Rank invariance and automorphisms of generalized Kac-Moody superalgebras

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Automorphisms of finite dimensional simple Lie algebras are products of inner automorphisms and diagram automorphisms [FH]. Unlike the finite dimensional case, for a Kac-Moody algebra, given a Cartan decomposition, the Borel subalgebra generated by the positive root spaces and the one generated by the negative root spaces are not in general conjugate under the action of inner automorphisms (or in other words of the corresponding Kac-Moody group) [K4, §5.9] [KP]. Hence the Chevalley involution has to be considered. The aim of this paper is to give the decomposition of an arbitrary automorphism of an infinite dimensional generalized Kac-Moody superalgebra. In order to do so, we show that Cartan subalgebras of generalized Kac-Moody superalgebras are conjugate under the action of inner automorphisms. This well-known result of Kac-Peterson [KP] for Kac-Moody algebras therefore remains valid in the more general context. Furthermore, we show that all infinite dimensional generalized Kac-Moody superalgebras G are characterized by a unique Cartan matrix. This matrix is of generalized Kac-Moody type, i.e. it satisfies the usual conditions $(a)-(d)$ of §1. In other words, there are two conjugacy classes of Borel subalgebras, one corresponding respectively to positive and negative root spaces of a giving Cartan decomposition. This also generalizes a classical result of Kac-Peterson for Kac-Moody algebras [KP] and implies that not only the rank of a Cartan subalgebra of an infinite dimensional generalized Kac-Moody superalgebra, but also the rank of of the Kac-Moody sub-superalgebra generated by all the simple finite type (real) root spaces, are well defined.

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These results suggest that the structure of infinite dimensional generalized Kac-Moody superalgebras is closer to that of Kac-Moody algebras than to finite dimensional Lie superalgebras.

Finite dimensional simple Lie superalgebras were extensively studied by many. In particular, their automorphisms were described by Serganova [S] and it is well known that in general, there are more than two conjugacy classes of Borel sub-superalgebras. Hence for finite dimensional simple Lie superalgebras, the situation differs much from the Lie algebra one. However, we show that if the finite dimensional classical Lie superalgebra [K2] G is a generalized Kac-Moody superalgebra, then except when the derived Lie superalgebra $G' = [GG]$ is isomorphic to the 8 dimensional classical Lie superalgebra $A(1,0)$, it has a unique Cartan matrix of generalized Kac-Moody type associated to it. In other words, except in that one case, there are at most two conjugacy classes of Borel subalgebras corresponding to a Cartan decomposition of generalized Kac-Moody type, as in the infinite dimensional case.

The uniqueness of the Cartan matrix also has a consequence in the Moonshine context. It follows that the moonshine module constructed by Frenkel, Lepowsky and Meurman [FLM] cannot be constructed as a lattice vertex algebra. That the construction of the Monster vertex algebra could not be as simple as that of the fake monster vertex algebra was suspected but not proved.

1 Preliminaries

We first need to fix some notations and to remind the reader of a few basic facts about generalized Kac-Moody superalgebras which will be needed later on. For details, see [Ra1, sections 1 and 2].

As usual $G_{\overline{0}}$ and $G_{\overline{1}}$ will denote the even and odd part of the Lie superalgebra G.

The base field $\mathbf{F}$ will be either $\mathbf{R}$ or $\mathbf{C}$. Let I be a finite set $\{1, \ldots, n\}$ or a countable one identified with $\mathbf{Z}_+$ and S be a subset of I. The set S indexes the generators contained in $G_{\overline{1}}$.

Let H be a $\mathbf{F}$-vector space with a non-degenerate symmetric bilinear form $(.,.)$, containing non-trivial vectors h_i indexed by the set I.

Set $a_{ij} := (h_i, h_j)$ and assume that

(a) $a_{ij} = a_{ji}$;
(b) $a_{ij} \leq 0$ if $i \neq j$;
(c) $\frac{2a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$;
(d) $\frac{a_{ij}}{a_{ii}} \in \mathbf{Z}$ if $a_{ii} > 0$ and $i \in S$.

The Lie superalgebra $\tilde{G} = \tilde{G}(A, S, H)$ is generated by the vector space H considered as an even abelian subalgebra and elements $e_i, f_i,\ i \in I$, satisfying the defining relations:
(i) $[e_i, f_j] = \delta_{ij} h_i$;
(ii) $[h, e_j] = (h, h_j) e_j,\ h \in H$;
(iii) $[h, f_j] = -(h, h_j) f_j,\ h \in H$;
(iv) $\deg e_i = \deg f_i = \bar{0}$ if $i \notin S$;
(v) $\deg e_i = \deg f_i = \bar{1}$ if $i \in S$.

Following Moody's definition [M], we define the generalized Kac-Moody superalgebra $G = G(A, S, H)$ by generators and relations.

Definition 1.1. *The generalized Kac-Moody superalgebra $G = G(A, S, H)$ with generalized symmetric Cartan matrix $A = (a_{ij})_{i,j \in I}$ and Cartan subalgebra H is the Lie superalgebra generated by H, e_i, and f_i, $i \in I$ and with defining relations (i)-(v) and with the following extra ones:*
(vi)[1] $(\operatorname{ad} e_i)^{\frac{-2a_{ij}}{a_{ii}}+1} e_j = 0 = (\operatorname{ad} f_i)^{\frac{-2a_{ij}}{a_{ii}}+1} f_j$ *for all $i \in I$ such that $a_{ii} > 0$ and $j \in I$;*
(vii) $[e_i, e_j] = 0 = [f_i, f_j]$ *if* $a_{ij} = 0$.

