Modularity of rigid Calabi-Yau threefolds over $\mathbb{Q}$

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Abstract

We prove modularity for a huge class of rigid Calabi-Yau threefolds over $\mathbb{Q}$. In particular we prove that every rigid Calabi-Yau threefold with good reduction at 3 and 7 is modular.

1 Introduction

Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Recall that (see [17], [18]) X is a smooth projective threefold over $\mathbb{Q}$, and satisfies

1. $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$,
2. $\bigwedge^3 \Omega^1_X \cong \mathcal{O}_X$, and
3. $h^{2,1}(X) = 0$.

The first two of these conditions define a Calabi-Yau threefold, while the third signifies rigidity. In a certain sense, rigid Calabi-Yau varieties are the natural generalization of elliptic curves to higher dimensions.

We approach the issue of modularity for rigid Calabi-Yau threefolds by using Galois representations. For other formulations, see [17], [18]. Here is how it works: Given a prime ℓ, the action of $G_{\mathbb{Q}}$ on the étale cohomology group $H^3_{\text{ét}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)$ defines a continuous, odd, two dimensional representation

$$\rho_{X,\ell} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell).$$

The representation $\rho_{X,\ell}$ is unramified at a prime not equal to ℓ if X has good reduction there; its determinant is equal to ϵ_ℓ^3 (where ϵ_ℓ is the ℓ-adic cyclotomic character). If X has good reduction at ℓ and $\ell > 3$, it

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follows from the comparison theorem of Fontaine-Messing (cf. [7]) that the representation $\rho_{X,\ell}$ is crystalline (locally at ℓ). Moreover, from the definition of rigid Calabi-Yau it also follows that this local ℓ-adic representation has Hodge-Tate weights $\{0, 3\}$.

If X has good reduction at p, we set

$$t_3(p) = \mathrm{tr}\rho_{X,\ell}(\mathrm{Frob}_p), \quad \ell \neq p.$$

Of course, the definition is independent of which ℓ we choose for calculating the trace. Moreover, the quantity $t_3(p)$ is relatively easy to calculate: one only needs to 'count points' on the mod p reduction of X (see Remark 2.9 of [18] for details).

The Riemann hypothesis, proved by Deligne as part of the Weil conjectures, states that the roots of the characteristic polynomial of $\rho_{X,\ell}(\mathrm{Frob}_p)$, $p \neq \ell$ a prime of good reduction, both have the same complex absolute value. If $\rho_{X,\ell}$ was reducible, its constituents should be ℓ-adic characters of $G_{\mathbb{Q}}$, and so we must have

$$\rho_{X,\ell} \sim \begin{pmatrix} \mu\epsilon_\ell^a & * \\ 0 & \mu^{-1}\epsilon_\ell^b \end{pmatrix}$$

where $a, b \in \mathbb{Z}$ satisfies $a + b = 3$ and μ is a finite order character. But this clearly contradicts the Riemann hypothesis as $a \neq b$. We can therefore conclude that $\rho_{X,\ell}$ is irreducible.

Conjecture 1.1. *Let X be a rigid Calabi-Yau threefold over $\mathbb{Q}$. There is then a weight 4 newform f on $\Gamma_0(N)$ such that*

$$\rho_{X,\ell} \sim \rho_{f,\ell}$$

for some (and hence every) prime ℓ.

We haven't specified what the level should be in the statement of the conjecture, but it is clear that N should only be divisible by primes where X has bad reduction. Specifying the level at primes of bad reduction is more problematic: in fact, results of de Jong (see in particular Deligne's corollary in [1], Prop. 6.3.2) imply that when ℓ ranges through all primes of good reduction of X, there exists a bound for the (prime to ℓ part of the) conductor of $\rho_{X,\ell}$, and this minimal upper bound will turn out to be the level of the associated modular form. However, this value is not effectively computable. As for evidence in support of the conjecture, it is known to be true for all rigid Calabi-Yau threefolds that we know of. (However, we must add that 'what we know' contains only a handful of examples!) In the form stated above, the conjecture forms a part of (or rather follows from) the Fontaine-Mazur conjecture on modularity of Galois representations.

