\sigma G$. Since G is demi-complete, the continuity of ψ on $\Gamma_\sigma G$ will follow from the continuity of $\psi_{|H}$ with respect to $\sigma(\Gamma G, G)_{|H}$.

Denote by $K^\natural{}_{|H} := \{g^\natural{}_{|H} : g \in K\}$. Since K is countably relatively compact in G^+, we have that $K^\natural{}_{|H}$ is countably relatively compact in $C_p(H, \mathbb{T})$. Taking into account that H is angelic-producing, the space $C_p(H, \mathbb{T})$ is angelic and therefore $K^\natural{}_{|H}$ is *relatively compact* in $C_p(H, \mathbb{T})$. It follows that the closure of $K^\natural{}_{|H}$ in $\mathbb{T}^H$ is contained in $C_p(H, \mathbb{T})$. Since ψ belongs to the closure of $K^\natural$ in $\mathbb{T}^{\Gamma G}$, we have that $\psi_{|H}$ belongs to the closure of $K^\natural{}_{|H}$ in $\mathbb{T}^H$. Consequently, $\psi_{|H} \in C_p(H, \mathbb{T})$ and this means that $\psi_{|H}$ is continuous with respect to $\sigma(\Gamma G, G)_{|H}$. □

Next we deal with locally convex vector groups. A locally convex vector group is defined as a vector space endowed with a topology which makes addition continuous, and which has a local basis for the neutral element formed by convex symmetric subsets. For this class of objects -which is between the classes of locally convex spaces and of locally quasi-convex groups- there is full equivalence between completeness and (*) [9, Corollary 4.4]. Therefore a complete locally convex vector group is demi-complete and by Theorem 4.1 we have:

Corollary 4.2. *Let G be a complete locally convex vector group, then G^+ is a g-space.*

Note that Theorem 4.1 gives a property weaker than angelicity for a big class of groups. A better result cannot be expected, even for the additive group of a complete locally convex space. In fact, an uncountable product of real lines is not angelic.

We do not know whether Corollary 4.2 remains true for any complete locally quasi-convex group, however we will show next that, even with milder assumptions, the assertion holds for nuclear groups. The definition as well as a thorough study of nuclear groups is presented in [5]. We only summarize here that the class of nuclear groups contains all LCA groups, all nuclear locally convex spaces considered in their group structure, and is closed by the operations of taking subgroups, Hausdorff quotients, arbitrary products and countable direct sums. Observe also that the dual of a metrizable nuclear group is again nuclear, but the property does not hold in general. The following assertions will be used in the sequel:

Lemma 4.3. *Let G be a nuclear group. Then,*

(a) Every closed subgroup of G is closed in G^+ [5, (8.6)].

(b) Every compact subset of G^+ is also compact in G [6].

(c) Every relatively countably compact subset of G^+ is totally bounded in G [6].

If a group G is such that any closed totally bounded subset of G is compact, we shall say that G is *von-Neumann complete.* Clearly any complete group is von-Neumann complete.

The following is a stronger version of Corollary 4.2.

Proposition 4.4. *If G is a Hausdorff von-Neumann complete nuclear group, then G^+ is a g-space. Even more is true: if K is any relatively countably compact subset of G^+, then K is relatively compact not only in G^+, but in G too.*

Proof. By Lemma 4.3(c) K is totally bounded in the original topology of G. Since G is von-Neumann complete, K is relatively compact in G. On the other hand, G^+ is Hausdorff, due to the fact that G is a Hausdorff nuclear group, and therefore a DS-group. Thus, we get that K is relatively compact in G^+. □

Notice that Proposition 4.4 is not true for **all** nuclear groups. In Example 4, we describe a totally bounded Hausdorff topological Abelian group $\mathbf{M}$, such that $\mathbf{M}^+ = \mathbf{M}$ is not a g-space.

5 NSS-groups

In this section we shall study in more details the class of groups for which Theorem 3.5 is applicable. A topological group G is said to have *no small* *subgroups* (or to be an NSS-group) if there is a neighborhood of the neutral element, which contains no nontrivial subgroup of G. This is a significant property, being linked with the solution of the Vth problem of Hilbert. As proved by A. Gleason, D. Montgomery and L. Zippin, a locally compact group is a Lie group if and only if it has no small subgroups. Reasonably, the property may also have interest out of the class of locally compact groups, and it has appeared in our work in a natural way, when searching Bohr angelic groups.