The cardinality of the indexing set I is called the rank of the Lie superalgebra G. We will show that this concept is well defined.

Remarks about Cartan matrices. If the matrix A is indecomposable and D is a diagonal matrix with positive entries, then the generalized Kac-Moody superalgebras $G(A, S, H)$ and $G(DA, S, H)$ are isomorphic. The matrix DA need not be symmetric and satisfies conditions $(b)-(d)$ and if it is not symmetric, condition (a) has to be replaced by $a_{ij} = 0$ if $a_{ji} = 0$. Note that one can also take a matrix D with negative entries. To include this possibility, the definition of a Cartan matrix A needs to be extended and modified in an obvious manner. Multiplying by -1 does not change anything: as long as all non-diagonal entries are of the same sign, whether they are taken to be non-positive or non-negative is only a matter of convention. Hence we need the concept of equivalent matrices.

Definition 1.2. *Two matrices B and C of the same size are said to be equivalent, if there exist non-singular matrices P and D with D diagonal, such that $C = DPBP^{-1}$.*

The generalized Kac-Moody superalgebras corresponding to equivalent Cartan matrices are isomorphic, given the same Cartan subalgebra H and indexing set S.

[1] Note that there is a mistake in the statement of condition (vi) in [Ra2, §5.1].

Let G be the generalized Kac Moody superalgebra. Let Q be the root lattice with basis α_i, $i \in I$ and bilinear form given by the generalized Cartan matrix A: $(\alpha_i, \alpha_j) = a_{ij}$. Let $Q^+ = \{\sum_{i \in I} k_i \alpha_i : k_i \in \mathbf{Z}_+\}$. For $\alpha \in Q^+$ (resp. $-\alpha \in Q^+$), let G_α be the subspace of G generated by all elements $[\dots[e_{i_3}[e_{i_2}, e_{i_1}]]\dots]$ (resp. $[\dots[f_{i_3}[f_{i_2}, f_{i_1}]]\dots]$), where $\alpha_1 + \alpha_2 + \alpha_3 + \cdots = \alpha$ (resp. $-\alpha$). let Δ denote the set of roots, i.e. the non-zero elements $\alpha \in Q$ for which $G_\alpha \neq 0$, Δ^+ the set of positive roots, Δ_0 the set of even roots and Δ_0^+ the set of even positive roots. The Cartan decomposition holds for the generalized Kac-Moody superalgebra G [K4, §1]:

$$G = (\oplus_{-\alpha \in \Delta^+} G_\alpha) \oplus H \oplus (\oplus_{\alpha \in \Delta^+} G_\alpha).$$

Let $N^+ := \oplus_{\alpha \in \Delta^+} G_\alpha$ and $N_0^+ = N^+ \cap G_{\overline{0}}$.

We give a simpler definition for infinite type roots that the one given in [Ra1, Definition 2.3] for the earlier one excluded roots of norm 0.

Definition 1.3. *A root $\alpha \in \Delta$ is said to be of finite type if for any root $\beta \in \Delta$, $n\alpha + \beta$ is a root for only finitely many integers n. Otherwise it is said to be of infinite type.*

So we can reformulate [Ra1, Lemma 2.4 and Proposition 2.6] as follows:

Proposition 1.4. *A root $\alpha \in \Delta^+$ is of infinite type if for any root $\beta \notin \{\alpha, \frac{1}{2}\alpha, 2\alpha\}$ for which $(\alpha, \beta) < 0$, $n\alpha + \beta$ are roots for all positive integers n, unless α and β are both positive or both negative, β is of finite type and norm 0 and $\alpha - \beta$ is a root.*

In the case of a generalized Kac-Moody algebra of rank at least two, the former are the real roots and the latter the imaginary ones. All roots of positive norm are of finite type as in the algebra case.

Remark. By (vii), the support of a root α is always connected.

Let α_i be a simple root of positive norm, then r_i will denote the corresponding reflection when α_i is even, and the reflection corresponding to $2\alpha_i$, when α_i is odd. Set W_1 to be the group generated by these reflections. Roots of negative norm might be of finite type. The full Weyl group W is the group generated by all reflections corresponding to non-zero norm roots of finite type.

Note that when the derived Lie superalgebra $[G, G]$ has infinite dimension, or equivalently when the set of roots Δ is infinite, the group W_1 is the full Weyl group W for then G does not contain roots of finite type having negative norm [Ra1, Corollary 2.5]. When G contains roots of the latter type, W_1 is a proper subgroup of W.

Lemma 1.5. *For any root $\alpha \in \Delta^+$, if α has positive norm, then there exits $w \in W_1$, such that $w(\alpha)$ or $\frac{1}{2}w(\alpha)$ is a simple root. If α is an odd root of norm 0 and finite type, then exits $w \in W_1$ such that $w(\alpha)$ is a simple root.*

For the proof of this result, see [Ra1, Lemma 2.2]. There is an omission in the statement in [Ra1]: as W_1 contains the reflection r_i corresponding to $2\alpha_i$ when the simple root α_i is an odd root of positive norm, a positive root α of positive norm could be conjugate to $2\alpha_i$ rather than α_i since $r_i(\alpha) < 0$ if and only if $\alpha = \alpha_i$ or $2\alpha_i$.