In this article, we prove the modularity conjecture for all rigid Calabi-Yau threefolds verifying some local conditions. More precisely, we shall prove the following:

Theorem 1.2. *Let X be a rigid Calabi-Yau threefold over $\mathbb{Q}$. Suppose X satisfies one of the following two conditions:*

1. *X has good reduction at 3 and 7; or,*
2. *X has good reduction at 5 and some prime $p \equiv \pm 2 \pmod 5$ with $t_3(p)$ not divisible by 5.*

Then X is modular.

Remark 1.3. *This result applies to all but one (class) of the known examples of rigid Calabi-Yau threefolds.*

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2 Modularity of ℓ-adic representations

More generally, let us consider irreducible continuous representations

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell^{ac})$$

which are odd and unramified outside finitely many primes. We also assume that $\ell \geq 5$, and that the determinant of ρ is ϵ_ℓ^3. By choosing a stable lattice, we can reduce ρ modulo ℓ and obtain a unique semi-simple continuous representation

$$\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_\ell^{ac}).$$

Following Wiles, one then deduces the modularity of ρ from that of $\overline{\rho}$.

2.1

We shall now describe some results under which such a deduction is possible. We fix ρ and $\overline{\rho}$ as above (in the beginning of the main section). Throughout this subsection, we shall also suppose that

Assumption 2.1. *ρ is crystalline at ℓ with Hodge-Tate weights $\{0, 3\}$.*

The first result we describe is due to Skinner and Wiles (Theorem A, Section 4.5 in [11]).

Theorem 2.2. *If $\overline{\rho}$ is reducible, then ρ is modular.*

The only point to be noted is that if $\overline{\rho}$ is reducible, then ρ is necessarily ordinary at ℓ, and so the quoted result applies. This follows immediately from Proposition 9.1.2 of [2], which shows that if $\rho|_{D_\ell}$ is irreducible and crystalline then its reduction is irreducible. (See also [6] for a similar argument in the case $\ell = 3$, $w = 1$ where w is the non-zero Hodge-Tate weight.)

Theorem 2.3. *Suppose that ρ is ordinary at ℓ, and also suppose that $\overline{\rho}$ is irreducible and modular. Then ρ is modular.*

This follows immediately from Theorem 5.2 of Skinner and Wiles in [12]: since ρ is ordinary at ℓ and has determinant ϵ_ℓ^3, we have

$$\overline{\rho}|_{I_\ell} \sim \begin{pmatrix} \epsilon_\ell^3 \pmod{\ell} & * \\ 0 & 1 \end{pmatrix}$$

where I_ℓ is the inertia group at ℓ. In particular, it is D_ℓ-distinguished (i.e., the two diagonal characters in the residual representation are distinct). As we have started out with the condition that $\overline{\rho}$ is modular, the conclusion above is the one given by Theorem 5.2 of [12].

The above two results deal with the case when the representation ρ is ordinary at ℓ. For the case when ρ is crystalline but not ordinary, we have the following result of Taylor, [16]:

Theorem 2.4. *Suppose that ρ is not ordinary at ℓ. Let F be a totally real field of even degree such that $Gal(F/\mathbb{Q})$ is solvable, and such that ℓ splits completely in F. Assume the following:*

- *There is a Hilbert modular form of weight 4 and level 1 on $GL_2(\mathbb{A}_F)$ which gives $\overline{\rho}$.*
- *$\overline{\rho}$ restricted to the absolute Galois group of $F\left(\sqrt{(-1)^{(\ell-1)/2}\ell}\right)$ is irreducible.*

Then ρ is modular.

Proof. Using Theorem 2.6 and Theorem 3.2 of [16], we see that $\rho|_{G_F}$ is 'modular'. Solvable base change results of Langlands then enable us to deduce the modularity of ρ from that of $\rho|_{G_F}$. □

As for the auxiliary totally real F that appears in the above theorem, we have the following 'level lowering' result of Skinner and Wiles, [13].

