ETALE HOMOTOPY TYPES OF MODULI STACKS OF ALGEBRAIC CURVES WITH SYMMETRIES

PAOLA FREDIANI AND FRANK NEUMANN

Abstract. Using the machinery of etale homotopy theory à la Artin-Mazur we determine the etale homotopy types of moduli stacks over $\bar{\mathbb{Q}}$ parametrizing families of algebraic curves of genus $g \geq 2$ endowed with an action of a finite group G of automorphisms, which comes with a fixed embedding in the mapping class group Γ_g, such that in the associated complex analytic situation the action of G is precisely the differentiable action induced by this specified embedding of G in Γ_g.

Introduction

Let $\mathcal{M}_g$ be the moduli stack of families of proper smooth algebraic curves of genus g with $g \geq 2$. In a fundamental paper Oda [O] determined the etale homotopy type of $\mathcal{M}_g \otimes \bar{\mathbb{Q}}$, the moduli stack representing the restriction of the moduli functor to the subcategory of schemes over $\bar{\mathbb{Q}}$. The etale homotopy type a la Artin-Mazur of the moduli stack $\mathcal{M}_g \otimes \bar{\mathbb{Q}}$ is given as the profinite Artin-Mazur completion of the Eilenberg-MacLane space $K(\Gamma_g, 1)$, where Γ_g is the Teichmüller modular or mapping class group of compact Riemann surfaces of genus g. In fact, Oda proved this result for all moduli stacks $\mathcal{M}_{g,n}$ of families of proper smooth algebraic curves of genus g with n distinct ordered points and $2g + n > 2$. In this more general context, the group Γ_g is replaced with the Teichmüller modular group $\Gamma_{g,n}$ of compact Riemann surfaces of genus g with n punctures. These results of Oda on the etale fundamental group of the moduli stack of algebraic curves are of great importance for the geometry and arithmetic of moduli spaces of algebraic curves and for the study of geometric Galois actions as they allow to relate the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to the profinite completion $\Gamma_{g,n}^{\wedge}$ of the mapping class groups $\Gamma_{g,n}$ which is important for realizing Grothendieck's ideas on a "Lego-Teichmüller game" [G2]. We refer

1991 *Mathematics Subject Classification.* Primary 14F35, secondary 14H10.

Key words and phrases. algebraic curves, etale homotopy theory, algebraic stacks, moduli of algebraic curves, Teichmüller theory.

to the excellent articles in the book edited by Schneps and Lochak [SL] and the article by Matsumoto [Ma] for good overviews on this fascinating cycle of ideas.

In this article we embark in a different direction. We study the etale homotopy types of certain moduli stacks $\mathcal{M}_{g,G}$ parametrizing families of proper smooth algebraic curves of genus $g \geq 2$ with certain prescribed fixed finite subgroups of automorphisms. It turns out that a similar theorem like that of Oda for $\mathcal{M}_g$ holds with some modifications also in this context. We outline very briefly the argument here. We will study the following general situation: Let G be a finite group with a fixed embedding in the mapping class group Γ_g with $g \geq 2$. Let now $\mathcal{M}_{g,G}$ be the moduli stack of those families of proper smooth algebraic curves of genus $g \geq 2$ which have G as subgroup of its automorphism groups and which are "keeping track of the differentiable action of G as a subgroup of Γ_g in the associated complex analytic situation". More precisely, the moduli stack $\mathcal{M}_{g,G}$ of algebraic curves with symmetries is defined as the category fibred in groupoids over the category of schemes over $\mathbb{Q}$ defined by its category of sections as follows: For a scheme S over $\mathbb{Q}$, the objects of $\mathcal{M}_{g,G}(S)$ are families of algebraic curves over S with an action of the finite group G such that the differentiable action of G on the associated complex analytic family of Riemann surfaces of genus g is precisely the one induced by the specified embedding of G in Γ_g. The morphisms of $\mathcal{M}_{g,G}(S)$ are those morphisms of families of algebraic curves over S, which are compatible with these data. It turns out that $\mathcal{M}_{g,G}$ is in fact, like $\mathcal{M}_g$, an algebraic stack in the sense of Deligne and Mumford [DM].

In order to determine the etale homotopy type $\{\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}\}_{et}$ of the stack $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$ we compare the algebraic with the associated complex analytic situation over the complex numbers. There we determine using Teichmüller theory the analytic or classical topological homotopy type of the complex analytification $\mathcal{M}^{an}_{g,G}$ of the stack $\mathcal{M}_{g,G}$ parametrizing families of compact Riemann surfaces of genus $g \geq 2$ with differentiable action of G as given by the fixed embedding of G in Γ_g. Moduli spaces of Riemann surfaces with symmetries and their related Teichmüller theory were studied systematically from a complex geometry point of view by Gonzaléz-Díez and Harvey [GDH]. They studied irreducible subvarieties of the moduli space of Riemann surfaces of genus $g \geq 2$ and their normalizations characterized by specifying a finite subgroup G of the mapping class group whose action on the surfaces is fixed geometrically. From their results it follows that in our notation, the complex analytification $\mathcal{M}^{an}_{g,G}$ of the stack $\mathcal{M}_{g,G}$ is basically the orbifold given as the quotient $T^G_g/\Gamma_{g,G}$, where T^G_g is the fixed point locus of the classical Teichmüller space T_g, representing the moduli functor for families of marked compact Riemann surfaces of genus g and where the modular group $\Gamma_{g,G}$ is the normalizer of the specified finite group G in the mapping class group Γ_g. Especially, it follows that the stack $\mathcal{M}^{an}_{g,G}$ is actually a normal complex analytic space, the normalization of a substack $\mathcal{M}'^{an}_{g,G}$ of the analytification $\mathcal{M}^{an}_g$ of the moduli stack $\mathcal{M}_g$.

The solution of the Nielsen realization problem by Kerkhoff [K] and others (cf. also the articles by Tromba [Tr] and Catanese[Ca]) implies that the Teichmüller space T^G_g is in fact a contractible topological space, which allows to determine the classical homotopy type of the stack $\mathcal{M}^{an}_{g,G}$ and using a general comparison theorem comparing the etale and classical homotopy types over the complex numbers essentially due to Artin-Mazur [AM] (cf. also Cox [C] and Friedlander [F]) we finally derive our main theorem.

Theorem. *There is a weak homotopy equivalence of pro-simplicial sets*

$$\{\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}\}^{\wedge}_{et} \simeq K(\Gamma_{g,G}, 1)^{\wedge}.$$

where $\Gamma_{g,G}$ *is the normalizer of the group* G *in the mapping class group* Γ_g *and* $^\wedge$ *denotes Artin-Mazur profinite completion.*

An interesting special case is given by the moduli stack $\mathcal{H}_g$ of families of hyperelliptic curves, i.e., families of algebraic curves with a hyperelliptic involution, which corresponds to the moduli stack $\mathcal{M}_{g,G}$ where G is simply the group $\mathbb{Z}/2$. This particular case was already mentioned by Oda [O].

The paper is organized as follows: In the first chapter we define homotopy types of Deligne-Mumford stacks using the machinery of Artin-Mazur [AM]. We also compare the etale and classical homotopy types in the algebraic and complex analytic context. In the second chapter we collect the necessary facts from Teichmüller theory for families of Riemann surfaces with symmetries, introduce the analytic stacks $\mathcal{M}^{an}_{g,G}$ and determine their classical homotopy types. Finally, in the third chapter we introduce the algebraic stacks $\mathcal{M}_{g,G}$ of families of algebraic curves with symmetries and determine their etale homotopy types using the results from the second and first chapter.

Acknowledgements. The first author likes to thank all the people at the Centre de Recerca Matemàtica, Bellaterra, Spain and the members of the Barcelona Algebraic Topology Group for much support and a wonderful stay, while holding a post-doc research grant as part of the EU research training network Modern Homotopy Theory.
Both authors like to thank Fabrizio Catanese for his interest and support

and the people of the former Algebraic Geometry Seminar in Göttingen, where discussion on this work was initiated.

1. Homotopy Types of Deligne-Mumford stacks

1.1. **Homotopy types for locally connected topoi.** We briefly recall the construction of homotopy types for locally connected topoi following Artin-Mazur [AM] and Cox [C]. We use the language of topoi here following Moerdijk [Mo] and Zoonekynd [Z].

Let $\mathfrak{E}$ be a topos and let $\emptyset$ be its initial object. An object X of $\mathfrak{E}$ is *connected*, if whenever $X = X_1 \coprod X_2$, either X_1 or X_2 is the initial object $\emptyset$. The topos $\mathfrak{E}$ is *locally connected* if every object of $\mathfrak{E}$ is a coproduct of connected objects.

Let Δ be the *category of simplices* i.e. the category whose objects are sets $[n] = \{0, 1, 2, \ldots n\}$ and whose morphisms are non-decreasing maps. Let also Δ_n be the full subcategory of Δ of all sets $[k]$ with $k \leq n$.

Let $\mathfrak{E}$ be a locally connected topos. Define π to be the *connected component functor*

$$\pi : \mathfrak{E} \to (Sets)$$

associating to any object X of $\mathfrak{E}$ its set $\pi(X)$ of connected components.

A *simplicial object* in $\mathfrak{E}$ is a functor $X_\bullet : \Delta^{op} \to \mathfrak{E}$. The category of all simplicial objects in the topos $\mathfrak{E}$ will be denoted by $\Delta^{op}\mathfrak{E}$.

The *restriction* or *n-truncation functor*

$$(-)^{(n)} : \Delta^{op}\mathfrak{E} \to \Delta_n^{op}\mathfrak{E}$$

has left and right adjoint functors, the *n-th skeleton* and *n-th coskeleton functor*

$$\mathrm{sk_n},\ \mathrm{cosk_n} : \Delta^{\mathrm{op}}_{\mathrm{n}}\mathfrak{E} \to \Delta^{\mathrm{op}}\mathfrak{E}.$$

A *hypercovering* of the topos $\mathfrak{E}$ is a simplicial object $U_\bullet$ such that the morphisms

$$U_0 \to *$$

$$U_{n+1} \to \mathrm{cosk_n(U_\bullet)_{n+1}}$$

are epic, where $*$ is the final object of $\mathfrak{E}$.