Since the generalized Kac-Moody algebra G has no non-trivial ideal intersecting the Cartan subalgebra H trivially [Ra1, Lemma 2.1], the following two results are immediate.

Corollary 1.6. *The centre of the generalized Kac-Moody superalgebra G is contained in the Cartan subalegebra H.*

Corollary 1.7. *The even part $G_{\overline{0}}$ of the generalized Kac-Moody superalgebra G is a generalized Kac-Moody algebra.*

Proof. The invariant, consistent, super-symmetric bilinear form induced on G by the bilinear form on H [K4, Theorem 2.2] is non-degenerate since its non-degenerate on H and any ideal intersecting H trivially is trivial. The result then follows by [Ra1, Theorem 3.3]. □

We fix some more notation before proceeding to the main part of the paper. Let $J = I$ when $|I| = 1$ and $a_{11} \neq 0$ and $J = \{i \in I | a_{ii} > 0\}$ otherwise. So J is the set indexing the simple roots of finite type with non-zero norm. Let G_J be the Kac-Moody sub-superalgebra of G generated by the elements e_i and f_i for $i \in J$. We will refer to the cardinal of the set J as the Kac-Moody rank of G. We will show that this is a well defined notion.

Let G_0 be the Lie subalgebra of G generated by the Cartan subalgebra H, the elements e_i, f_i belonging to the even part $G_{\overline{0}}$ of the Lie superalgebra G and by the elements $[e_i, e_i]$, $[f_i, f_i]$, where e_i, f_i belong to the odd part $G_{\overline{1}}$ and the simple root α_i is of finite type with non-zero norm. This is clearly a generalized Kac-Moody (even) subalgebra of G_J.

Define U^+ (resp. U^-) as the Lie sub-superalgebra generated by the root spaces G_α where there exits some index i in the support of the root $\alpha \in \Delta^+$ (resp. $-\alpha \in \Delta^+$) such that the simple α_i is either of infinite type or of finite type with norm 0.

2 Conjugacy of Cartan subalgebras of the Lie superalgebra G

We first state an obvious fact:

Lemma 2.1.

$$G = U^+ \oplus (G_J + H) \oplus U^-$$

as a direct sum of vector spaces.

Proof. This is a direct consequence of the elimination Theorem [Bou]. □

Proposition 2.2. *If the derived Lie superalgebra $[G, G]$ has infinite dimension, then*

(i) every element of G which act ad-locally finitely on G is contained in the Lie sub-superalgebra $G_J + H$;

(ii) all Cartan subalgebras of G are contained in the Lie sub-superalgebra $G_J + H$.

Proof. Without loss of generality, we may assume that the Cartan matrix of G is indecomposable. Suppose that K is a Cartan subalgebra of G not contained in $G_J + H$. By definition, K is an even subalgebra of G.

Since K is a Cartan subalgebra, it acts on G in a locally finite way, i.e. for any element $x \in K$ and $v \in G$, there is a finite dimensional subspace of G containing $(adx)^n v$ for all $n \in \mathbf{Z}_+$. Let x be an element of K not contained in $G_J + H$. Write

$$x = u^+ + v + u^-,$$

where $u^+ \in U^+$, $u^- \in U^-$ and $v \in G_J + H$.

We first assume that $u^+ \neq 0$. So there exists positive roots $\beta_1, \ldots, \beta_r \in \Delta$ containing either infinite type simple roots or finite type simple roots with zero norm in their support and root vectors $u_i \in G_{\beta_i}$ such that $u = \sum_{i=1}^r u_i$.

Suppose that there exists a simple root α_i such that for some $j = 1, \ldots, r$, $(\beta_j, \alpha_i) < 0$.

Since K is an even subalgebra, all roots β_j are even. So by Corollary 1.7, the roots β_j, $1 \leq j \leq r$ are of infinite type as they have non-positive norm and are even and by assumption, $[G, G]$ being is infinite dimensional so that there are no roots of negative norm and finite type [Ra1, Corollary 2.5]. This implies that $(adu_j)^n e_i \neq 0$ for all $n \in \mathbf{Z}_+$ [Ra1, Lemma 2.4].

We claim that there are infinitely many vectors $(adx)^n(e_i)$, $n \in \mathbf{Z}_+$ which are linearly independent. Let $k \in \{1, \ldots, r\}$ be such that β_k is of maximum height with the property that $\beta_k + \alpha_i$ is a root. Then for each positive integer

n, $(adu^+)^n(e_i)$ is the sum of root vectors, exactly one of which corresponds to the root $n\beta_k + \alpha_i$. Hence there are infinitely many linearly independent vectors $(adu^+)^n(e_i)$, $n \in \mathbf{Z}_+$. Indeed for all $n \in \mathbf{Z}_+$, the component of $(adu^+)^n(e_i)$ in the $n\beta_k + \alpha_i$ root space cannot be a linear combination of components of $(adu^+)^r(e_i)$, $r \leq n-1$ since all the components of u^- belong to negative root spaces and those of v belong to root spaces generated by simple finite type root spaces and to the Cartan subalgebra H, it follows that the claim holds. This contradicts the fact that K acts in a locally finite way.