Theorem 2.5. *If $\overline{\rho}$ is irreducible and modular, then we can find a totally real solvable extension F, and a Hilbert modular form of weight 4, level 1 on $GL_2(\mathbb{A}_F)$ which gives $\overline{\rho}$. (Note that $\overline{\rho}$ is 'crystalline'.) Moreover, given a finite set of primes Σ where $\overline{\rho}$ is unramified, we can insist that the primes in $\Sigma \cup \{\ell\}$ split completely in F.*

Proof. This is essentially proved in [13] (they give a complete proof for the case when the weight is 2).

Firstly, note that (cf. [4]) we can find a weight 4 newform f on $\Gamma_0(N)$ with N prime to ℓ and divisible only by primes where $\overline{\rho}$ is ramified such that

$$\overline{\rho} \sim \rho_{f,\ell} \pmod{\ell}.$$

We can therefore find an even degree, totally real, solvable extension $F_1/\mathbb{Q}$ in which all the primes in $\Sigma \cup \{\ell\}$ split completely, and a Hilbert modular form f_1 over F_1 of level $\mathfrak{n}$, weight 4, such that

- $\overline{\rho}_{f_1,\ell} \sim \overline{\rho}|_{G_{F_1}}$, and
- $\mathfrak{n}$ is an ideal dividing N (and so is prime to ℓ).

It is now immediate that the result we want follows immediately from a higher weight version of the main theorem of [13]. The proof of the main theorem of [13] for the higher weight case works in exactly the same way as for the weight 2 case. The only modification, or something that is to be checked, required is Lemma 2 of [13]; and, this can be found in [15] (with a slight modification, the result we need is Lemma 2, *ibid.*). □

2.2

We now turn to the other serious issue of when the residual representation is modular. Conjecturally, the answer is always yes for odd, absolutely irreducible, continuous representations (Serre's conjecture); our knowledge covers only a handful of residue fields.

Theorem 2.6. *Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_5)$ be an absolutely irreducible, continuous representation with determinant equal to $\epsilon_5^3 \pmod 5$. Then $\overline{\rho}$ is modular.*

Proof. Let $\widetilde{\rho} := \overline{\rho} \otimes \epsilon_5^{-1} \pmod 5$. Then $\widetilde{\rho}$ is a continuous Galois representation with values in $GL_2(\mathbb{F}_5)$ and determinant equal to the mod 5 cyclotomic character $\epsilon_5 \pmod 5$. By Theorem 1.2 of [10], we can find an elliptic curve E over $\mathbb{Q}$ such that $\overline{\rho}_{E,5}$, the Galois representation given by the 5-torsion points of E, is equivalent to $\widetilde{\rho}$. Since all elliptic curves over $\mathbb{Q}$ are modular (by [3]), it follows that $\widetilde{\rho}$ is modular. The required result then follows. □

For Galois representations with values in $GL_2(\mathbb{F}_7)$, we have the following (see [9], and Theorem 9.1 of [8]).

Theorem 2.7. *Let $\overline{\rho} : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{F}_7)$ be an absolutely irreducible, continuous, odd representation. If the projective image of $\overline{\rho}$ is insoluble, we also assume that:*

- *The projective image of inertia at 3 has odd order.*
- *The determinant of $\overline{\rho}$ restricted to the inertia group at 7 has even order.*

Then $\overline{\rho}$ is modular.

The result applies, for example, if $\overline{\rho}$ has determinant $\epsilon_7^3 \pmod 7$ and is unramified at 3.

The two results above do not specify the weight, or the level. For the applications we have in mind, one needs to be able to produce newforms of the 'right weight and right level' giving rise to the residual representation $\overline{\rho}$. In view of [11], we might as well assume that $\overline{\rho}$ is absolutely irreducible. In this case, one can indeed produce newforms of the 'right weight and right level' (see Theorem 1.1 of [4]).

3 Proof of the main theorem

We now apply the results of the last section. Let us recall that we are given a rigid Calabi-Yau threefold X defined over $\mathbb{Q}$.

