A uniform proof on the weak Hilbert's 16th problem for n=2 *

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Abstract

The weak Hibert's 16th problem for $n = 2$ was solved by Horozov and Iliev (1994 *Proc. London Math. Soc.* **69** 198–244), Zhang and Li (1997 *Adv. Math.* **26** 445–460), Gavrilov (2001 *Invent. Math.* **143** 449–497), and Li and Zhang (2002 *Nonlinearity* **15** 1775–1992), by using different methods for different cases. The aim of this paper is to give a uniform and easier proof for all cases. The proof is restricted to the real domain, combines geometric and analytical methods, and uses deformation arguments.

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1 Introduction

1.1 The Weak Hilbert's 16th problem and its solution for $n = 2$

Let $H(x, y)$ be a real polynomial of degree $n + 1$, and suppose that the corresponding Hamiltonian vector field

$$X_H = H_y \frac{\partial}{\partial x} - H_x \frac{\partial}{\partial y} \tag{1.1}$$

has at least one center. A natural problem is asking for a maximal number of limit cycles, bifurcated from the period annulus of the center for the perturbed system

$$X_\epsilon = X_H + \epsilon Y_\epsilon, \tag{1.2}$$

where

$$Y_\epsilon = Y_1(x, y, \epsilon)\frac{\partial}{\partial x} + Y_2(x, y, \epsilon)\frac{\partial}{\partial y},$$

Y_1 and Y_2 are polynomials in x and y of degree n, and their coefficients depend analytically on the parameter ϵ. It is well known that this number is closely related to the least upper bound of the number of zeros of the *Abelian integral*

$$I(h) = \int_{\delta(h)} Y_2(x, y, 0)dx - Y_1(x, y, 0)dy = -\iint_{\mathrm{Int}(\delta(h))} \mathrm{div}(Y_0)\, dxdy, \tag{1.3}$$

where $\delta(h) \subset H^{-1}(h)$ is an oval.

The *weak Hilbert's 16th problem* is asking for a least upper bound $Z(n)$ of the number of zeros of $I(h)$ for a fixed n and for all possible H, Y_0.

It is known that $Z(n)$ is finite (see [14] and [18]), but there is no precise estimate of it. For the case $n = 2$ this problem was completely solved. The result can be stated as follows using the above notation.

Theorem 1.1 *Z(2)=2.*

We briefly recall the relevant results. If X_H has at least one center then, without loss of generality, the cubic Hamiltonian function can be transformed into the form (see [7])

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3, \tag{1.4}$$

where (a,b) belongs to the region

$$G = \left\{(a,b) \in \mathbb{R}^2 : -\frac{1}{2} \leq a \leq 1,\ 0 \leq b \leq (1-a)\sqrt{2a+1}\right\}. \tag{1.5}$$

If $(a,b) \in \partial G$, then X_H is degenerate (i.e. X_H belongs to the intersection of the Hamiltonian class with other integrable class(es)) and $I(h)$ has at most one zero, the cyclicity of its period annulus is determined by the second- or third- order Melnikov function, see [13]. The cyclicity of the period annulus (or annuli) is 3 for the Hamiltonian triangle case (see [12]), and 2 for all other 7 cases (see [6],[11],[22],[23] and [3]).

The set $G \backslash \partial G$ is divided into three disjoint regions G_i, $i = 1, 2, 3$, by the following two curves

$$l_2 = \left\{(a,b) \in \mathbb{R}^2 : b = \sqrt{-4a(2a+1)},\ -\frac{1}{2} < a < 0\right\} \tag{1.6}$$

and

$$l_\infty = \left\{(a,b) \in \mathbb{R}^2 : b = 2\sqrt{a^3},\ 0 < a < \frac{1}{2}\right\}. \tag{1.7}$$

If $(a,b) \in G\backslash\partial G = G_1 \cup l_2 \cup G_2 \cup l_\infty \cup G_3$, then X_H is generic (i.e. X_H belongs only to the Hamiltonian class), and the least upper bound of the zeros of $I(h)$ gives the cyclicity of the period annulus (see also [13]).

Note that along the curve l_2 two singularities of X_H coincide, and that one singularity of X_H tends to infinity when (a,b) tends to the curve l_∞; X_H has one, two or three saddle points for $(a,b) \in G_1, G_2$ or G_3, respectively; X_H has two period annuli for $(a,b) \in G_2$ and one period annulus in the other cases. A partial result of Theorem 1.1 was first proved by Horozov and Iliev [7] for $(a,b) \in G_3$, then by Gavrilov [5] for $(a,b) \in G_1 \cup G_2$. Since a basic assumption in [7] and [5] is that $H(x,y)$ has 4 distinct critical values, the cases $(a,b) \in l_2 \cup l_\infty$ must be considered separately. Papers [21] and [16] independently gave different proofs for $(a,b) \in l_\infty$, and a recent work [15] solved the problem for $(a,b) \in l_2$. Note that the conclusion of Theorem 1.1 can be extended to the maximal number of limit cycles of X_ϵ in the case $(a,b) \in G_1 \cup G_2 \cup G_3$ by a result of Roussarie [17].

The methods used in the above mentioned papers are quite different; for example, [5] uses some tools from complex analysis and algebraic topology. The aim of this paper is to provide a uniform and easier proof for all the generic cases. We will restrict our discussion to the real domain.

1.2 Outline of the uniform proof

In the normal form (1.4), X_H has a center at the origin $O(0,0)$ surrounded by a period annulus, which ends at a homoclinic orbit of the saddle $S(0,1)$.

Let $\delta(h) \subset H^{-1}(h)$ be an oval for $h \in \left(0, \frac{1}{6}\right)$, $\delta(h)$ shrinks to the center O as $h \to 0+0$, and $\delta(h)$ expands to the homoclinic loop as $h \to \frac{1}{6} - 0$. Since both $Y_1(x, y, 0)$ and $Y_2(x, y, 0)$ are quadratic, it is not difficult to see that (1.3) can be written as

$$I(h) = \iint_{\mathrm{Int}(\delta(h))} (\alpha + \beta x + \gamma y)\, dx\, dy, \tag{1.8}$$

where α, β and γ are arbitrary constants. Following the notations of [7], we define

$$\begin{array}{ll} M(h) = \iint_{\mathrm{Int}(\delta(h))} dx\, dy, & X(h) = \iint_{\mathrm{Int}(\delta(h))} x\, dx\, dy, \\ Y(h) = \iint_{\mathrm{Int}(\delta(h))} y\, dx\, dy, & K(h) = \iint_{\mathrm{Int}(\delta(h))} xy\, dx\, dy. \end{array}$$

Note that $M(h)$ is the area of $\mathrm{Int}(\delta(h))$ and $M'(h)$ is the period of $\delta(h)$, so $M(h) > 0$ and $M'(h) > 0$ for $h \in \left(0, \frac{1}{6}\right)$. Hence (1.8) can be written as

$$I(h) = \alpha M(h) + \beta X(h) + \gamma Y(h) = M(h)\left[\alpha + \beta P(h) + \gamma Q(h)\right], \tag{1.9}$$

where

$$P(h) = \frac{X(h)}{M(h)}\ , \quad Q(h) = \frac{Y(h)}{M(h)}. \tag{1.10}$$

The following results are easily obtained from the definitions of $I(h)$, $P(h)$ and $Q(h)$. The second point of statement (3) was first proved in Theorem 2.4 of [7], see also lemma 4.1 of [15].

Lemma 1.1 *For any $(a, b) \in G \backslash \partial G$ we have*

(1) $I(0) \equiv 0$ *for any constants* α, β *and* γ.

(2) $P(0) = \lim_{h \to 0+0} P(h) = 0$ *and* $Q(0) = \lim_{h \to 0+0} Q(h) = 0$.

(3) $P(h) < 1$ *for* $h \in [0, 1/6]$ *and* $Q(h) < 0$ *for* $h \in (0, 1/6]$.

(4) $P, Q \in C^{\infty}\left[0, \frac{1}{6}\right) \bigcup C^0\left[0, \frac{1}{6}\right]$.

Thus we can define the *centroid curve* (see [7]) by

$$\Sigma_{a,b} = \left\{ \big(P, Q\big)(h) : 0 \le h \le \frac{1}{6} \right\}, \tag{1.11}$$

where $(a, b) \in G \backslash \partial G$. Obviously, for any constants α, β and γ the number of zeros of $I(h)$ for $h > 0$ equals to the number of intersection points (counting multiplicities) of the curve $\Sigma_{a,b}$ with the straight line

$$L_{\alpha\beta\gamma}: \quad \alpha + \beta P + \gamma Q = 0 \tag{1.12}$$

in the (P, Q)–plane, where $\beta^2 + \gamma^2 \neq 0$.

Definition 1.1 *A plane curve is called sectorial, if it is smooth, and when running it, the tangential vector rotates an angle less than* π.

Therefore, if X_H has only one period annulus, then Theorem 1.1 follows from the following result.

Theorem 1.2 *For any* $(a, b) \in G \backslash \partial G$ *the curve* $\Sigma_{a,b}$ *is sectorial, and is strictly convex with non–zero curvature.*

Since the Picard–Fuchs equation of $M(h), X(h), Y(h)$ and $K(h)$ is of order 4, it is very difficult to get the global information of the curve $\Sigma_{a,b}$, except some of its local properties for h near 0 and near 1/6, for more details on Ficard–Fuchs equations see section 2. We note that for h near 1/6, which corresponds to the saddle loop, we need to use the expansion of $M(h)$, $X(h)$ and $Y(h)$ in the form of $c_1 + c_2(h-1/6)\ln(1/6-h) + c_3(h-1/6) + o(h-1/6)$, see (1.8) of [7], for example. The following result was proved in section 3 of [7] and lemma 7 of [5].

Lemma 1.2 *For any* $(a, b) \in G \backslash \partial G$ *the curvature of* $\Sigma_{a,b}$ *near its two endpoints is non–zero.*

This result is equivalent to say that for generic quadratic Hamiltonian systems the order of the Hopf bifurcation and of the homoclinic bifurcation is at most two, and it basically follows from the Bautin's theory [1] and a result due to Horozov and Iliev [8]. Note that this result for h near 0 can be obtained directly by computations from the Picard–Fuchs equation (see lemma 2.2(2) below).

Taking derivative on $I(h)$ twice, we get

$$I''(h) = \alpha M''(h) + \beta X''(h) + \gamma Y''(h) = M''(h)\Big[\alpha + \beta\nu(h) + \gamma\omega(h)\Big], \quad (1.13)$$

where

$$\nu(h) = \frac{X''(h)}{M''(h)}, \quad \omega(h) = \frac{Y''(h)}{M''(h)}. \quad (1.14)$$

Note that $M''(h) \neq 0$ for quadratic Hamiltonian vector fields (see [2]). We define the curve in the (ν, ω)–plane

$$\Omega_{a,b} = \Big\{\big(\nu, \omega\big)(h) : 0 \le h \le \frac{1}{6}\Big\}. \quad (1.15)$$

Hence the number of zeros of $I''(h)$ for $h > 0$ equals to the number of intersection points (counting multiplicities) of the curve $\Omega_{a,b}$ with the straight line

$$L'_{\alpha\beta\gamma} : \quad \alpha + \beta\nu + \gamma\omega = 0 \quad (1.16)$$

in the (ν, ω)–plane.

We identify the (P,Q)–plane with the (ν,ω)–plane, hence the two lines $L_{\alpha\beta\gamma}$ and $L'_{\alpha\beta\gamma}$ are identified. We will prove the following basic facts.

Lemma 1.3 *For any $(a,b)\in G\backslash\partial G$ the following statements hold, which imply the regularity of the curve $\Omega_{a,b}$.*

(1) $\big[\omega'(h)\big]^2+\big[\nu'(h)\big]^2\neq 0$ *for* $h\in\big(0,\frac{1}{6}\big)$, *and*
(2) $(\nu,\omega)(h_1)\neq(\nu,\omega)(h_2)$ *for* $h_1\neq h_2$ *and* $h_1,h_2\in[0,\frac{1}{6}]$.

To prove theorem 1.2, we suppose the contrary: for some $(a,b)\in G\backslash\partial G$ the curve $\Sigma_{a,b}$ has zero curvature at some points, and we denote by $(P,Q)(h^*)$ the nearest such a point to the endpoint $(P,Q)(0)$. By lemma 1.2, $h^*\in(0,\frac{1}{6})$. Now we denote the arc of $\Sigma_{a,b}$ from $h=0$ to $h=h^*$ by $\Sigma^*_{a,b}$. We will prove the following property of $\Sigma^*_{a,b}$.

Lemma 1.4 *For any $(a,b)\in G\backslash\partial G$, the curve $\Sigma_{a,b}$ is smooth. In particular, along $\Sigma^*_{a,b}$ (for $h\in[0,h^*)$) we have*

$$\frac{d^2Q}{dP^2}>0,\quad k_0(a,b)\leq\frac{dQ}{dP}\leq k_1(a,b), \tag{1.17}$$

where

$$k_0(a,b)=\frac{b}{a-1}<0,\quad k_1(a,b)=\frac{-Q(\frac{1}{6})}{1-P(\frac{1}{6})}>0. \tag{1.18}$$

If such h^* does not exist, then (1.17) holds globally along $\Sigma_{a,b}$ ($h\in[0,1/6]$).

