Tilting modules and Gorenstein rings

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In his pioneering work [30], Miyashita extended classical tilting theory to the setting of finitely presented modules of finite projective dimension over an arbitrary ring. More recently, tilting theory has been generalized to arbitrary modules of finite projective dimension [2], [3], [6], et al. Encouraged by this developement, in the present paper, we apply infinite dimensional tilting theory to the setting of modules over Iwanaga-Gorenstein rings.

We start with pointing out the central role played by resolving subcategories of mod$R$ for an arbitrary ring R. In Theorem 2.2 we prove that these subcategories correspond bijectively to tilting classes of finite type in Mod$R$, as well as to cotilting classes of cofinite type in RMod. This generalizes a classical result of Auslander-Reiten [9] characterizing cotilting classes in mod$R$ over an artin algebra R. Moreover, it provides for a bijection between certain cotilting left modules and tilting right modules which extends the duality between finitely generated cotilting and tilting modules over artin algebras.

We then consider Iwanaga-Gorenstein rings, that is (not necessarily commutative) two-sided noetherian rings of finite self-injective dimension on both sides. We show that the first finitistic dimension conjecture holds true for any Iwanaga-Gorenstein ring (Theorem 3.2). We prove that Gorenstein injectivity, and

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flatness, can be tested by finitely generated modules of finite projective dimension (Corollary 3.5). We also show that Iwanaga-Gorenstein rings are characterized among noetherian rings by the property that Gorenstein injective modules on either side form a tilting class (Theorem 3.4). For an artin algebra R, we obtain that R is Iwanaga-Gorenstein if and only if the class of Gorenstein projective right modules is induced by a finitely generated cotilting module (Corollary 3.8).

In the final section, we provide for an explicit construction of tilting and cotilting modules over any commutative 1-Gorenstein ring R (Example 4.1). By [32], it is consistent with ZFC that these are the only tilting and cotilting modules up to equivalence in case R is a small Dedekind domain.

1 Preliminaries

Let R be a ring. Denote by Mod$R$ the category of all (right R-) modules, and by mod$R$ the subcategory of all modules possessing a projective resolution consisting of finitely generated modules. The elements of mod$R$ are sometimes referred to as the modules of type FP_∞, cf. [12, VIII, §4]; if R is right coherent then these are exactly the finitely presented modules.

We denote by $\mathcal{P}$ and $\mathcal{I}$ the class of all modules of finite projective and injective dimension, respectively. For $n < \infty$, we denote by $\mathcal{P}_n$ ($\mathcal{I}_n$, $\mathcal{F}_n$) the class of all modules of projective (injective, flat) dimension $\leq n$.

Finally, we put $\mathcal{P}^{<\infty} = \mathcal{P} \cap \mathrm{mod}R$, $\mathcal{P}_n^{<\infty} = \mathcal{P}_n \cap \mathrm{mod}R$, and $\mathcal{I}_n^{<\infty} = \mathcal{I}_n \cap \mathrm{mod}\, R$.

A. Resolving subcategories. Let $\mathcal{C}$ be a class of finitely generated modules. Then $\mathcal{C}$ is *resolving* provided that
(R1) $\mathcal{C}$ contains all finitely generated projective modules,
(R2) $\mathcal{C}$ is closed under direct summands and extensions, and
(R3) $X \in \mathcal{C}$ whenever there is an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ with $Y, Z \in \mathcal{C}$.

Notice that every resolving class $\mathcal{C}$ is contained in mod$R$. Moreover, the property of being resolving can be tested in a weaker form:

Lemma 1.1 Let R be a ring.

1. Let $\mathcal{C} \subseteq \text{mod}R$. Then $\mathcal{C}$ is resolving iff $\mathcal{C}$ satisfies conditions (R1), (R2) and (R3') $X \in \mathcal{C}$ whenever there is an exact sequence $0 \longrightarrow X \longrightarrow P \longrightarrow Y \longrightarrow 0$ such that P is finitely generated projective and $Y \in \mathcal{C}$.

2. The class mod$R$ is resolving.

3. Assume $\mathcal{C} \subseteq \mathcal{P}_1^{<\infty}$. Then $\mathcal{C}$ is resolving iff $\mathcal{C}$ satisfies conditions (R1) and (R2).

Proof: 1. Clearly, any resolving class $\mathcal{C}$ satisfies (R3'). Conversely, assume $\mathcal{C}$ satisfies (R1), (R2) and (R3'). In order to prove (R3), consider an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ with $Y, Z \in \mathcal{C}$. By assumption, there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow Z \longrightarrow 0$ with P finitely generated projective and $K \in \mathcal{C}$. Consider the pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & L & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathcal{C}$ is closed under extensions, the left column gives $L \in \mathcal{C}$. Since P is projective, the upper row splits, X is a direct summand of L, and hence $X \in \mathcal{C}$. This proves condition (R3).

2. Condition (R1) is obvious. Schanuel's lemma gives closure of mod$R$ under direct summands, and the horseshoe lemma implies closure under extensions - so (R2) holds. Schanuel's lemma then gives (R3'), and part 1. applies.

3. If $\mathcal{C} \subseteq \text{mod}R \cap \mathcal{P}_1$, then (R3') follows from (R1). □

B. Orthogonal classes. For a class $\mathcal{C} \subseteq \text{Mod}R$, we define $\mathcal{C}^{\perp_1} = \text{KerExt}^1_R(\mathcal{C}, -)$, ${}^{\perp_1}\mathcal{C} = \text{KerExt}^1_R(-, \mathcal{C})$, $\mathcal{C}^{\perp} = \bigcap_{i\geq 1}$ Ker $\text{Ext}^i_R(\mathcal{C}, -)$, and ${}^{\perp}\mathcal{C} = \bigcap_{i\geq 1}$ Ker $\text{Ext}^i_R(-, \mathcal{C})$. Similarly, we define the classes $\mathcal{C}^{\intercal_1}$, ${}^{\intercal_1}\mathcal{C}$, $\mathcal{C}^{\intercal}$, and ${}^{\intercal}\mathcal{C}$, replacing Ext by Tor.

Observe that ${}^{\perp}\mathcal{C}$ and ${}^{\intercal}\mathcal{C}$ satisfy conditions (R1), (R2), (R3) above. So, it follows from Lemma 1.1.2 that ${}^{\perp}\mathcal{C} \cap \text{mod}\, R$ and ${}^{\intercal}\mathcal{C} \cap \text{mod}\, R$ are resolving.

We collect here some further properties which we will often use in the sequel.

Lemma 1.2 If $\mathcal{C} \subseteq \text{mod}\, R$, then $\mathcal{C}^{\perp}$ is closed under direct limits, and $\mathcal{C}^{\intercal}$ is closed under direct products.

Proof: See [19, 10.2.4 and 3.2.26].□

Theorem 1.3 Let $\mathcal{C} \subseteq \text{mod}\, R$ be a resolving class, and denote by $\varinjlim \mathcal{C}$ the class of all modules that are direct limits of an inductive system of modules from $\mathcal{C}$. Then the following hold true:
(1) $\mathcal{C}^{\perp} = \mathcal{C}^{\perp_1}$, and $\mathcal{C}^{\intercal} = \mathcal{C}^{\intercal_1}$.
(2) ${}^{\perp}(\mathcal{C}^{\perp}) \subseteq \varinjlim \mathcal{C} = {}^{\intercal}(\mathcal{C}^{\intercal})$.
(3) $\mathcal{C} = {}^{\perp}(\mathcal{C}^{\perp}) \cap \text{mod}\, R = \varinjlim \mathcal{C} \cap \text{mod}\, R = {}^{\intercal}(\mathcal{C}^{\intercal}) \cap \text{mod}\, R$.

Proof: see [2, 1.2], [7, 2.3] and [27].□

We shall denote by $S^* = \text{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$ the dual of the module S. As we recall in the next lemma this definition of dual module has useful homological properties. For particular rings, other notions with the same properties can be considered: the k-dual module in case R is an Artin k-algebra, and the W-dual module where W is any injective cogenerator for Mod$R$ in case R is commutative.