X has good reduction at 3 and 7

We consider the 7-adic representation

$$\rho_{X,7} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_7).$$

Note that it is crystalline at 7, and unramified at 3. We also know that it is absolutely irreducible.

If the residual representation is absolutely reducible, then $\rho_{X,7}$ is modular by Theorem 2.2. So suppose that the residual representation is absolutely irreducible. The residual representation is modular by Theorem 2.7. Moreover, we know (from [4]) that there is weight 4 newform of level coprime to 7 and 3 which gives rise to $\overline{\rho}$. The theorem in the ordinary case follows from Theorem 2.3.

Finally, suppose that we are in the crystalline but not ordinary case. Let ω_2 denote the second fundamental character of the inertia group at 7. We then have (cf. [2], Chapter 9)

$$\overline{\rho}_{X,7}|_{I_7} \sim \left(\omega_2^3 \oplus \omega_2^{7.3}\right),$$

and so the projectivization of the image of inertia is a cyclic group of order 8. One can then check the following: $\overline{\rho}_{X,7}$ restricted to $\mathbb{Q}\left(\sqrt{-7}\right)$ is irreducible. Making use of the (essential) fact that we are working with a prime as large as 7, the description above of $\overline{\rho}_{X,7}|_{I_7}$ and the inequality $(\ell+1)/\gcd(w,\ell+1) > 2$ valid for $\ell = 7$ and $w = 3$ (w denotes the non-zero Hodge-Tate weight), one deduces the absolute irreducibility of $\overline{\rho}$ when restricted to the Galois group of $\mathbb{Q}_7(\sqrt{-7})$. The proof (for general ℓ and w) follows an idea of Ribet to deal with dihedral primes for Galois representations, and is given in full detail in [6] for the case $\ell = 3,\ w = 1$. Alternatively, one can manipulate in the group $PSL_2(\mathbb{F}_7)$ (which we leave for the interested reader).

Now take F to be a totally real, even degree, solvable extension of $\mathbb{Q}$ in which 7 splits completely and $\overline{\rho}|_{G_F}$ arises from a Hilbert modular form of level 1. This can be done by Theorem 2.5. Finally, we deduce that ρ is modular by applying Theorem 2.4 (that $\overline{\rho}$ restricted to the absolute Galois group of $F(\sqrt{-7})$ is irreducible follows from the preceding paragraph as 7 splits completely in F).

X has good reduction at 5

In this case, we make use of the 5-adic representation $\rho_{X,5}$. As before, we know that $\rho_{X,5}$ is absolutely irreducible. Moreover, as before, we only need to consider the cases when $\overline{\rho}_{X,5}$ is absolutely irreducible. Using Theorem 2.6 and Theorem 1.1 of [4], we see that there is a newform of weight 4 and level coprime to 5 which gives rise to $\overline{\rho}$. One then deals with the ordinary case as before using Theorem 2.3.

So suppose that we are in the crystalline but not ordinary case. The image of inertia at 5 is then cyclic of order 8; the image of a decomposition group at 5 is non-abelian of order 16. Thus if F is a totally real field in which 5 splits completely, we will still have

$$\overline{\rho}_{X,5}\big|_{G_F}$$

absolutely irreducible.

Suppose that the restriction of $\overline{\rho}_{X,5}$ to the Galois group of $G_{F(\sqrt{5})}$ is absolutely reducible. Since $F(\sqrt{5})$ is totally real, this is the same as reducibility. In particular, we can assume that the image of $\overline{\rho}_{X,5}(G_{F(\sqrt{5})})$ is the subgroup of $GL_2(\mathbb{F}_5)$ consisting of elements

$$\begin{pmatrix} x & 0 \\ 0 & \pm x^{-1} \end{pmatrix} \quad \text{with} \quad x \in \mathbb{F}_5^\times,$$

and hence that the image of G_F is generated by the above subgroup and $\left(\begin{smallmatrix} 0 & 1 \\ 3 & 0 \end{smallmatrix}\right)$. It follows that elements with determinant equal to 2 or 3 have trace 0.