We denote the set of tangent lines of $\Sigma_{a,b}$ (resp. $\Omega_{a,b}$) by $T_{\Sigma_{a,b}}$ (resp. $T_{\Omega_{a,b}}$), that is

$$T_{\Sigma_{a,b}}=\{\xi_h:\text{the tangent line to }\Sigma_{a,b}\text{ at }(P,Q)(h),h\in[0,1/6]\}, \tag{1.19}$$

$$T_{\Omega_{a,b}}=\{\eta_h:\text{the tangent line to }\Omega_{a,b}\text{ at }(\nu,\omega)(h),h\in[0,1/6]\}. \tag{1.20}$$

We will prove

Lemma 1.5 *For any $(a,b)\in G\backslash\partial G$ we have:*

(1) $\xi_t\cap\Omega_{a,b}\neq\emptyset$ *for any* $t\in(0,\frac{1}{6})$.
(2) $\xi_0\cap\Omega_{a,b}=\{(\nu,\omega)(0)\}$, $\xi_{\frac{1}{6}}\bigcap\Omega_{a,b}=\{(\nu,\omega)(\frac{1}{6})\}$, *and the crossing is transversal.*
(3) $\big\{(\nu,\omega)(0)\cup(\nu,\omega)(\frac{1}{6})\big\}\bigcap\xi_h=\emptyset$ *for any* $\xi_h\in T_{\Sigma^*_{a,b}}$ *and any* $h\in(0,h^*]$.
(4) $\Sigma_{a,b}$ *and* $\Omega_{a,b}$ *have no common tangent line.*

Proof of theorem 1.2 assuming lemmas 1.1-1.5 We will prove the non–existence of the hypothetical value h^*. Hence, by Lemma 1.4, $\Sigma_{a,b}$ is sectorial and strictly convex with non–zero curvature. Now, we start the proof of the non–existence. We move $\xi_t \in T_{\Sigma^*_{a,b}}$ along $\Sigma^*_{a,b}$ as t increases from 0 to h^* and consider the number of intersection points of $\xi_t \cap \Omega_{a,b}$. By lemma 1.5(1) and (2) and lemma 1.2, $\xi_t \cap \Omega_{a,b}$ consists of one point for $0 < t \ll 1$, counting its multiplicity. As we have supposed that $\Sigma^*_{a,b}$ has a zero curvature at $h = h^* \in (0, 1/6)$ for some $(a, b) \in G \backslash \partial G$, hence by taking $L_{\alpha\beta\gamma} = \xi_{h^*}$ the function $I(h)$ has at least a triple zero at $h = h^*$ plus a zero at $h = 0$ (lemma 1.1 (1)), this implies $\xi_{h^*} \cap \Omega_{a,b}$ consists of at least two points. By lemma 1.5(3), the two endpoints of $\Omega_{a,b}$ keep on the different sides of ξ_h for all $h \in (0, h^*]$. Hence, we must find a $h' \in (0, h^*)$ such that $\xi_{h'} \in T_{\Sigma^*_{a,b}}$ is also tangent to $\Omega_{a,b}$, in order to increase the number of intersection points of $\xi_t \cap \Omega_{a,b}$ as t increases from 0 to h^*, and this contradicts lemma 1.5(4). □

For $(a, b) \in G_2$, X_H has two period annuli, hence there are two centroid curves $\Sigma^i_{a,b}$ for $i = 1, 2$. To finish the proof of theorem 1.1, we also need the following result, which was first proved in [9] by using the results of [1] and [8], and we will give a new direct proof.

Theorem 1.3 *For any $(a, b) \in G_2$ both centroid curves are strictly convex with non–zero curvature, and any straight line cuts $\Sigma^1_{a,b} \cup \Sigma^2_{a,b}$ at most at two points, counting the multiplicities.*

The paper is organized as follows. We give some preliminaries in section 2, and we prove lemma 1.3 in section 3, lemmas 1.4 and 1.5 in sections 4, and theorem 1.3 in section 5.

2 Preliminaries

For $(a, b) \in G \backslash \{\partial G \cup l_\infty\}$ by using a standard method one can obtain the following Picard–Fuchs equation of order 4, satisfied by $X(h)$, $Y(h)$, $M(h)$ and $K(h)$ (see lemma 3.3 of [7]):

$$\begin{aligned}
&-6bhM' + bX' - (a+1)Y' - 2a(a+1)K' + 4bM = 0,\\
&(6\lambda h + a + 1)Y' + (4a(a+1)^2 - \lambda)K' + b(a+1)M - 6\lambda Y = 0,\\
&b\lambda(6h-1)X' + a(\lambda - 2a(a+1))Y' + ((4a^2+3a+1)\lambda -\\
&\qquad 8a^3(a+1)^2)K' - 6b\lambda X + b(\lambda - 2a^2(a+1))M = 0,\\
&((1-6h)\lambda^2 - (8a^3+12a^2+5a+1)\lambda + 16a^3(a+1)^3)K' +\\
&\qquad a(4a(a+1)^2 - \lambda)\cdot(Y' + bM) + \lambda(-(a+1)bX + 8\lambda K +\\
&\qquad (4a^2(a+1) - \lambda)Y) = 0,
\end{aligned} \tag{2.1}$$

where $' = \frac{d}{dh}$, and $\lambda = 4a^3 - b^2$. Note that $\lambda = 0$ corresponds to $(a, b) \in l_\infty$, and in this case the last three equations in (2.1) are not independent. In the second part of the thesis [20], a parallel Picard–Fuchs equation of order 3 was obtained for the case $\lambda = 0$ ($K(h)$ and $K'(h)$ do not appear). Hence, the results, parallel to Lemma 2.1 and equation (2.5) in this section, are obtained for $\lambda = 0$. It is natural that all these results are limits as $\lambda \to 0$ from the corresponding results here. Hence the discussions in this paper are valid for all $(a, b) \in G\backslash\partial G$. See also the remark 3.5 of [10].

For simplicity we use the following notations

$$\begin{aligned} &\lambda_1(a, b) = (1-a)^2(2a+1) - b^2, \quad \lambda_2(a, b) = (3a+1)^2 + b^2, \\ &\lambda_3(a, b) = (3a-1)^2 + 5b^2 + 4. \end{aligned} \tag{2.2}$$

Note that $\lambda_1(a, b) > 0$ for $(a, b) \in G\backslash\partial G$, see (1.5). The following result follows from (2.1) (see the proof of lemma 2 of [4] or lemma 3.3 of [7]).

Lemma 2.1 *For* $0 < h \ll 1$ *we have*

$$\begin{aligned} P(h) &= \tfrac{1}{2}(1-a)h + \tfrac{1}{72}[-5(11a+1)b^2 \\ &\quad -(a-1)(63a^2+18a+55)]h^2 + O(h^3), \\ Q(h) &= -\tfrac{b}{2}h + \tfrac{b}{72}(-55b^2 - 183a^2 + 42a + 5)h^2 + O(h^3). \end{aligned}$$

Lemma 2.2 *For* $(a, b) \in G\backslash\partial G$ *we have*

(1) $\displaystyle \lim_{h \to 0+0} \frac{dQ}{dP} = \frac{b}{a-1} < 0.$

(2) $\displaystyle \lim_{h \to 0+0} \frac{d^2Q}{dP^2} = \frac{20}{3} \frac{b\lambda_1(a, b)}{(1-a)^3} > 0.$

(3) $\displaystyle \lim_{h \to \frac{1}{6}-0} \frac{dQ}{dP} = -\frac{Q(\frac{1}{6})}{1 - P(\frac{1}{6})} > 0.$

Proof. Statements (1) and (2) are easily deduced from lemma 2.1. The statement (3) can be proved in the same way as in lemma 4.3 of [15]. □

Taking derivatives with respect to h in the first three equations of (2.1), and removing M', we can express X'', K'' through M'', Y'' as follows

$$\begin{aligned} X'' &= d_1(h)M'' + d_2(h)Y'', \\ K'' &= d_3(h)M'' + d_4(h)Y'', \end{aligned} \tag{2.3}$$

where

$$d_1(h) = \frac{6\lambda_1(a,b)h}{L(h)},$$
$$d_2(h) = \frac{[12(3a^2+2a+1)b^2 - 24a^3(3a+1)(a-1)]h + (a-1)\lambda_2(a,b)}{bL(h)},$$
$$d_3(h) = \frac{-6b(a+1)(6h-1)h}{L(h)},$$
$$d_4(h) = \frac{6[12(4a^3-b^2)h + b^2 - 6a^3 - 3a^2 + 1]}{L(h)},$$
$$L(h) = 12(a^3 - 6a^2 - 3a - b^2)h + \lambda_2(a,b). \tag{2.4}$$

Note that $L(0) > 0$, $L(1/6) = \lambda_1(a,b) > 0$ (see (2.2) and (1.5)), hence the linear function $L(h) \neq 0$ for all $h \in [0, 1/6]$.

Taking derivatives in (2.1) with respect to h once more, and using (2.3) we get

$$T(h)\frac{d}{dh}\begin{pmatrix} M'' \\ Y'' \end{pmatrix} = \begin{pmatrix} e_1(h) & e_2(h) \\ e_3(h) & e_4(h) \end{pmatrix}\begin{pmatrix} M'' \\ Y'' \end{pmatrix}, \tag{2.5}$$

where

$$T(h) = -6bh(6h-1)L(h)\bar{T}(h),$$
$$\bar{T}(h) = 36(4a^3-b^2)^2h^2 - 6[b^4 + 2(6a^2+3a+1)b^2 + 8a^3(3a+1)]h + \lambda_2(a,b),$$

and

$$e_i(h) = \sum_{k=0}^{4} e_{ik}h^k,$$

with

$$e_{10} = -6b\lambda_2^2(a,b),$$
$$e_{11} = 6b\lambda_2(a,b)[7b^4 + (83a^2+46a+31)b^2 + 4(3a+1)(12a^3 - a^2 + 10a + 3)],$$
$$e_{12} = -36b[23b^6 - 2(26a^3 - 203a^2 - 124a - 39)b^4 + (84a^6 - 572a^5 + 1619a^4 + 1964a^3 + 1254a^2 + 408a + 43)b^2 + 8a(12a^5 - 4a^4 + 40a^3 + 9a^2 + 10a + 5)(3a+1)^2],$$
$$e_{13} = 864b[8b^6 - (39a^3 - 103a^2 - 59a - 13)b^4 + a(85a^5 - 303a^4 + 174a^3 + 278a^2 + 125a + 25)b^2 - 16a^4(3a+1)(a^4 - 10a^3 - 10a - 5)],$$
$$e_{14} = 10368b(3a^3 - 10a^2 - 2b^2 - 5a)(4a^3 - b^2)^2,$$

$$
\begin{aligned}
e_{20} =& -(5b^2+9a^2-6a+5)\lambda_2^2(a,b),\\
e_{21} =& 12\lambda_2(a,b)[(-a^2-a+15)b^4+(-9a^4-72a^3+94a^2+52a+15)b^2+\\
& a(3a+1)(36a^4-27a^3+57a^2-a-1)],\\
e_{22} =& -216(7a^2+7a+10)b^6+432(14a^5-28a^4-23a^3-67a^2-31a-5)b^4-\\
& 216a(a-1)(3a+1)(12a^5-56a^4+11a^3-69a^2-35a-7)b^2-\\
& 1728a^4(3a^4-3a^3+14a^2+a+1)(3a+1)^2,\\
e_{23} =& -1728(4a^3-b^2)[(6a^2+6a+5)b^4-2a(12a^4-6a^3-7a^2-12a-3)b^2+\\
& a^4(3a+1)(3a^3-37a^2-15a-15)],\\
e_{24} =& -20736a(a+1)(4a^3-b^2)^3,\\
e_{30} =& 0,\\
e_{31} =& -6b^2\lambda_2^2(a,b),\\
e_{32} =& 36b^2[(-4a-1)b^4+(14a^4-22a^3+18a+6)b^2+(3a+1)\cdot\\
& (36a^5+26a^4+53a^3+57a^2+19a+1)],\\
e_{33} =& 432b^2[(3a+2)b^4-2(5a^4-9a^3-12a^2-a+1)b^2-\\
& a(4a^6+98a^5+117a^4+72a^3+58a^2+30a+5)],\\
e_{34} =& 2592b^2(a+1)(b^2+a^3+10a^2+5a)(4a^3-b^2),\\
e_{40} =& 0,\\
e_{41} =& 6b(5b^2+16a^2-4a)\lambda_2^2(a,b),\\
e_{42} =& -36b[25b^6-2(52a^3-175a^2-92a-15)b^4+\\
& (204a^6-868a^5+973a^4+1228a^3+474a^2+96a+5)b^2+\\
& 4a(3a+1)(144a^6+48a^5+236a^4+145a^3+9a^2-5a-1)],\\
e_{43} =& 864b[10b^6-(69a^3-95a^2-49a-5)b^4+a(155a^5-489a^4-66a^3+\\
& 142a^2+55a+11)b^2-16a^4(3a^5-61a^4-35a^3-27a^2-20a-4)],\\
e_{44} =& 5184b(4a^3-b^2)^2(-5b^2+3a^3-34a^2-17a).
\end{aligned}
$$

From the definition of $\omega(h)$ (see (1.14)), we have

$$\omega'(h) = \frac{(Y''(h))'}{M''(h)} - \frac{(M''(h))'}{M''(h)}\omega(h).$$

Combining this fact with (2.5), we obtain a 2–dimensional system of equations

$$\dot{h} = T(h), \qquad \dot{\omega} = \phi(h, \omega), \tag{2.6}$$

where $\phi(h,\omega) = -e_2(h)\omega^2 + (e_4(h) - e_1(h))\omega + e_3(h)$, and the dot denotes the derivative with respect an arbitrary variable s.

Remark 2.1 *We note that $T(h) \neq 0$ for $h \in (0, \frac{1}{6})$, hence system (2.6) has no singularities for $h \in (0, \frac{1}{6})$. In fact, we have shown that $L(h)$ has no zeros for $h \in (0, \frac{1}{6})$. If $(a, b) \in G_1$ then $\bar{T}(h)$ has no real roots. If $(a, b) \in G_2 \cup G_3 \cup l_2 \cup l_\infty$ then the roots of $\bar{T}(h)$ correspond to other singularities of X_H, besides the center O and the saddle S. By the monotonic property of the level curves of the Hamiltonian vector field and the relative positions of the singularities, we immediately obtain that the roots of $\bar{T}(h)$ must be greater than $\frac{1}{6}$.*

By Remark 2.1 and direct computation we obtain the following result.

Lemma 2.3 *For $h \in [0, \frac{1}{6}]$ system (2.6) has 4 singularities : two improper nodes at $(0,0)$ and $\left(\frac{1}{6}, 0\right)$, two hyperbolic saddles at $(0, \omega_0)$ and $\left(\frac{1}{6}, \omega_1\right)$, where*

$$\omega_0 = \frac{-6b}{\lambda_3(a,b)} < 0, \quad \omega_1 = \frac{-6b(2a+1)}{5b^2 - 82a^3 - 93a^2 - 36a - 5}. \tag{2.7}$$

When $5b^2 - 82a^3 - 93a^2 - 36a - 5 \to 0$, the singularity $(\frac{1}{6}, \omega_1)$ goes to infinity.