If S is a set and X an object of the topos $\mathfrak{E}$ define $T \otimes X := \coprod_{s\in S} X$. If $S_\bullet$ is a simplicial set and $X_\bullet$ a simplicial object of $\mathfrak{E}$ define

$$S_\bullet \otimes X_\bullet : \Delta^{op} \to \mathfrak{E}$$

to be the simplicial object given by $(S_\bullet \otimes X_\bullet)_n = S_n \otimes X_n$.

Let $\Delta[m] = \mathrm{Hom}_\Delta(-, [\mathrm{m}])$ be the standard m-simplicial set i.e., the functor $\Delta[m] : \Delta^{op} \to (Sets)$ represented by the set $[m]$.

Two morphisms $f, g : X_\bullet \to Y_\bullet$ are *strictly homotopic* if there is a *strict homotopy* $H : X_\bullet \otimes \Delta[1] \to Y_\bullet$ such that the following diagram is commutative

$$\begin{array}{ccc} X_\bullet = X_\bullet \otimes \Delta[0] & & \\ \downarrow{\scriptstyle id\otimes d^0} & \searrow^{f} & \\ X_\bullet \otimes \Delta[1] & \xrightarrow{H} & Y_\bullet \\ \uparrow{\scriptstyle id\otimes d^1} & \nearrow_{g} & \\ X_\bullet = X_\bullet \otimes \Delta[0] & & \end{array}$$

Two morphisms $f, g : X_\bullet \to Y_\bullet$ are *homotopic*, if they can be related by a chain of strict homotopies. *Homotopy* is the equivalence relation generated by strict homotopy.

Let $\mathcal{HR}(\mathfrak{E})$ be the *homotopy category of hypercoverings of* $\mathfrak{E}$, i.e., the category of hypercoverings of $\mathfrak{E}$ and their morphisms up to homotopy. It turns out that the opposite category $\mathcal{HR}(\mathfrak{E})^{op}$ is actually a filtering category. The proof of [AM], Corollary 8.13 applies verbatim.

Let also $\mathcal{H}$ be the category of simplicial sets $\Delta^{op}(Sets)$ and their morphisms up to homotopy, i.e., $\mathcal{H}$ is the *homotopy category of simplicial sets* $\mathcal{H}(\Delta^{op}(Sets))$. This category is actually equivalent to the homotopy category of CW-complexes (cf. [BK], VIII.3).

Further let $\mathrm{pro} - \mathcal{H}$ be the *category of pro-objects* in the category $\mathcal{H}$, i.e., the category of (contravariant) functors $X : \mathcal{I} \to \mathcal{H}$ from some filtering index category $\mathcal{I}$ to $\mathcal{H}$. We will write normally $X = \{X_i\}_{i\in\mathcal{I}}$ to indicate that we think of pro-objects X as inverse systems of objects of $\mathcal{H}$ (cf. [AM], Appendix).

Let $\mathfrak{E}$ be a locally connected topos. The connected component functor $\pi : \mathfrak{E} \to (Sets)$ induces a functor

$$\Delta^{op}\pi : \Delta^{op}\mathfrak{E} \to \Delta^{op}(Sets).$$

Passing to homotopy categeories and restricting to hypercoverings of $\mathfrak{E}$ gives a functor

$$\pi : \mathcal{HR}(\mathfrak{E}) \to \mathrm{pro} - \mathcal{H}.$$

Now we can define the homotopy type of a locally connected topos.

Definition 1.1. Let $\mathfrak{E}$ be a locally connected topos. The *homotopy type* of $\mathfrak{E}$ is given as the following pro-object in the homotopy category of simplicial sets:

$$\{\mathfrak{E}\} = \{\pi(U_\bullet)\}_{U_\bullet \in \mathcal{HR}(\mathfrak{E})}.$$

This construction is actually functorial with respect to morphisms of topoi, the associated functor $\{-\}$ is also called the *Verdier functor*.

If $\mathfrak{E}$ is a locally connected topos and x a point of the topos $\mathfrak{E}$, i.e., a morphism of topoi $x : \mathfrak{Set} \to \mathfrak{E}$, one can also define *homotopy groups* $\pi_n(\mathfrak{E}, x)$ for $n \geq 1$ following Artin-Mazur [AM] (cf. also [Mo] and [Z]). In general these homotopy groups $\pi_n(\mathfrak{E}, x)$ turn out to be pro-groups.

1.2. **Etale homotopy types of Deligne-Mumford stacks.** We will now define the etale homotopy type for the topos of sheaves on the small etale site of a Deligne-Mumford stack following Oda [O]. The source for the etale homotopy theory of simplicial schemes we will use here as well is the book of Friedlander [F] and the article by Cox [C]. For the definitions and properties of stacks we refer to the book of Laumon and Moret-Bailly [LMB], to the appendix of the article of Vistoli [V] and to the original article of Deligne-Mumford [DM]. For a good overview we also recommend the article of Fantechi [Fa].

Let us first recall that an *algebraic Deligne-Mumford stack* $\mathcal{S}$ is a contravariant functor from the big etale site of the category of schemes to the category of groupoids such that the diagonal morphism $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is quasi-compact, separated and representable by schemes and such that there exists a surjective etale 1-morphism $x : X \to \mathcal{S}$ where X is a scheme [DM]. From a slightly different viewpoint a Deligne-Mumford stack is just an etale groupoid in the category of schemes, i.e., a groupoid $R \rightrightarrows U$ in the category (Sch) of schemes such that the two structure morphisms $R \rightrightarrows U$ are etale morphisms (cf. [V], Appendix).

Let $\mathcal{S}$ be an algebraic Deligne-Mumford stack. Then we can consider its small etale site $\mathcal{S}_{et}$. The objects are etale 1-morphisms $x : X \to \mathcal{S}$, where X is a scheme, morphisms are morphisms over $\mathcal{S}$, i.e., commutative diagrams of 1-morphisms of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \mathcal{S} & \end{array}$$

and the coverings are the etale coverings of the schemes.

An important property of algebraic Deligne-Mumford stacks is the existence of fibre products $X \times_{\mathcal{S}} Y$ in $\mathcal{S}_{et}$ [DM].

We let $\mathfrak{E}_{et} = \mathfrak{Sh}(\mathcal{S}_{et})$ be the associated etale topos, i.e., the category of sheaves on the small etale site of the algebraic stack $\mathcal{S}$. It turns out that this topos $\mathfrak{E}_{et}$ is actually a locally connected topos (cf. [Z], 3.1.)

So we can define the etale homotopy type of an algebraic Deligne-Mumford stack.

Definition 1.2. Let $\mathcal{S}$ be an algebraic Deligne-Mumford stack. The *etale homotopy type* $\{\mathcal{S}\}_{et}$ of $\mathcal{S}$ is the pro-object in the homotopy category of simplicial sets given as

$$\{\mathcal{S}\}_{et} = \{\mathfrak{E}_{et}\}.$$

The Artin-Mazur homotopy groups $\pi_n(\mathfrak{E}_{et}, x)$ will be also denoted as $\pi_n^{et}(\mathcal{S}, x)$ and called the *etale homotopy groups* of the stack $\mathcal{S}$. For $n = 1$ this gives the etale fundamental group $\pi_1(\mathcal{S}, x)$ of $\mathcal{S}$, which is discussed in detail in Zoonekynd [Z].

The following generalization in the context of algebraic stacks of the homotopy descent theorem for simplicial schemes of Cox [C], Theorem IV.2 (cf. also [O], Theorem 3), is straightforward and allows to determine the etale homotopy type of a Deligne-Mumford stack directly from that of a hypercovering $X_\bullet$, i.e., a simplicial scheme over $\mathcal{S}$ which can often be constructed directly.

Theorem 1.3. *If $\mathcal{S}$ is an algebraic Deligne-Mumford stack with etale site $\mathcal{S}_{et}$ and $X_\bullet$ a simplicial scheme which is a hypercovering of $\mathfrak{E}_{et}$, then there is a weak homotopy equivalence of pro-simplicial sets*

$$\{\mathcal{S}\}_{et} \simeq \{X_\bullet\}_{et}.$$

where $\{X_\bullet\}_{et}$ is the etale homotopy type of the simplicial scheme $X_\bullet$.

The etale homotopy type $\{X_\bullet\}_{et}$ of the simplicial scheme $X_\bullet$ over $\mathcal{S}$ is defined here as in Cox [C], Chap. III, where its relation with the rigid etale homotopy type of Friedlander [F] is also explained.

The existence of such a hypercovering follows basically from the existence of a surjective, etale morphism $x : X \to \mathcal{S}$ where X is a scheme, which is intrinsic in the definition of a Deligne-Mumford stack $\mathcal{S}$. The hypercovering can then be constructed via itered fiber products along this morphism.

1.3. **Analytic Homotopy Types of Deligne-Mumford stacks.** We will consider now Deligne-Mumford stacks in the context of complex analytic spaces. An excellent overview on analytic stacks in general and analytification as well as GAGA type theorems can be found in Toen [T], Chap. 5. An *analytic Deligne-Mumford stack* is a contravariant functor from the category of complex analytic spaces to the category of groupoids such that the diagonal morphism $\Delta : \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ is finite and representable by complex analytic spaces and such that there exists a surjective etale 1-morphism $x : X \to \mathcal{T}$ where X is here a complex analytic space. Again, we could think of an analytic Deligne-mumford stack as an etale groupoid in the category $(AnSp)$ of complex analytic spaces.

For such an analytic stack $\mathcal{T}$ we can define its site of local isomorphisms $\mathcal{T}_{cl}$ given by local isomorphisms $x : X \to \mathcal{T}$, where X is a complex analytic space and morphisms are morphisms of analytic spaces over $\mathcal{T}$ and the coverings are given by families of local isomorphisms (cf. Mumford [M], and Cox [C], Chap. IV, §3).

We let $\mathfrak{E}_{cl} = \mathfrak{Sh}(\mathcal{T}_{cl})$ be the associated topos of local isomorphisms, i.e., the category of sheaves on the small site of local isomorphisms of the analytic stack $\mathcal{T}$. It is again locally connected and we can define the analytic homotopy type of $\mathcal{T}$.