Lemma 1.4 Let R be a ring, $A, D \in \mathrm{Mod}R$, $B \in R\mathrm{Mod}$, and $C \in \mathrm{mod}R$. Let $i \geq 1$. Then

1. $\mathrm{Ext}^i_R(A, B^*) \cong (\mathrm{Tor}^R_i(A, B))^*$ and
2. $\mathrm{Tor}^R_i(C, D^*) \cong (\mathrm{Ext}^i_R(C, D))^*$.

Proof: See e.g. [19, 3.2.1 and 3.2.13]. □

C. Cotorsion pairs. A pair of classes of modules $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is *complete* provided that for each module M there is an exact sequence $0 \to M \to B \to A \to 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (or, equivalently, provided that for each module M' there is an exact sequence $0 \to B' \to A' \to M' \to 0$ with $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$).

For example, $(\mathcal{P}_n, \mathcal{P}_n{}^{\perp})$, $({}^{\perp}\mathcal{I}_n, \mathcal{I}_n)$ and $(\mathcal{F}_n, \mathcal{F}_n{}^{\perp})$ are complete cotorsion pairs for any ring R, cf. [19, §7.4], [31] and [17].

A cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in Mod$R$ is of *finite type* provided there exists a class $\mathcal{S} \subseteq \mathrm{mod}R$ such that $\mathcal{B} = \mathcal{S}^{\perp}$. The class $\mathcal{B}$ is then definable, that is, it is closed under direct products, direct limits, and pure submodules.

Dually, a cotorsion pair $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ in RMod is of *cofinite type* provided there is a class $\mathcal{S} \subseteq \mathrm{mod}R$ such that $\mathcal{A} = \mathcal{S}^{\intercal}$. Equivalently, $\mathcal{A} = {}^{\perp}\mathcal{S}^*$ where $\mathcal{S}^* = \{S^* \mid S \in \mathcal{S}\}$.

2 Tilting modules of finite type

Let $n < \infty$. A module T is *n-tilting* provided
(T1) $T \in \mathcal{P}_n$,

(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and all sets I, and
(T3) there exist $r \geq 0$ and a long exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0$ such that $T_i \in \mathrm{Add}T$ for each $0 \leq i \leq r$.

A class of modules $\mathcal{T}$ is *n-tilting* provided there is an n-tilting module such that $\mathcal{T} = T^\perp$. In this case, $\mathfrak{C} = ({}^\perp\mathcal{T}, \mathcal{T})$ is a complete cotorsion pair, cf. [16]. If $\mathfrak{C}$ is of finite type, then $\mathcal{T}$ and T are called *n-tilting of finite type*.

Clearly, $\mathcal{T}$ is n-tilting of finite type in case $T \in \mathrm{mod}R$. However, there are examples of tilting classes of finite type not induced by any $T \in \mathrm{mod}R$: if R is an artin algebra of finitistic dimension $n < \infty$ such that $\mathcal{P}^{<\infty}$ is not contravariantly finite, then $\mathcal{P}^{<\infty\perp}$ is an n-tilting class of finite type, but there is no n-tilting module $T \in \mathrm{mod}R$ such that $T^\perp = \mathcal{P}^{<\infty\perp}$, cf. [6, 25]. Another instance of this fenomenon occurs in Example 4.1 below. In fact, no example appears to be known of an n-tilting class which is not of finite type, or even not definable.

Dually, a module C is *n-cotilting* provided that
(C1) $C \in \mathcal{I}_n$,
(C2) $\mathrm{Ext}^i_R(C^I, C) = 0$ for each $i \geq 1$ and all sets I, and
(C3) there exists $r \geq 0$ and a long exact sequence $0 \longrightarrow C_r \longrightarrow \cdots \longrightarrow C_0 \longrightarrow W \longrightarrow 0$ where W is an injective cogenerator for $\mathrm{Mod}R$ and $C_i \in \mathrm{Prod}C$ for each $0 \leq i \leq r$.

A class of modules $\mathcal{F}$ is *n-cotilting* provided there is an n-cotilting module such that $\mathcal{F} = {}^\perp C$. In this case, $\mathfrak{C} = (\mathcal{F}, \mathcal{F}^\perp)$ is a complete cotorsion pair, cf. [2]. If $\mathfrak{C}$ is of cofinite type, then $\mathcal{F}$ and C are called *n-cotilting of cofinite type*.

Tilting classes and cotilting classes can be characterized as follows.

Theorem 2.1 [2, 4.2 and 4.3]
(1) A class of modules $\mathcal{T}$ is n-tilting if and only if

(a) $({}^\perp\mathcal{T}, \mathcal{T})$ is a complete cotorsion pair,

(b) $\mathcal{T}$ is closed under arbitrary direct sums, and $X \in \mathcal{T}$ whenever there is an exact sequence $0 \longrightarrow Z \longrightarrow Y \longrightarrow X \longrightarrow 0$ with $Y, Z \in \mathcal{T}$,

(c) ${}^{\perp}\mathcal{T} \subseteq \mathcal{P}_n$.

(2) A class of modules $\mathcal{F}$ is n-cotilting if and only if

(a) $(\mathcal{F}, \mathcal{F}^{\perp})$ is a complete cotorsion pair,

(b) $\mathcal{F}$ is closed under arbitrary direct products, and $X \in \mathcal{F}$ whenever there is an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ with $Y, Z \in \mathcal{F}$,

(c) $\mathcal{F}^{\perp} \subseteq \mathcal{I}_n$.

In fact, the Theorem above was inspired by a previous result due to Auslander and Reiten [9, Theorem 5.5] which characterizes cotilting classes in mod$R$ – in case R is an artin algebra – as the contravariantly finite resolving subcategories $\mathcal{F}$ of mod$R$ satisfying $\mathcal{F}^{\perp} \cap \operatorname{mod} R \subseteq \mathcal{I}_n^{<\infty}$.

The characterization gets simpler if we consider (co)tilting classes of (co)finite type in the category of arbitrary modules. In fact, we will now show that there is a bijective correspondence between n-tilting classes of finite type in Mod$R$ and the resolving subclasses in $\mathcal{P}_n^{<\infty}$, and there is a similar correspondence for n-cotilting classes of cofinite type in RMod. The point is that there are more approximations available in Mod$R$ - for instance, any cotorsion pair cogenerated by a set of modules is complete, cf.[16].

Theorem 2.2 Let R be a ring and $n < \infty$.

1. The n-tilting classes of finite type in Mod$R$ correspond bijectively to the resolving subclasses of $\mathcal{P}_n^{<\infty}$. The correspondence is given by the mutually inverse assignments $\alpha : \mathcal{T} \mapsto {}^{\perp}\mathcal{T} \cap \operatorname{mod} R$ and $\beta : \mathcal{C} \mapsto \mathcal{C}^{\perp}$.

2. The n-cotilting classes of cofinite type in RMod correspond bijectively to the resolving subclasses of $\mathcal{P}_n^{<\infty}$. The correspondence is given by the mutually inverse assignments $\gamma : \mathcal{F} \mapsto {}^{\intercal}\mathcal{F} \cap \operatorname{mod} R$ and $\delta : \mathcal{C} \mapsto \mathcal{C}^{\intercal}$.

3. Let $\mathcal{S} \subseteq \mathcal{P}_n^{<\infty}$. Then $\mathcal{T} = \mathcal{S}^\perp$ is an n-tilting class of finite type, and $\mathcal{F} = \mathcal{S}^\intercal$ is the corresponding n-cotilting class of cofinite type, that is, $\mathcal{F} = \delta\alpha(\mathcal{T})$.