We recall that an improper node is a node such that all the orbits arrive to or exit from it in one direction.

Let

$$C_\omega = \left\{ (h, \omega) : 0 \le h \le \frac{1}{6},\ \omega = \omega(h) \text{ is defined in (1.14)} \right\}. \tag{2.8}$$

The following lemma can be proved in the same way as the proof of lemma 3.1 in [15], except statement (2) which is a consequence of lemma 3.5 below. Statement (1) of the next lemma shows that C_ω is the unstable manifold from the saddle $(0, \omega_0)$ to the improper node $(1/6, 0)$ of system (2.6).

Lemma 2.4 (1) $\lim_{h\to 0+0} \omega(h) = \omega_0$, $\lim_{h\to \frac{1}{6}-0} \omega(h) = 0$.
(2) $\omega(h) < 0$ *for* $h \in (0, \frac{1}{6})$.
(3) $\lim_{h\to 0+0} \nu(h) = \nu_0$, $\lim_{h\to \frac{1}{6}-0} \nu(h) = 1$, *where*

$$\nu_0 = \frac{6(1-a)}{\lambda_3(a,b)} > 0. \tag{2.9}$$

(4) *We have*

$$\lim_{h\to 0+0}\omega'(h)=\frac{5}{2}\frac{bf_1(a,b)}{(\lambda_3(a,b))^2},\qquad \lim_{h\to 0+0}\nu'(h)=\frac{5}{2}\frac{f_2(a,b)}{(\lambda_3(a,b))^2},$$

$$\lim_{h\to 0+0}[\omega''(h)\nu'(h)-\omega'(h)\nu''(h)]=\frac{175}{6}\frac{b\lambda_1(a,b)\lambda_2(a,b)f_3(a,b)}{(\lambda_3(a,b))^3},$$

where

$$\begin{aligned}
f_1(a,b) &= 7b^4+(42a^2+60a-70)b^2-\\
&\quad (189a^4-180a^3+174a^2+12a-67),\\
f_2(a,b) &= (7a-67)b^4+(162a^3-270a^2-58a+70)b^2+\\
&\quad (a-1)(3a+1)\cdot(9a^3-27a^2-21a+7),\\
f_3(a,b) &= 55b^4+(126a^2-204a-106)b^2-81a^4+\\
&\quad 324a^3+162a^2-204a+55.
\end{aligned}$$

(5) *We have*

$$\lim_{h\to\frac{1}{6}}\frac{\nu'(h)}{\omega'(h)}=-\frac{(2a+1)(3a+1)}{b}.$$

□

From (2.3) and definition (1.14) we obtain the expression of $\nu(h)$ as a function of h and $\omega(h)$ as follows

$$\nu(h)=d_1(h)+d_2(h)\,\omega(h), \tag{2.10}$$

where $d_i(h)=d_i(h;a,b)$, $i=1,2$ are given in (2.4).

We consider the following transformation from the (h,ω)–plane to the (ν,ω)–plane:

$$\nu=d_1(h)+d_2(h)\omega,\qquad \omega=\omega. \tag{2.11}$$

It is easy to see that (2.11) maps the straight line $\{(h,\omega):h=h_0\}$ ($h_0\in[0,1/6]$) in the (h,ω)–plane to a straight line in the (ν,ω)–plane. In particular, maps $\{(h,\omega):h=0\}$ to L_0, and maps $\{(h,\omega):h=\frac{1}{6}\}$ to L_3, where

$$L_0=\Big\{(\nu,\omega):\nu=\frac{a-1}{b}\omega\Big\},\ L_3=\Big\{(\nu,\omega):\nu=-\frac{(2a+1)(3a+1)}{b}\omega+1\Big\}. \tag{2.12}$$

We note that if $a=0$ then L_0 is parallel to L_3, and if $a\neq 0$ then $L_0\cap L_3=\{(\hat{\nu},\hat{\omega})\}$, where

$$\hat{\nu}=\frac{a-1}{6a(a+1)},\qquad \hat{\omega}=\frac{b}{6a(a+1)}. \tag{2.13}$$

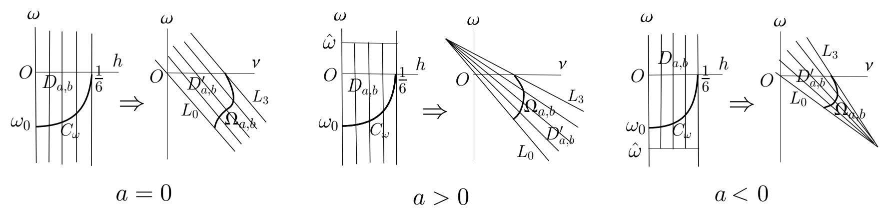

Figure 1. From $D_{a,b}$ to $D'_{a,b}$ through the transformation (2.11).

Let

$$D_{a,b} = \{(h,\omega) \in \mathbb{R}^2 : 0 \le h \le 1/6;\ -\infty < \omega < \infty \text{ if } a = 0,$$
$$-\infty < \omega < \hat{\omega} \text{ if } a > 0,\ \hat{\omega} < \omega < \infty \text{ if } a < 0\}.$$

Correspondingly, let $D'_{a,b}$ be the region in the (ν,ω)–plane, which is the strip $-\frac{1}{b}\omega \le \nu \le -\frac{1}{b}\omega + 1$ if $a = 0$; and is the sector limited by the two straight lines L_0 and L_3 with vertex at $(\hat{\nu},\hat{\omega})$ if $a \ne 0$. Note that these two lines are included in $D'_{a,b}$, but the vertex $(\hat{\nu},\hat{\omega})$ is not, see figure 1.

The Jacobian of transformation (2.11)

$$\frac{D(\nu,\omega)}{D(h,\omega)} = d_1'(h) + d_2'(h)\omega = -\frac{6\lambda_1(a,b)\lambda_2(a,b)[6a(a+1)\omega - b]}{bL^2(h)}$$

is non–zero if $a = 0$, and is zero only for $\omega = \hat{\omega}$ if $a \ne 0$. Hence, we immediately have the following result.

Lemma 2.5 *For any $(a,b) \in G\backslash\partial G$ the transformation (2.11) from $D_{a,b}$ to $D'_{a,b}$ is a smooth diffeomorphism. Hence, system (2.6) in $D_{a,b}$ becomes the smooth system*

$$\dot{\nu} = \varphi_1(\nu,\omega), \qquad \dot{\omega} = \varphi_2(\nu,\omega), \tag{2.14}$$

in $D'_{a,b}$.

From remark 2.1, lemmas 2.3 and 2.5 we obtain the following result.

Lemma 2.6 *For $(a.b) \in G\backslash\partial G$ we have*

(1) *Any orbit of system (2.6), especially C_ω, is transversal to all lines $\{h = h_0, h_0 \in [0,1/6)\}$ in $D_{a,b}$.*

(2) *Any orbit of system (2.14), especially $\Omega_{a,b}$, is transversal to all straight lines between L_0 and L_3 in $D'_{a,b}$, the lines are parallel if $a = 0$, or are in the sector region with vertex $(\hat{\nu},\hat{\omega})$ if $a \ne 0$, see figure 1.*

We denote by $L^*_{\alpha\beta\gamma}$ the part of the straight line $L'_{\alpha\beta\gamma}$ in the (ν,ω)–plane, which is contained into $D'_{a,b}$. Let $C_U = \{(h,\omega) : 0 \le h \le 1/6,\ \omega = U(h)\}$ where

$$U(h) = U(h; a, b, \alpha, \beta, \gamma) = \frac{Z(h)}{N(h)} \equiv \frac{z_1 h + z_0}{n_1 h + n_0}, \tag{2.15}$$

with

$$\begin{aligned} z_1 &= 6b[2(b^2 + 3a - a^3 + 6a^2)\alpha - \lambda_1(a,b)\beta], \\ z_0 &= -b\alpha\lambda_2(a,b), \\ n_1 &= 12b(a^3 - 6a^2 - 3a - b^2)\gamma + [12(3a^2 + 2a + 1)b^2 - \\ &\qquad 24a^3(3a+1)(a-1)]\beta, \\ n_0 &= \lambda_2(a,b)[(a-1)\beta + b\gamma]. \end{aligned}$$

Lemma 2.7 *For any $(a,b) \in G\backslash\partial G$ and any constants α, β and γ, $L^*_{\alpha\beta\gamma}$ is tangent to an orbit of system (2.14) of order k (in particular to $\Omega_{a,b}$, at a point $(\nu,\omega)(h_0)$ for $h_0 \in (0,1/6)$), if and only if C_U is tangent to the corresponding orbit of system (2.6) of order k (in particular to C_ω, at $(h_0, \omega(h_0))$).*

Proof. Under the transformation (2.11) the line $L'_{\alpha\beta\gamma}$ becomes

$$\alpha + \beta\nu + \gamma\omega = \frac{N(h)\omega - Z(h)}{bL(h)} = 0, \tag{2.16}$$

where $L(h) \neq 0$ for $h \in [0,1/6]$ is given in (2.4), and the linear functions $N(h)$ and $Z(h)$ are defined in (2.15). If $N(h_0) \neq 0$, then for h near h_0 we can rewrite the above equality as

$$\alpha + \beta\nu + \gamma\omega = \frac{N(h)}{bL(h)}[\omega - U(h)] = 0.$$

This means that the transformation (2.11) maps the straight line $L^*_{\alpha\beta\gamma}$ to the curve C_U, and the lemma is proved for h near h_0 by lemma 2.5. Next, we show that we can skip all zero points of $N(h)$ for $h \in [0,1/6]$. In fact, if $N(h_0) = 0$ but $Z(h_0) \neq 0$, then the equation (2.16) is not satisfied, we do not need to consider it. If $N(h_0) = Z(h_0) = 0$, then the resultant of $N(h)$ and $Z(h)$ must be zero. By a direct computation and using $(a,b) \in G\backslash\partial G$, we obtain

$$\beta\,[6a(a+1)\alpha + (a-1)\beta + b\gamma] = 0.$$

If $\beta = 0$, then $N(h) = bL(h)\gamma$ and $Z(h) = -bL(h)\alpha$, this contradicts the non–zero property of $L(h)$ for $h \in [0,1/6]$. If $6a(a+1)\alpha + (a-1)\beta + b\gamma = 0$, then $L^*_{\alpha\beta\gamma} \in D'_{a,b}$ is parallel to L_0 and L_3 when $a = 0$, or passes through the vertex $(\hat{\nu}, \hat{\omega})$ of the sector when $a \neq 0$, see (2.12) and (2.13). By lemma 2.6, there is no orbit of system (2.14) tangent to it, and the assumption of the lemma is not satisfied. □

Lemma 2.8 *For any $(a,b) \in G \backslash \partial G$ and any constants α, β, and γ, there exist at most four points on $L^*_{\alpha\beta\gamma}$, counting their multiplicities, such that at each of these points the vector field (2.14) is tangent to $L^*_{\alpha\beta\gamma}$. In particular, if one of the endpoints of $L^*_{\alpha\beta\gamma}$ is $(\nu,\omega)(0)$ or $(\nu,\omega)(1/6)$, then the endpoint is included in these tangent points. If two of the endpoints of $L^*_{\alpha\beta\gamma}$ are $(\nu,\omega)(0)$ and $(\nu,\omega)(1/6)$, then the endpoints are included in these tangent points.*

Proof. By lemma 2.7 we only need to consider the number of tangent points on C_U (corresponding to $L^*_{\alpha\beta\gamma}$) with respect to the vector field (2.6) in the (h,ω)-plane. By using (2.6) and (2.15) we obtain

$$\dot{\omega} - U'(h)\dot{h}\Big|_{\omega=U(h)} = \phi(h, U(h)) - U'(h)T(h) = \frac{b^2 L^2(h)\, F(h)}{N^2(h)}, \qquad (2.17)$$

where $F(h) = F(h; a, b, \alpha, \beta, \gamma)$ is a polynomial in all its arguments, and of degree 4 in h. Besides, $F(h)$ has the factor h or $(h - \frac{1}{6})$ if $L^*_{\alpha\beta\gamma}$ has the endpoint $(\nu,\omega)(0)$ or $(\nu,\omega)(1/6)$, respectively. Note that we may suppose that $N(h) \neq 0$ for $h \in [0, 1/6]$, see the proof of lemma 2.7. □

3 Proof of lemma 1.3 and related results

We first give a proof of lemma 1.3, after we also prove some results which will be useful for additional studies.

Proof of Lemma 1.3 From (2.10) we see that the transformation (2.11) maps C_ω to $\Omega_{a,b}$, and by lemmas 2.3 and 2.4, C_ω, satisfying $\dot{h} \neq 0$ for $h \in (0, 1/6)$, is a regular curve. Hence, by lemma 2.5, to prove the regularity of $\Omega_{a,b}$ it is enough to show that C_ω stays in $D_{a,b}$, i.e. C_ω does not meet the straight line $\{\omega = \hat{\omega}\}$ in the (h,ω)–plane for $a \neq 0$. From (1.5), (2.7) and (2.13) we have that, for all $(a,b) \in G \backslash \partial G$, $\omega_0 < 0$, $\omega_1 > 0$ if $a > 0$, and $a\hat{\omega} > 0$ if $a \neq 0$, and

$$\omega_0 - \hat{\omega} = -\frac{5}{6} \frac{b\lambda_2(a,b)}{a(a+1)\lambda_3(a,b)} > 0, \text{ if } a < 0,$$

$$\hat{\omega} - \omega_1 = \frac{5}{6} \frac{b\lambda_1(a,b)}{a(a+1)[82a^3 + 93a^2 + 36a + 5(1-b^2)]} > 0, \text{ if } a > 0.$$

Hence, if $C_\omega \cap \{\omega = \hat{\omega}\} \neq \emptyset$, then by lemma 2.3 there are at least two points on the line $\{\omega = \hat{\omega}\}$ for $h \in (0, \frac{1}{6})$, at which the vector field (2.6) is tangent to this line, see figure 2.

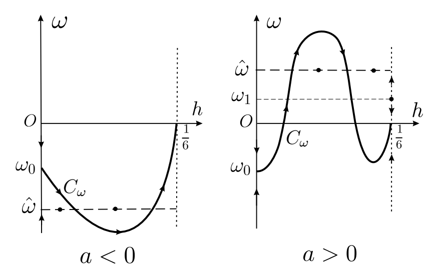

Figure 2. Hypothetical positions for C_ω and the straight line $\omega = \hat{\omega}$.