Definition 1.4. Let $\mathcal{T}$ be a Deligne-Mumford analytic stack. The *analytic* or *classical homotopy type* $\{\mathcal{T}\}_{cl}$ of $\mathcal{T}$ is the pro-object in the homotopy category of simplicial sets given as

$$\{\mathcal{T}\}_{cl} = \{\mathfrak{E}_{cl}\}.$$

For an analytic Deligne-Mumford stack $\mathcal{T}$ we have actually a very explicit description of the classical homotopy type $\{\mathcal{T}\}_{cl}$ using hypercoverings and the underlying topological spaces. Namely, if $\mathcal{T}$ is an analytic Deligne-Mumford stack and $X_\bullet$ is a simplicial analytic space, which is also a hypercovering of the topos $\mathfrak{E}_{cl}$, then similarly as in the case af algebraic Deligne-Mumford stacks there is a weak homotopy equivalence

$$\{\mathcal{T}\}_{cl} \simeq \{X_\bullet\}_{cl}$$

where the classical homotopy type $\{X_\bullet\}_{cl}$ of the small site of local isomoprphisms of the hypercovering $X_\bullet$ is given as $\{X_\bullet\}_{cl} = \Delta\mathrm{Sin}(\mathrm{X}_\bullet)$ where $\mathrm{Sin}(\mathrm{X}_\bullet)$ is the bisimplicial set given in bidegree s, t by $\mathrm{Sin_t}(\mathrm{X_s})$ and Δ is the diagonal functor (cf. [C], Chap. IV, §3 and [F], chap. 8).

Using the canonical homotopy equivalence

$$\Delta\mathrm{Sin}(\mathrm{X}_\bullet) \xrightarrow{\simeq} \mathrm{Sin}(|\mathrm{X}_\bullet|)$$

where $|X_\bullet|$ is here the geometric realization of the simplicial space $X_\bullet$ we have that

$$\{\mathcal{T}\}_{cl} \simeq \mathrm{Sin}(|\mathrm{X}_\bullet|).$$

Therefore we could have defined the classical homotopy type of an analytic Deligne-Mumford stack directly using a complex analytic hypercovering. In [O] it is shown that this is independent of the actual choice of the hypercovering $X_\bullet$.

Let now $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ be an embedding of the algebraic closure of the rationals into the complex numbers, then for any algebraic Deligne-Mumford stack $\mathcal{S}$ over $\bar{\mathbb{Q}}$ let $\mathcal{S}^{an}$ be the associated analytic Deligne-Mumford stack (cf. [T], Chapitre 5). Similarly for any scheme X over $\bar{\mathbb{Q}}$, let X^{an} denote the complex analytic space associated with the $\mathbb{C}$-valued points $X(\mathbb{C})$ of X.

As the etale and analytic homotopy types are determined by a hypercovering of their respective topoi, it is important to have a good comparison theorem. We have the following general comparison theorem between etale and classical homotopy types of Deligne-Mumford stacks. We refer to Artin-Mazur [AM], §3 and Friedlander [F], Chap. 6 for the Artin-Mazur profinite completion functor and its properties.

Theorem 1.5. *Let $\mathcal{S}$ be an algebraic Deligne-Mumford stack over $\bar{\mathbb{Q}}$ and $X_\bullet$ a simplicial scheme which is of finite type over $\bar{\mathbb{Q}}$ and a hypercovering of the topos $\mathfrak{E}_{et} = \mathfrak{Sh}(\mathcal{S}_{et})$. If $X_\bullet^{an}$ is the associated simplicial complex analytic space of $X_\bullet$, then there is weak homotopy equivalence of pro-simplicial sets*

$$\{\mathcal{S}\}_{et}^{\wedge} \simeq \mathrm{Sin}(|\mathrm{X}_\bullet^{\mathrm{an}}|)^{\wedge}.$$

where $^\wedge$ denotes the Artin-Mazur profinite completion functor on the homotopy category of simplicial sets.

Proof. This follows directly from the above considerations and the general comparison theorem for homotopy types of simplicial schemes (cf. Cox [C], Theorem IV.8 or Friedlander [F], Theorem 8.4), because the etale homotopy type of a Deligne-Mumford stacks is completely determined by a hypercovering simplicial scheme. □

2. Moduli stacks of families of Riemann surfaces with symmetries and their homotopy types

2.1. **Recollections on classical Teichmüller theory.** We will first recall some notions and fundamental facts from the Teichmüller theory of families of Riemann surfaces (cf. also [O]).

Let C_g be a compact connected Riemann surface of genus g and Π_g be the abstract group with generators

$$\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$$

and relation $[\alpha_1, \beta_1]...[\alpha_g, \beta_g] = 1$. The abstract group Π_g is isomorphic to the fundamental group of the Riemann surface C_g. Recall that for the group $\mathrm{Out(\Pi_g)}$ of outer automorphisms of Π_g we have $\mathrm{Out(\Pi_g)} = \pi_0(\mathrm{Diff(\Sigma_g)})$, where Σ_g is a differentiable model for the Riemann surface of genus g.

The group $\pi_0(\mathrm{Diff(\Sigma_g)})$ acts properly discontinuously on the Teichmüller space $T_g = M(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$, where $M(\Sigma_g)$ is the moduli space of complex analytic structures on Σ_g and $\mathrm{Diff}_0(\Sigma_g)$ the topological group of all diffeomorphisms which are isotopic to the identity. The action of $\pi_0(\mathrm{Diff(\Sigma_g)})$ on $M(\Sigma_g)$ is defined in the following way. Given an element f of $\mathrm{Diff(\Sigma_g)}$ the induced element on $M(\Sigma_g)$ is given by $f^* : M(\Sigma_g) \to M(\Sigma_g)$, where for a complex analytic structure $X \in M(\Sigma_g)$, $f^*(X)$ is that complex analytic structure of Σ_g for which the map $f : (\Sigma_g, f^*(X)) \to (\Sigma_g, X)$ is either holomorphic or antiholomorphic, depending on whether f is orientation preserving or not.

We denote by $\mathrm{Diff^+(\Sigma_g)}$ the topological group of all orientation preserving diffeomorphisms of Σ_g and by $\Gamma_g = \pi_0(\mathrm{Diff^+(\Sigma_g)})$ the mapping class group of genus g.

Let now C_g be a compact connected Riemann surface of genus $g \geq 2$ and c_0 a point in C_g. Algebraically $H^2(\Pi_g, \mathbb{Z}) \cong \mathbb{Z}$ and we choose a generator e for $H^2(\Pi_g, \mathbb{Z})$.

We consider now any isomorphism $\phi : \pi_1(C_g, c_0) \xrightarrow{\cong} \Pi_g$ such that the induced isomorphism of cohomology groups

$$H^2(C_g, \mathbb{Z}) = H^2(\pi_1(C_g, c_0), \mathbb{Z}) \cong H^2(\Pi_g, \mathbb{Z})$$

maps the orientation class of C_g to the given generator e of $H^2(\Pi_g, \mathbb{Z})$. We say that two such isomorphisms ϕ_1 and ϕ_2 are equivalent if and only if there exists an inner automorphism θ of Π_g such that $\phi_2 = \theta \circ \phi_1$. An equivalence class $[\phi]$ of such isomorphisms is called a *marking* of the Riemann surface C_g and the pair $(C_g, [\phi])$ a *marked* Riemann surface of genus g.

In order to study the moduli problem for families of marked Riemann surfaces, we recall the following basic definitions. In what follows we will always assume that $g \geq 2$.

Definition 2.1. Let S be a complex analytic space. A *family of complex analytic curves of genus g over S* is a proper, smooth surjective holomorphic morphism $p : C \to S$ between complex analytic spaces such that for every point s in S the fibre $C_s = p^{-1}(s)$ is a compact connected Riemann surface of genus g.

The family version of a marked Riemann surfaces is given by introducing locally constant families of markings.

Definition 2.2. Let $p : C \to S$ be a family of complex analytic curves of genus g over S. A *locally constant family of markings* of p is an isomorphism up to conjugation of local systems of groups

$$\phi : \Pi_1(C/S) \cong \Pi_g \times S$$

where $\Pi_1(C/S)$ is the local system of the fundamental groups $\pi_1(C_s, c_0)$ of the fibers C_s and $\Pi_g \times S$ is the constant local system of groups over S with fibers Π_g such that ϕ induces an isomorphism $R^2p_*\mathbb{Z} \cong H^2(\Pi_g, \mathbb{Z}) \times S$ compatible with the orientations at each fiber C_s. A *family of marked complex analytic curves of genus g* is a pair $(p : C \to S, \phi)$, where $p : C \to S$ is a family of complex analytic curves over a complex analytic space S and ϕ a locally constant family of markings of p.

Definition 2.3. Let $(p : C \to S, \phi)$ and $(p' : C' \to S, \phi')$ be two families of marked complex analytic curves of genus g over S. An *isomorphism* between $(p : C \to S, \phi)$ and $(p' : C' \to S, \phi')$ is an S-isomorphism $\alpha : C \to C'$ which is compatible with the families ϕ and ϕ' of markings.

We can now describe the moduli problem for families of marked Riemann surfaces.

Definition 2.4. Let $\mathcal{T}_g : (AnSp) \to (Sets)$ be the contravariant functor from the category of complex analytic spaces to the category of sets, which associates to every complex analytic space S the set of isomorphism classes of families $(p : C \to S, \phi)$ of marked complex analytic curves of genus g over S and to every holomorphic morphism $f : S' \to S$ the map between isomorphism classes induced by the base change with f.

From classical Teichmüller theory we know that this functor is representable, i.e., the moduli problem has a fine solution (cf. for example [G1]).

Theorem 2.5. *The moduli problem for families of marked Riemann surfaces has a fine solution, i.e., the moduli functor $\mathcal{T}_g$ is representable by a complex analytic space T_g, called the Teichmüller space.*

2.2. **Teichmüller theory for families of marked Riemann surfaces with symmetries.** Now we will consider more generally the moduli problem for families of marked complex analytic curves of genus g with a given fixed subgroup of the automorphism group.

Given a Riemann surface C of genus g, let G be a subgroup of its automorphism group Aut(C). We can view G as a subgroup of the group of the outer automorphisms of Π_g, $\mathrm{Out}(\Pi_g) = \mathrm{Aut}(\Pi_g)/\mathrm{Inn}(\Pi_g)$, since every complex automorphism which acts as the identity on the first homology group of C must be the identity.

If we now look at a finite subgroup G of the mapping class group $\Gamma_g = \pi_0(\mathrm{Diff}^+(\Sigma_g))$, as a subgroup of $\mathrm{Out}(\Pi_g)$, we can consider the fixed point locus T_g^G of the action of G on the Teichmüller space T_g.