So

$$(\beta_j, \alpha_i) \geq 0 \qquad \forall j = 1, \ldots, r, i \in I. \tag{1}$$

Let α_i be a simple root either of infinite type or of finite type and norm 0 in the support of β_j. Then condition (1) forces $(\beta_j, \alpha_i) = 0$ for all $j = 1, \ldots, r$. Hence the simple root α_i has norm 0 and is orthogonal to all the simple roots in the support of each root β_j, $j = 1, \ldots, r$. Since the support of a root is connected and β_j is an even root, we can deduce that $\beta_j = \alpha_i$ is a simple root of infinite type. As β_j is simple, $(\beta_j, \alpha_k) \leq 0$ for any simple root α_k of the generalized Kac-Moody superalgebra G. Hence by (1), the simple root β_j is orthogonal to all the simple roots of G. As the Cartan matrix is assumed to be indecomposable, it follows that $j = 1$ and $I = \{b_1\}$. So G is a Heisenberg algebra of rank 1. Therefore the Cartan subalgebra H is unique and in particular $K = H$, contradicting assumptions. So $u^+ = 0$.

If $u^- \neq 0$, then similar arguments lead to a contradiction. It follows that $x \in G_J + H$ and this forces the Cartan subalgebra K to be contained in $G_J + H$, contradicting assumptions and proving the result. □

Proposition 2.2 implies the first main result of this paper. We remind the reader that the derived Lie group of the Lie algebra G_0 consists of the inner automorphisms.

Theorem 2.3. *All Cartan subalgebras of the generalized Kac-Moody superalgebra G are conjugate under the action of the derived Lie group of G_0.*

Proof. We first suppose that the derived Lie superalgebra $[G, G]$ is infinite dimensional. Let H_1 be a Cartan subalgebra of G. Then $H_1 \leq H + G_J$ by Proposition 2.2. Let K be a finite subset of the indexing set J and $H_K = H_1 \cap (H + G_K)$. Write Z for the centre of the Lie superalgebra $H + G_K$. By Corollary 1.6 applied to the generalized Kac-Moody superalgebra $H + G_K$, $Z \leq H_K \cap H$. For $i \in K$, let V_i^+ (resp. V_i^-) be the H_K-submodule of $H + G_K$ generated by the vector e_i (resp. f_i) and ϕ_i^+ (resp. ϕ_i^-) be the Lie algebra homomorphism from H_K to $gl(V_i^+)$ (resp. $gl(V_i^-)$). Then $Z = (\cap_{i\in K} \ker \phi_i^+) \cap (\cap_{i\in K} \ker \phi_i^-)$. As the action of H_K on

G is locally finite, the modules V_i^+ and V_i^- are finite dimensional. Hence the subspaces $H_K/(\ker \phi_i^+)$ and $H_K/(\ker \phi_i^-)$ are finite dimensional. So the subspace Z has finite co-dimension in H_K since the set K is finite. Let L be a complement of Z in H_1. As L is finite dimensional, with obvious modifications for the superalgebra case, the Conjugacy Theorems of Peterson-Kac for Kac-Moody algebras [KP] can directly be extended to $L + G_K$. Thus, the subalgebra L of $H + G_J$ is conjugate to a subalgebra of H under the action of some element g of the derived Lie group of G_0. Since Z is central in $H+G_K$ and contained in H, it follows that H_K is conjugate to a subalgebra of H under the action of this element g. A symmetric argument applied to H shows that H_K is conjugate to H under the action of G_0. Similarly $H \cap (H_1 + G_K)$ is conjugate to H_1 under the action of the derived Lie group of G_0. Hence H and H_1 are conjugate under this action, proving the result.

When the derived Lie superalgebra $[G, G]$ is finite dimensional, the argument above shows that we may assume G to be finite dimensional. The result is then a well known result in the Lie algebra case; and has been proved in [S] in the context of Lie superalgebras. □

The next result is a also a basic one needed for the uniqueness of the Cartan matrix associated to the generalized Kac-Moody superalgebra G.

Theorem 2.4. *If the set of roots Δ is infinite, then all bases of Δ are conjugate to the base Π or $-\Pi$ under the action of the Weyl group W.*

Proof. Note that as there are infinitely many roots, the group W_1 is the full Weyl group W. Let Π be the base of Δ defined in §1 and Π_2 be another set of simple roots in Q with respect to which the symmetric Cartan matrix is $B = (b_{ij})$. We may assume that the Cartan matrix A (with respect to Π) is indecomposable. It is then clear that the Cartan matrix B with respect to Π_2 is also indecomposable. If the odd root α is of infinite type or has positive norm, then 2α is an even root. By Corollary 1.7, the Lie algebra $G_{\overline{0}}$ is a generalized Kac-Moody algebra and the set $(\Pi \cap \Delta_0) \cup \{2\alpha : \alpha \in \Pi - \Delta_0, (\alpha, \alpha) \neq 0\}$ can be completed to get a base $\tilde{\Pi}$ of Δ_0. Similarly the base Π_2 gives a base $\tilde{\Pi}_2$ of Δ_0. By [Ka4, Proposition 5.9 and §11.10], we may assume that $\tilde{\Pi}_2 = \tilde{\Pi}$.

We show that $\Pi_2 \leq \Delta^+$ or $\leq \Delta^-$. If the root $\alpha \in \Pi_2$ is not odd of norm 0, then α (if α is even) or 2α (if α is odd) is in $\tilde{\Pi}_2$, and so $\alpha \in \Delta^+$.

So suppose that the root $\alpha \in \Pi_2$ is odd of norm 0 and that $-\alpha \in \Delta^+$. As it belongs to a base Π_2, it is of finite type as there are no non-trivial ideals intersecting the Cartan subalgebra H trivially [Ra1, Lemma 2.1].