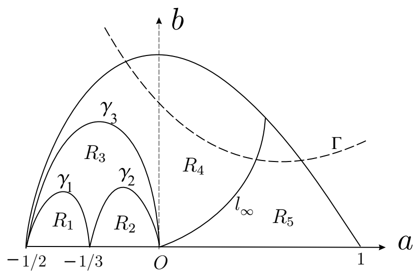

Figure 3. Partition of G by γ_1, γ_2, γ_3 and l_∞.

From (2.6) we have

$$\phi(h, \hat{\omega}) = \frac{5}{36} \frac{b^2 L^2(h) \psi(h)}{a^2(a+1)^2}, \tag{3.1}$$

where $L(h) \neq 0$ for $h \in (0, 1/6)$ (see (2.4)), $\psi(h) = \psi(h; a, b) = s_2 h^2 + s_1 h + s_0$, and

$$\begin{aligned} s_2 &= 72a(a+1)(4a^3 - b^2), \\ s_1 &= -12[(a^2 + a + 1)b^2 + a(3a+1)(2a^2 + a + 1)], \\ s_0 &= (3a+1)^2 + b^2. \end{aligned}$$

A computation shows that

$$\begin{aligned} \Delta = s_1^2 - 4 s_2 s_0 = 144[&(a^4 + 2a^3 + 5a^2 + 4a + 1)b^4 + 2a(2a+1) \cdot \\ &(3a^4 + 2a^3 + 9a^2 + 8a + 2)b^2 + a^2(a-1)^2(2a+1)^2(3a+1)^2]. \end{aligned} \tag{3.2}$$

In the region G (see (1.5)) $\Delta = 0$ defines 3 curves γ_i, $i = 1, 2, 3$, all of which are located in the region of $a < 0$. γ_1, γ_2, γ_3 and l_∞ (see (1.7)) divide G into 5 subregions R_i, $i = 1, 2, \cdots, 5$, shown in figure 3.

It is not difficult to check that for $(a, b) \in \gamma_1$, γ_2 and γ_3, $\psi(h)$ has a double zero h belonging to $(0, 1/6)$, $(-\infty, 0)$ and $(1/6, \infty)$, respectively. If $(a, b) \in R_3$, then $\Delta < 0$ and $\psi(h)$ has no real zeros. If $(a, b) \in R_1$, R_2 and R_4, then $\psi(h)$ has two zeros in $(0, 1/6)$, $(-\infty, 0)$ and $(1/6, \infty)$, respectively. If $(a, b) \in l_\infty$, then $\psi(h)$ has only one zero. If $(a, b) \in R_5$, then $\psi(h)$ has one zero in $(1/6,\infty)$ and other in $(-\infty,0)$. Consequently, the only case we need to consider is $(a,b)\in R_1$. We show that even in this case, one still has $C_\omega\cap\{\omega=\hat\omega\}=\emptyset$, and this is equivalent to prove that in (ν,ω)–plane the curve $\Omega_{a,b}$ does not meet the point $(\hat\nu,\hat\omega)$. Let $R=R_1\cup\gamma_1\cup R_3$. We claim that for any $(a,b)\in R$, we have the following properties:

(A) The tangent vector of $\Omega_{a,b}$ at $(\nu,\omega)(0)$, $\eta_0=(\nu'(0),\omega'(0))$, pointing to the region $D'_{a,b}$, is transversal to the straight line L_0, which joints the origin to the point $(\nu,\omega)(0)$. And η_0 is located clockwise from the vector v_{01}, going from the point $(\nu,\omega)(0)$ to the point $(\nu,\omega)(1/6)=(1,0)$.

(B) The tangent line to $\Omega_{a,b}$ at $(\nu,\omega)(\frac{1}{6})$, $\eta_{\frac{1}{6}}$, is just the line L_3, contained in the boundary of $D'_{a,b}$.

(C) Near the endpoint $(\nu,\omega)(0)$, the curve $\Omega_{a,b}$ keeps its convexity, and is located between η_0 and v_{01}.

Now we prove the claim. From (2.7), (2.9) and lemma 2.4(4), for all $(a,b)\in G\backslash\partial G$ we have

$$\begin{vmatrix} \nu(0) & \omega(0) \\ \nu'(0) & \omega'(0) \end{vmatrix} = \frac{180b\lambda_1(a,b)}{(\lambda_3(a,b))^2}>0,$$

and for $(a,b)\in R$,

$$\begin{vmatrix} \nu'(0) & \omega'(0) \\ 1-\nu(0) & -\omega(0) \end{vmatrix} = \frac{5}{2}\frac{b(21a^2-18a+5-7b^2)\lambda_2(a,b)}{(\lambda_3(a,b))^2}>0,$$

because the branch of the hyperbola $\Gamma:\ \{7b^2=21a^2-18a+5\}$, which satisfies $b>0$, does not meet any curve γ_i, see figure 2. Hence, statement (A) follows. (We remark here that if (a,b) moves above the curve Γ, then the property (A) does not hold, figure 6 shows this case.) Statement (B) follows from lemma 2.4(5), and statement (C) follows from the expression $\nu''(0)\omega'(0)-\nu'(0)\omega''(0)$ in lemma 2.4(4), where we rewrite $f_3(a,b)$ as

$$55\Big[\Big(b^2-\frac{53}{55}\Big)^2+1-\Big(\frac{53}{55}\Big)^2\Big]+6(21a^2-34a)b^2-a(81a^3+204)+162a^2(2a+1)>0,$$

for $-\frac{1}{2}<a<0$. Hence, the claim is proved.

We show that if $(a,b) \in R$, then $\Omega_{a,b}$ is located above η_0, hence it never meets the point $(\hat{\nu}, \hat{\omega})$. We call the following technique a *deformation–inflection principle*, and we will use it several times later on.

Since $(a,b) = (-\frac{1}{3}, \frac{2}{3}) \in R$, the curve $\Omega_{-\frac{1}{3},\frac{2}{3}}$ is convex (lemma 3.3 of [15]), and the tangent lines $\eta_h \in T_{\Omega_{-\frac{1}{3},\frac{2}{3}}}$ (see (1.20)) cut the line L_0 below the point $(\nu,\omega)(0)$ for all $h \in (0, 1/6]$. If there is a $(a_1, b_1) \in R$ such that Ω_{a_1,b_1} does not stay above the tangent line η_0, then by the properties (A)-(C) for h near 0 and 1/6 the tangent line $\eta_h \in T_{\Omega_{a_1,b_1}}$ still cuts L_0 below $(\nu,\omega)(0)$, but we must find some $h \in (0, 1/6)$, such that $\eta_h \in T_{\Omega_{a_1,b_1}}$ cuts L_0 above $(\nu,\omega)(0)$. Since the slope of the tangent lines of a curve takes its minimum or maximum at an inflection point of the curve, by the deformation of $\Omega_{a,b}$ as $(a,b) \in R$ varies continuously from $(-\frac{1}{3}, \frac{2}{3})$ to (a_1, b_1), we must find a value $(a_2, b_2) \in R$, such that Ω_{a_2,b_2} has an inflection point $(\nu,\omega)(h_0)$, and the tangent line, say $L_{\alpha\beta\gamma}$, of Ω_{a_2,b_2} at this point passes through the point $(\nu,\omega)(0)$, see figure 4. We show that this is impossible. In fact, by lem-

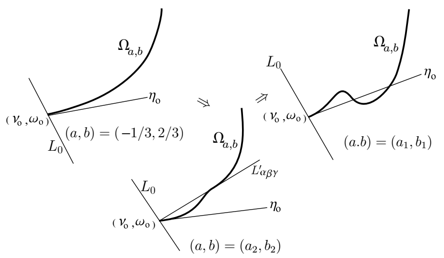

Figure 4. Deformation of $\Omega_{a,b}$.

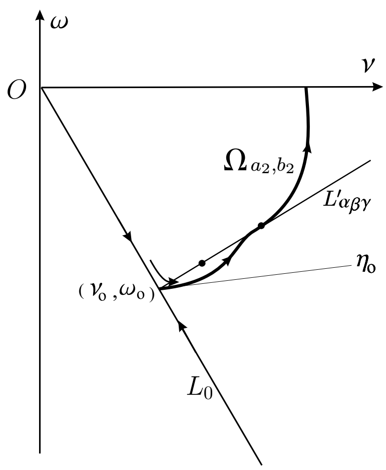

Figure 5. Tangent points on $L'_{\alpha\beta\gamma}$.

mas 2.7 and 2.8, for the corresponding function $U(h) = U(h; a_2, b_2; \alpha, \beta, \gamma)$ the two curves C_ω and C_U must be tangent at $(h_0, \omega(h_0))$ of order at least 3. Since $\beta^2 + \gamma^2 \neq 0$, without loss of generality we suppose $\gamma \neq 0$, then by a rescaling we can change to $\gamma = 1$. The condition that $L'_{\alpha\beta\gamma}$ passes through $(\nu,\omega)(0)$ implies $\alpha = -(\nu_0\beta + \omega_0)$. In this case (2.17) becomes

$$\dot{\omega} - U'(h)\dot{h}\Big|_{\omega=U(h)} = \frac{30b^2 L(h)^2\, h\, V(h)}{N(h)^2(\lambda_3(a,b))^2}, \qquad (3.3)$$

where $V(h) = V(h,\beta;a,b)$ is a polynomial in all its arguments, of degree 3 in h and of degree 2 in β. By the above analysis, $h = h_0 \in (0,1/6)$ is at least a double zero of $V(h)$ for some $\beta = \beta_0$, i.e. for $(a,b) = (a_2,b_2)$, (h_0,β_0) is a solution of the system of equations

$$V(h,\beta;a,b) = 0, \;\; V'_h(h,\beta;a,b) = 0. \tag{3.4}$$

Note that h_0 is the only (possible) solution of (3.4) for a fixed β, since V has degree 3 in h. If we also have $V''(h_0) = 0$, then Ω_{a_2,b_2} is tangent to $L_{\alpha\beta\gamma}$ at the point $(\nu,\omega)(h_0)$ of order at least 4 (i.e. h_0 is at least a triple zero of $V(h)$). Since (ν_0,ω_0) is a saddle, and Ω_{a_2,b_2} is one of its unstable separatrices, we must find at least one more point on $L'_{\alpha\beta\gamma}$ between (ν_0,ω_0) and $(\nu,\omega)(h_0)$, such that the vector field (2.14) is also tangent to $L'_{\alpha\beta\gamma}$ at this point (see figure 5). This is called the saddle property of Ω_{a_2,b_2} at (ν_0,ω_0). This contradicts the fact that $V(h)$ is of degree 3 in h. If $V''(h_0) \neq 0$, then eliminating β from (3.4) we obtain $v(h;a,b) = 0$, where v is also a polynomial in h, a and b. So h_0 must be a simple zero of $v(h;a_2,b_2)$. Computations show that for all $(a,b) \in R$ we have

$$v(0;a,b) = (\lambda_2(a,b))^4 f_3(a,b) > 0,$$
$$v(\tfrac{1}{6};a,b) = 5(2a+1)^6(\lambda_1(a,b))^5[b^2 + (2a+1)(11a^2+2a+11)] > 0,$$

where $f_3(a,b) > 0$ for $-1/2 < a < 0$ as it was shown above. Hence $v(h;a_2,b_2) = 0$ has even number (counting multiplicity) of solutions for $h \in (0,1/6)$. This means that besides Ω_{a_2,b_2} has an inflection point $(\nu,\omega)(h_0)$ and the tangent line at this point passes through $(\nu,\omega)(0)$ with slope $-\beta_0$, there is another orbit of system (2.14) with an inflection point $(\nu,\omega)(h'_0)$, and the tangent line at this point also passes through $(\nu,\omega)(0)$ with slope $-\beta'_0$. But direct calculation shows that $v(h;-1/3,2/3) = 0$ has no solution for $h \in (0,1/6)$ (by the method described in Appendix A). Therefore, when $(a,b) \in R$ varies continuously from $(-1/3,2/3)$ to (a_2,b_2), we must find a $(a_3,b_3) \in R$, such that there is an orbit γ of system (2.14), and a straight line L, passing through the point $(\nu,\omega)(0)$, with the property that γ is tangent to L at a point of order 5, or of order 4 plus a crossing point nearby. Hence, there exist at least 4 points (counting their multiplicities) on $L \cap (D'_{a,b} \backslash \partial D'_{a,b})$, at each of these points the vector field (2.14) is tangent to L. This also contradicts the fact that the function $V(h)$ in (3.3) is a polynomial in h of degree 3. □

Lemma 3.1 *For $(a.b) \in G \backslash \partial G$ we have*

(1) $L_0 = \xi_0$, *and* $\xi_0 \cap \Omega_{a,b} = \{(\nu,\omega)(0)\}$.

(2) $\Sigma_{a,b} \cap L_0 = \{(P,Q)(0)\}$, *and* $\Sigma_{a,b} \backslash \{(P,Q)(0)\}$ *stays above* L_0.

(3) *In* $D'_{a,b}$ *the line* $L_3 = \eta_{1/6}$ *is always located on the right hand side of the straight line* L_2, *joining the two endpoints of* $\Omega_{a,b}$.

Proof. Statement (1) follows from lemma 2.2(1) and lemma 2.6(2), see also the definitions (2.12) and (1.19). If the statement (2) is not true, then we take $L_0 = L_{\alpha\beta\gamma}$, by lemmas 1.1(1) and 2.2 the Abelian integral $I(h)$ has a double zero at $h = 0$ plus one more zero at some $h_1 \in (0, 1/6]$, hence $I''(h)$ has at least one zero $h_2 \in (0, 1/6)$, i.e. $(\nu, \omega)(h_2) \in L_0 \cap \Omega_{a,b}$. This contradicts statement (1). By (2.12), (2.7) and (2.9) we have

$$\begin{vmatrix} 1-\nu(0) & -\omega(0) \\ -(2a+1)(3a+1) & b \end{vmatrix} = \frac{5b\lambda_2(a,b)}{\lambda_3(a,b)} > 0.$$

Therefore, statement (3) follows. □

By (2.7) and (2.9) we find that the slope of the straight line L_2, passing through $(\nu, \omega)(0)$ and $(\nu, \omega)(1/6) = (1, 0)$, is

$$k_2(a,b) = \frac{6b}{5b^2 + 9a^2 - 1}. \tag{3.5}$$

The next lemma gives the fixed relative position of the lines L_2 and L_1, passing through $(\nu, \omega)(1/6) = (1, 0)$ and $(P, Q)(1/6)$.