The complex analytic space T_g^G is a non empty complex submanifold of the Teichmüller space T_g and it is also contractible. This follows from the solution of the Nielsen realization problem first proven by Kerkhoff [K] and in the following form due to Tromba [Tr] and slightly generalized by Catanese [Ca].

Proposition 2.6. *Given a finite subgroup G of the mapping class group $\pi_0(\mathrm{Diff}(\Sigma_g))$, the fixed point locus T_g^G of the action of G on the Teichmüller space T_g is nonempty and connected, indeed diffeomorphic to an euclidean space.*

We observe that T_g^G parametrizes the isomorphism classes of Riemann surfaces of genus g with a holomorphic action of G, which is differentiably equivalent to the given one on the differentiable model Σ_g of the Riemann surface C.

The submanifold T_g^G is itself a Teichmüller space. In fact, let Σ_g be a differentiable model for a fixed Riemann surface C of genus g and let us consider the quotient Σ_g/G by the action of G. This quotient $\Sigma_{g'} := \Sigma_g/G$ is a Riemann surface of a genus g' and the projection $p : \Sigma_g \to \Sigma_{g'}$ is a ramified covering. We denote by $B = \{p_1, ..., p_r\} \subset \Sigma_{g'}$ the branch locus of p. Let $T_{g',r}$ now be the Teichmüller space of the punctured surface $\Sigma^* := \Sigma_{g'} - B$.

A point in T_g^G is just an isomorphism class $[(C, \phi, H)]$, where ϕ is the marking of the Riemann surface C and $H = \nu(G)$ for the given embedding $\nu : G \hookrightarrow \mathrm{Aut(C)}$ of G in Aut(C). For each such point $[(C, \phi, H)] \in T_g^G$, we denote by $C' = C/H$ the quotient under the action of G and by C^* the punctured surface obtained from C' by removing the branch locus of the projection $C \to C'$. The marking $\phi : \pi_1(C) \xrightarrow{\cong} \Pi_g = \pi_1(\Sigma_g)$ induces a marking $\phi^* : \pi_1(C^*) \to \pi_1(\Sigma^*)$, which defines a map

$$\Psi : T_g^G \to T_{g',r}, \quad [(C, \phi, H)] \mapsto [(C^*, \phi^*)].$$

We have the following Theorem (cf. [GDH], [H], [Kr]).

Theorem 2.7. *The spaces T_g^G and $T_{g',r}$ are biholomorphically equivalent via the mapping Ψ.*

We also like to remind that any finite group G actually has an embedding in the mapping class group of a certain Riemann surface.

Proposition 2.8. *Given a finite group G, there is an embedding of G in a mapping class group Γ_g of a Riemann surface C with genus g for some $g \geq 2$.*

Proof. Let us assume that the group G is generated by t elements $\delta_1, ..., \delta_t$ and choose an integer $g' \geq 2$ such that $g' \geq t$. Let us define a homomorphism ψ from the group $\Pi_{g'}$ to the free group $F_{g'}$ with g' generators $\gamma_1, ..., \gamma_{g'}$ defined by $\gamma_i := \psi(\alpha_i) = \psi(\beta_i)$ for $i = 1, ..., g'$.

Let us finally define an epimorphism $\varphi : F_{g'} \to G$ by sending γ_i to δ_i for $i = 1, ..., t$, and such that $\varphi(\gamma_j)$ for $j > t$ can be chosen whatever we want.

The composition $\varphi \circ \psi : \Pi_{g'} \to G$ is then an epimorphism and if we choose a point $[C', \phi]$ in the Teichmüller space $T_{g'}$, i.e., we choose a marking $\phi : \pi_1(C') \cong \Pi_{g'}$, we can consider the unramified Galois covering C associated to the kernel of the epimorphism $\varphi \circ \psi$. The genus of the Riemann surface C is then by the Hurwitz formula given as $g = |G|(g' - 1) + 1$ and the group G is contained in the group of automorphisms Aut(C) of C and we have an embedding of G in the mapping class group Γ_g. □

Let us consider an interesting class of examples.

Example. (cf. [Ca1]) Consider the set $B \subset \mathbb{P}^1_{\mathbb{C}}$ consisting of the three real points $B := \{-1, 0, 1\}$. Let us choose the following generators α, β, γ of the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} - B, 2)$:

α goes from 2 to $-1 - \epsilon$ along the real line and passes through $+\infty$, then it makes a full turn counterclockwise along the circumference with centre -1 and radius ϵ, thes goes back to 2 along the same way on the real line.

γ goes from 2 to $1 + \epsilon$ along the real line, then it makes a full turn counterclockwise along the circumference with centre $+1$ and radius ϵ, then goes back to 2 along the same way on the real line.

β goes from 2 to $+1+\epsilon$ along the real line, it makes a half turn counterclockwise around the circumference with centre $+1$ and radius ϵ, then it goes on the real line reaching $+\epsilon$, it makes a full turn counterclockwise around the circumference with centre 0 and radius ϵ, it goes back to $1-\epsilon$ along the real line, it makes a half turn clockwise around the circumference with centre $+1$ and radius ϵ and finally it proceeds along the real line returning to 2.

Now it is immediate to see that α and γ are free generators of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - B, 2)$ and $\alpha\beta\gamma = 1$. Therefore we have fixed an isomorphism of $\pi_1(\mathbb{P}^1_{\mathbb{C}} - B, 2)$ with the group $K := <\alpha,\ \beta,\ \gamma \mid \alpha\beta\gamma = 1>$.

Let now G be a finite group generated by two elements $a,\ b$. We have a surjection $\pi : K \to G$ given by $\pi(\alpha) = a$, $\pi(\beta) = b$, and we define $c := \pi(\gamma)$. Then it follows that $abc = 1$.

The choice of the isomorphism $\pi_1(\mathbb{P}^1_{\mathbb{C}} - B, 2) \cong K$ and of the epimorphism $\pi : K \to G$ determines a Galois covering $C \to \mathbb{P}^1_{\mathbb{C}}$ branched on B with Galois group G.

Therefore G is contained in Aut(C) and we have an embeddingof G in Γ_g where g is the genus of C. So we can consider the fixed point set T_g^G of the action of G on the Teichmüller space T_g and by theorem 2.7 we know that T_g^G is isomorphic to $T_{0,3}$ which consists of only one point.

Observe that if we set $m,\ n,\ p$ to be the periods of the respective elements $a,\ b,\ c$, then by the Hurwitz formula we have:

$$2g - 2 = |G|(1 - 1/m - 1/n - 1/p).$$

Now if we consider for instance the action on C of the cyclic group H generated by a, we find an embedding of H in Γ_g and consequently a fixed point locus T_g^H of the action of H on T_g.

An example of such a curve C is the Fermat curve of degree n in the projective space $\mathbb{P}^2_{\mathbb{C}}$.

$C = \{(z_0, z_1, z_2) \mid z_0^n + z_1^n + z_2^n = 0\}$ is a Galois cover of the projective line $\mathbb{P}^1_{\mathbb{C}} = \{(x_0, x_1, x_2) \mid x_0 + x_1 + x_2 = 0\}$ under the map $\pi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$, given by $\pi(z_0, z_1, z_2) = (z_0^n, z_1^n, z_2^n)$. The map π is ramified in three points and the Galois group of the covering is the group $G = (\mathbb{Z}/n)^2$ of diagonal projectivities with entries the n-th roots of unity.

As it is observed in [Ca1], the curve C can be seen as a covering of $\mathbb{P}^1_{\mathbb{C}}$ branched in three points in two different ways. In fact, we can consider the quotient of C by the full group of automorphisms of C, which is a semidirect product of the normal subgroup G by the symmetric group exchanging the three coordinates.

This can be easily seen directly, but we also notice that if G and G' are two finite groups with $G \subset G' \subset \Gamma_g$, we have an inclusion of the fixed point loci in the Teichmüller space T_g as $T_g^{G'} \subset T_g^G$.

Therefore in the case of the Fermat curve, we have $G \subset \mathrm{Aut(C)} \subset \Gamma_\mathrm{g}$, where $g = (n-1)(n-2)/2$ is the genus of C and thus $T_g^{\mathrm{Aut(C)}} \subset T_g^G \cong T_{0,3}$, which is only a point, i.e., the curve C with the chosen marking, and since they are both non empty we have $T_g^{\mathrm{Aut(C)}} = T_g^G \cong T_{0,3}$. So $C/\mathrm{Aut(C)}$ is isomorphic to $\mathbb{P}^1_\mathbb{C}$ and the map $C \to C/\mathrm{Aut(C)}$ is branched in three points.

Now we introduce the moduli functor for families of marked complex analytic curves of genus $g \geq 2$ with a given fixed finite subgroup of the automorphism group as symmetry group.

Let G be a finite group and let us assume that there exists an embedding of G in the mapping class group Γ_g, i.e., there is an injection $i : G \hookrightarrow \Gamma_g$.

Definition 2.9. A *family of marked complex analytic curves of genus g with symmetry group G* is a triple $(p : C \to S, \phi, \nu : G \to \mathrm{Aut_S(C)})$, where $p : C \to S$ is a family of complex analytic curves of genus g over a complex analytic space S, ϕ a locally constant family of markings of p and $\nu : G \to \mathrm{Aut_S(C)}$ a group homomorphism injective in the fibres, with G acting in such a way that for all points s of S, if we denote by $\nu_s(G)_*$ the induced action of $\nu(G)$ on the fundamental group of the fibre C_s, and by $\psi : \mathrm{Aut}(\Pi_\mathrm{g}) \to \mathrm{Out}(\Pi_\mathrm{g})$ the natural projection, we have $\psi(\phi_s \circ \nu_s(G)_* \circ \phi_s^{-1}) = i(G) \subset \Gamma_g$.