In the sequel we shall use the following two assertions whose proof derives easily from the fact that $\mathbb{T}_+ := \{t \in \mathbb{T} : Re(t) \geq 0\}$ does not contain any nontrivial subgroup of $\mathbb{T}$ (thus, $\mathbb{T}$ is an NSS-group).

Lemma 5.1. *Let M be a nonempty subset of G. Then,*

(a) If M contains a subgroup H, then $\varphi(H) = \{1\}$ for all $\varphi \in M^{\triangleright}$.

(b) If $M^{\triangleright}$ contains a subgroup L, then $\varphi(M) = \{1\}$ for all φ in L.

The equivalence we state next was known to hold for locally compact Abelian Hausdorff groups (LCA groups) [3, Proposition 7.9]. We extend it to a larger class, which contains all the groups whose closed subgroups are also Bohr-closed, and in particular all the nuclear groups (Lemma 4.3(a)).

Proposition 5.2. *For a topological Abelian group G, consider the next two claims:*

(a) G contains a compactly generated dense subgroup.

(b) The dual $G^\wedge$ has no small subgroups.

Then, (a) implies (b). The converse also holds, provided that any dense subgroup of G^+ is also dense in G.

Proof. (a) $\Longrightarrow$ (b) Let K be a compact subset of G which spans a dense subgroup S. Suppose L is a subgroup contained in $K^{\triangleright}$. For any φ in L, by Lemma 5.1 (b), $\varphi(K) = \{1\}$ and hence, $\varphi(S) = \{1\}$. By the density of S, also $\varphi(G) = \{1\}$, which means that L contains only the null character. Thus, $K^{\triangleright}$ is a 0-neighborhood in $G^\wedge$, which contains no nontrivial subgroup of $G^\wedge$.

For the second statement, let now K be a compact subset of G such that $K^{\triangleright}$ does not contain nontrivial subgroups. Denote by S the subgroup of G generated by K. Since $K^\perp$ is a subgroup contained in $K^{\triangleright}$, $K^\perp = \{1\}$, and so $S^\perp = \{1\}$. By Proposition 2.2(c), S is dense in G^+ and by the assumption made, S is dense also in G. □

Proposition 5.3. *Let G be a topological Abelian group and U a neighborhood of the neutral element of G. Then, $U^{\triangleright}$ separates the points of G if and only if the quasi-convex hull of U contains no nontrivial subgroups.*

Proof. The quasi-convex hull of U always contains the subgroup $\bigcap_{\varphi \in U^{\triangleright}} \varphi^{-1}(1)$.
So, under the assumption that $Q(U)$ does not contain nontrivial subgroups, $\bigcap_{\varphi \in U^{\triangleright}} \varphi^{-1}(1) = \{0\}$. In particular, for every $g \neq 0$, there exists $\varphi \in U^{\triangleright}$ such that $\varphi(g) \neq 1$.
Conversely, if $Q(U)$ contains a nontrivial subgroup H, then $\varphi(H) = 1$ for all $\varphi \in U^{\triangleright}$. Thus, $U^{\triangleright}$ does not separate the points of G. □

The next statement is a dual version of Proposition 5.2.

Theorem 5.4. *Let G be a reflexive group such that every closed subgroup of its dual $G^{\wedge}$ is Bohr-closed [4] (in particular, G may be a Čech-complete nuclear group). The following assertions are equivalent:*

(a) G contains a neighborhood U of the neutral element of G such that $U^{\triangleright}$ separates the points of G.

(b) G has no small subgroups.

(c) $G^{\wedge}$ contains a dense compactly generated subgroup.

Proof. The equivalence between (a) and (b) can be obtained from Proposition 5.3 taking into account that a reflexive group is locally quasi-convex. On the other hand, applying Proposition 5.2 to $G^{\wedge}$ we obtain that (c) is equivalent to (b). □

6 Angelicity and the Schur property

Recall that a group G is said *to respect compactness* if every Bohr compact subset of G is compact [29]. So, by (b) of Lemma 4.3, any nuclear group respects compactness.

Next we will prove that for the groups which are angelic in their Bohr topology, to respect compactness can be expressed in terms of sequences. We need the following notion: a topological Abelian group G has the *Schur property* if every convergent sequence of G^{+} is convergent in G. More information on the Schur property for groups is given in [15], [22] and [26].