Lemma 3.2 *For any* $(a, b) \in G \backslash \partial G$ *we have that in the half–plane* $\{(\nu, \omega) : \omega < 0\}$, L_2 *is always located on the right hand side of* L_1, *see figure 6.*

Proof. Note that both L_2 and L_1 pass through the same point $B(1, 0)$, and that the slope $k_1(a, b)$ of L_1, see (1.18), is always positive. Thus, by (3.5) the conclusion of the lemma is true if $g_1(a, b) = 5b^2 + 9a^2 - 1 \le 0$. So, in the following we prove $0 < k_1(a, b) < k_2(a, b)$ for $(a, b) \in G' \subset G \backslash \partial G$, where $g_1(a, b) > 0$, i.e. above the curve $\zeta_1 : \{g_1(a, b) = 0\}$, see figure 7.

By the definitions of $P(h)$ and $Q(h)$, this is equivalent to prove

$$\iint_{\mathrm{Int}(\delta(\frac{1}{6}))} (y - k_2(a,b)(x-1))\, dxdy > 0, \ (a,b) \in G'. \tag{3.6}$$

In polar coordinates (r, θ) given by $x = 1 + r\cos\theta$, $y = r\sin\theta$, the oval $\delta(\frac{1}{6})$ is given by

$$r = r(\theta) = \frac{3(\cos^2\theta - (2a+1)\sin^2\theta)}{2(b\sin^3\theta + 3a\sin^2\theta\cos\theta - \cos^3\theta)},$$

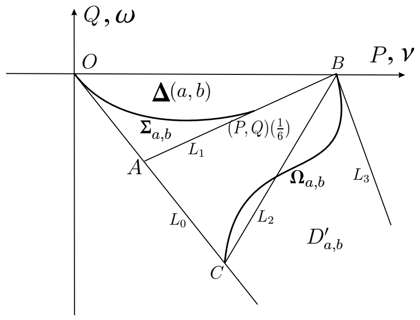

Figure 6. The relative positions of $\Sigma_{a,b}$, $\Omega_{a,b}$ and L_i.

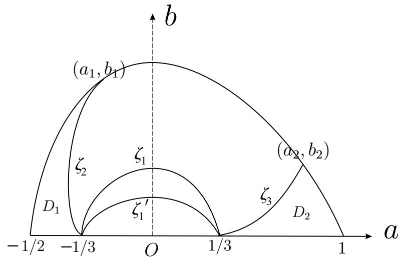

Figure 7. The regions D_1 and D_2.

where $\theta \in \tan^{-1}(-t_a, t_a)$, $t_a = (2a+1)^{(-1/2)}$. Let $t = \tan\theta$, then the double integral (3.6) is positive, if and only if

$$J(a,b) = \int_{-t_a}^{t_a} \left(\frac{(2a+1)t^2 - 1}{bt^3 + 3at^2 - 1} \right)^3 (k_2(a,b) - t)\, dt > 0. \tag{3.7}$$

From (3.7) it is obvious that if $k_2(a,b) \geq t_a$, then we immediately have $J(a,b) > 0$. From (3.5) we have $(k_2(a,b))^2 - (t_a)^2 = g_2(a,b)/((2a+1)(g_1(a,b))^2)$, where

$$g_2(a,b) = -25b^4 - (90a^2 - 72a - 46)b^2 - (3a-1)^2(3a+1)^2. \tag{3.8}$$

The locus of $g_2(a,b) = 0$ in G is a piecewise smooth curve with three pieces ζ_2, ζ_1' and ζ_3, and two vertices at $(a,b) = (\pm 1/3, 0)$, shown in figure 7. Note that only ζ_2 and ζ_3 stay in G'. The above analysis shows that $J(a,b) > 0$ if (a,b) belongs to the region between ζ_2 and ζ_3, and above ζ_1 (ζ_2 and ζ_3 are included). Hence, we only need to prove $J(a,b) > 0$ for (a,b) belongs to the curved triangle regions D_1 and D_2, which are located in G' and on the left hand side of ζ_2 or on the right hand side of ζ_3, respectively.

Case (A): $(a,b) \in D_1$. We will prove that $P(1/6) < \nu_0$ for $(a,b) \in D_1$, which is stronger than $k_1(a,b) < k_2(a,b)$, see lemma 3.1(2) and figure 6.

By the definition of $P(1/6)$, we need to prove

$$\iint_{\mathrm{Int}(\delta(\frac{1}{6}))} (x - \nu_0)\, dxdy < 0, \ (a,b) \in D_1. \tag{3.9}$$

From (2.9) we have $\nu_0 > 0$ and $1 - \nu_0 = (5b^2 + 9a^2 - 1)/\lambda_3(a,b) > 0$ for $(a,b) \in D_1$. Hence, the straight line $L = \{(x,y) : x = \nu_0\}$ cuts $\mathrm{Int}(\delta(\frac{1}{6}))$, located in the strip $-1/2 < x < 1$, into two parts P_L (on the left hand side of L) and P_R (on the right hand side of L). We denote by $\bar{P}_R$ the reflection of P_R with respect to the straight line L. Then (3.9) follows from $\bar{P}_R \subset P_L$, which is, by (1.4), a consequence of

$$H(\nu_0 + u, y) - H(\nu_0 - u, y) = \frac{4uJ(u, y_i(u))}{(\lambda_3(a,b))^2} > 0,$$

where $u \in (0, 1-\nu_0)$ is positive, $y_1(u) < 0 < y_2(u)$ satisfying $(u, y_i(u)) \in \mathrm{Int}(\delta(\frac{1}{6}))$, and

$$J(u, y) = (3ay^2 - u^2)(\lambda_3(a,b))^2 + 18(1-a)(5b^2 + 9a^2 - 1).$$

We first prove that $J(u, y_i(u)) > 0$ for u near $1 - \nu_0$ and near 0. Note that $y_i(u) \to 0$ as $u \to 1 - \nu_0$, and by (2.9)

$$J(1-\nu_0, 0) = (5b^2 + 9a^2 - 1)(-5b^2 - 9a^2 - 18a + 19) > 0,$$

which is obviously satisfied for $(a, b) \in D_1$. To prove $J(0, y_i(0)) > 0$, we eliminate y from $H(\nu_0, y) - 1/6 = 0$ and $J(0, y_i(0)) = 0$, then obtain $K_1(a, b) = 0$, which is impossible because $(a, b) \in D_1$ (see example 6.1 of appendix A). This means $J(0, y_i(0))$ has a fixed sign for all $(a, b) \in D_1$, and we can choose a special value of (a, b) to determine it. It is proved in Lemma 4.1 of [15] that $J(u, y_i(u)) > 0$ for all $u \in (0, 1-\nu_0)$ and $(a, b) \in l_2$. Note that $l_2 \cap D_1 \neq \emptyset$.

Thus, we have obtained two facts: (1) $J(u, y_i(u)) > 0$ for u near 0 and near $1 - \nu_0$, and for all $(a, b) \in D_1$. (2) There are some points (a, b) in D_1, for which $J(u, y_i(u)) > 0$ for all $u \in (0, 1-\nu_0)$. Hence, if there is a point (a_1, b_1) in D_1, such that $J(u, y_i(u)) = 0$ has roots for some $u \in (0, 1-\nu_0)$, then, by continuity, we must find another point (a_2, b_2) in D_1, such that $J(u, y_i(u)) = 0$ has (at least) a double root for some $u \in (0, 1-\nu_0)$. We shall show that this is impossible. In fact, eliminating y from $H(u+\nu_0, y) = 1/6$ and $J(u, y) = 0$ gives $J_1(u; a, b) = 0$, where J_1 is a polynomial in u, a and b. Then we eliminate u from $J_1(u; a, b) = 0$ and $\partial J_1(u; a, b)/\partial u = 0$, and obtain $K_1(a, b)K_2(a, b) = 0$, where $K_1(a, b) \neq 0$ for $(a, b) \in D_1$ as mentioned above, and $K_2(a, b) \neq 0$ for $(a, b) \in D_1$ is proved in example 6.2 of appendix A. This contradiction finishes the proof for case (A).

Case (B): $(a, b) \in D_2$. It is easy to verify that $D_2 \subset G_3$. Recall that $G_3 = \{(a, b) \in \mathbb{R}^2 : 0 < b < 2\sqrt{a^3} \text{ for } 0 < a \leq 1/2;\ 0 < b < (1-a)\sqrt{2a+1} \text{ for } 1/2 \leq a < 1\}$. Hence, $bt^3 + 3at^2 - 1 = 0$ has 3 real roots t_1, t_2 and t_3, satisfying

$$t_1 < t_2 < -t_a < 0 < t_a < t_3, \quad bt_j^3 + 3at_j^2 - 1 = 0, \tag{3.10}$$

and

$$t_1 + t_2 + t_3 = -\frac{3a}{b}, \quad t_1t_2 + t_2t_3 + t_3t_1 = 0, \quad t_1t_2t_3 = \frac{1}{b}. \qquad (3.11)$$

By (3.5) and using repeatedly (3.10) and (3.11), we obtain

$$\left(\frac{(2a+1)t^2 - 1}{bt^3 + 3at^2 - 1}\right)^3 (k_2(a,b) - t) = \frac{1}{\kappa(a,b)} \sum_{i,j=1}^{3} \frac{\alpha_{ij}}{(t - t_j)^i}, \qquad (3.12)$$

where $\kappa(a,b) = 27g_1(a,b)(4a^3 - b^2)^3 > 0$ for $(a,b) \in D_2$, and

$$\alpha_{ij} = \alpha_{ij}(a,b) = (4a^3 - b^2)^{[\frac{i}{2}]}[(2a+1)(t_j^2 - t_a^2)]^{i-1}(\xi_{j2}t_j^2 + \xi_{j1}t_j + \xi_{j0}),$$

where $t_j = t_j(a,b)$ are the roots of (3.10), $[p]$ denotes the integer part of p, and

$$\xi_{12} = -10b^8 + (195a^3 + 105a^2 + 75a + 9)b^6 - (350a^6 - 1077a^5 - 1203a^4 - 698a^3 - 144a^2 + 15a + 5)b^4 - 2a^3(807a^5 - 315a^4 - 198a^2 - 746a^3 + 51a + 17)b^2,$$

$$\xi_{11} = 5(3 - 5a)b^7 + 3(90a^4 - 109a^3 - 41a^2 - 17a - 3)b^5 - 2a(205a^6 - 1542a^5 - 840a^4 - 220a^3 - 27a^2 - 6a - 2)b^3 - 2a^4(1845a^5 - 945a^4 - 1742a^3 - 450a^2 + 105a + 35)b,$$

$$\xi_{10} = 5a(a+3)b^6 - 3a(105a^4 + 214a^3 + 116a^2 + 26a + 3)b^4 + a^2(640a^6 - 147a^5 - 1929a^4 - 1654a^3 - 378a^2 + 57a + 19)b^2 + 32a^5(a-1)(9a^2 - 1)(2a+1)^2,$$

$$\xi_{22} = (5a - 8)b^5 - (65a^4 + 85a^3 + 99a^2 + 37a + 2)b^3 - a^2(9a^2 - 1)(13a^2 + 26a + 9)b,$$

$$\xi_{21} = -(65a^2 + 86a + 20)b^4 + (125a^5 - 133a^4 - 325a^3 - 113a^2 + 10a + 4)b^2 + a^3(25a + 11)(a-1)(9a^2 - 1),$$

$$\xi_{20} = 10b^5 + (5a^3 + 180a^2 + 81a + 16)b^3 - a(3a+1)(51a^3 - 107a^2 - 9a + 9)b,$$

$$\xi_{32} = (15a^2 + 27a + 6)b^4 + 3a(5a^3 + 13a^2 - a - 1)b^2,$$

$$\xi_{31} = 5b^5 - (5a^3 - 42a^2 + 1)b^3 - a^2(45a^3 - 63a^2 - a + 3)b,$$

$$\xi_{30} = -(10a+11)b^4+(10a^4 - 22a^3 - 27a^2+2a+1)b^2+2a^3(a-1)(9a^2-1).$$

From (3.7) and (3.12) we have

$$\kappa(a,b)\,J(a,b) = \sum_{j=1}^{3}\left\{\alpha_{1j}\ln\frac{t_j-t_a}{t_j+t_a}+\frac{2\,\alpha_{2j}\,t_a}{t_j^2-t_a^2}-\frac{2\,\alpha_{3j}\,t_a\,t_j}{(t_j^2-t_a^2)^2}\right\}. \qquad (3.13)$$

Note that on the left hand side of (3.12) the numerator is a polynomial in t of degree lower than that of the denominator, we must have $\alpha_{11}+\alpha_{12}+\alpha_{13}=0$. By calculation, (3.13) can be simplified as

$$\alpha_{11}\ln\frac{A_1}{A_2}+\alpha_{13}\ln\frac{A_3}{A_2}+B,$$

where $A_j = (t_j-t_a)/(t_j+t_a)$, and

$$B = 6\sqrt{2a+1}(4a^3-b^2)(5a^3+36a^2-9a+10b^2)\lambda_2(a,b)\,b > 0.$$

From now on we simplify the notation using $\alpha_j = \alpha_j(a,b)$ to denote α_{1j}. Therefore, by using (3.11) we have

$$\prod_{j=1}^{3}\alpha_j(a,b) = (4a^3-b^2)^4\lambda_1(a,b)\,K_3(a,b),$$

where $K_3(a,b)\neq 0$ for $(a,b)\in D_2$, see example 6.3 of appendix A. It is easily to determine that

$$\alpha_1 > 0, \quad \alpha_2 < 0, \quad \alpha_3 < 0, \quad \text{if } (a,b)\in D_2. \qquad (3.14)$$

Thus, $J(a,b) > 0$ if $(a,b)\in D_2$ is equivalent to

$$\widetilde{J}(a,b) = -\frac{\alpha_1}{\alpha_3}\ln\frac{A_1}{A_2}-\ln\frac{A_3}{A_2}-\frac{B}{\alpha_3} > 0, \quad \text{if } (a,b)\in D_2, \qquad (3.15)$$

which follows from the following two statements:

(i) $\widetilde{J}(a,0)=0$ for $a\in(1/3,1)$; and

(ii) $\partial\widetilde{J}(a,b)/\partial b > 0$ for $(a,b)\in D_2$.