To be more precise, for every element $h \in G$ and every point s of S, $\nu_s(h)$ yields an automorphism of the fibre C_s, and therefore we have an isomorphism $\nu_s(h)_* : \pi_1(C_s, c_0) \xrightarrow{\cong} \pi_1(C_s, \nu_s(h)(c_0))$, for a basepoint c_0. Let us fix a path γ between $\nu_s(h)(c_0)$ and c_0 and the corresponding isomorphism of fundamental groups $f_\gamma : \pi_1(C_s, \nu_s(h)(c_0)) \xrightarrow{\cong} \pi_1(C_s, c_0)$ sending an element δ to $\gamma^{-1}\delta\gamma$. Then if we take a representative ϕ_s of a marking ϕ, say $\phi_s : \pi_1(C_s, c_0) \xrightarrow{\cong} \Pi_g$, we require in the definition that $\psi(\phi_s \circ f_\gamma \circ \nu_s(h)_* \circ \phi_s^{-1}) = i(h')$ for some element $h' \in G$. Clearly this does not depend on the chosen representative of the marking. Furthermore if we choose another path γ' between $\nu_s(h)(c_0)$ and c_0 and the corresponding isomorphism of fundamental groups $f_{\gamma'}$, we see that

$$(\phi_s \circ f_{\gamma'} \circ \nu_s(h)_* \circ \phi_s^{-1}) \circ (\phi_s \circ f_\gamma \circ \nu_s(h)_* \circ \phi_s^{-1})^{-1} = \phi_s \circ (f_{\gamma'} \circ f_\gamma^{-1}) \circ \phi_s^{-1}$$

is in $\mathrm{Inn}(\Pi_\mathrm{g})$, and so everything is well defined.

We are now going to define the moduli functor $\mathcal{T}_g^G$ from the category of analytic spaces to the category of sets as follows.

Definition 2.10. Let $\mathcal{T}_g^G : (AnSp) \to (Sets)$ be the contravariant functor from the category of complex analytic spaces to the category of sets, which associates to every complex analytic space S the set of isomorphism classes of families $(p : C \to S, \phi, \nu : G \to \mathrm{Aut_S(C)})$ of marked complex analytic curves with symmetry group G over S, where isomorphism between the triples $(p, \phi, \nu : G \to \mathrm{Aut_S(C)})$ and $(p', \phi', \nu' : G \to \mathrm{Aut_S(C')})$ means isomorphism of the couples (p, ϕ) and (p', ϕ') (we do not require them to be equivariant). To every holomorphic morphism $f : S' \to S$ the functor $\mathcal{T}_g^G$ associates the map between isomorphism classes induced by the base change with f.

The fundamental statement is now, that the moduli problem for families of marked complex analytic curves of genus g with symmetry group G has a fine solution.

Theorem 2.11. *The moduli functor $\mathcal{T}_g^G$ is representable by the complex analytic space T_g^G.*

Proof. The fixed point locus T_g^G is a complex submanifold of T_g and we claim that if $\alpha : \mathcal{U}_g \to T_g$ is the universal family on T_g and $j : T_g^G \hookrightarrow T_g$ the embedding of T_g^G in T_g, then the fibre product $T_g^G \times_{T_g} \mathcal{U}_g =: \mathcal{U}_g^G$ defines a universal family on T_g^G.

Since $j : T_g^G \hookrightarrow T_g$ is an embedding, then also the induced map $f : \mathcal{U}_g^G \hookrightarrow \mathcal{U}_g$ is an embedding.

Now, given a family $p : C \to S$ together with a marking ϕ and a homomorphism $\nu : G \to \mathrm{Aut_S}(\mathcal{C})$ representing an isomorphism class (p, ϕ, ν) in $\mathcal{T}_g^G(S)$, we have a map $\gamma : S \to T_g$ obtained by forgetting the action of the group G. As γ factors through T_g^G via a map $h : S \to T_g^G$ we have $\gamma = j \circ h$.

Then, since $\alpha : \mathcal{U}_g \to T_g$ is a universal family, we know that C is given as the the fibre product $S \times_{T_g} \mathcal{U}_g$, and we get a cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{\beta} & \mathcal{U}_g \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \alpha} \\ S & \xrightarrow{\gamma} & T_g \end{array}$$

where β factors through a map $k : C \to \mathcal{U}_g^G$. Also the following diagram

$$\begin{CD} \mathcal{U}_g^G @>{f}>> \mathcal{U}_g \\ @VVV @VV{\alpha}V \\ T_g^G @>{j}>> T_g \end{CD}$$

is cartesian and so we get a big commutative diagram

$$\begin{CD} C @>{k}>> \mathcal{U}_g^G @>{f}>> \mathcal{U}_g \\ @V{p}VV @VVV @VV{\alpha}V \\ S @>{h}>> T_g^G @>{j}>> T_g \end{CD}$$

such that the diagram on the right is cartesian and the one obtained by taking the composition maps is also cartesian. Since the map $f : \mathcal{U}_g^G \hookrightarrow \mathcal{U}_g$ is injective, it is immediate to verify the universal property in order to show that also the diagram on the left

$$\begin{CD} C @>{k}>> \mathcal{U}_g^G \\ @V{p}VV @VVV \\ S @>{h}>> T_g^G \end{CD}$$

is cartesian.

The uniqueness of the map h is a direct consequence of the universality of the family $\mathcal{U}_g \to T_g$.

So we have proven that $\mathcal{U}_g^G \to T_g^G$ is a universal family and therefore the moduli functor $\mathcal{T}_g^G$ is representable by the complex analytic space T_g^G. $\square$

Because of the representability of the moduli functor $\mathcal{T}_g^G$, the identity morphism $id : T_g^G \to T_g^G$ defines the universal family of marked complex analytic curves with symmetry group G over T_g^G, which, as the proof above shows, is given as a triple $(\mathcal{U}_g^G \to T_g^G, \Phi, \nu : G \to \mathrm{Aut}_{\mathrm{T}_g^G}(\mathcal{U}_g^G))$, where T_g^G is the fixed point set of the classical Teichmüller space $\mathcal{T}_g$ and $\mathcal{U}_g^G$ the fibre product $T_g^G \times_{T_g} \mathcal{U}_g$ along the classical universal family $\mathcal{U}_g \to T_g$ given by the identity morphism $id : T_g \to T_g$ via the representability of the moduli functor $\mathcal{T}_g$ (cf. Theorem 2.5).

2.3. **The moduli stack of complex analytic curves with symmetries and its analytic homotopy type.** We will now study the moduli stack of families of complex analytic curves of genus g with a fixed finite group G of automorphisms and determine its analytic homotopy type as introduced in the first chapter. We will always assume here again that $g \geq 2$. Forgetting markings we get the concept of families of complex analytic curves with symmetries.

Definition 2.12. Let S be a complex analytic space and G a finite group with fixed embedding in the mapping class group Γ_g. A *family of complex analytic curves of genus g over S with symmetry group G* is a pair $(p : C \to S, \nu : G \to \mathrm{Aut_S(C)})$, where $p : C \to S$ is a family of complex analytic curves of genus g over S and $\nu : G \to \mathrm{Aut_S(C)}$ is a group homomorphism such that the group $\nu(G)$ of S-automorphisms is acting faithfully on the fibres C_s and this action is differentiably equivalent to the one given by the embedding of G in Γ_g.

We define now the analytic moduli stack $\mathcal{M}^{an}_{g,G}$ of *families of complex analytic curves of genus g with symmetry group G*.

Definition 2.13. Let G be a finite group embedded in the mapping class group Γ_g. The moduli stack $\mathcal{M}^{an}_{g,G}$ is the category fibred in groupoids over the category (AnSp) of complex analytic spaces defined by its groupoid of sections $\mathcal{M}^{an}_{g,G}(S)$ as follows: for a complex analytic space S the objects of $\mathcal{M}^{an}_{g,G}(S)$ are families $(p : C \to S, \nu : G \to \mathrm{Aut_S(C)})$ of complex analytic curves of genus g over S with symmetry group G and the morphisms are G-equivariant S-isomorphisms, i.e., morphisms between $(p : C \to S,\ \nu : G \to \mathrm{Aut_S(C)})$ and $(p' : C' \to S,\ \nu' : G \to \mathrm{Aut_S(C')})$ are S-isomorphisms $\phi : C \to C'$ such that $\nu(G) = \phi^{-1} \circ \nu'(G) \circ \phi$.

Consider now the universal family $(\mathcal{U}^G_g \to T^G_g, \Phi, \nu : G \to \mathrm{Aut}_{\mathrm{T^G_g}}(\mathcal{U}^{\mathrm{G}}_{\mathrm{g}}))$ on T^G_g, where Φ is the universal marking, then by forgetting the universal marking, this canonical data defines an object in the category of sections $\mathcal{M}^{an}_{g,G}(T^G_g)$ of the stack $\mathcal{M}^{an}_{g,G}$.

Considering the complex analytic space T^G_g as a stack, i.e., as the stack associated to the complex analytic space T^G_g, and forgetting the universal marking induces a morphism of stacks $\pi : T^G_g \to \mathcal{M}^{an}_{g,G}$. It is not hard to see from this that $\mathcal{M}^{an}_{g,G}$ is actually an analytic Deligne-Mumford stack as introduced in the first chapter. As we shall see in the next chapter $\mathcal{M}^{an}_{g,G}$ is in fact the complex analytification of an algebraic Deligne-Mumford stack.

Now we determine the fibre product $Isom(\pi,\pi) := T_g^G \times_{\mathcal{M}_{g,G}^{an}} T_g^G$ of the morphism π of stacks with itself. We have the following diagram

$$\begin{array}{ccc} Isom(\pi,\pi) = T_g^G \times_{\mathcal{M}_{g,G}^{an}} T_g^G & \longrightarrow & T_g^G \\ \downarrow & & \downarrow \pi \\ T_g^G & \xrightarrow{\pi} & \mathcal{M}_{g,G}^{an} \end{array}$$

Both projections from $Isom(\pi,\pi)$ to T_g^G are local isomorphisms for the classical topology (cf. [M]).

For each point $x \in T_g^G$ the data over x has no non trivial automorphism, therefore the map $Isom(\pi,\pi) \to T_g^G \times T_g^G$ is actually an immersion.

We know that two points $x, y \in T_g$ define the same object over the stack $\mathcal{M}_g^{an}$ if and only if the fibres C_x and C_y are isomorphic curves and for the markings Φ_x of C_x and Φ_y of C_y we have that $\Phi_y = \theta \circ \Phi_x$, where $[\theta] := \psi(\theta)$ is an element of the mapping class group $\Gamma_g = A(\Pi_g)/\mathrm{Inn}(\Pi_g)$. (cf. [O]). Here $A(\Pi_g)$ is the subgroup of $\mathrm{Aut}(\Pi_g)$ given by the automorphisms inducing the identity in the second cohomology $H^2(\Pi_g, \mathbb{Z}) \cong \mathbb{Z}$.