If the Lie superalgebra G has finite growth, then it is an affine Lie superalgebra [K3] since there are infinitely many roots. Affine generalized

Kac-Moody Lie superalgebras have no odd root of finite type having norm 0 [Ra1, §2] and so there is nothing to show. Hence we may assume that the growth of the Lie superalgebra G is not finite.

Then by [Ra1, Theorem 3.3], either the matrix B or $-B$ satisfies conditions $(a)-(d)$. Furthermore, by [Ra1, Lemma 3.17], there are roots of infinite type and non-zero norm or else the indexing set I is infinite and all roots are of finite type.

Suppose first that there is a positive (with respect to Π_2) root β of infinite type with non-zero norm. So $(\beta,\beta)<0$. It follows that the matrix B satistfies conditions $(a)-(d)$ for otherwise by [Ra1, Lemma 2.2], infinite type roots would have non-negative norm. Also, depending on whether the root β is even or odd, β or $2\beta \in \tilde{\Pi}_2 = \tilde{\Pi}$, and thus $\beta \in \Delta^+$. As the matrix B is indecomposable, we may assume that $(\alpha,\beta)\neq 0$, and so $(\alpha,\beta)<0$ by condition (b) satisfied by B. On the other hand, as $-\alpha \in \Delta^+$, by [Ra1, Lemma 2.2], there exists $w \in W_1$ such that $w(-\alpha)\in \Pi$, and so $(\beta,-\alpha)=(w(\beta),w(-\alpha))\leq 0$ since $w(\beta)\in\Delta^+$. This contradiction implies that $\alpha\in\Delta^+$.

Suppose next that the indexing set I is infinite and all roots are of finite type. Then, there are roots of positive norm and finite type in Π_2, and so the matrix B again satisfies conditions $(a)-(d)$. Since there are no roots of negative norm, and positive roots of positive norm with respect to Π_2 are sums of roots of positive norm belonging to the base Π_2 [Ra1, Lemma 2.2] (and so are sums of roots in Π_2 different from α), $\Pi_2=\{\alpha=\beta_0,\beta_1,\dots,\}$, where for all integers $n\geq 0$, $\sum_{i=0}^{n}\beta_i$ is an odd root of norm 0, and β_n is a root of positive norm and so is in Δ^+. Hence there is a minimal integer $n>0$ such that $\sum_{i=0}^{n}\beta_i\in\Delta^+$ and $-\sum_{i=0}^{j}\beta_i\in\Delta^+$ for all $j<n$. Let r be the reflection with respect to the root β_n. However these are not both positive roots, contradicting [Ra1, Lemma 2.2]. This again shows that $\alpha\in\Delta^+$.

Therefore, the base $\Pi_2\leq\Delta^+$ and so $\Pi_2=\Pi$, proving the result. □

We are now ready to state the result on Borel sub-superalgebras and on the invariance of the Cartan matrix as it is an immediate consequence.

Theorem 2.5. *Suppose that the derived Lie superalgebra $[G,G]$ is infinite dimensional. Then*

(1) there is a unique symmetric Cartan matrix associated to the generalized Kac-Moody superalgebra G;
(2) all Borel subalgebras are conjugate to B^+ or B^- under the action of the Weyl group W;
(3) the rank and Kac-Moody rank are well defined.

Remark. Suppose that the Lie superalgebra $[G, G]$ has finite dimension. It is well known that in general a simple finite dimensional classical Lie superalgebra has Cartan decompositions giving non-equivalent symmetric Cartan matrices [K2]. For example, both $\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ are Cartan matrices of $A(1,1)$. However, from the list of finite dimensional generalized Kac-Moody superalgebras in [Ra1, §2] and the list of all their simple root systems in [K2, §2.5.4], it follows that G is caracterised by a unique Cartan matrix satisfying conditions $(a)-(d)$ except when $[G, G]$ is isomorphic to $A(1,0)$ when, up to equivalence, there are two such matrices associated to G: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (with $S = I$) and $\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ (with $S = \{1\}$). Furthermore, the Borel sub-superalgebras B^+ and B^- are not W-conjugate unless G is a Lie algebra or $[G, G]$ is isomorphic to $B(0, n)$ or to $B(1, 1)$. Though in the latter case, there still are two conjugacy classes of Borel subalgebras (one with Cartan matrix A and the other $-A$). If $[G, G]$ is isomorphic to $A(1,0)$, then there are three conjugacy classes of Borel subalgebras.

We next give two applications of Theorem 2.4 and 2.5.

a) The Monster vertex algebra

The crucial first step towards the proof of the Moonshine Theorem was the construction of a $\mathbf{Z}$-graded moonshine module V for the Monster simple group [FLM]. It has a rich algebraic structure. One of its most important characteristics is that it is a vertex algebra [FLM], [B1]. It would be useful to have a different construction of the Monster vertex algebra for its present one is very complex. In particular this may help to find an adequate integral form on V for modular moonshine questions [Ry] [BR] [B4] [B5].

Unlike the Monster vertex algebra, the Fake Monster vertex algebra has a nice simple construction. It is the lattice vertex algebra associated to the even unimodular non-degenerate Lorentzian lattice $II_{25,1}$ of rank 26 [B1], [B2].