Now we shall prove these two statements. Note that $\kappa(a,0) = 1728a^9(9a^2-1) > 0$ for $a\in(1/3,1)$. It is easy to see that if $b\to 0$, then $t_1\to-\infty$, $bt_1\to -3a$, $t_2\to -1/\sqrt{3a}$ and $t_3\to 1/\sqrt{3a}$. This implies

$$-\alpha_3(a,0) = 32a^5(1-a)(9a^2-1)(2a+1)^2 > 0, \quad \text{if } a\in(1/3,1).$$

Hence the claim (i) follows from $J(a,0) = 0$, which is obviously true by (3.5) and (3.7). To prove the claim (ii), we first find a simple form for $\partial\widetilde{J}(a,b)/\partial b$. We simply denote the derivative with respect to b by $'$. Note that $\alpha_j' = \partial\alpha_j/\partial b + (\partial\alpha_j/\partial t_j)(\partial t_j/\partial b)$. Using repeatedly (3.10) and (3.11), we obtain

$$\Delta_1(a,b) \equiv \alpha_1\alpha_3' - \alpha_1'\alpha_3 = 4(4a^3 - b^2)^{(5/2)}K_4(a,b)/\sqrt{3} > 0,$$

$$\Delta_2(a,b) \equiv \alpha_1(\ln(A_1/A_2))' + \alpha_3(\ln(A_3/A_2))' = 2\sqrt{2a+1}K_5,$$

where $K_4(a,b) > 0$ for $(a,b) \in D_2$, see example 6.4 of appendix A, and $K_5 = -10b^6 + (175a^3 + 130a^2 + 65a + 14)b^4 + a(1127a^4 + 1270a^3 + 412a^2 - 38a - 19)b^2 + 32a^4(2a+1)(9a^2-1)$. Hence, multiplying $\partial\widetilde{J}(a,b)/\partial b$ by $\alpha_3^2/\Delta_1(a,b)$, we obtain

$$J^*(a,b) = \ln\frac{A_1}{A_2} + \frac{B\alpha_3' - \alpha_3(\Delta_2(a,b) + B')}{\Delta_1(a,b)}.$$

To prove $J^*(a,b) > 0$ for $(a,b) \in D_2$, we repeat the same argument as above, i.e. to show

(I) $\lim_{b\to 0}(1-a)J^*(a,b) > 0$ for $(a,b) \in D_2$; and

(II) $\partial\widetilde{J}^*(a,b)/\partial b > 0$ for $(a,b) \in D_2$.

A calculation gives for $\bar{J}(a) = \lim_{b\to 0}(1-a)J^*(a,b)$ the following

$$(1-a)\left(\ln\frac{1-a}{5a+1+2\sqrt{3a(2a+1)}}\right.$$
$$\left. + \frac{2\sqrt{3a(2a+1)}(3a+1)(45a^3+387a^2+35a-35)}{-1845a^5+945a^4+1742a^3+450a^2-105a-35}\right).$$

It is not difficult to find that $\bar{J}(1/3) \approx 0.1358$, $\bar{J}(1) = 0$, and $\bar{J}(a)/(1-a)$ has a unique maximal value at $a \approx 0.9543$ for $a \in (1/3, 1)$. Thus, the claim (I) is proved. We use the factor $(1-a)$ here, since the straight line $\{(a,b) : a = 1\}$ is out of D_2. Making derivative on $J^*(a,b)$ with respect to b, we obtain

$$\frac{\partial J^*(a,b)}{\partial b} = \frac{24\sqrt{3}\,b\,(2a+1)^{7/2}(-\alpha_3(a,b))K_6(a,b)}{\sqrt{4a^3-b^2}\lambda_1(a,b)K_4^2(a,b)} > 0, \text{ if } (a,b) \in D_2,$$

where $K_6(a,b) > 0$ for $(a,b) \in D_2$, see example 6.5 of appendix A, and $-\alpha_3(a,b) > 0$ was shown in (3.14). Hence, the claim (II) is proved. Consequently, (ii) is also proved. □

We denote the intersection point of the two straight lines L_0 and L_1 by $A(\nu_A, \omega_A)$. lemma 3.2 implies that

$$0 < \nu_A < \nu_0 \text{ for } (a,b) \in G \backslash \partial G. \tag{3.16}$$

Lemma 3.3 *For any $(a,b) \in G\backslash\partial G$ if $\Omega_{a,b}$ has an inflection point, then the tangent line of $\Omega_{a,b}$ at this point does not pass through the point A, where $\{A\} = L_0 \cap L_1$.*

Proof. We suppose the contrary, i.e. there is a straight line $L : \alpha + \beta\nu + \gamma\omega = 0$, passing through the point A, and L is tangent to $\Omega_{a,b}$ at an inflection point $(\nu, \omega)(\bar{h})$ for some $(a,b) \in G\backslash\partial G$. We first give a restriction on the slope $-\beta/\gamma$ of the line L as follows

$$k_0(a,b) < -\beta/\gamma < k_1(a,b), \tag{3.17}$$

where $k_0(a,b) < 0 < k_1(a,b)$ are the slopes of L_0 and L_1, respectively, see (1.18). At the end of the proof we will give a reason for this restriction. Hence, we will not consider the vertical line passing through the point A, so we suppose $\gamma \neq 0$, and without loss of generality we take $\gamma = 1$. Therefore, we have $\alpha = \alpha(\beta) \equiv -(\beta + b/(a-1))\nu_A$.

By lemmas 2.7, 2.8 and (2.17), $\bar{h} \in (0, 1/6)$ must be a double zero of the function $F(h) = F(h; a, b, \alpha(\beta), \beta, 1)$, F is a polynomial in all its arguments, and has degree 4 in h. Eliminating β from

$$F(h) = 0, \ F'(h) = 0, \tag{3.18}$$

we obtain

$$b^4(a-1)^4(\lambda_1(a,b))^4 h^2 F_1(\nu_A, a) F_2(h; \nu_A, a, b) = 0, \tag{3.19}$$

where the first three factors are non–zero for $(a,b) \in G\backslash\partial G$, F_2 is also a polynomial in all its arguments with a long expression, but we will only use it at $h = 0$ and $h = 1/6$, and

$$F_1(\nu_A, a) = 6a(a+1)\nu_A - (a-1).$$

If $a = 0$ then $F_1 = 1$; if $a \neq 0$, then $F_1 = 6a(a+1)(\nu_A - \hat{\nu})$, where $\hat{\nu}$ is defined in (2.13). Note that $\hat{\nu} < 0$ if $a > 0$, and $\hat{\nu} > \nu_0$ if $a < 0$. Hence, by (3.16), we have $F_1(\nu_A, a) > 0$ for all $(a,b) \in G\backslash\partial G$. Next, again by the inequality (3.16) we have

$$F_2(0; \nu_A, a, b) = 10\lambda_2(a,b)\lambda_3(a,b)\nu_A^2(\nu_0 - \nu_A) > 0.$$

We also have

$$F_2(1/6;\nu_A,a,b)=\frac{(2a+1)^6}{12}(\lambda_1(a,b))^5(F_1(\nu_A,a))^2F_3(\nu_A,a,b)>0,$$

where $F_3(x,a,b)=p(a,b)x+q(a,b)$, and

$$p(a,b)=6[(47a^+47a+10)b^2+(7a+2)(2a+1)(11a^3+39a^2+25a+5)],$$
$$q(a,b)=(1-a)[7b^2+(2a+1)(77a^2+38a+5)].$$

Since $F_3(0,a,b)=q(a,b)>0$ and

$$F_3(\nu_0,a,b)=\frac{35(1-a)\lambda_2(a,b)[b^2+(2a+1)(11a^2+26a+11)]}{\lambda_3(a,b)}>0,$$

by using (3.16) and the linearity of F_3 in x, we obtain $F_3(\nu_A,a,b)>0$.

Now we go back to the proof of the lemma. We have supposed that $\bar{h}$ is a solution of (3.18). If $F''(\bar{h})=0$, then the straight line L is tangent to $\Omega_{a,b}$ at $(\nu,\omega)(\bar{h})$ of order at least 4, and by the saddle property at $(\nu,\omega)(0)$, the vector field (2.14) must be tangent to L also at some point between $(\nu,\omega)(0)$ and $(\nu,\omega)(\bar{h})$, hence $F(h)$ has four zeros for $h\in(0,1/6)$. On the other hand, we consider the direction of the vector field (2.14) at the two intersection points of L with the boundary of domain $D'_{a,b}$, i.e the straight lines L_0 and L_3, see (2.12). Since the vector field (2.14) is induced from (2.6), by lemmas 2.3 and 2.5 and the condition (3.16), A is between the improper unstable node $(0,0)$ and the saddle $(\nu,\omega)(0)$ on the invariant line L_0, so at this point the direction of the vector field (2.14) is downward with respect to L. On the invariant line L_3, $B(1,0)$ is an improper stable node, and the other singular point (a saddle), say M, may be above or below B, but never meet $\{D\}=L\cap L_3$, since there is already four tangent points (counting their multiplicities) of (2.14) along L. If M is above B or below D, then the vector field (2.14) is upward with respect to L at D, this implies an odd number of tangent points (counting their multiplicities) along L, and this contradicts the fact that $F(h)$ is a polynomial in h of degree four (lemma 2.8). If M is between B and D, then the stable manifold $W^s(M)$ of M in $D'_{a,b}$ must cut L. Since, by lemma 2.4(1), $W^s(M)$ can not meet $\Omega_{a,b}$, the unstable manifold of the saddle point $(\nu,\omega)(0)$. Hence, we may use $W^s(M)$ instead of L_3, and this leads to the same contradiction.

If $F''(\bar{h})\neq 0$, then $\bar{h}$ must be a simple zero of the function (3.19). But as we have proved equation (3.19) has an even number of solutions for $h\in(0,1/6)$; a direct computation shows that it has no solution for $(a,b)=(-1/3,2/3)$. Thus, by the same deformation technique used in the proof of lemma 1.3, we would find (a',b'), β' and $h'\in(0,1/6)$, such that at a point $(\nu, \omega)(h')$ an orbit of (2.14) is tangent to L (with slope $-\beta'$ and passing through the point A) of order 5 or order 4 plus a simple intersection point nearby, and this provides the same contradiction.

We remark that the above proof is based on the restriction (3.17). Since the point A is on L_0, the assumption $k_0(a,b) < -\beta/\gamma$ is necessary to guarantee the intersection of L with $\Omega_{a,b}$, but if $-\beta'/\gamma' \geq k_1(a,b)$ (i.e. the line L is located above L_1 in $D'_{a,b}$), then the corresponding point $\{M'\} = L \cap L_3$ would be above B, and the argument we used before is not valid in this case, because the directions of the vector field (2.14) are downward with respect to L at its two endpoints. But lemma 3.4 shows that during the deformation process any line L, passing through the point A and tangent to the vector field (2.14) with multiplicity of tangency three, never pass the position L_1, so the assumption $-\beta/\gamma < k_1(a,b)$ is reasonable, and the proof is finished. □

Lemma 3.4 *For any $(a,b) \in G \backslash \partial G$, the tangency of any orbit of vector field (2.14) along the line L_1 is less than three.*

Proof. We use lemmas 2.7, 2.8 and (2.17) once more. In this special case we may take $\gamma = 1$ and $\alpha = -\beta$, since L_1 passes through the point $B(1,0)$ and its slope $k_1(a,b) = -\beta(a,b)$ satisfies $0 < -\beta(a,b) < +\infty$. In this case the function $F(h) = (6h-1)R(h)$, where $R(h) = R(h; a, b, \beta)$ is a polynomial in all its arguments, and has degree 3 in h, where $\beta = \beta(a,b)$. We claim that

$$\begin{aligned} R(0) &= -\beta(\lambda_2(a,b))^2[(5b^2 + 9a^2 - 1)\beta + 6b] > 0, \\ R(\tfrac{1}{6}) &= (2a+1)(\lambda_1(a,b))^2[(2a+1)(3a+1)\beta - b]^2 > 0. \end{aligned} \tag{3.20}$$

Now we prove the claim for $(a,b) \in G \backslash \partial G$. Since $-\beta = k_1(a,b) > 0$, we certainly have $R(0) > 0$ if $5b^2 + 9a^2 - 1 \leq 0$. If $5b^2 + 9a^2 - 1 > 0$ then, by (3.5), the last factor in $R(0)$ becomes $(5b^2 + 9a^2 - 1)(k_2(a,b) - k_1(a,b))$, which is also positive by lemma 3.2 (see the beginning of the proof of this lemma). Concerning $R(\frac{1}{6}) > 0$, we need only to note that the slope of L_3 is $-b/((2a+1)(3a+1))$, and that L_3 is an invariant straight line of system (2.14). Hence the claim is proved.

The lemma is true if we prove that $R(h)$ has no double zero (or a zero with higher multiplicity) for $h \in (0, 1/6)$. It is hard to give a direct proof, since the expression of $\beta(a,b)$ is complicated and not algebraic. First we prove this fact for $a = 0$. Since the proof is technical, we put it in the appendix B. Now we extend the conclusion from $a = 0$ to all $G \backslash \partial G$ by a homotopy method.