If now two points $x, y \in T_g^G$ i.e., two isomorphism classes of marked complex analytic curves of genus g with symmetry group G, define the same object over the stack $\mathcal{M}_{g,G}^{an}$, there must exists an isomorphism of the curves $h : C_x \xrightarrow{\cong} C_y$ which conjugates $\nu_x(G)$ and $\nu_y(G)$, i.e., $h^{-1}\nu_y(G)h = \nu_x(G)$, where $\nu_x : G \to \mathrm{Aut}(C_x)$ and $\nu_y : G \to \mathrm{Aut}(C_y)$ are the induced actions of G on the fibres C_x and C_y and furthermore $\psi(\Phi_x(\nu_x(G))_*\Phi_x^{-1}) = i(G)$ and $\psi(\Phi_y(\nu_y(G))_*\Phi_y^{-1}) = i(G)$, where $i : G \to \Gamma_g$ is the fixed embedding of the group G in the mapping class group Γ_g.

Since $\theta = \Phi_y \circ h_* \circ \Phi_x^{-1}$, we get immediately

$$\begin{aligned} \theta\psi^{-1}(i(G))\theta^{-1} &= \Phi_y h_* \Phi_x^{-1} \psi^{-1}(i(G)) \Phi_x h_*^{-1} \Phi_y^{-1} \\ &= \Phi_y h_* (\nu_x(G))_* h_*^{-1} \Phi_y^{-1} = \Phi_y(\nu_y(G))_* \Phi_y^{-1} = \psi^{-1}(i(G)). \end{aligned}$$

And so we finally have

$$[\theta] i(G) [\theta]^{-1} = i(G).$$

Therefore we have proven that for each point $x \in T_g^G$, and if we identify G with $i(G) \subset \Gamma_g$, the fibre $p^{-1}(x)$ of the projection $p : Isom(\pi,\pi) \to T_g^G$ is isomorphic to the cartesian product $T_g^G \times \Gamma_{g,G}$, where the group $\Gamma_{g,G} := \{\alpha \in \Gamma_g \mid \alpha G \alpha^{-1} = G,\}$ is the normalizer of the finite group G in the mapping class group Γ_g. In fact, the isomorphism is explicitely given as

$$p^{-1}(x) \xrightarrow{\cong} T_g^G \times \Gamma_{g,G},$$
$$(x,y) \mapsto (x, [\Phi_y \circ \Phi_x^{-1}] = [\theta]).$$

By induction on the number of factors, we easily get an isomorphism

$$T_g^G \times_{\mathcal{M}_{g,G}^{an}} T_g^G \ldots \times_{\mathcal{M}_{g,G}^{an}} T_g^G \cong T_g^G \times \Gamma_{g,G} \times \ldots \times \Gamma_{g,G}.$$

Now we determine the analytic homotopy type of the moduli stack $\mathcal{M}_{g,G}^{an}$ of complex analytic curves with symmetries.

Proposition 2.14. *Let* $\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)$ *be the Čech nerve associated to the locally isomorphic surjective covering morphism* $\pi : T_g^G \to \mathcal{M}_{g,G}^{an}$. *Then its geometric realization* $|\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)|$ *is homotopy equivalent to the classifying space* $|B\Gamma_{g,G}|$ *of the group* $\Gamma_{g,G}$.

Proof. From the induction argument above, we see first that the m-simplex $\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)_m$ of the Čech nerve $\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)$ is given by the $(m+1)$-tuple fiber product of copies of T_g^G over $\mathcal{M}_{g,G}^{an}$, which as we showed is isomorphic to $T_g^G \times \Gamma_{g,G} \times \Gamma_{g,G} \times \cdots \Gamma_{g,G}$. So by definition, we get

$$\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G) \cong \mathrm{T}_g^G \times \mathrm{B}\Gamma_{g,G},$$

where $B\Gamma_{g,G}$ is the classifying simplicial set of the discrete group $\Gamma_{g,G}$. Therefore after geometric realization we have a homeomorphism of topological spaces

$$|\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)| \cong \mathrm{T}_g^G \times |\mathrm{B}\Gamma_{g,G}|.$$

From Proposition 2.6 we know that the space T_g^G as an euclidean space is indeed contractible, so we finally get the conclusion of the proposition. □

The following corollary is therefore an immediate consequence from the considerations about analytic homotopy types of the first chapter as the Čech nerve $\mathrm{cosk}_0^{\mathcal{M}_{g,G}^{an}}(\mathrm{T}_g^G)$ is a simplicial analytic space and a hypercovering of the topos $\mathfrak{Sh}(\mathcal{M}_{g,G}^{an}{}_{cl})$.

Corollary 2.15. *There is a weak homotopy equivalence*

$$\{\mathcal{M}_{g,G}^{an}\}_{cl} \simeq B\Gamma_{g,G}.$$

Let us introduce another analytic moduli stack $\mathcal{M}'^{an}_{g,G}$, which is closely related to the moduli stack $\mathcal{M}_{g,G}^{an}$ (cf. also [GDH]).

Definition 2.16. Let G be a finite group embedded in the mapping class group Γ_g. The moduli stack $\mathcal{M'}^{an}_{g,G}$ is the category fibred in groupoids over the category $(AnSp)$ of complex analytic spaces defined by its groupoid of sections $\mathcal{M'}^{an}_{g,G}(S)$ as follows: for a complex analytic space S the objects of $\mathcal{M'}^{an}_{g,G}(S)$ are families $(p : C \to S, \nu : G \to \mathrm{Aut_S(C)})$ of complex analytic curves of genus g over S with symmetry group G and the morphisms are S-isomorphisms, not necessarily equivariant, i.e., morphisms between $(p : C \to S,\ \nu : G \to \mathrm{Aut_S(C)})$ and $(p' : C' \to S,\ \nu' : G \to \mathrm{Aut_S(C')})$ are just S-isomorphisms $\phi : C \to C'$.

While the moduli stack $\mathcal{M}^{an}_{g,G}(S)$ is not an analytic substack of $\mathcal{M}^{an}_g$, the moduli stack $\mathcal{M'}^{an}_{g,G}$ is and by forgetting the action we get a natural morphism of stacks $f : \mathcal{M}^{an}_{g,G} \to \mathcal{M'}^{an}_{g,G}$. We also have the following commutative diagram:

$$\begin{array}{ccccc} T_g^G & \longrightarrow & & \longrightarrow & T_g \\ \downarrow{\scriptstyle \pi} & & & & \downarrow \\ \mathcal{M}^{an}_{g,G} & \xrightarrow{f} & \mathcal{M'}^{an}_{g,G} & \longrightarrow & \mathcal{M}_g \end{array}$$

where the map π is the morphism of stacks as defined above.

Similarly as above we can see that two points $x, y \in T_g^G$, i.e., two isomorphism classes of marked complex analytic curves of genus g with symmetry group G $(C_x, \Phi_x, \nu_x : G \hookrightarrow \mathrm{Aut(C_x)})$ and $(C_y, \Phi_y, \nu_y : G \hookrightarrow \mathrm{Aut(C_y)})$ give the same image in the stack $\mathcal{M'}^{an}_{g,G}$ if and only if $C_x \cong C_y$ and $\Phi_y = \theta \circ \Phi_x$ where $[\theta] \in \Gamma_g$. But we also must have that $(\Phi_x)^{-1} \circ G \circ \Phi_x = (\nu_x(G))_* =: K$ and $(\Phi_y)^{-1} \circ G \circ \Phi_y = (\nu_y(G))_* = (\theta \circ \Phi_x)^{-1} \circ G \circ (\theta \circ \Phi_x) =: K'$, where K, K' are subgroups of $\mathrm{Aut(C_x)}$, since we have identified C_x with C_y. Now if (C_x, Φ_x, ν_x) and (C_y, Φ_y, ν_y) give different images in $\mathcal{M}^{an}_{g,G}$, then $[\theta] \notin \Gamma_{g,G}$ and so $[\theta]^{-1} G [\theta] \neq G$, hence $K \neq K'$ are two different subgroups of $\mathrm{Aut(C_x)}$.

This shows therefore that the moduli stacks $\mathcal{M}^{an}_{g,G}$ and $\mathcal{M'}^{an}_{g,G}$ are different if and only if there exists a Riemann surface C of genus g with symmetry group G whose automorphism group contains two subgroups which are conjugated topologically, but not holomorphically.

From the results of González-Díez and Harvey [GDH] it also follows that $\mathcal{M}^{an}_{g,G}$ can actually be described as an orbifold given as the quotient $T_g^G/\Gamma_{g,G}$, where T_g^G is the fixed point locus of the classical Teichmüller space T_g and where the modular group $\Gamma_{g,G}$ is the normalizer of the embedded finite group G in the mapping class group Γ_g. Actually $\mathcal{M}_{g,G}^{an}$ is a normal complex analytic space, the normalization of the irreducible subvariety $\mathcal{M'}_{g,G}^{an}$ of $\mathcal{M}_g^{an}$.

Example 2.17. If we take the group $G = \mathbb{Z}/2$ and C is a hyperelliptic complex analytic curve with hyperelliptic involution $\tau : C \to C$, then τ is the only automorphism of the curve C of order two such that the quotient $C/ < \tau >$ is isomorphic to the complex projective line $\mathbb{P}^1(\mathbb{C})$, therefore we have $\mathcal{M}_{g,<\tau>}^{an} = \mathcal{M'}_{g,<\tau>}^{an}$ and the moduli stack $\mathcal{M}_{g,<\tau>}^{an}$ of hyperelliptic complex analytic curves of genus g is actually an analytic substack of the moduli stack $\mathcal{M}_g^{an}$.

3. Moduli stacks of families of algebraic curves with symmetries and their etale homotopy types

3.1. **The Deligne-Mumford stack of algebraic curves with symmetries.** In this section we will study moduli stacks of algebraic curves with symmetries. Let (Sch) here always be the category of schemes over $\mathbb{Q}$. First we recall some basic notions (cf. [MFK], [DM]).

Definition 3.1. Let $g \geq 2$ and S be a scheme (over $\mathbb{Q}$). A *family of algebraic curves of genus g over S* or *algebraic curve over S* is a morphism $p : C \to S$ of schemes such that p is proper and smooth and the geometric fibers C_s of p are 1-dimensional smooth connected schemes of genus g. We will also denote an algebraic curve over S simply by C/S.