[4] It is sufficient to assume that every dense subgroup of $\Gamma_{\sigma} G$ is dense in $G^{\wedge}$.

Proposition 6.1. *Let G be a dually separated topological Abelian group. Consider the folowing assertions:*

(a) G respects compactness.

(b) G has the Schur property.

Then,

(1) (a) implies (b).

(2) If G has the Schur property and K is a sequentially compact subset of G^+, then K is sequentially compact in G.

(3) If G has the Schur property and K is a compact subset of G^+, then K is compact in G provided at least one of the following conditions is satisfied:

(C1) G^+ is angelic.

(C2) G^+ is a Š-space and G is a g_w-space.

(C3) G^+ is a Š-space and G is locally quasi-convex.

Proof. (1) Take a sequence (x_n) in G which is Bohr-convergent to $x \in G$. The set $S = \{x_n\} \cup \{x\}$ is Bohr compact, and by (a) also compact. Let $y \in G$ be a cluster point of (x_n) in G. Then y is also a cluster point of (x_n) in G^+. Since G^+ is separated, we get $y = x$. This and compactness of $S = \{x_n\} \cup \{x\}$ imply $x_n \to x$ in G.

(2) In fact, take a sequence (x_n) in K. Since K is Bohr sequentially compact, (x_n) has a Bohr-convergent subsequence (x_{p_n}). Since G has the Schur property, (x_{p_n}) converges in G too. Therefore K is sequentially compact in the original topology of G.

(3) Suppose (C1) is satisfied. By Lemma 1.3(d), G is angelic too. From this and (2) we get that K is compact.

Suppose (C2) is satisfied. Then K is Bohr-sequentially compact and by (2) it is sequentially compact. This implies that K is a countably compact subset of G. Since g is a g_w-space, K is relatively compact in G. The Bohr compactness of K implies that K is a *closed* subset of G. Consequently K is compact in G.

Suppose (C3) is satisfied. As in the previous case, we conclude that K is is sequentially compact. Then K is *totally bounded* as a subset of G [16, 1.2(3), p.7]. Now the Bohr compactness of K, together with local quasi-convexity of G imply that K is a *complete* subset of G. Consequently, K, as a complete and totally bounded subset, is compact.

□

Corollary 6.2. *Let G be a dually separated topological Abelian group, such that G^+ is angelic (or G^+ is a Š-space and G is a g_w-space; or else, G^+ is a Š-space and G is locally quasi-convex). Then the following statements are equivalent:*

(i) G respects compactness.

(ii) G has the Schur property.

Remark The implication $(ii) \Rightarrow (i)$ may not be true in general even for the additive group of a complete Hausdorff locally convex space (a way of constructing a corresponding example is indicated in [32, Example 6] (see also, [15, Example 19.19]).

7 Some examples

Example 1. *A compact metrizable Abelian group G, such that for every neighborhood V of the neutral element, $V^{\triangleright}$ does not separate the points of G.*

The product of countably many copies of $\mathbb{T}$, say $G := \mathbb{T}^{\mathbb{N}}$, is compact metrizable. However, the polar of any neighborhood of zero $V \subset G$ does not separate points. In fact, let $V = \bigcap_{i \in F} p_i^{-1}(U_i)$, being $U_i \in \mathcal{B}_{\mathbb{T}}(1)$ for all i in a finite set $F \subset \mathbb{N}$. Clearly V contains the subgroup $P := \{1\} \times \cdots \times \mathbb{T} \times \{1\} \times \cdots$, where the factors $\mathbb{T}$ are located in the j-coordinates, for $j \notin F$. If $\varphi \in V^{\triangleright}$, $\varphi(P) = 1$ since $\{z \in \mathbb{T} : Re(z) \geq 0\}$ does not contain subgroups distinct from $\{1\}$. Thus, $V^{\triangleright}$ does not separate the points of G.

Example 2. *A noncompact nonmetrizable Abelian group G, which contains a neighborhood U of the neutral element such that $U^{\triangleright}$ separates the points of G.*

Let $G := \omega\mathbb{R}$ be the direct sum of countably many copies of $\mathbb{R}$ with the topology induced by the box topology of $\mathbb{R}^{\mathbb{N}}$. The fact that G is nonmetrizable can be easily checked. On the other hand, G meets the requirements of Theorem 5.4. Firstly, closed subgroups of G are Bohr-closed due to the fact that G is strongly reflexive [5, (17.1)]. Secondly, neighborhoods like $U := ((-1,1) \times (-1,1) \times \cdots) \cap \omega\mathbb{R}$, do not contain nontrivial subgroups.