Note that for all $(a,b) \in G$, along the line $\{a = 0\}$ the coordinate b takes its maximum at $b = 1$ and its minimum at $b = 0$. We suppose that for some

$(a_1, b_1) \in G \backslash \partial G$, $R(h)$ has a double (or higher) zero $h_1 \in (0, 1/6)$. We take two points $(0, b')$ and $(0, b'')$ in $G \backslash \partial G$, such that $b' < b_1 < b''$, and construct a family of smooth curves $\gamma_\mu = \{(a_\mu(b), b) : b' \le b \le b''\}$ in $G \backslash \partial G$, such that $a_\mu(b') = a_\mu(b'') = 0$ for $0 \le \mu \le 1$, γ_0 is the segment on $\{a = 0\}$, and $(a_1, b_1) \in \gamma_1$. Next, let

$$S(h; b, \mu) = R(h; a_\mu(b), b, \beta(a_\mu(b), b)).$$

Then, the above assumption means that $(h; b, \mu) = (h_1; b_1, 1)$ is a solution of the following equations

$$S(h; b, \mu) = 0, \; S'_h(h; b, \mu) = 0. \tag{3.21}$$

If we choose $a_\mu(b)$ properly, then we have $S'_b(h_1; b_1, 1) \neq 0$. We claim that $S''_{hh}(h_1; b_1, 1) \neq 0$. In fact, we already have $S_h(h_1; b_1, 1) = S'_h(h_1; b_1, 1) = 0$, if $S''_{hh}(h_1; b_1, 1) = 0$, then an orbit of system (2.14) is tangent to L_1 at $(\nu, \omega)(h_1)$ with order 4 (could not be more, since $R(h)$ is of degree 3 in h), hence on both sides near this point on L_1 the vector field (2.14) has different directions with respect to L_1. On the other hand, from lemmas 2.3 and 2.5 we know that on L_1 in $D'_{a,b}$ near the points $(\nu, \omega)(0)$ and $(\nu, \omega)(1/6) = (1, 0)$, the vector field (2.14) have the same direction with respect to L_1 (downward). This contradicts the fact that the degree of $R(h)$ is 3.

From these results we conclude that the Jacobian

$$\left.\frac{D(S, S'_h)}{D(h, b)}\right|_{(h_1; b_1, 1)} = \left|\begin{array}{cc} S'_h & S'_b \\ S''_{hh} & S''_{hb} \end{array}\right|_{(h_1; b_1, 1)} = -S''_{hh} S'_b|_{(h_1; b_1, 1)} \neq 0.$$

Hence, by the Implicit Function Theorem, we obtain a solution for equations (3.21): $h = h(\mu), b = b(\mu)$ in a neighborhood of $\mu = 1$ ($\mu < 1$). Repeating this procedure, we may extend this solution continuously to the direction of $\mu < 1$, and we obtain a curve $\Gamma : \{h = h(\mu), b = b(\mu)\}$ in (h, b, μ)–space, which can reach the boundary of the region $\{0 \le h \le 1/6, 0 \le \mu \le 1, b' \le b \le b''\}$, like the extension of a solution for an initial problem of an ordinary differential equation. But (3.20) implies that Γ cannot meet the faces $\{h = 0, 0 \le \mu \le 1, b' \le b \le b''\}$ and $\{h = 1/6, 0 \le \mu \le 1, b' \le b \le b''\}$; the result for $a = 0$ means that Γ cannot meet the faces $\{\mu = 0, 0 \le h \le 1/6, b' \le b \le b''\}$, $\{b = b', 0 \le \mu \le 1, 0 \le h \le 1/6\}$ and $\{b = b'', 0 \le \mu \le 1, 0 \le h \le 1/6\}$. Hence, there is no place where Γ can go, and this is a contradiction. □

Lemma 3.5 *For any $(a, b) \in G \backslash \partial G$ the curve $\Omega_{a,b}$ is located in $D'_{a,b}$ and on the right hand side of the line L_1, except the endpoint $(\nu, \omega)(1/6) = (1, 0) \in L_1$, see figure 6.*

Proof. By lemmas 2.4 and 3.2 the endpoint $C = (\nu, \omega)(0)$ of $\Omega_{a,b}$ is always located on the right hand side of L_1, and in the region $D'_{a,b}$ the tangent line L_3 of $\Omega_{a,b}$ at the other endpoint $B(1, 0)$ is always located on the right hand side of L_2 (lemma 3.1 (3)), hence is on the right hand side of L_1 (lemma 3.2). On the other hand, the conclusion of the lemma is true for $(a, b) = (-1/3, 2/3)$, since $\Omega_{-1/3,2/3}$ is globally convex (lemma 3.3 of [15]). Hence, if it is not true for some $(a, b) \in G\backslash\partial G$, then by the deformation–inflection principle (see the proof of lemma 1.3), we must find another $(a, b) \in G\backslash\partial G$, for which $\Omega_{a,b}$ has an inflection point, and the tangent line to $\Omega_{a,b}$ at this point passes through the point $\{A\} = L_0 \cap L_1$, in contradiction with lemma 3.3. □

4 Proofs of lemmas 1.4 and 1.5

Before proving lemmas 1.4 and 1.5 we need some auxiliary results.

Lemma 4.1 *In the identified $(P, Q)-$ and $(\nu, \omega)-$plane, if a straight line $L_{\alpha\beta\gamma}$ intersects $\Sigma_{a,b}$ at least at two points (counting their multiplicities, but the left endpoint $(0, 0)$ of $\Sigma_{a,b}$ is not included), then $L_{\alpha\beta\gamma}$ must meet $\Omega_{a,b}$ at some point $(\nu, \omega)(h)$ with $h \in (0, 1/6)$.*

Proof. For this (α, β, γ), the Abelian integral $I(h)$ (see (1.9)) has at least 3 zeros for $h \in [0, 1/6]$, since $I(0) = 0$ (lemma 1.1). Hence $I''(h)$ has at least one zero for $h \in (0, 1/6)$. This gives the desired conclusion, see (1.13). □

We denote by $\Delta(a, b)$ the closed region bounded by the triangle with vertices O, A and B (see figure 6).

Lemma 4.2 *For any $(a, b) \in G\backslash\partial G$ the curve $\Sigma_{a,b}$ is located inside $\Delta(a, b)$.*

Proof. Lemma 1.1 (3) and lemma 3.1 (2) show that $\Sigma_{a,b}$ is below $\{Q = 0\}$ and above L_0, except the endpoint $(P, Q)(0) = (0, 0) \in L_0 \cap \{Q = 0\}$. Now we show that $\Sigma_{a,b}$ is also above L_1, except the endpoint $B = (P, Q)(1/6) \in L_1$. If this is not true, then $L_1 \cap \Sigma_{a,b}$ consists of at least two points, since $(P, Q)(1/6) \in L_1$ and $(P, Q)(0) = (0, 0)$ is above L_1. Hence, by lemma 4.1, L_1 would intersect $\Omega_{a,b}$ at some point $(\nu, \omega)(h)$ for $h \in (0, 1/6)$, and this contradicts lemma 3.5. □

Proof of lemma 1.4 By lemma 1.1 (4), $P'(h)$ and $Q'(h)$ are continuous for $h \in (0, 1/6)$. If $(P'(h_0))^2 + (Q'(h_0))^2 \neq 0$, then dQ/dP exists and is continuous at $(P,Q)(h_0)$. We suppose that $P'(h_0) = Q'(h_0) = 0$, then by definitions (1.10) and (1.14) for h near h_0 we have

$$\begin{aligned}\frac{Q'(h)}{P'(h)} &= \frac{Y'(h)M(h) - M'(h)Y(h)}{X'(h)M(h) - M'(h)X(h)} \\ &= \frac{(Y'(h)M(h) - M'(h)Y(h)) - (Y'(h_0)M(h_0) - M'(h_0)Y(h_0))}{(X'(h)M(h) - M'(h)X(h)) - (X'(h_0)M(h_0) - M'(h_0)X(h_0))} \\ &= \frac{Y''(\theta_1)M(\theta_1) - M''(\theta_1)Y(\theta_1)}{X''(\theta_2)M(\theta_2) - M''(\theta_2)X(\theta_2)} = \frac{M(\theta_1)M''(\theta_1)(\omega(\theta_1) - Q(\theta_1))}{M(\theta_2)M''(\theta_2)(\nu(\theta_2) - P(\theta_2))},\end{aligned}$$

where θ_1 and θ_2 are between h and h_0. We note that $M(h)M''(h) \neq 0$ for any $h \in (0, 1/6)$. Hence, we have

$$\lim_{h\to h_0} \frac{Q'(h)}{P'(h)} = \frac{\omega(h_0) - Q(h_0)}{\nu(h_0) - P(h_0)}.$$

This limit exists because the curves $\Sigma_{a,b} = \{(P,Q)(h)\}$ and $\Omega_{a,b} = \{(\nu,\omega)(h)\}$ are separated by the line L_1 (lemmas 3.5 and 4.2, see figure 6), hence the numerator and the denominator on the right hand side of above equality cannot be zero at the same time. This proves the smoothness of $\Sigma_{a,b}$

From lemma 2.2 we see that the curve $\Sigma_{a,b}$ is convex for h near 0, and along it dQ/dP is increasing from $k_0(a,b) = b/(a-1)$, until the first possible zero–curvature point $(P,Q)(h^*) \in \Sigma_{a,b}$, and $d^2Q/dP^2 > 0$ for $h \in [0, h^*)$. We claim that $dQ/dP < k_1(a,b)$ for any point on $\Sigma_{a,b}$. In fact, if this is not true, then we would find a point on $\Sigma_{a,b}$ (it is in $\Delta(a,b)$ by lemma 4.2), such that the tangent line at this point is parallel to L_1, hence it has no intersection with the curve $\Omega_{a,b}$ (lemmas 3.5 and 4.2), and this contradicts lemma 4.1. If such h^* does not exist, (1.17) certainly holds globally for $\Sigma_{a,b}$. □

Proof of lemma 1.5 The statement (1) is a special case of lemma 4.1. The first point of statement (2) was proved in lemma 3.1(1); the second point of this statement follows from lemma 2.2(3) and lemma 3.5. To prove statement (3), we note that $\Sigma^*_{a,b}$ is convex, hence $(\nu,\omega)(0) \cap \xi_h = \emptyset$ for $h \in (0, h^*]$ is obviously true. If $(\nu,\omega)(1/6) \cap \xi_h \neq \emptyset$ for some $h \in (0, h^*]$, then, by lemmas 3.5 and 4.2, $B = (\nu,\omega)(1/6)$ is the only intersection point of $(\nu,\omega)(1/6) \cap \xi_h$, and this contradicts Lemma 4.1.

Finally, we prove statement (4). By lemma 3.1(3) in $D'_{a,b}$ the tangent line $\eta_{1/6} = L_3$ of $\Omega_{a,b}$ is always on the right hand side of L_2, and by lemma 4.2, $\Sigma_{a,b}$ stays in the triangle region $\triangle(a,b)$, which is on the left hand side of L_1, so on the left hand side of L_2 (lemma 3.2). Hence, $\Sigma_{a,b} \cap \eta_h = \emptyset$ for $0 < 1/6 - h \ll 1$, see figure 6. We consider the motion of the tangent line η_h, running on $\Omega_{a,b}$, as h decreases from 1/6. Since $\Omega_{a,b}$ is on the right hand side of L_1, there are only two possibilities for η_h also tangent to $\Sigma_{a,b}$: either η_h passes over the point $\{A\} = L_0 \cap L_1$ upwards, and enters $\Delta(a,b)$, or η_h passes the points $B(1,0)$, $O(0,0)$, and over $\Sigma_{a,b}$ downwards, and get a tangent position. The later case contradicts lemma 2.6(2). In the former case, by using the deformation–inflection principle, varying (a,b) from the present value to $(-1/3, 2/3)$, we would find a $(\bar{a}, \bar{b}) \in G \backslash \partial G$, such that $\Omega_{\bar{a},\bar{b}}$ has an inflection point and the tangent line at this point passes through the point A, contradicting lemma 3.3. □

5 Proof of theorem 1.3

For $(a,b) \in G_2$, the Hamiltonian vector field X_H has two centers C, C', two saddles S, S', two saddle loops γ, γ', and the corresponding periodic annuli $D(\gamma)$, $D(\gamma')$. Hence we have two centroid curves $\Sigma \subset D(\gamma)$ and $\Sigma' \subset D(\gamma')$. Since the convexity does not change under affine transformations, we can move C or C' (resp. S or S') to $(0,0)$ (resp. (1,0)) and obtain the normal form (1.4) by an affine transformation, so from theorem 1.2 we conclude that both Σ and Σ' are strictly convex. Note that X_H is a quadratic system, the four singularities form a quadrilateral with C and C' as a pair of opposite vertices and S and S' as another opposite pair (see, for example [19]), if we exchange C to C' and S to S' by doing an affine transformation, then we must reverse the direction of one coordinate axis (or with a rotation π), hence Σ and Σ' must be one convex and one concave.

Now we denote by L_c (resp. L_s) the straight line passing through C and C' (resp. S and S'); by O the intersection point of L_c and L_s; by Δ (resp. Δ') the interior of the triangle with vertices C, S and O (resp. C', S' and O). Next we denote by t_c the straight half–line which is tangent to Σ at C and points to the direction of the convexity; by t_s the straight half–line from S to another endpoint Z of Σ (the centroid point of $D(\gamma)$); by M the intersection point of t_c and t_s. By lemma 1.5(2), t_s is tangent to Σ at Z (note that the point $(\nu,\omega)(1/6)$ corresponds to a saddle), and by lemma 1.4, M is located on the same side of the convexity of Σ. We similarly define the straight half–lines t'_c, t'_s and let $\{M'\} = t'_c \cap t'_s$, see figure 8.

As it has been pointed out by Horozov and Iliev [9] to finish the proof of theorem 1.3, it is enough to show that for any $(a,b) \in G_2$, $M \in \Delta$ and $M' \in \Delta'$. This follows from the claims: (1) t_s and t'_s are located on different sides of L_s; and (2) t_c and t'_c are located on different sides of L_c.