Proof: 1. For any n-tilting class $\mathcal{T}$ we have ${}^\perp\mathcal{T} \subseteq \mathcal{P}_n$ by Theorem 2.1, so $\alpha(\mathcal{T})$ is a resolving subclass of $\mathcal{P}_n^{<\infty}$. Next, let $\mathcal{C}$ be a resolving subclass of $\mathcal{P}_n^{<\infty}$. Then $\mathcal{C}^\perp$ is closed under direct sums by Lemma 1.2, and $({}^\perp(\mathcal{C}^\perp), \mathcal{C}^\perp)$ is a complete cotorsion pair by [16]. Moreover, $\mathcal{C}^\perp$ is closed under cokernels of monomorphisms, and ${}^\perp(\mathcal{C}^\perp) \subseteq \mathcal{P}_n$ by [2, Lemma 2.2(a)]. So $\beta(\mathcal{C}) = \mathcal{C}^\perp$ is an n-tilting class (of finite type) by Theorem 2.1.

Now, let $\mathcal{T}$ be an n-tilting class of finite type. Take $\mathcal{S} \subseteq \text{mod}R$ such that $\mathcal{S}^\perp = \mathcal{T}$. Then $\mathcal{S} \subseteq \alpha(\mathcal{T})$, so $\beta\alpha(\mathcal{T}) \subseteq \mathcal{T}$. On the other hand, $\alpha(\mathcal{T}) \subseteq {}^\perp\mathcal{T}$, so $\mathcal{T} \subseteq ({}^\perp\mathcal{T})^\perp \subseteq \beta\alpha(\mathcal{T})$, and $\beta\alpha =$id. Conversely, if $\mathcal{C}$ is a resolving subclass of $\mathcal{P}_n^{<\infty}$, then Theorem 1.3.3 proves that $\alpha\beta =$id.

2. Let $\mathcal{F}$ be an n-cotilting class of cofinite type, and consider $\mathcal{S} \subseteq \text{mod}R$ such that $\mathcal{F} = \mathcal{S}^\intercal (= {}^\perp\mathcal{S}^*)$. Then $\mathcal{S}^* \subseteq \mathcal{F}^\perp \subseteq \mathcal{I}_n$ by Theorem 2.1. So $\mathcal{S} \subseteq \mathcal{F}_n$, and $\gamma(\mathcal{F}) \subseteq {}^\intercal(\mathcal{S}^\intercal) \subseteq {}^\intercal({\mathcal{F}_n}^\intercal) = \mathcal{F}_n$ by [7, 2.7]. Since $\gamma(\mathcal{F}) \subseteq \text{mod}R$, each $F \in \gamma(\mathcal{F})$ has a partial projective resolution $0 \longrightarrow Q \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow F \longrightarrow 0$ where P_i $(i < n)$ are finitely generated projective, and Q is (finitely presented and) flat. Then Q is projective, and $F \in \mathcal{P}_n$. This proves that $\gamma(\mathcal{F})$ is a resolving subclass of $\mathcal{P}_n^{<\infty}$.

On the other hand, if $\mathcal{C}$ is a resolving subclass of $\mathcal{P}_n^{<\infty}$, then $\mathcal{C}^\intercal = {}^\perp\mathcal{C}^*$ gives rise to a complete cotorsion pair $(\mathcal{C}^\intercal, (\mathcal{C}^\intercal)^\perp)$ by [17, Corollary 10]. Moreover, $\mathcal{C}^* \subseteq \mathcal{I}_n$, so $(\mathcal{C}^\intercal)^\perp = ({}^\perp\mathcal{C}^*)^\perp \subseteq \mathcal{I}_n$ by [2, Lemma 2.2(b)]. Finally, $\mathcal{C}^\intercal$ satisfies (R3), and it is closed under direct products by Lemma 1.2. It now follows from Theorem 2.1 that $\delta(\mathcal{C})$ is an n-cotilting class (of cofinite type).

The proof for $\gamma\delta$=id and $\delta\gamma$=id is analogous to that in 1.

3. Let $\mathcal{C}$ be the smallest resolving subclass of $\mathcal{P}_n^{<\infty}$ containing $\mathcal{S}$. Then $\mathcal{T} = \mathcal{S}^\perp = \mathcal{C}^\perp$, and so $\mathcal{F} = \mathcal{S}^\intercal = \mathcal{C}^\intercal$ by Lemma 1.4.1. By part 1. and 2. we know that $\mathcal{T} = \beta(\mathcal{C})$ is a tilting class and $\mathcal{F} = \delta(\mathcal{C})$ is a cotilting class. Moreover, $\mathcal{F} = \delta\alpha\beta(\mathcal{C}) = \delta\alpha(\mathcal{T})$. $\square$

Besides the correspondence for the classes of modules, there is also a correspondence for the individual modules:

Proposition 2.3 Let R be a ring and $n < \infty$. Let T be an n-tilting module of finite type in Mod$R$. Let $\mathcal{T} = T^{\perp}$, and let $\mathcal{F} = \delta\alpha(\mathcal{T})$ be the corresponding n-cotilting class.

Then $C = T^*$ is an n-cotilting module of cofinite type in RMod with ${}^{\perp}C = \mathcal{F} = T^{\intercal}$. Moreover, $M \in \mathcal{T}$ iff $M^* \in \mathcal{F}$, for each $M \in$Mod$R$.

Proof: Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the long exact sequences in (T1) and (T3) for T, we obtain conditions (C1) and (C3) for C.

By Theorem 2.2.1, we further know that $\mathcal{S} = \alpha(\mathcal{T}) \subseteq \mathcal{P}_n^{<\infty}$ is resolving and $\mathcal{S}^{\perp} = \mathcal{T} = T^{\perp}$. In particular, $\mathcal{S}^{\perp} \cap (R\mathrm{Mod})^* = T^{\perp} \cap (R\mathrm{Mod})^*$, and hence $\mathcal{S}^{\intercal} = T^{\intercal}$ by Lemma 1.4.1.

Consider $M \in$Mod$R$. By (the left-sided version of) Lemma 1.4.1, for each $i > 0$, $\mathrm{Ext}^i_R(M^*, C) = 0$ iff $\mathrm{Tor}^R_i(T, M^*) = 0$. It follows that $M^* \in {}^{\perp}C$ iff $M^* \in \mathcal{S}^{\intercal}$ iff $M \in \mathcal{S}^{\perp} = \mathcal{T}$, by Lemma 1.4.2. Taking $M = T^{(I)}$ for an arbitrary set I, we see that condition (C2) for C is equivalent to (T2) for the module T.

Finally, we have $\mathcal{F} = \delta\alpha\beta(\mathcal{S}) = \delta(\mathcal{S}) = \mathcal{S}^{\intercal}$, while ${}^{\perp}C = T^{\intercal} = \mathcal{S}^{\intercal}$. $\square$

Let us notice some immediate consequences of Theorem 2.2 in particular settings:

Examples 2.4 (i) Let R be a ring. Then 1-tilting (1-cotilting) classes of finite (cofinite) type correspond bijectively to subclasses of $\mathcal{P}_1^{<\infty}$ satisfying conditions (R1) and (R2). This follows by Lemma 1.1.3.

(ii) Let R be a von Neumann regular ring. Then mod$R \subseteq \mathcal{P}_0$. So, a module T is tilting of finite type (cotilting of cofinite type) iff T is a projective generator (injective cogenerator). Moreover, Mod$R$ is the only tilting (cotilting) class of finite (cofinite) type.

(iii) Let R be a Prüfer domain. Then $\text{mod}R \subseteq \mathcal{P}_1$, cf. [22, V.2.7]. So, any tilting module of finite type has projective dimension ≤ 1, and any cotilting module of cofinite type has injective dimension ≤ 1.

(iv) Let R be an elementary divisor domain (a semilocal Prüfer domain, for example), see [22, III.§6]. A class $\mathcal{T} \subseteq \text{Mod}R$ is tilting of finite type iff there is a set $S \subseteq R$ such that $\mathcal{T}$ coincides with the class of all S-divisible modules (that is, of all modules M such that $sM = M$ for all $s \in S$, or equivalently, $\text{Ext}^1_R(R/sR, M) = 0$ for all $s \in S$).