Now, an important property of the moonshine module V is that it is a vertex algebra on which the Virasoro algebra acts with central charge 24, and an important aspect of a lattice vertex algebra constructed from an even non-degenerate lattice is that the Virasoro algebra acts on it with central charge equal to the rank of the lattice. Therefore, as the Monster vertex algebra is closely connected to the Leech lattice (i.e. the unique, up to isomorphism, positive definite even lattice of rank 24 not containing vectors of norm 2), it is natural to ask whether it can be constructed as a lattice vertex algebra V_L, where L is a Niemeier lattice [CS] (i.e. a positive definite even lattice of rank 24) as the Virasoro algebra acts on the V_L with central charge 24. We remind the reader that the Virasoro algebra is the unique (up to isomorphism) central extension by a 1 dimensional centre of the Lie algebra of all derivations of the algebra of Laurent polynomials $\mathbf{C}[t,t^{-1}]$. It is generated by a central element c and elements L_n, $n \in \mathbf{Z}$, satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c.$$

Furthermore, there is a natural non-degenerate bilinear form on the vertex algebra V_L such that the adjoint of the operator L_n is L_{-n} for $n \in \mathbf{Z}$.

The monster Lie algebra G is constructed as follows: consider the tensor product of the moonshine module V with the lattice vertex algebra $V_{II_{1,1}}$. Tensor products of vertex algebras are vertex algebras. The quotient of the subalgebra $V_1 = \{v \in V \otimes V_{II_{1,1}} : L_0 v = v; L_n v = 0, n \geq 1\}$ of $V \otimes V_{II_{1,1}}$ by the radical of the restriction of the bilinear form on $V \otimes V_{II_{1,1}}$ to V_1 is the monster Lie algebra. This is a generalized Kac-Moody algebra [B3]. The Fake Monster Lie algebra is the generalized Kac-Moody algebra constructed in a similar way from the lattice vertex algebra $V_{II_{25,1}}$.

We show that due to Theorem 2.5, the Moonshine module does not as simple a construction as a lattice vertex algebra V_L. It had seemed doubtful that it could, but this had not yet been proved.

Corollary 2.6. *If L is a Niemeier lattice, then V_L is not isomorphic to the Moonshine module V.*

Proof. Suppose that the moonshine module is isomorphic to the lattice vertex algebra V_L, where L is a Niemeier lattice.

The direct sum $L \oplus II_{1,1}$ is an even unimodular Lorentzian lattice of rank 26. Up to isomorphism there is a unique lattice $II_{25,1}$ with this property. It is the root lattice of the Fake Monster Lie algebra [B2]. Therefore the tensor product vertex algebra $V' = V_L \otimes V_{II_{1,1}}$ is isomorphic to the Fake Monster vertex algebra. It therefore follows that the monster Lie algebra G is isomorphic to the Fake Monster Lie algebra. Hence by Theorem 2.5, they both have the same Cartan matrix (up to reordering). As this is false, it shows that these generalized Kac-Moody algebras are not isomorphic and proves the result. □

b) Automorphisms of generalized Kac-Moody superalgebras

We finally give the decomposition of an arbitrary automorphism of the generalized kac-Moody superalgebra G.

Let ω be the Chevalley automorphism of period 4 of G [K3]: i.e.

$$\omega(e_i) = \begin{cases} f_i, & \text{if } i \in S; \\ -f_i, & \text{otherwise;} \end{cases} \quad \omega(f_i) = -e_i \ \forall i \in I, \quad \omega(h) = -h \ \forall h \in H.$$

It is the Chevalley involution if G is a generalized Kac-Moody superalgebra.

We remind the reader that a diagram automorphism θ of G satisfies $\theta e_i = e_{(i)}$ and $\theta f_i = f_{(i)}$ for all $i \in I$, where is a bijection on the indexing set I keeping the Cartan matrix invariant, i.e. $a_{i,j} = a_{i,j}$ and such that $(S) = S$.

Consider the natural map from the root lattice Q to the Cartan subalgebra H: $\alpha_i \mapsto h_i, \quad i \in I$. In general this map is not injective: Suppose $h_i = h_j$, $i \neq j \in I$. Then $a_{ki} = a_{kj}$ for all $k \in I$, i.e. the i-th and j-th columns of the Cartan matrix are equal. In particular this implies that the simple roots α_i and α_j are of infinite type or are both odd roots of norm 0 and finite type. When this is the case, it may be that $h_i = h_j$. However as $i \neq j$, $\alpha_i \neq \alpha_j$. Note that this map is always injective when G is a Kac-Moody superalgebra since then all the simple roots have positive norm.

Set

$$G_{h_i} = \{x \in G : [h,x] = (h_i,h)x\}, \qquad G_{-h_i} = \{x \in G : [h,x] = -(h_i,h)x\};$$

and

$$I_i = \{j \in I : a_{ki} = a_{kj} \forall k \in I\}.$$

Then $G_{h_i} = \sum_{j \in I_i} G_{\alpha_j}$ and $G_{-h_i} = \sum_{j \in I_i} G_{-\alpha_j}$ can have dimension greater than 1 when $i \in I - J$.

Note that the Weyl group W clearly acts on the subspace of H generated by the elements h_i, $i \in I$.