We are now ready to make use of the condition that $t_3(p)$ is not divisible by 5. Choose an F which is totally real and solvable as given by Theorem 2.5. We can choose such an F with 5 and p splitting completely in F. Since $t_3(p)$ is not divisible by 5, the preceding argument shows that F satisfies both the conditions of Theorem 2.4. It follows that $\rho_{X,5}$ is modular.

Remark 3.1. *We are grateful to Fred Diamond for pointing out that one could prove modularity without appealing to base change. The relevant result, dealing with modular representations over $\mathbb{Q}$, is proved in [5].*

Remark 3.2. *What we proved, purely in terms of Galois representations, is the following:*
Let $\ell = 5$ or 7, and let

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{Q}_\ell^{ac})$$

be an irreducible continuous representation, unramified outside finitely many primes, and having determinant ϵ_ℓ^3. Assume the following:

1. *ρ is crystalline at ℓ with Hodge-Tate weights $\{0, 3\}$.*
2. *The residual representation*

$$\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_\ell^{ac})$$

   takes values in $GL_2(\mathbb{F}_\ell)$.
3. *If $\ell = 7$, then ρ is unramified at 3.*
4. *If $\ell = 5$, then there is some prime $p \equiv \pm 2 \pmod 5$ where ρ is unramified and the trace of Frobenius at p is not divisible by 5.*

Then ρ is modular.

4 Applications to known examples

As mentioned in the introduction, all known rigid Calabi-Yau threefolds are modular. There are about thirty examples known; the levels for which one can construct a rigid Calabi-Yau (of that level) are

$$6,\ 8,\ 9,\ 12,\ 21,\ 25,\quad 50,$$

a family coming from conifolds (see Section 5.4 of [18]), and those of type III_0 (see Section 5.5 of [18]) We are grateful to Helena Verrill and Noriko Yui for pointing these out; for details and equations for the rigid Calabi-Yau threefolds, see [17], [18] and the references there. It is a pleasant surprise that the result of this article covers all known rigid Calabi-Yau threefolds except for the level 9 case.

Let us check that the hypotheses of Theoerem 1.2 are satisfied by these known Calabi-Yau threefolds. In principle, we should write down explicit models and verify the conditions (for such explicit models, see [17], [18]. However, we already know that these examples are modular—we just want to check that the result applies often. We get this information by looking at tables of newforms. What we mean is the following: let N be a positive integer, and let $CY(N)$ be the set of rigid Calabi-Yau threefolds whose L-series (or q-series) expansions agree with newforms of weight 4, level N, at least up to, say, twenty terms. We know somehow from explicit calculations that if $X \in CY(N)$, then X has good reduction outside primes dividing N. To see if our result would apply to the threefolds in $CY(N)$, we need to see if $(21, N) = 1$ or $5 \not| N$. In the second case, we would to look at a few $t_3(p)$: we know how to calculate them, and they agree with what comes from modular forms.

For example, Theorem 1.2 with the first condition will apply to the cases for which the level comes out to be 8, 25 or 50. This is easy to check: for instance, the quintic of Chad Schoen given in section 5.1 of [17] is easily seen to have good reduction at 3 and 7. For the cases when the level is 6, 12, or 21, one looks at tables of newforms with weight 4 (see W. Stein's website, [14]). We need only look for newforms with q-expansions having $\mathbb{Z}$ coefficients. We find one newform of level 6 (with $t_3(7) = -16$), one of level 12 (with $t_3(7) = 8$), and two of level 21 (they have $t_3(13) = -34,\ -62$ respectively); in all cases, we can apply theorem 1.2 with the second set of conditions (the true values of these traces for the corresponding examples of rigid Calabi-Yau threefolds have been computed by many people and of course they agree with the above values).

We leave it to the reader to apply the result and check for modularity of the family coming from conifolds and those of type III_0 using the information given in Section 5.4 of [18].

Remark 4.1. *While our theorem gives nothing new at present, and while it might not always work, it seems simpler to apply than other results. It also could establish modularity for an infinite family (provided that there exists such a family)!*

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