This is because each finitely presented module is a direct sum of cyclically presented modules [22, V.3.4]. So a resolving subclass $\mathcal{C} \subseteq \text{mod}R$ is determined by the set $S = \{s \in R \mid R/sR \in \mathcal{C}\}$, and Theorem 2.2.3 applies.

Similarly, a class $\mathcal{F} \subseteq \text{Mod}R$ is cotilting of cofinite type iff there is a set $S \subseteq R$ such that $\mathcal{F}$ coincides with the class of all S-torsion-free modules (that is, of all modules M such that $sm \neq 0$ for all $0 \neq m \in M$ and $0 \neq s \in S$, or equivalently, $\text{Tor}^R_1(R/sR, M) = 0$ for all $s \in S$). □

We remark that resolving subclasses of $\mathcal{P}_1^{<\infty}$ have good approximation properties in the noetherian setting:

Proposition 2.5 Let R be a right noetherian ring. Let $\mathcal{C}$ be a subclass of $\mathcal{P}_1^{<\infty}$ which is closed under direct summands and extensions and contains R. Then $\mathcal{C}$ is covariantly finite.

Proof: The class $\mathcal{C}$ is resolving by Lemma 1.1.3. Put $\mathcal{D} = \mathcal{C}^\intercal$. Then ${}^\intercal\mathcal{D} = \varinjlim \mathcal{C}$ by Theorem 1.3. By assumption, $\mathcal{D}$ is closed under submodules, and each module in $\mathcal{D}$ is a direct limit of finitely presented modules in $\mathcal{D}$. It follows that ${}^\intercal\mathcal{D} = {}^\intercal(\mathcal{D} \cap \text{mod}R)$. The latter class is closed under direct products by Lemma 1.2, so [15, 4.2] shows that $\mathcal{C}$ is covariantly finite. □

Tilting classes of finite type can also be characterized as certain definable classes:

Proposition 2.6 Let R be a ring.

1. Let $\mathcal{T}$ be a tilting class and T be a tilting module such that $T^{\perp} = \mathcal{T}$. Then $\mathcal{T}$ is of finite type iff $\mathcal{T}$ is definable and $T \in \varinjlim ({}^{\perp}\mathcal{T} \cap \operatorname{mod}R)$.
2. Assume R is right noetherian. Let $\mathcal{F}$ be a 1-cotilting (torsion-free) class and C be a 1-cotilting module such that ${}^{\perp}C = \mathcal{F}$. Then $\mathcal{F}$ is of cofinite type iff $C^* \in \varinjlim ({}^{\intercal}\mathcal{F} \cap \operatorname{mod}R)$.

Proof: 1. By Theorem 2.1, the class $\mathcal{C} = {}^{\perp}\mathcal{T} \cap \operatorname{mod}R$ is a resolving subclass of $\mathcal{P}_n^{<\infty}$ where n is the projective dimension of T. If $\mathcal{T}$ is of finite type, then by Theorem 2.2.1 we have $\mathcal{T} = \mathcal{C}^{\perp}$, hence $\mathcal{T}$ is definable. Moreover, $T \in {}^{\perp}(\mathcal{C}^{\perp}) \subseteq \varinjlim \mathcal{C}$ by Theorem 1.3.

Conversely, put $\mathcal{A} = \mathcal{C}^{\perp}$. Then $\mathcal{T} \subseteq \mathcal{A}$, both $\mathcal{A}$ and $\mathcal{T}$ are definable, and $\mathcal{A}$ is tilting of finite type by Theorem 2.2.1. In order to prove that $\mathcal{A} = \mathcal{T}$, it suffices to show that $\mathcal{A}$ and $\mathcal{T}$ contain the same pure-injective modules.

Let $P \in \mathcal{A}$ be pure-injective. By assumption, there is a direct system $(C_\sigma \mid \sigma \in I)$ of modules in $\mathcal{C}$ such that $T = \varinjlim_{\sigma \in I} C_\sigma$, and $\operatorname{Ext}^i_R(C_\sigma, P) = 0$ for all $\sigma \in I$ and $i > 0$. Then $\operatorname{Ext}^i_R(T, P) \cong \varprojlim_{\sigma \in I} \operatorname{Ext}^i_R(C_\sigma, P) = 0$ $(i > 0)$, hence $P \in \mathcal{T}$.

2. First, we prove that ${}^{\perp_1}C = {}^{\perp_1}C^{**}$. Let $F \in \operatorname{mod}R$. By Lemma 1.4, we have $\operatorname{Ext}^1_R(F, C^{**}) = 0$ iff $\operatorname{Tor}^R_1(F, C^*) = 0$ iff $\operatorname{Ext}^1_R(F, C) = 0$, so ${}^{\perp_1}C \cap \operatorname{mod}R = {}^{\perp_1}C^{**} \cap \operatorname{mod}R$.

By [11], C is pure-injective, so C is a direct summand in C^{**}, hence ${}^{\perp_1}C^{**} \subseteq {}^{\perp_1}C$.

Conversely, let $M \in {}^{\perp_1}C$. Since ${}^{\perp_1}C$ is a torsion-free class and R is right noetherian, $M = \varinjlim_{\sigma \in I} C_\sigma$ for a directed system $(C_\sigma \mid \sigma \in I)$ of finitely presented modules in ${}^{\perp_1}C$. Then $C_\sigma \in {}^{\perp_1}C^{**}$ for all $\sigma \in I$, and $\operatorname{Ext}^1_R(M, C^{**}) \cong \varprojlim_{\sigma \in I} \operatorname{Ext}^i_R(C_\sigma, C^{**}) = 0$.

This proves that $\mathcal{F} = {}^{\perp_1}C = {}^{\perp_1}C^{**} = (C^*)^{\intercal_1}$.

Let now $\mathcal{C} = {}^{\intercal}\mathcal{F} \cap \operatorname{mod}R$. If $\mathcal{F}$ is of cofinite type, then by Theorem 2.2.2. we have $\mathcal{C}^{\intercal} = \mathcal{F} = (C^*)^{\intercal_1}$, and $C^* \in {}^{\intercal_1}(\mathcal{C}^{\intercal_1}) = \varinjlim \mathcal{C}$ by Theorem 1.3.

Conversely, assume $C^* \in \varinjlim \mathcal{C}$. Since $\varinjlim \mathcal{C} = {}^{\intercal_1}(\mathcal{C}^{\intercal_1})$, we have $\mathcal{F} = (C^*)^{\intercal_1} = \mathcal{C}^{\intercal_1}$, so $\mathcal{F}$ is of cofinite type. □

Since the classes $\mathcal{P}_0^{<\infty}$ and $\mathcal{P}_n^{<\infty}$ are resolving there always exists the least, and the largest, n-tilting class of finite type (and similarly for the n-cotilting classes). While the largest tilting class is induced by any progenerator, the least class may not be induced by any tilting module in mod$R$ (cf. [6, 4.4]). Further examples of tilting classes of finite type not induced by tilting modules in mod$R$ will be exhibited in the next two sections.

3 Iwanaga-Gorenstein rings

Recall that a ring R is said to be **Gorenstein**, or Iwanaga-Gorenstein [19], if R is (left and right) noetherian and both idim R_R and idim ${}_RR$ are finite. It is well known that idim R_R and idim ${}_RR$ then coincide [19, 9.1.8], and R is called n-Gorenstein if this number is n. Over an n-Gorenstein ring, one has $\mathcal{P} = \mathcal{P}_n = \mathcal{I} = \mathcal{I}_n$, see [19, 9.1.10]. In particular, the big finitisitic dimension Findim$R$, that is, the supremum of the projective dimensions attained on $\mathcal{P}$, equals n. We are now going to see that also the little finitistic dimension, that is, the supremum of the projective dimensions attained on $\mathcal{P}^{<\infty}$, equals n.