We choose a set of representatives from each class of indices I_i: $\hat{I} = \{i \in I : i \leq j, \quad \forall j \in I_i\}$. The vectors e_j (resp. f_j), $j \in I_i$ form a basis of G_{h_i} (resp G_{-h_i}). For simplicity of notation later, we choose an ordering of $I_i = \{j_1, \dots, j_{n_i}\}$, where $n_i = \dim G_{h_i}$ and write $e_i^{(s)} = e_{j_s}$ and $f_i^{(s)} = f_{j_s}$. For each $i \in \hat{I}$, for each bijective linear map μ_i of the vector space $G_{h_i} \oplus G_{-h_i}$ satisfying $\mu_i(G_{h_i}) = G_{h_i}$, $\mu_i(G_{-h_i}) = G_{-h_i}$, and $[\mu_i(e_j), \mu_i(f_j)] = \mu_i([e_j, f_j])$ for all $j \in I_i$, let $\hat{\mu_i}$ be the automorphism of G given by

$$\hat{\mu_i}(e_j) = \begin{cases} \mu_i(e_j), & \text{if } j \in I_i, \\ e_j, & \text{otherwise;} \end{cases} \qquad \hat{\mu_i}(f_j) = \begin{cases} \mu_i(f_j), & \text{if } j \in I_i, \\ f_j, & \text{otherwise.} \end{cases}$$

First, a preliminary result. We write $\operatorname{Aut} G$ for the group of automorphisms of the generalized Kac-Moody algebra G, and $In(G)$ for the derived Lie group of G_0.

Lemma 2.7. *(i) For fixed $i \in \hat{I}$, the set $A_i = \{\hat{\mu_i} : \mu_i \in GL(G_{h_i})\}$ is a group isomorphic to the general linear group $GL(n_i)$, where $n_i = \dim G_{h_i}$.*

(ii) *The set D of diagram automorphisms is a group.*

(iii) *The group of inner automorphisms $In(G)$ is normal in G.*

(iv) *For any $i, j \in \hat{I}$, $[A_i, In(G)] = 1 = [A_i, A_j]$, and A_i is normal in $A_i.D$.*

(v) *(v) The Chevalley automorphism ω commutes with every element in A_i, D and $In(G)$.*

Proof. (i) is obvious and (ii) and (iii) are standard results (the proof of the semisimple finite dimensional case applies). (v) and the second equality of (iv) are easy to see. With above notation, for $j \in \hat{I}$, for any $k \in J$, $\hat{\mu}_i([e_k, e_j^{(s)}]) = [e_k, \hat{\mu}_i(e_j^{(s)})]$ since $\dim G_{h_k} = 1$. The same holds with e_k replaced by f_k. Therefore $\hat{\mu}_i$ commutes with every inner automorphism.

Since $\theta^{-1}\hat{\mu}_i\theta = \hat{\mu}_{{}^{-1}i}$, A_i is normal in $A_i.D$. □

We are now ready to give the decomposition of an arbitrary automorphism of a generalized Kac-Moody superalgebra.

Theorem 2.8. *Suppose that the set of roots Δ is infinite. Let ϕ be an automorphism of the generalized Kac-Moody superalgebra G. Then there is a diagram automorphism ϕ_1, an inner automorphism ϕ_2 of G, and for each $i \in \hat{I} - J$, automorphisms μ_i of G such that*

$$\phi = (\prod_{i \in \hat{=} I - J} \mu_i)\omega^i \phi_1 \phi_2 \quad \textit{for some integer} \quad 0 \leq i \leq 3.$$

This decomposition (in the given order) is unique.

Proof. Let ϕ be an automorphism of the generalized Kac-Moody superalgebra G. Then $\phi(H)$ is a Cartan subalgebra of G. So by Theorem 2.3, there exists an inner automorphism (i.e. an element of the derived Lie group G_0) τ_1 such that $\psi_1 = \phi\tau_1^{-1}$ fixes H. Then for all elements $h \in H$,

$$[\psi_1(h), \psi_1(e_i)] = \psi_1([h, e_i]) = (h, h_1)\psi_1(e_i) \quad \forall i \in I.$$

Thus $\psi_1(e_i)$ is a root vector. Furthermore $\psi_1([G, G]) = [G, G]$ as its a Lie algebra homomorphism. Hence ψ_1 fixes the subspace of H generated by the elements h_i as $< h_i : i \in I >= H \cap [G, G]$. Clearly the $\psi_1(e_i)$, $\psi_1(f_i)$ and H are generators of the Lie superalgebra G satisfying relations $(i) - (vii)$ of Definition 1.1. Also ψ_1 induces a linear bijection of the root lattice Q keeping the set of roots Δ invariant and mapping the base Π to another base of Δ. Thus Theorem 2.4 implies that this isomorphism of the root lattice is given by an element $w \in W$. Now, there is an inner automorphism τ_2 of the generalized Kac-Moody superalgebra G inducing the Weyl group element w (the argument remains the same as in the case of finite dimensional semisimple Lie algebras: see [FH]). Hence $\psi_2 = \psi_1\tau_2^{-1}$ is an automorphism of G mapping the Cartan subalgebra to itself and

$$\psi_2(e_i) \in \mathbf{C}e_{\hat{\psi}_2(\alpha_i)}, \quad \psi_2(f_i) \in \mathbf{C}e_{\hat{\psi}_2(-\alpha_i)}, \quad \psi_2(h_i) = h_{\hat{\psi}_2(\alpha_i)} = d \ \forall i \in I.$$

Set $\phi_1 = \tau_2\tau_1$. Since $\omega(h) = -h$ for all $h \in H$, the above implies that there exists some $0 \leq l \leq 3$, and a diagram automorphism ϕ_2 of G such that

$$\phi\phi_1^{-1}\omega^l\phi_2^{-1}(G_\alpha) = G_\alpha, \quad \forall \alpha \in \Delta.$$

The decomposition of ϕ follows.