Here geometric fibers means as usual scheme theoretic fibers over points, i.e for a point s of S we define $C_s := C \times_S \mathrm{Spec}\,\mathrm{k(s)}$, with $k(s) = \mathcal{O}_{S,s}/\mathfrak{m}_\mathfrak{s}$. And the genus is given cohomologically as $g = \dim H^1(C_s, \mathcal{O}_{C_s})$. Over the complex numbers $\mathbb{C}$ geometric fibers are just complex smooth projective curves and morphisms are regular maps and in the associated complex analytic case geometric fibers are just compact Riemann surfaces of topological genus g.

Definition 3.2. Let S be a scheme (over $\mathbb{Q}$) and G a finite group. An *action of the group G* on a family of algebraic curves $p : C \to S$ over S is a morphism of group schemes over S

$$\mu : G_S \to \mathrm{Aut_S(C)}$$

of the constant group scheme $G_S := G$ over S to $\mathrm{Aut_S(C)}$, which is injective in the fibres C_s.

Let us now again consider a finite group G with a fixed embedding $i : G \hookrightarrow \Gamma_g$ in the mapping class group Γ_g. We define now the moduli stack $\mathcal{M}_{g,G}$ of *algebraic curves with symmetries*.

Definition 3.3. The moduli stack $\mathcal{M}_{g,G}$ is the category fibred in groupoids over the category (Sch) of schemes over $\mathbb{Q}$ defined by its groupoid of sections $\mathcal{M}_{g,G}(S)$ as follows: for a scheme S over $\mathbb{Q}$ the objects of $\mathcal{M}_{g,G}(S)$ are families of algebraic curves $p : C \to S$ over S such that the fibres are connected smooth algebraic curves of genus $g \geq 2$ endowed with an action of the group G on $p : C \to S$ satisfying the following property. Let C_s be the fibre over a point s of S. It is a scheme over $\mathrm{Spec}(\mathbb{Q})$ and we let C_s^{an} to be the complex analytic space associated with the $\mathbb{C}$-valued points $C_s(\mathbb{C})$ of C_s. We require that the differentiable action of G on C_s^{an} is the one given by the embedding of G in Γ_g that we have fixed. The morphisms are G-equivariant S-isomorphisms, i.e., morphisms between $(p : C \to S, \nu : G \to \mathrm{Aut_S(C)})$ and $(p' : C' \to S,\ \nu' : G \to \mathrm{Aut_S(C')})$ are S-isomorphisms $\phi : C \to C'$ such that $\nu(G) = \phi^{-1} \circ \nu'(G) \circ \phi$.

We will also consider the following moduli stack, studied by Tufféry [Tu1], [Tu2] which we will compare with $\mathcal{M}_{g,G}$ via forgetting the embedding of the finite group G in the mapping class group Γ_g.

Definition 3.4. Let G be a finite group. The moduli stack $\mathcal{M}_g[G]$ is the category fibred in groupoids over the category (Sch) of schemes over $\mathbb{Q}$ defined by its groupoid of sections $\mathcal{M}_g[G](S)$ as follows: for a scheme S over $\mathbb{Q}$ the objects of $\mathcal{M}_g[G](S)$ are families of algebraic curves $p : C \to S$ over S with a group action of the group G and the morphisms are the G-equivariant S-isomorphisms.

The moduli stack $\mathcal{M}_{g,G}$ is a substack of $\mathcal{M}_g[G]$ and its main property is the following.

Proposition 3.5. *The moduli stack $\mathcal{M}_{g,G}$ is a Deligne-Mumford stack.*

Proof. Let $\mathcal{M}_g \otimes \mathbb{Q}$ be the moduli stack which represents the restriction of the Deligne-Mumford stack $\mathcal{M}_g$ of all families of algebraic curves of genus $g \geq 2$ (cf. [DM]) to the subcategory of schemes over $\mathbb{Q}$.

We claim that the morphism of stacks $\mathcal{M}_g[G] \to \mathcal{M}_g \otimes \mathbb{Q}$ induced by forgetting the action of the group G is finite and representable (cf. [Tu2]), thus the stack $\mathcal{M}_g[G]$ is a Deligne-Mumford stack and the same holds then for $\mathcal{M}_{g,G}$ as being a substack of $\mathcal{M}_g[G]$.

Recall that a morphism $\mathcal{N} \to \mathcal{M}$ of stacks is said to be representable if for every morphism $Y \to \mathcal{M}$ of stacks, where Y is (the stack associated to) a scheme, the fibre product $Y \times_{\mathcal{M}} \mathcal{N}$ is isomorphic to (the stack associated to) a scheme.

Therefore, in order to prove that the morphism of stacks $g : \mathcal{M}_g[G] \to \mathcal{M}_g \otimes \mathbb{Q}$ given by forgetting the action of G is representable, we must prove that given a morphism $f : Y \to \mathcal{M}_g \otimes \mathbb{Q}$, with Y a scheme, any object in the fibre product $Z = Y \times_{\mathcal{M}_g \otimes \mathbb{Q}} \mathcal{M}_g[G]$ has no nontrivial automorphisms.

The objects of Z are triples (x, z, α) where $\alpha : f(x) \to g(z)$ is a morphism in a fibre of $\mathcal{M}_g \otimes \mathbb{Q}$. A morphism $(x, z, \alpha) \to (x', z', \alpha')$ is a pair of morphisms $(\beta_1 : x \to x'; \beta_2 : z \to z')$ in the fibres of Y and $\mathcal{M}_g[G]$ respectively such that $g(\beta_2) \circ \alpha = \alpha' \circ f(\beta_1) : f(x) \to g(z')$.

So, assume that an object (x, z, α) of Z has an automorphism $(\beta_1 : x \to x; \beta_2 : z \to z)$ as above, then clearly $\beta_1 = id$, and we must have $g(\beta_2) \circ \alpha = \alpha$, so $g(\beta_2) = id$. Now, since z is a family $C \to S$ of curves with a G-action and β_2 is an equivariant automorphism of $C \to S$, we have that $g(\beta_2) = \beta_2$, because g only forgets the action of G. Thus we have shown $\beta_2 = g(\beta_2) = id$ and (x, z, α) has no nontrivial automorphism.

This proves that the stack Z is indeed a scheme and the morphism of stacks $g : \mathcal{M}_g[G] \to \mathcal{M}_g \otimes \mathbb{Q}$ is a representable morphism. □

3.2. **The etale homotopy type of the moduli stack of algebraic curves with symmetries.** Now we can prove our main theorem on the etale homotopy type of the moduli stack of families of algebraic curves of genus g with a prescribed symmetry group using the above considerations and the general comparison theorem for etale homotopy types of Cox and Friedlander as stated in the first section.

Let G be again a finite group with an embedding in the mapping class group Γ_g of genus g. Let now $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$ be the restriction of the moduli stack $\mathcal{M}_{g,G}$ to the subcategory of schemes over $\bar{\mathbb{Q}}$. Fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ of the algebraic closure of the rationals in the complex numbers. The analytic stack $\mathcal{M}^{an}_{g,G}$ is precisely the complex analytification $(\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}})^{an}$ of the Deligne-Mumford stack $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$ and we can now determine the etale homotopy type of $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$.

Theorem 3.6. *There is a weak homotopy equivalence of pro-simplicial sets*

$$\{\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}\}^{\wedge}_{et} \simeq K(\Gamma_{g,G}, 1)^{\wedge}.$$

where $\Gamma_{g,G} = \mathrm{Norm}_{\Gamma_g}(\mathrm{G})$ *is the normalizer of the group* G *in the mapping class group* Γ_g.

Proof. The moduli stack $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$ is a Deligne-Mumford stack, so there exists a scheme X of finite type over $\bar{\mathbb{Q}}$ and an etale surjective covering morphism $x : X \to \mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$. The Čech nerve $\mathrm{cosk}_0^{\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}}}(\mathrm{X})$ for this morphism defines a hypercovering of the stack $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$. Similarly $\mathrm{cosk}_0^{(\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}})^{\mathrm{an}}}(\mathrm{X}^{\mathrm{an}})$ is a hypercovering of the analytic stack $\mathcal{M}_{g,G}^{an}$. Here X^{an} denotes the associated complex analytic space of the covering scheme X over $\bar{\mathbb{Q}}$.

The homotopy descent theorem (Theorem 1.3) shows that there are weak equivalences of pro-simplicial sets:

$$\{\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}\}_{et} \simeq \{\mathrm{cosk}_0^{\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}}}(\mathrm{X})\}_{\mathrm{et}} \simeq \{\mathrm{cosk}_0^{\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}}}(\mathrm{X}) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}\}_{\mathrm{et}}.$$

and the comparison theorem for simplicial schemes (cf. Cox [C], Theorem IV.8) shows that there is also a weak equivalence after profinite completions:

$$\{\mathrm{cosk}_0^{\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}}}(\mathrm{X}) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}\}_{\mathrm{et}}^{\wedge} \simeq \mathrm{Sin}(|\mathrm{cosk}_0^{(\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}})^{\mathrm{an}}}(\mathrm{X}^{\mathrm{an}})|)^{\wedge}.$$

As $\mathrm{cosk}_0^{(\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}})^{\mathrm{an}}}(\mathrm{X}^{\mathrm{an}})$ is a hypercovering of the analytic stack $\mathcal{M}_{g,G}^{an}$ we have a weak equivalence:

$$\{\mathcal{M}_{g,G}^{an}\}_{cl} \simeq \mathrm{Sin}(|\mathrm{cosk}_0^{(\mathcal{M}_{g,G}\otimes\bar{\mathbb{Q}})^{\mathrm{an}}}(\mathrm{X}^{\mathrm{an}})|)$$

From the determination (Corollary 2.15) of the classical homotopy type of $\mathcal{M}_{g,G}^{an}$ we get therefore at the end the following weak homotopy equivalence:

$$\{\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}\}_{et}^{\wedge} \simeq B\Gamma_{g,G}^{\wedge} = K(\Gamma_{g,G}, 1)^{\wedge}.$$

This finally proves the main theorem. □

If the group G is the trivial group we recover as a special case the main theorem of Oda (cf. [O], Theorem 1) for the moduli stack $\mathcal{M}_g$ of all families of algebraic curves of genus $g \geq 2$.

Another interesting special case is given by the moduli stack $\mathcal{H}_g$ parametrizing only families of hyperelliptic curves of genus g.

Definition 3.7. Let $g \geq 2$ and S be a scheme (over $\mathbb{Q}$). A family of algebraic curves $p : C \to S$ of genus g over S is called *hyperelliptic* if there exists an S-involution $\tau : C \to C$ such that $C/ < \tau > \cong \mathbb{P}^1_S$.