To this end, we first have to recall the following result due to Auslander and Buchweitz. If $\mathcal{X} \subseteq \text{Mod}R$, then $\widehat{\mathcal{X}}$ will denote the subcategory of all modules M_R admitting a finite $\mathcal{X}$-resolution, that is, a long exact sequence $0 \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$ with $r \geq 0$ and $X_i \in \mathcal{X}$ for $i = 0, \cdots, r$.

Proposition 3.1 [8, 1.8] Let R be a noetherian ring such that ${}_RR$ has finite injective dimension, and let $\mathcal{X} = {}^{\perp}R \cap \text{mod}\,R$. Then for each $X \in \widehat{\mathcal{X}}$, there exist short exact sequences

$$0 \longrightarrow K_X \longrightarrow M_X \xrightarrow{g_X} X \longrightarrow 0 \text{ with } K_X \in \mathcal{P}^{<\infty}, \text{ and } M_X \in \mathcal{X},$$

$$0 \longrightarrow X \xrightarrow{f_X} B_X \longrightarrow C_X \longrightarrow 0 \text{ with } B_X \in \mathcal{P}^{<\infty}, \text{ and } C_X \in \mathcal{X}.$$

Moreover, if R is a Gorenstein ring, then $\widehat{\mathcal{X}} = \mathrm{mod}\, R$.

Theorem 3.2 If R is an n-Gorenstein ring, then Findim$R$ = findim$R = n$, and every module of finite projective dimension is a direct summand of a $\mathcal{P}^{<\infty}$-filtered module.

Proof: Let R be n-Gorenstein, so $\mathcal{P} = \mathcal{P}_n = \mathcal{I} = \mathcal{I}_n$. Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$. We claim that every module N in $\mathcal{P} \cap \mathcal{B}$ is injective. By the Baer Lemma, it is enough to show that $\mathrm{Ext}^1_R(R/I, N) = 0$ for every right ideal I of R. Now, setting $X = R/I$ and $\mathcal{X} = {}^{\perp}R \cap \mathrm{mod}\, R$, we know from Proposition 3.1 that there is a short exact sequence $0 \longrightarrow X \xrightarrow{f_X} B_X \longrightarrow C_X \longrightarrow 0$ with $B_X \in \mathcal{P}^{<\infty}$, and $C_X \in \mathcal{X}$. This yields a long exact sequence $\cdots \mathrm{Ext}^1_R(B_X, N) \rightarrow \mathrm{Ext}^1_R(R/I, N) \rightarrow \mathrm{Ext}^2_R(C_X, N) \cdots$ where $\mathrm{Ext}^1_R(B_X, N) = 0$ since $N \in \mathcal{B}$, and $\mathrm{Ext}^2_R(C_X, N) = 0$ because $N \in \mathcal{P}$ and $\mathcal{X}^{\perp}$ contains all projective modules and is coresolving, hence contains $\mathcal{P}$.
Now, since findim$R \leq n$, we know from [6, Theorem 2.6] that $(\mathcal{A}, \mathcal{B})$ is a tilting cotorsion pair induced by a tilting module T, and that $\mathcal{A} \subseteq \mathcal{P}$. Then Add$T = \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{P} \cap \mathcal{B}$ consists of injective modules. In particular, Add$T$ is closed under cokernels of monomorphisms. By [6, Theorem 3.2] we then know that $\mathcal{A} = \mathcal{P}$ and that our statement holds true. $\square$

The following result was first proven in [28, 4.7]. Further proofs are given in [29] and [1]. We now obtain yet another proof.

Corollary 3.3 Let R be a Gorenstein ring and let $0 \rightarrow R_R \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots I_n \rightarrow 0$ be a minimal injective resolution of R_R. Then every indecomposable injective right module is isomorphic to a direct summand of some I_k with $0 \leq k \leq n$.

Proof: We have just seen that $(\mathcal{P}, \mathcal{B})$ is a tilting cotorsion pair whose kernel $\mathcal{P} \cap \mathcal{B}$ is the class of the injective modules. Observe that the module $I = \bigoplus_{i=0}^{n} I_k$ is a tilting module with $I^{\perp} = \mathcal{B}$, see [2, proof of 4.1]. Then $\mathrm{Add}I = \mathcal{P} \cap \mathcal{B}$. In particular, every indecomposable injective module belongs to $\mathrm{Add}I$ and thus is isomorphic to an indecomposable direct summand of some I_k by Azumaya's Decomposition Theorem. □

Recall from [18] and [19] that a module M over an arbitrary ring R is said to be **Gorenstein injective** if it can be written as $M = \mathrm{Ker}\, f$ for some long exact sequence

$$\cdots E_1 \to E_0 \to E^0 \xrightarrow{f} E^1 \to \cdots$$

of injective modules which stays exact under $\mathrm{Hom}_R(I,\ )$ for any injective module I_R. The **Gorenstein projective** modules are defined dually. Furthermore, M is said to be **Gorenstein flat** if it can be written as $M = \mathrm{Ker}\, f$ for some long exact sequence

$$\cdots F_1 \to F_0 \to F^0 \xrightarrow{f} F^1 \to \cdots$$

of flat modules which stays exact under $I \otimes_R -$ for any injective module I_R. We denote by $\mathcal{GI}$, $\mathcal{GP}$, and $\mathcal{GF}$ the class of all Gorenstein injective, Gorenstein projective, and Gorenstein flat modules, respectively.

If R is a Gorenstein ring, then we know from [19, 11.2.3 and 10.1.2] that $(\mathcal{P}, \mathcal{GI})$ is a complete cotorsion pair whose kernel $\mathcal{P} \cap \mathcal{GI}$ is just the class of all injective modules. Similarly, $(\mathcal{GP}, \mathcal{P})$ is a complete cotorsion pair whose kernel $\mathcal{GP} \cap \mathcal{P}$ is just the class of all projective modules, see [19, 11.5.10 and 10.2.3].

The results above now show that over a Gorenstein ring, $(\mathcal{P}, \mathcal{GI})$ coincides with the tilting cotorsion pair of finite type cogenerated by $\mathcal{P}^{<\infty}$. Let us determine the corresponding cotilting cotorsion pair of cofinite type.

Theorem 3.4 The following statements are equivalent for a noetherian ring R.
(1) R is Gorenstein.
(2) The Gorenstein injective modules in $\mathrm{Mod}R$ and in $R\mathrm{Mod}$ form a tilting class.

(3) The Gorenstein injective modules in Mod$R$ form a tilting class and the Gorenstein flat modules in Mod$R$ form a cotilting class.

Proof: (1)⇒(2),(3): We know by Theorem 3.2 that $\mathcal{GI} = (\mathcal{P}^{<\infty})^{\perp}$ is a tilting class of finite type in Mod$R$. Under the bijection of Theorem 2.2, the class $\mathcal{GI}$ is mapped to the cotilting class $(\mathcal{P}^{<\infty})^{\intercal}$ in RMod, which by Lemma 1.4.1 consists of the left modules M such that M^* belongs to $\mathcal{GI}$. By [19, 10.3.8] the latter are precisely the Gorenstein flat left modules. The statements on the Gorenstein flat right modules and on the Gorenstein injective left modules hold by symmetry.
(2),(3)⇒(1): If $\mathcal{GF}$ is a cotilting class in Mod$R$, then by [2, Lemma 2.2(b)] there is an integer $n \geq 0$ such that every module X_R has a resolution $0 \to F_n \to \cdots \to F_1 \to F_0 \to X \to 0$ with Gorenstein flat modules $F_0, \ldots, F_n$. We show that this implies idim$R_R \leq n$. Recall that idimR_R is the supremum of the flat dimensions attained on the injective left modules, [19, 9.1.6]. So, let ${}_RE$ be an injective module. By [19, 10.3.1] all Gorenstein flat right modules belong to ${}^{\intercal}E$, thus by dimension shift we have $\mathrm{Tor}^R_{i+n}(X,E) \cong \mathrm{Tor}^R_i(F_n,E) = 0$ for any module X and any $i \geq 0$, and the claim is proven.
Assume now that $\mathcal{GI}$ is a tilting class in Mod$R$. Then by [2, Lemma 2.2] there is an integer $n \geq 0$ such that ${}^{\perp}\mathcal{GI} \subseteq \mathcal{P}_n$. Now, let E_R be an injective module. By [19, 10.1.3] all Gorenstein injective modules belong to $E^{\perp}$, hence we infer that $E \in {}^{\perp}(E^{\perp}) \subseteq {}^{\perp}\mathcal{GI}$ has (projective and) flat dimension at most n. As above, we conclude that idim${}_RR \leq n$.
By symmetry, we finally obtain that idim$R_R < \infty$ provided that the Gorenstein injective left modules form a tilting class. □

Enochs and Jenda have proven in [20] that the Gorenstein injective modules over a Gorenstein ring can be characterized as the modules X such that $\mathrm{Ext}^i_R(L,X) = 0$ for all $i > 0$ and all countably generated modules L of finite projective dimension. Our results above improve this characterization.