Since $In(G)$ is normal in $\mathrm{Aut}\, G$, to prove uniqueness, we only need to check that $(< \omega > (\bigsqcup A_i).D) \cap In(G) = 1$. Suppose that $\phi = \omega^l\mu\theta$, where $\mu \in \bigsqcup A_i$ and $\theta \in D$ is an inner automorphism. As ϕ keeps the Cartan subalgebra H invariant, there is an element $w \in W$ corresponding to ϕ. Either $w(\Pi) = \Pi$ or $-\Pi$. For any root α, the height of $w(\alpha)$ or of $-w(\alpha)$ is the same as that of α.

Suppose first that there are roots of infinite type. By Corollary 1.7, the even part $G_{\overline{0}}$ is a generalized Kac-Moody algebra and has a base $\tilde{\Pi} \supseteq \Pi$. By [K4, 11.13.3], any element $\alpha \in Q^+$ with connected support containing an even number of indices in S satisfying $(\alpha, \alpha_i) \leq 0$ is a positive even root of infinite type provided that it is not a multiple $m\beta$ for $m \geq 2$ and $\beta \in \tilde{\Pi}$ with $(\beta, \beta) \leq 0$. So for all $\alpha \in K$, $w(\alpha) > 0$, and so $w(\Pi) = \Pi$. Thus $(w(\alpha), w(\alpha_i)) \leq 0$ for all $i \in I$ implies that $(w(\alpha), \alpha_j) \leq 0$ for all $j \in I$. Hence $w(\alpha) = \alpha$ by [K4, Proposition 3.12.b]. This implies that $w = 1$.

Suppose next that all roots are of finite type and that the indexing set I is countably infinite. Let $w = r_{i_1} \dots r_{i_m}$ be a reduced expression for w. Then as any finite subset of the indexing set I generates a finite subset of Δ, there is a simple root $\alpha_j \in \Pi$ such that $(\alpha_j, \alpha_{i_k}) = 0$ for all $1 \leq k \leq m$. Thus $w(\alpha_j) = \alpha_j$ and so $w(\Pi) = \Pi$. However, $w(\alpha_{i_m}) < 0$ by [K4, Lemma 3.11.b]. This contradiction forces $w = 1$.

Hence in all cases, ϕ acts as the identity on G. From this it easily follows that $l = 0$ and $\mu = 1 = \theta$. □

Remarks. 1. In the above definition of diagram automorphisms, we could have included the maps μ_i by considering the entire indexing set I rather than $\hat{I}$. However, as the following example of the Monster Lie algebra shows the above method is more convenient. Indeed in practice, the roots are more naturally considered as elements of the Cartan subalgebra H than of an abstract root lattice Q, where the roots are linearly independent.

2. When $i \in J$, i.e. the corresponding simple root is of finite type, we do not have to include the automorphism μ_i. Indeed, the map given by $e_i \mapsto ce_i$ for non-zero constants c is accounted for by inner automorphisms.

As an immediate consequence of Lemma 2.7 and Theorem 2.8, we can deduce that:

Theorem 2.9. *When* $\dim([G,G]) = \infty$*, the automorphism group of the generalized Kac-Moody superalgebra* G *is the direct product of a central subgroup and of a semidirect product:*

$$\mathrm{Aut}\, G = <\omega> \times \left(D \ltimes \left(\bigsqcup A_i \times In(G) \right) \right).$$

Example. To construct the Monster Lie algebra M, we take the Cartan subalgebra $H = \mathbf{C^2}$, with bilinear form given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$. The set $\hat{I}$ (as defined above) is countably infinite ($S = \emptyset$). Let $h_0 = (1,-1)$, $h_i = (1,i)$ for any integer $i > 0$. Then, the element (m,n) of Z^2 is a root of M with multiplicity $c(mn)$, where for each $n \in \mathbf{Z}$, $c(n)$ is the coefficient of q^n in the q-expansion of the modular invariant $j(q) = q^{-1} + c(1)q + c(2)q^2 + \cdots = \sum_n c(n)q^n$. The vectors $(1,n)$ are the simple roots and have multiplicity $c(n)$.

Let us denote the bilinear form on H by $u.v$, for $u,v \in H$. As $(1,n)\cdot(1,n) = -2n$, the only simple root of finite type is $(1,-1)$. Therefore the Weyl group W is cyclic of order 2 and G_J is isomorphic to sl_2. Hence the group $Inn(M)$ of inner automorphisms of M is isomorphic to SL_2. The group of diagram automorphisms is trivial. Indeed $(1,n).(1,n) = -2n$. Hence the identity map is the only bijection of the set $\hat{I}$ keeping the bilinear form invariant, since under such a map the norm of the image of $(1,n)$ must be the same as the norm of $(1,n)$.

For $i > 0$, the groups A_i are isomorphic to the general linear group $gl_{c(i)}$ and the Chevalley involution has order 2. It follows that the group of automorphisms of G is isomorphic to:

$$\mathrm{Aut}\, M \cong \mathbf{Z^2} \times \mathbf{Z^2} \times \bigsqcup_{i>0} gl_{c(i)}.$$

The root spaces $G_{(1,n)}$ are representations of the Monster group. Thus the Monster group is a subgroup of $\cap_{i>0} A_i$.

Remark. The finite dimensional case is classical when G is a Lie algebra and has been treated in [S] when G is a simple finite dimensional Lie superalgebra. Hence it need not be included in this paper.

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