Example 3. *(1) A compact Hausdorff Abelian group which is not sequentially compact, and therefore, not angelic.*

Let $G := \mathbb{T}^{\mathbb{R}}$. Clearly G is compact Hausdorff. Now we want to find a sequence without convergent subsequences. Take a bijection $g\colon \mathbb{R} \to \mathbb{N}^{\mathbb{N}}$, and put $g(r) = (j_1^r, j_2^r, \ldots, j_n^r, \ldots)$. Define $x_n = (x_n(r))_{r\in\mathbb{R}} \in \mathbb{T}^{\mathbb{R}}$ by

$$x_n(r) := \begin{cases} i \in \mathbb{T} & if \quad n = j_k^r \text{ for some odd } k \\ 1 \in \mathbb{T} & si \quad n \notin \{j_1^r, j_3^r, \ldots, j_{2p+1}^r, \ldots\} \end{cases}.$$

The sequence $\{x_1, x_2, \ldots, x_n, \ldots\} \subset \mathbb{T}^{\mathbb{R}}$ does not have convergent subsequences; in fact, were $\{x_{l_m}\} \subset \{x_n\}$ a convergent subsequence then also $\{x_{l_m}(r)\}$ would be convergent for any $r \in \mathbb{R}$. Suppose $s \in \mathbb{R}$ is such that $g(s) = (l_m)$. Then $\{x_{l_m}(s)\} = \{i, 1, i, 1, \ldots\}$ is not convergent in $\mathbb{T}$, which contradicts the previous statement.

(2) *A sequentially compact Hausdorff Abelian group S which is not compact.*

Let $G := \mathbb{T}^{\mathbb{R}}$. For $x \in G$ denote $supp(x) = \{r \in \mathbb{R} : x(r) \neq 1\}$ and

$$S := \{x \in G : card(supp(x)) \leq \aleph_0\}.$$

Then S is a dense subgroup of G. It follows that S with the topology induced from G is sequentially compact, since the range of any sequence can be embedded in a countable product of copies of $\mathbb{T}$, which is obviously metrizable. (The space S is frequently called a Σ-product).

Example 4. *A totally bounded Hausdorff Abelian group* $\mathbf{M}$ *which is not countably compact, but contains a dense sequentially compact subgroup. Therefore,* $\mathbf{M}$ *is not a g-space.*

Let K be the discrete multiplicative group $\{-1, 1\}$ and let $G := K^{\mathbb{R}}$. Fix a sequence I_n, $n = 1, 2, \ldots$ of disjoint uncountable subsets of $\mathbb{R}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$. Let

$$\mathbf{M} := \left\{ x \in G : \exists n \in \mathbb{N},\, card\left(supp(x) \bigcap \bigcup_{k=n}^{\infty} I_k \right) \leq \aleph_0 \right\}.$$

As in the previous example, denote by S the Σ-product of $\mathbb{R}$-copies of K. Evidently $S \subset \mathbf{M}$, and $\mathbf{M}$ is a totally bounded subgroup of the compact group G. The closure of S in $\mathbf{M}$ is precisely $\mathbf{M}$ (since S is dense in G). Therefore $\mathbf{M}$ contains a dense sequentially compact subgroup.

Let us prove that $\mathbf{M}$ is not countably compact. For each natural n, put $J_n = \cup_{k=1}^n I_k$ and define a mapping $f_n : \mathbb{R} \to K$ as follows: $f_n(r) = -1$, $\forall r \in J_n$ and $f_n(r) = 1$ otherwise. Clearly $f_n \in \mathbf{M}$ for all $n \in \mathbb{N}$ and $f_n \to -\mathbf{1}$ pointwise (i.e., in G), while $-\mathbf{1} \notin \mathbf{M}$. Thus, the sequence $\{f_n\}$ does not have any cluster point in $\mathbf{M}$.

Remark. The idea of Example 4 is taken from [16, Example 1.2(9)]; where it is shown that the closure of a sequentially compact convex subset of a (non-complete) locally convex space may not be countably compact. We have corrected a gap contained in the arguments of the mentioned reference.

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