Corollary 3.5 Let R be a Gorenstein ring, and let X_R be a right R-module.
(1) X is Gorenstein injective if and only if $\mathrm{Ext}^i_R(L, X) = 0$ for all $i > 0$ and all finitely generated modules L_R of finite projective dimension.
(2) X is Gorenstein flat if and only if $\mathrm{Tor}^R_i(X, L) = 0$ for all $i > 0$ and all finitely generated modules ${}_RL$ of finite projective dimension.

It is well known that over a Gorenstein ring, $\mathcal{GP} = \mathcal{GF}$ if and only if R is artinian. We now characterize the artinian case in terms of cotilting modules.

Proposition 3.6 The following statements are equivalent for a Gorenstein ring R.
(1) R is right (or left) artinian.
(2) $(\mathcal{GP}, \mathcal{P})$ is a cotilting cotorsion pair.
(3) R_R is a cotilting module and ${}^{\perp}R_R = \mathcal{GP}$.

Proof: Let R be n-Gorenstein. Since $(\mathcal{GP}, \mathcal{P})$ is a complete cotorsion pair with $\mathcal{P} = \mathcal{I} = \mathcal{I}_n$, we know from [2, Theorem 4.2] that $(\mathcal{GP}, \mathcal{P})$ is a cotilting cotorsion pair iff its kernel $\mathcal{GP} \cap \mathcal{P}$ is closed under products. In other words, (2) just means that the class of the projective right modules is closed under products. By a theorem of Chase [14], the latter is equivalent to the fact that R is right perfect (and left coherent), which means that R is right (or left) artinian. So, we have shown (1)⇔(2).
We now show (1)⇒(3): First of all, (1) implies that R is Σ-pure-injective, and so we even have Prod$R$ =Add$R$. Then for any injective cogenerator W_R the projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to W \to 0$ yields an exact sequence as in (C3). Furthermore, all products of copies of R are projective, which proves (C2). Finally, we know that idim$R_R = n$, yielding (C1). So R_R is a cotilting module. Moreover, ${}^{\perp}R = {}^{\perp}\mathrm{Add}R$, and the latter class coincides with ${}^{\perp}\mathcal{P} = \mathcal{GP}$.□

Note that in general for a Gorenstein ring R, the class ${}^{\perp}R$ need not coincide with the class of Gorenstein flat modules, and R need not be a cotilting module.

Example 3.7 (1) Let $R = \mathbb{Z}$. Then $R^{I\!N}$ is flat, hence Gorenstein flat, but it does not belong to ${}^{\perp}R$ which is the class of Whitehead groups [21, Chapter XII, 1.3]. Moreover, $\mathcal{GP} = \mathcal{P}_0$. So, the question whether $\mathcal{GP} = {}^{\perp}R$ is the Whitehead problem the solution of which depends on set theoretic assumptions, see [21, Chapter XIII] .
(2) Let p be a prime number and $R = \mathbb{J}_p$ the ring of all p-adic integers. Then ${}^{\perp}R = \mathcal{GF}$ is the class of all torsion-free $\mathbb{J}_p$-modules, and it is a cotilting class which properly contains Cogen$R$, see [5, 2.6] and [4, 2.4]. Thus R_R is not a cotilting module. □

Let us now turn to artin algebras.

Corollary 3.8 An artin algebra R is Gorenstein if and only if there is a cotilting module $C \in \operatorname{mod} R$ such that $\mathcal{GP} = {}^{\perp}C$.

Proof: The only-if part follows from Proposition 3.6. For the if-part, we use [9, Theorem 5.5] stating that under the given assumption every $X \in \operatorname{mod} R$ admits a finite $\mathcal{GP} \cap \operatorname{mod} R$-resolution. By [19, 10.2.6], the class $\mathcal{GP} \cap \operatorname{mod} R$ consists of the reflexive modules G_R such that $G \in {}^{\perp}R_R$ and $\operatorname{Hom}_R(G, R) \in {}^{\perp}({}_RR)$, in other words, of the modules of Gorenstein dimension zero. So, we conclude that every finitely presented right module has finite Gorenstein dimension, which by [10, 24] means that R is Gorenstein.□

Note that $\mathcal{GP} \cap \operatorname{mod} R$ is always a resolving class which is contained in $\mathcal{X} = {}^{\perp}R_R \cap \operatorname{mod} R$, see [19, 10.2.8 and 10.2.2]. Moreover, if R is an artin algebra, then $\mathcal{GP}$ is a precovering class by [26]. However, we don't know whether $\mathcal{X}$ is contravariantly finite in $\operatorname{mod} R$ unless R is Gorenstein, cf. Proposition 3.1.

Recall from [3] that a module M satisfying condition (T1) and (T2) in the definition of a tilting module is called a **partial**

tilting module. Furthermore, a module N is a **complement** of M if $N \oplus M$ is a tilting module. **Partial cotilting** modules and their complements are defined dually.

Lemma 3.9 Let R be an artin algebra with duality $D\colon \mathrm{mod}\,R \to R\mathrm{mod}$. The following statements are equivalent.
(1) $\mathrm{idim}_R R = n < \infty$.
(2) $D({}_RR)^\perp$ is a tilting class in Mod$R$.
(3) ${}^\perp({}_RR)$ is a cotilting class in RMod.

Proof: (1)⇒(2): The injective right R-module $W = D({}_RR)_R$ has projective dimension at most n and obviously satisfies condition (T2). So, it is a finitely presented partial tilting module, and by [3, Corollary 2.2] it has a complement M such that $M \oplus W$ is a tilting module with $(M \oplus W)^\perp = W^\perp$.
(2)⇒(3): By Proposition 2.2 we know that $D({}_RR)^\intercal = {}^\perp D^2({}_RR) = {}^\perp({}_RR)$ is a cotilting class.
(3)⇒(1): Of course ${}_RR \in ({}^\perp\, {}_RR)^\perp$, and by [2, Lemma 2.2] there is an integer $n \geq 0$ such that $({}^\perp\, {}_RR)^\perp \subseteq \mathcal{I}_n$. □

Proposition 3.10 Assume that R is an artin algebra such that ${}_RR$ has finite injective dimension. The following statements are equivalent.
(1) R is Gorenstein.
(2) $D({}_RR)^\perp = (\mathcal{P}^{<\infty})^\perp$.
(3) The partial tilting module $D({}_RR)$ has a complement of finite injective dimension.
(4) The partial cotilting module ${}_RR$ has a complement of finite projective dimension.
(5) Every (finitely presented) module in $D({}_RR)^\perp$ is generated by $D({}_RR)$.
(6) Every (finitely presented) module in ${}^\perp({}_RR)$ is cogenerated by ${}_RR$.
(7) $\mathcal{GP} = {}^\perp({}_RR)$.