We define the moduli stack $\mathcal{H}_g$ of hyperelliptic curves of genus g as follows.

Definition 3.8. The moduli stack $\mathcal{H}_g$ is the category fibred in groupoids over the category (Sch) of schemes over $\mathbb{Q}$ defined by its groupoid of sections $\mathcal{H}_g(S)$ as follows: For a scheme S over $\mathbb{Q}$ the objects of $\mathcal{H}_g(S)$ are families of hyperelliptic curves $p : C \to S$ of genus g over S and the morphisms are S-isomorphisms equivariant with respect to the hyperelliptic involution.

The moduli stack of hyperelliptic curves $\mathcal{H}_g$ is a Deligne-Mumford stack. In fact, $\mathcal{H}_g$ is also one of the *Hurwitz stacks* studied by Fulton [Fu], §6, namely $\mathcal{H}_g$ is the stack $\mathcal{H}^{2,2g+2}$ of 2-sheeted simple coverings of $\mathbb{P}^1_S$ with $2g + 2$ branching points, where S is a scheme (over $\mathbb{Q}$). It would be interesting to study the etale homotopy types of more general Hurwitz stacks.

In our words the stack $\mathcal{H}_g$ of hyperelliptic curves is simply the moduli stack $\mathcal{M}_{g,G}$, where $G =< \tau >$ the group generated by a hyperelliptic involution τ as discussed above, i.e., $G \cong \mathbb{Z}/2$.

In this particular case the group $\Gamma_{g,G}$ is then just the *hyperelliptic mapping class group group* Γ_g^h, which is the centralizer $\mathrm{Cent}_{\Gamma_g}(< \tau >)$ of a hyperelliptic involution τ in the mapping class group Γ_g.

It also turns out, as already mentioned by Oda [O] that in the case of hyperelliptic curves we do not need the profinite Artin-Mazur completion of the Eilenberg-MacLane space $K(\Gamma_g^h, 1)$, because the hyperelliptic mapping class group Γ_g^h is a *good* group in the sense of Serre [S], which essentially means that the Galois cohomology of its profinite completion coincides with the corresponding discrete group cohomology. Therefore there is a weak homotopy equivalence between $K(\Gamma_g^h, 1)$ and its profinite Artin-Mazur completion as was shown by Artin and Mazur [AM], 6.9. And we can deduce now directly from our main theorem:

Corollary 3.9. *Let $\mathcal{H}_g$ be the moduli stack of hyperelliptic curves of genus g. There is a weak homotopy equivalence of pro-simplicial sets*

$$\{\mathcal{H}_g \otimes \bar{\mathbb{Q}}\}^{\wedge}_{et} \simeq K(\Gamma_g^h, 1).$$

where Γ_g^h is the hyperelliptic mapping class group.

From the discussion above, the following open question seems to be of some general interest:

Question. Let G be a finite group embedded in the mapping class group Γ_g of genus g and $g \geq 2$. For wich pairs (g, G) the groups $\Gamma_{g,G} = \mathrm{Norm}_{\Gamma_g}(\mathrm{G})$ are *good* groups in the sense of Serre or equivalently by the theorem of Artin-Mazur [AM], 6.9 when is the Eilenberg-MacLane space $K(\Gamma_{g,G}, 1)$ weakly equivalent to its profinite completion $K(\Gamma_{g,G}, 1)^\wedge$?

In Oda [O] the similar question, attributed to Deligne and Morava was discussed for the mapping class groups $\Gamma_{g,n}$.

Using the Grothendieck short exact sequences of etale fundamental groups as in [Z], Corollary 6.6. (cf. also [O], [Ma]) we can derive finally also the following corollary of the main theorem relating the etale fundamental groups of the moduli stacks $\mathcal{M}_{g,G}$, the groups $\Gamma_{g,G}$ and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Theorem 3.10. *Let G be a finite group embedded in the mapping class group Γ_g of genus $g \geq 2$. Let x be any point in the stack $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$, then there is a short exact sequence of profinite groups*

$$1 \to \Gamma_{g,G}^{\wedge} \to \pi_1^{et}(\mathcal{M}_{g,G}, x) \to \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

It follows also from the main theorem that all higher etale homotopy groups of the stacks $\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}$ in the sense of Artin-Mazur [AM] are trivial, i.e., we have $\pi_n^{et}(\mathcal{M}_{g,G} \otimes \bar{\mathbb{Q}}, x) = 0$ for all $n \geq 2$.

References

[AM] M. Artin, B. Mazur, *Etale Homotopy*, Lecture Notes in Math. **100**, Springer-Verlag, Berlin (1969).

[BK] A. K. Bousfield, D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. **304**, Springer-Verlag, Berlin (1972).

[Ca] F. Catanese, Fibred surfaces, varieties isogeneous to a product and related moduli spaces, *American Journal of Math.* **122** (2000), 1-44.

[Ca1] F. Catanese, Moduli spaces of surfaces and real structures, to appear in *Annals of Maths.*, `arXiv:math.AG/0103071`.

[Co] A. F. Costa, S. M. Natanzon, Topological classification of $\mathbb{Z}_p^m$ actions on surfaces, *Michigan Maths. J.* **50** (2002), 451-460.

[C] D. A. Cox, Homotopy theory of simplicial schemes, *Compositio Math.* **39** (1979), 263-296.

[DM] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, *Publ. Math. IHES* **36** (1969), 75-109.

[Fa] B. Fantechi, Stacks for everybody, in: C. Casacuberta, R. Miró-Roig, J. Verdera, S. Xambó (eds.), *European Congress of Mathematics, Vol. I (Barcelona, 10-14 July, 2000)*, Progr. Math. **201**, Birkhäuser, Basel (2001), 349-359

[F] E. M. Friedlander, *Etale Homotopy of Simplicial Schemes*, Annals of Math. Studies **104**, Princeton University Press (1982).

[Fu] W. Fulton, Hurwitz schemes and irreducibility of moduli of algebraic curves, *Annals of Math.* **90** (1969), 542-575.

[GDH] G. González-Díez, W. J. Harvey, Moduli of Riemann surfaces with symmetry, in: *Discrete groups and geometry (Birmingham, 1991)*, London Math. Soc. Lecture Note Series 173, Cambridge University Press (1992), 75-93.

[GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York (1978) (Reprinted 1994).

[G1] A. Grothendieck, Techniques de construction en géométrie analytique, Séminaire H. Cartan, 1960/61, exposés 7, 9-17, Paris (1962).

[G2] A. Grothendieck, Esquisse d'un Programme, in: *Geometric Galois Actions 1. Around Grothendieck's Esquisse d' un Programme* ed. L. Schneps, P. Lochak, London Math. Soc. Lecture Note Series 242, Cambridge University Press (1997), 5-48.

[H] W. J. Harvey, On branch loci in Teichmüller space, *Trans. Amer. Math. Soc.* **153** (1971), 387-399.

[K] S. P. Kerckhoff, The Nielsen realization problem, *Annals of Math.* **117** (1983), 235-265.

[Kr] S. Kravetz, On the geometry of Teichmüller spaces and the structure of their modular groups, *Ann. Acad. Sci. Fenn.* **278** (1959), 1-35.

[LMB] G. Laumon, L. Moret-Bailly: *Champs algébriques*, Erg. der Mathematik 3. Folge, Band **39**, Springer-Verlag, Berlin (2000).

[LL] O. A. Laudal, K. Lønsted, Deformations of Curves I: Moduli for Hyperelliptic Curves, *Algebraic geometry (Proc. Sympos., Univ. Tromsø, 1977)*, Lecture Notes in Math. **687**, Springer-Verlag, Berlin (1978), 150-167.

[LK] K. Lønsted, S. L. Kleiman: Basics on families of hyperelliptic curves, *Compositio Math*, **38** (1979), 83-111.

[Ma] M. Matsumoto, Arithmetic fundamental groups and moduli of curves, School on Algebraic Geometry (Trieste, 1999), 355-383, ICTP Lect. Notes, **1**, ICTP, Trieste, 2000.

[Mo] I. Moerdijk, *Classifying Spaces and Classifying Topoi*, Lecture Notes in Math. **1616**, Springer-Verlag, Berlin (1995).

[MFK] D. Mumford, J. Fogarthy and F. Kirwan, *Geometric Invariant Theory*, Ergeb. der Math. Grenz. **34**, 3rd ed., Springer-Verlag, Berlin (1992).

[M] D. Mumford, Picard groups of moduli problems, *Arithmetical Algebraic Geometry* ed. O.F.G. Schilling, Harper & Row (1965), 33-81.

[O] T. Oda, Etale homotopy type of the moduli spaces of algebraic curves, in: *Geometric Galois Actions 1. Around Grothendieck's Esquisse d' un Programme* ed. L. Schneps, P. Lochak, London Math. Soc. Lecture Note Series 242, Cambridge University Press (1997), 85-96.

[SL] L. Schneps, P. Lochak (eds.), *Geometric Galois Actions 1. Around Grothendieck's Esquisse d' un Programme* ed. L. Schneps, P. Lochak, London Math. Soc. Lecture Note Series 242, Cambridge University Press (1997).

[S] J. P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. **5**, Springer-Verlag, Berlin (1965).

[T] B. Toen, K-théorie et cohomologie des champs algébriques: Théorèmes de Riemann-Roch, D-modules et théorèmes GAGA, PhD thesis, L' Université Paul Sabatier de Toulouse, 1999, `arXiv:math.AG/9908097`

[Tr] A. Tromba, Dirichlet's energy on Teichmüller's moduli space and the Nielsen realization problem, *Math. Zeit.* **222** (1996), 451-464.

[Tu1] S. Tufféry, Déformations de courbes avec automorphismes, *C. R. Acad. Sci. Paris*, t. **314** (1992), Série I, p. 379-382.

[Tu2] S. Tufféry, Déformations de courbes avec action de groupe, *Forum Math.* **5** (1993), 243-259.

[V] A. Vistoli, Intersection theory on algebraic stacks and their moduli spaces, *Inventiones Math.* **97** (1989), 613-670.

[Z] V. Zoonekynd, The fundamental group of an algebraic stack, preprint, `arXiv:math.AG/0111071`.

Dipartimento di Matematica, Università degli Studi di Pavia, Via Ferrata 1, I-27100 Pavia, Italy

E-mail address: `frediani@dimat.unipv.it`

Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, England, UK

E-mail address: `fn8@mcs.le.ac.uk`