Proof: From Theorem 2.2 we know that (1) implies (2). Moreover, (1) implies (3) and (5) since $D({}_RR)$ is a tilting module and hence satisfies $D({}_RR)^\perp \subseteq \mathrm{Gen}D({}_RR)$ by [2, Lemma 2.3]. Dually one obtains that (1) implies (4) and (6). Finally, (1) implies (7) by Proposition 3.6.
For the converse implications, first of all observe that (2) implies that findim$R$ is finite by Lemma 3.9 and [6, Theorem 2.6]. Then [9, 6.10] yields that idimR_R is finite.
Next, assume (3). Condition (T3) then implies that R_R has finite injective dimension.
Further, (7)⇒(5) since by definition $\mathcal{GP}\subseteq$Cogen ${}_RR$, and (5)⇒(3) because the partial tilting module $D({}_RR)$ is then even tilting by [23, Proposition in §3] or [2, Theorem 4.4]. The remaining implications are proven dually. □

4 Commutative 1-Gorenstein rings

In this section, we first consider in greater detail a construction of tilting modules of finite type over commutative 1-Gorenstein rings:

Example 4.1 Let R be a commutative 1-Gorenstein ring. Let P_i denote the set of all prime ideals in R of height i $(i = 0, 1)$. By a classical result of Bass, R has a minimal injective resolution of the form

$$0 \longrightarrow R \longrightarrow Q \longrightarrow \bigoplus_{p\in P_1} E(R/p) \longrightarrow 0$$

where $Q = \bigoplus_{p\in P_0} E(R/p)$.

Recall from Corollary 3.3 that $Q \oplus \bigoplus_{p\in P_1} E(R/p)$ is the tilting module inducing the class of Gorenstein injectives. We are now going to consider a family of tilting modules of finite type obtained as variations of that construction, which in some cases lead to a classification of all tilting classes of finite type, cf. Remark 4.2.

Let $P \subseteq P_1$. Let $F_P = \bigoplus_{p\in P} E(R/p)$ and $G_P = \bigoplus_{q\in P_1\setminus P} E(R/q)$. Consider the module R_P defined by $R \subseteq R_P \subseteq Q$ and $R_P/R \cong F_P$. We have the exact sequence $0 \to R_P \to Q \to G_P \to 0$. Since the module G_P has flat dimension ≤ 1 and Q is flat (cf. [19, 9.1.10 and 9.3.3]), R_P is also flat. Define

$$T_P = R_P \oplus F_P.$$

We will prove that T_P is a 1-tilting module of finite type.

Clearly, T_P has projective dimension ≤ 1. We verify condition (T2) for T_P: since R_P is $\{R\} \cup \{E(R/p) \mid p \in P\}$-filtered, it suffices to prove that

$$\mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) = 0$$

for all $p \in P$ and any κ. Consider the exact sequence

$$0 \longrightarrow R_P^{(\kappa)} \longrightarrow Q^{(\kappa)} \longrightarrow G_P^{(\kappa)} \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(E(R/p), -)$, we get part of the induced long exact sequence

$$0 = \mathrm{Hom}_R(E(R/p), G_P^{(\kappa)}) \to \mathrm{Ext}^1_R(E(R/p), R_P^{(\kappa)}) \to \mathrm{Ext}^1_R(E(R/p), Q^{(\kappa)}) = 0$$

This proves (T2). The exact sequence $0 \to R \to R_P \to F_P \to 0$ yields condition (T3) for T_P.

Altogether, T_P is 1-tilting; so $\mathrm{Gen}(R_P) = \mathrm{Gen}(T_P) = T_P^{\perp_1} = F_P^{\perp_1}$ is a torsion class in $\mathrm{Mod}R$.

Let $p \in P$. Since R is 1-Gorenstein, Proposition 3.1 provides for a short exact sequence $0 \longrightarrow R/p \stackrel{\rho}{\longrightarrow} M_p \longrightarrow N_p \longrightarrow 0$ such that $M_p \in \mathcal{P}_1^{<\infty}$ and $N_p \in {}^{\perp_1}R$.

Let $(C_n \mid n < \omega)$ be a composition series in $E(R/p)$. By induction, we construct a direct system of short exact sequences $0 \longrightarrow C_n \longrightarrow D_n \longrightarrow E_n \longrightarrow 0$ $(n < \omega)$ and connecting maps (μ_n, ν_n, π_n) such that μ_n is the embedding of C_n into C_{n+1} and ν_n is a monomorphism with $\mathrm{Coker}(\nu_n) \cong M_p$ for each $n < \omega$.

For $n = 0$, we take the trivial exact sequence $0 \to 0 \to 0 \to 0 \to 0$. Next, consider the push-out of the monomorphism $C_n \to D_n$ and of μ_n:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & 0 \\
 & & {\scriptstyle \mu_n}\downarrow & & {\scriptstyle \alpha}\downarrow & & \| & & \\
0 & \longrightarrow & C_{n+1} & \longrightarrow & F_n & \xrightarrow{\rho} & E_n & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \beta}\downarrow & & & & \\
 & & R/p & = & R/p & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since D_n has projective (and injective) dimension ≤ 1, we have $\mathrm{Ext}^2_R(N_p, D_n) = 0$. It follows that the map $\mathrm{Ext}^1_R(\rho, D_n) : \mathrm{Ext}^1_R(M_p, D_n) \to \mathrm{Ext}^1_R(R/p, D_n)$ is surjective, so there is a pull-back diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & D_n & = & D_n & & & & \\
 & & {\scriptstyle \alpha}\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F_n & \longrightarrow & D_{n+1} & \longrightarrow & N_p & \longrightarrow & 0 \\
 & & {\scriptstyle \beta}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & R/p & \xrightarrow{\rho} & M_p & \longrightarrow & N_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Altogether, we get a commutative diagram with exact rows and columns that provides for the induction step of the construction

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \nu_n} & & \downarrow{\scriptstyle \pi_n} & & \\
0 & \longrightarrow & C_{n+1} & \longrightarrow & D_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R/p & \xrightarrow{\rho} & M_p & \longrightarrow & N_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Let $\mathfrak{D} : 0 \longrightarrow E(R/p) \longrightarrow D \longrightarrow E \longrightarrow 0$ be the direct limit of the system $0 \longrightarrow C_n \longrightarrow D_n \longrightarrow E_n \longrightarrow 0$ $(n < \omega)$. Since $E(R/p)$ is injective, $\mathfrak{D}$ splits. Since $\mathrm{Coker}(\nu_n) \cong M_p$ for each $n < \omega$, the module D is $\{M_p\}$-filtered. In particular, we have $E(R/p)^{\perp_1} \supseteq M_p^{\perp_1}$.

Let $\mathcal{S}_P = \{M_p \mid p \in P\}$. We have just shown that $T_P^{\perp_1} \supseteq \mathcal{S}_P^{\perp_1}$.

In order to prove that T_P is of finite type, it suffices to show that $T_P^{\perp_1} \subseteq M_p^{\perp_1}$ for all $p \in P$. For this, it is enough to prove that $\mathrm{Ext}^1_R(M_p, R_P) = 0$ for each $p \in P$ (then $\mathrm{Gen}(R_P) \subseteq M_p^{\perp_1}$). Applying $\mathrm{Hom}_R(-, R_P)$ to the exact sequence $0 \to R/p \to M_p \to N_p \to 0$, we get

$$
\begin{aligned}
\cdots \longrightarrow \mathrm{Ext}^1_R(N_p, R_P) \longrightarrow \mathrm{Ext}^1_R(M_p, R_P)& \\
\longrightarrow \mathrm{Ext}^1_R(R/p, R_P) \longrightarrow \cdots&
\end{aligned}
$$

Applying $\mathrm{Hom}_R(N_p, -)$ to the exact sequence $0 \to R \to R_P \to F_P \to 0$ and using the fact that $N_p \in {}^{\perp_1}R$, we get

$\mathrm{Ext}^1_R(N_p, R_P) = 0$. Since $\mathrm{Hom}_R(R/p, Q/R_P) = 0$, we have $\mathrm{Ext}^1_R(R/p, R_P) = 0$. It follows that $\mathrm{Ext}^1_R(M_p, R_P) = 0$.

This proves that $\mathcal{T}_P = T_P^{\perp_1} = \mathcal{S}_P^{\perp_1}$ is a 1-tilting class of finite type.

Applying Proposition 2.3, we can now describe the corresponding 1-cotilting modules, and 1-cotilting classes, of cofinite type.

We will use the duality $(-)^* = \mathrm{Hom}_R(-, W)$ induced by the injective cogenerator $W = \bigoplus_{q \in P_0 \cup P_1} E(R/q)$. For each $P \subseteq P_1$, we obtain a 1-cotilting module

$$C_P = T_P^* = D_P \oplus \prod_{p \in P} J_p$$

where $D_P = R_P^*$, and $J_p = \mathrm{Hom}_R(E(R/p), E(R/p))) = \hat{R}_p$ is the p-adic module. Since R_P is flat, the module D_P is injective.

We claim that $D_P \cong F_P \oplus \bigoplus_{q \in P_0} E(R/q)^{(\alpha_q)}$ where $\alpha_q \geq 1$ for all $q \in P_0$. Indeed, $D_P \cong \mathrm{Hom}_R(R_P, Q \oplus G_P) \oplus \mathrm{Hom}_R(R_P, F_P)$. But $\mathrm{Hom}_R(R_P, Q \oplus G_P) \cong \mathrm{Hom}_R(R, Q \oplus G_P) \cong Q \oplus G_P$, while $\mathrm{Hom}_R(R_P, F_P) \cong \mathrm{Hom}_R(Q, F_P)$. The latter module is flat and injective, hence isomorphic to a direct summand of $Q^{(J)}$ for a set J, and the claim follows.

The corresponding 1-cotilting class is $\mathcal{F}_P = {}^{\perp_1}C_P = \mathcal{S}_P^{\intercal_1}$ (cf. Theorem 2.2.3). □

Remark 4.2 (i) The construction of the module M_p in Example 4.1 can be made more explicit for any $p \in P$ satisfying $p \nsubseteq \cup_{q \neq p} q$ (By prime avoidance, the latter holds for all $p \in P$ in case $P_1 \setminus P$ is finite.) For such p there is a regular element $y_p \in p \setminus \bigcup_{p \neq q \in P_1} q$. Since R is 1-Gorenstein, y_p forms a maximal p-sequence in R and we can take $M_p = R/y_pR$. The proof is then the same as above except that $\mathrm{Ext}^1_R(M_p, R_P) = 0$ follows from $\mathrm{Hom}_R(M_p, R/q) = 0$ which in turn is a consequence of y_p being an R/q-regular element in y_pR.

(ii) The reasoning in Example 4.1 can substantially be simplified in case R is a Dedekind domain. Though in this case it can still happen that $p \subseteq \cup_{q \neq p} q$ (so part (i) may not apply), we have a much simpler choice: $M_p = R/p$. It follows that $\mathcal{T}_P$ is just the class of all modules that are p-divisible for each $p \in P$, and $\mathcal{F}_P$ is the class of all modules that are p-torsion-free for each $p \in P$. Moreover, the classes $\mathcal{F}_P$ ($P \subseteq P_1$) are the only cotilting classes of modules by [11] and [17, Corollary 17], and the classes $\mathcal{T}_P$ ($P \subseteq P_1$) are the only tilting classes of finite type, cf. [32, Theorem 5]. If R is a small Dedekind domain and V=L, then the classes $\mathcal{T}_P$ ($P \subseteq P_1$) are the only tilting classes of modules, [32, Theorem 12].

We finish this part by considering tilting modules and tilting classes of finite type under classical localization. Let R be a commutative ring, $S \subseteq R$ a multiplicative set, $T \in \mathrm{Mod}R$, and $\mathcal{T} \subseteq \mathrm{Mod}R$. Then R_S denotes the localization of R at S, $T_S \cong T \otimes_R R_S$ the localization of T at S, and $\mathcal{T}_S = \{N \in \mathrm{Mod}R_S \mid N \cong M \otimes_R R_S \text{ for some } M \in \mathcal{T}\}$. As usual, for a maximal ideal m of R, $T_m = T_{R \setminus m}$ and $\mathcal{T}_m = \mathcal{T}_{R \setminus m}$.

Proposition 4.3 Let R be a commutative ring. Let T be an n-tilting module of finite type and $\mathcal{T} = T^{\perp}$ be the corresponding n-tilting class.

1. Let S be a multiplicative subset of R. Then T_S is an n-tilting R_S-module of finite type, the corresponding n-tilting class being $\mathcal{T}_S = T_S^{\perp} = \mathcal{T} \cap \mathrm{Mod}R_S$.
2. Let $M \in \mathrm{Mod}R$. Then $M \in \mathcal{T}$ if and only if $M_m \in \mathcal{T}_m$ for all maximal ideals m of R.

Proof: By assumption, there is a resolving class $\mathcal{C} \subseteq \mathcal{P}_n^{<\infty}$ such that $T^{\perp} = \mathcal{C}^{\perp_1}$.

Let $\mathfrak{P}$ be a projective resolution of T in $\mathrm{Mod}R$. By [31], each syzygy of T in $\mathfrak{P}$ is isomorphic to a direct summand of a

$\mathcal{C}$-filtered module, and conversely, each module in $\mathcal{C}$ is isomorphic to a direct summand of a module filtered by the syzygies of T.

1. Applying the exact functor $- \otimes_R R_S$ to the long exact sequences in (T1) and (T3) for T, we obtain conditions (T1) and (T3) for T_S.

Further, each element of $\mathcal{C}_S$ has a projective resolution of length $\leq n$ consisting of finitely generated R_S-modules.

Localizing $\mathfrak{P}$ at S, we obtain a projective resolution $\mathfrak{Q} = \mathfrak{P} \otimes_R R_S$ of T_S in ModR_S. Since R_S is a flat module, each syzygy of T_S in $\mathfrak{Q}$ is isomorphic to a direct summand of a $\mathcal{C}_S$-filtered R_S-module, and conversely, each R_S-module in $\mathcal{C}_S$ is isomorphic to a direct summand of an R_S-module filtered by the syzygies of T_S. By [31], this just says that in ModR_S, $T_S^{\perp} = \mathcal{C}_S^{\perp_1}$.

Let $i \geq 1$ and let I be a set. Condition (T2) for T and (a generalization of) [19, 3.2.5] now give

$$0 = \operatorname{Ext}^1_R(C, T^{(I)}) \otimes_R R_S \cong \operatorname{Ext}^1_{R_S}(C_S, T_S^{(I)})$$

for each $C \in \mathcal{C}$. It follows that $T_S^{(I)} \in T_S^{\perp}$, so condition (T2) holds for the R_S-module T_S.

Similarly, [19, 3.2.5] gives $\mathcal{T}_S \subseteq \mathcal{C}_S^{\perp_1}$ $(= T_S^{\perp})$, and since $N \cong N \otimes_R R_S$ for any module $N \in$ ModR_S ($\subseteq$Mod$R$), we also get $N \in \mathcal{C}_S^{\perp_1}$ iff $N \in \mathcal{C}^{\perp_1}$. It follows that $T_S^{\perp} = \mathcal{T} \cap$ ModR_S, and clearly $\mathcal{T} \cap$ Mod$R_S \subseteq \mathcal{T}_S$.

Since $\mathcal{C}_S \subseteq$ modR_S, the equality $T_S^{\perp} = \mathcal{C}_S^{\perp_1}$ gives that T_S is of finite type.

2. By (a generalization of) [19, 3.2.15], we have

$$\operatorname{Ext}^1_R(C, M) \otimes_R R_m \cong \operatorname{Ext}^1_R(C, M_m)$$

for each $C \in \mathcal{C}$. Since $\mathcal{T} = \mathcal{C}^{\perp_1}$, $M \in \mathcal{T}$ iff $M_m \in \mathcal{C}^{\perp_1}$ for all maximal ideals m of R. The latter is equivalent to $M_m \in \mathcal{C}_m^{\perp_1} = \mathcal{T}_m$ for all maximal ideals m of R. $\square$

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