The claim (1) follows from the simple fact that for a quadratic system on any straight line there are at most two points at which the vector field is tangent to this line. Now L_s passes through the two singular points S

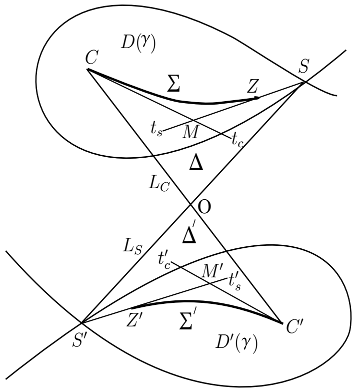

Figure 8. The relative positions of the two centroid curves.

and S', so it must stay outside $D(\gamma)$ and $D(\gamma')$. Otherwise, one orbit inside $D(\gamma)$ or $D(\gamma')$ would be tangent to it. On the other hand, the point Z (resp. Z') is inside $D(\gamma)$ (resp. $D(\gamma')$). Claim (2) can be verified by means of a direct calculation. For the normal form (1.4), $C(0,0)$ is a center and the slope of t_c is $k_0 = b/(a-1)$. From (1.5)–(1.7) we see that the condition for $(a,b) \in G_2$ is : $\xi_1 \equiv b^2 - 4a^3 > 0$, $\xi_2 \equiv b^2 + 4a(2a+1) > 0$ and $\xi_3 \equiv (1-a)^2(2a+1) - b^2 > 0$, where $0 < b < 1$, and $|a| < \frac{1}{2}$. The other center is $C'(x', y')$ with

$$x' = \frac{4a^2 + b^2 + b\sqrt{\xi_2}}{2\xi_1}, \quad y' = \frac{-(2ax'+1)}{b}.$$

Hence, the slope of L_c is $k' = y'/x'$, and we have

$$k_0 - k' = \frac{\eta_1 + \eta_2\sqrt{\xi_2}}{(1-a)(4a^2 + b^2 + b\sqrt{\xi_2})},$$

where $\eta_1 = b(2 - 6a^2 - b^2)$ and $\eta_2 = 2a - 2a^2 - b^2$. From $\xi_3 > 0$ we have $-b^2 > 3a^2 - 2a^3 - 1$, which implies $\eta_1 > b(1-2a)(a+1)^2 > 0$ because $|a| < \frac{1}{2}$ for $(a,b) \in G_2$. Hence, if $\eta_2 \geq 0$, then we have $k_0 - k' > 0$. If $\eta_2 < 0$, then a computation gives

$$\eta_1^2 - \eta_2^2\xi_2 = 4\xi_1\xi_3 > 0,$$

and we obtain the same conclusion. Since we may change t'_c by t_c through an affine transformation, and reverse one coordinate axis (or with a rotation π) as explained before, t'_c must stay on the other side of L_c. □

6 Appendix A

In this appendix we use symbolic computations (with Maple, for example) to solve two types of problems: (1) to find the exact number of zeros of a real polynomial of one variable in an interval, and (2) to prove that a real polynomial of two variables, say $f(x, y)$, has a fixed sign in a planar domain D with boundary defined also by a polynomial, say $g(x, y) = 0$. For type (1) problem, we use the Sturm Principle (for example, see the appendix of [15]), and it can be done easily and precisely by using the Maple command "sturm". For type (2) problem, we first determine the sign of $f(x, y)$ on the boundary of D. So we can eliminate one variable from $f(x, y) = 0$ and $g(x, y) = 0$, hence change to the type (1) problem. Then we check the value of $f(x, y)$ at each of its singular point inside D, i.e. at the solutions of $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$, and this also can be done by solving type (1) problem. In the following examples, there is no singular point in D at all.

Example 6.1 *$K_1(a, b) \neq 0$ for $(a, b) \in D_1$, where*

$$K_1(a,b) = 625a^3b^8 + 500a^2(9a^3 - 9a^2 - a + 9)b^6 + 30(405a^7 - 810a^6 + 540a^5 + 360a^4 - 679a^3 + 522a^2 - 162a + 144)b^4 + 4(3645a^9 - 10935a^8 + 15795a^7 - 10935a^6 - 3699a^5 + 19467a^4 - 16151a^3 + 891a^2 + 4698a - 216)b^2 + a(81a^5 - 162a^4 + 198a^3 - 144a^2 + a + 90)^2.$$

Proof. From (2.2) and (3.8) we find that the curved triangles D_1 and D_2 have the top vertices at (a_1, b_1) and (a_2, b_2), respectively, where a_i, for $i = 1, 2$, are the roots of the equation $5a^2 - 2a - 1 = 0$ ($a_1 \approx -0.2899$ and $a_2 \approx 0.6899$), see figure 7. On the other hand, we find a point $(a_3, b_3) \in \zeta_2$, which is the unique point in G such that $g_2(a, b) = \partial g_2/\partial b = 0$, and a_3 is the negative root of $45a^2 - 18a - 14 = 0$, $a_3 \approx -0.3925$. Hence ∂D_1 consists of 3 arcs, i.e. $\zeta_{11} : \{\lambda_1(a, b) = 0, -1/2 \leq a \leq a_1, b > 0\}$, $\zeta_2 : \{g_2(a, b) = 0, a_3 \leq a \leq a_1\}$ and $\zeta_{13} : \{b = 0, -1/2 \leq a \leq -1/3\}$. We first show that $K_1(a, b) > 0$ for $(a, b) \in \partial D_1$, except at one point on ζ_{11}, in which the locus of $K_1(a, b) = 0$, say C_{K_1}, is tangent to ζ_{11} with even order from exterior of D_1. We have that $C_{K_1} \cap \zeta_{13} = \emptyset$, this follows from the fact that $K_1(a, 0) = 0$ has no real roots for $a \in (-1/2, -1/3)$ (by the Sturm Principle). We eliminate b from $K_1(a, b) = 0$ and $\lambda_1(a, b) = 0$, and obtain the following polynomial in a, denoted by $R(K_1, \lambda_1; b)$,

$$(25a^7 - 30a^6 - 21a^5 + 8a^4 + 147a^3 - 210a^2 + 73a + 24)^2(5a^3 - 8a^2 + 2a + 3)^4.$$

Here we omit the constant coefficient and some obviously non–zero factors. The first factor above has no real roots for $a \in (-1/2, a_1)$, and the second factor has a unique real root at $a^* \approx -0.4520$. A further calculation shows that $(a^*, b^*) \in C_{K_1} \cap \zeta_{11}$. Similarly, we find that

$$R(K_1, g_2; b) = (13122a^{16} - 67797a^{15} + 263169a^{14} - 708102a^{13} + 1339092a^{12} - 1766259a^{11} + 1564983a^{10} - 834828a^9 + 152462a^8 + 204181a^7 - 313825a^6 + 223962a^5 - 17040a^4 - 74061a^3 + 24745a^2 + 10040a - 20)^2,$$

which has no real roots if $a \in (a_3, a_1)$. Thus, we have proved that $C_{K_1} \cap \partial D_1$ consists of a unique point (a^*, b^*). We consider

$$R(K_1, \partial K_1/\partial b; b) = (81a^5 - 162a^4 + 198a^3 - 144a^2 + a + 90)^2(253125a^{13} - 372600a^{12} - 44640a^{11} + 586665a^{10} - 774880a^9 + 392610a^8 + 895614a^7 - 833412a^6 - 369285a^5 + 389142a^4 + 208178a^3 + 54555a^2 + 14352a - 64)^2,$$

which has no real roots if $a \in (-1/2, a_1)$. Therefore, $C_{K_1} \cap D_1 = \emptyset$. Otherwise, $C_{K_1} \cap \partial D_1$ consists of at least two points. □

Example 6.2 *$K_2(a, b) \neq 0$ for $(a, b) \in D_1$, where*

$$K_2(a, b) = -125b^8 + 25(135a^3 + 108a^2 + 9a + 44)b^6 + 45(405a^5 + 468a^4 - 102a^3 - 8a^2 + 105a - 36)b^4 + (3a + 1)(10935a^6 + 11907a^5 - 9666a^4 - 2322a^3 + 7659a^2 - 4017a + 736)b^2 + (27a^3 + 15a^2 - 15a + 5)^3.$$

Proof. The proof is the same as the one of example 6.1. □

Example 6.3 *$K_3(a, b) \neq 0$ for $(a, b) \in D_2$, where*

$$K_3(a, b) = -1000b^{12} + 25(1715a^3 + 1605a^2 + 765a + 203)b^{10} - 10(56675a^6 + 77940a^5 + 73395a^4 + 62340a^3 + 32205a^2 + 6840a + 493)b^8 + 2(512000a^9 - 1156425a^8 - 1648755a^7 + 770675a^6 + 2420745a^5 + 1772325a^4 + 707495a^3 + 171585a^2 + 22275a + 1072)b^6 + 2(2764800a^{11} - 3657285a^{10} - 9977844a^9 - 1667403a^8 + 10167444a^7 + 10668018a^6 + 4577436a^5 + 764478a^4 - 62180a^3 - 43149a^2 - 5976a - 275)b^4 + (2a + 1)(9a^2 - 1)(552960a^{10} - 498771a^9 - 1139685a^8 - 726252a^7 - 219588a^6 + 41254a^5 + 80506a^4 + 28804a^3 + 1708a^2 - 875a - 125)b^2 + 256a^3(a - 1)(2a + 1)^5(9a^2 - 1)^3.$$

Proof. The curve ∂D_2 consists of three arcs, $\zeta_{21} : \{\lambda_1(a,b) = 0, a_2 \le a \le 1, b > 0\}$, $\zeta_3 : \{g_2(a,b) = 0, 1/3 \le a \le a_2\}$ and $\zeta_{22} : \{b = 0, 1/3 \le a \le 1\}$. The result is that $C_{K_3} \cap \partial D_2$ consists of two points: $(1/3, 0)$ and (a_2, b_2), but $C_{K_3} \cap D_2 = \emptyset$. This is due to the following two facts:

$$R(K_3, \lambda_1; b) = (5a^2 - 2a - 1)^2(5a^3 - 15a^2 + 15a - 1)^4$$

has one real root $a = a_2$ for $a \in (a_2, 1)$, and

$$\begin{aligned}R(K_3, g_2; b) = {} & (5a^2 - 2a - 1)^2(3a - 1)^6(112125312500a^{16}+\\ & 223853787500a^{15} - 55519461375a^{14} - 245390675750a^{13}+\\ & 253185034615a^{12} + 754642046424a^{11} + 599077857293a^{10}+\\ & 187715022998a^9 - 6451776045a^8 - 17927026996a^7 - 2066267389a^6+\\ & 906026502a^5+179959429a^4-23090960a^3-6171153a^2+207242a+72509)^2\end{aligned}$$

has no positive real roots except $a = 1/3$ and $a = a_2$. Finally,

$$\begin{aligned}R(\partial K_1/\partial a, \partial K_1/\partial b; b) = {} & (5a^3 - 15a^2 + 15a - 1)(451a^4 + 274a^3 + 18a^2-\\ & 26a - 5) \cdot (2069102592000000a^{21} + 8613178444800000a^{20}+\\ & 10100615918905625a^{19}-3222027416939375a^{18}-6720297366812925a^{17}+\\ & 24610652175653035a^{16} + 52156704775508180a^{15}+\\ & 25715967153523380a^{14}-29731254988165060a^{13}-54867195350199620a^{12}-\\ & 39768001626903810a^{11}-15539771104847410a^{10}-\\ & 2687628438744470a^9+313982842138170a^8 + 244742056538596a^7+\\ & 35687665492676a^6 - 3075758146260a^5 - 1405302792340a^4-\\ & 98361094607a^3 + 6545040681a^2 + 491820075a - 43162125)(3a - 1)\end{aligned}$$

has no real roots for $a \in (1/3, 1)$, hence $K_3(a, b)$ has no singular points in D_2. □

Example 6.4 $K_4(a, b) \neq 0$ *for* $(a, b) \in D_2$, *where*

$$\begin{aligned}K_4(a, b) = {} & 25(3 - a)b^8 - 5(760a^4 + 1548a^3 + 807a^2 + 60a + 9)b^6+\\ & (4100a^7 - 300a^6 + 13560a^5 + 33355a^4 + 30774a^3 + 11076a^2 + 638a-\\ & 147)b^4 - (2a + 1)(7380a^8 + 38070a^7 - 17135a^6 - 18926a^5 + 16045a^4+\end{aligned}$$

$11970a^3 + 1947a^2 - 138a - 45)b^2 + a(a-1)(9a^2-1)(2a+1)^2$

$(1845a^5 - 945a^4 - 1742a^3 - 450a^2 + 105a + 35).$

Proof. The proof is the same as the one of example 6.3. □

Example 6.5 *$K_6(a,b) > 0$ for $(a,b) \in D_2$, where*

$$K_6(a,b) = 125(a-3)b^8 - 5(350a^4 + 1860a^3 + 3423a^2 + 508a - 93)b^6 + 3(5250a^6 + 10150a^5 - 9725a^4 + 8230a^3 + 13114a^2 + 1124a - 239)b^4 - (47250a^8 - 167400a^7 + 10485a^6 + 123060a^5 - 42867a^4 + 840a^3 + 23367a^2 + 1516a - 603)b^2 - a(a-1)\cdot(9a^2-1)(28350a^6 - 12015a^5 - 42039a^4 + 9786a^3 + 17168a^2 + 5a - 871).$$

Proof. The proof of the non–zero property of $K_6(a,b)$ is the same as the one of example 6.3. There are four singular points of $K_6(a,b)$ for $a \in (1/3, 1)$, but they are not located in D_2. The sign of $K_6(a,b)$ can be determined by choosing any value of (a,b) in D_2. □

7 Appendix B

In this appendix we prove that the function $R(h; 0, b, \beta(0,b))$ has no double zero in $h \in (0, 1/6)$. We simply denote this function by $R_0(h)$, and denote $\beta(0,b)$ by $\beta(b)$.

Lemma 7.1 *We have that*

(1) $\beta(b) = -\dfrac{\int_{-1}^{1} f(t,b)tdt}{\int_{-1}^{1} f(t,b)dt}$, *where* $f(t,b) = \Big(\dfrac{1-t^2}{1-bt^3}\Big)^3$.

(2) $\beta'(b) < 0$.

(3) *$\beta(b)$ has the following "pseudo-convex" property, i.e. for any $c \in (0,1)$, we have $\beta(b)/b > \beta(c)/c$ for $0 < b < c$.*

Proof. Note that

$$\beta(a,b) = -k_1(a,b) = \frac{Q(1/6)}{1 - P(1/6)} = \frac{\iint_{\mathrm{Int}(\delta(\frac{1}{6}))} y dxdy}{\iint_{\mathrm{Int}(\delta(\frac{1}{6}))}(1-x)dxdy}.$$

We may use similar transformations as for the integration (3.6) in order to obtain statement (1). To prove statement (2), we let

$$g(t,u) = f_b'(u,b)f(t,b)t - f_b'(t,b)tf(u,b).$$

Then, the numerator of $\beta'(b)$ equals to

$$\begin{aligned}
&\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}(g(t,u)+g(u,t))dtdu \\
&= -\frac{3}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-t^2)^3(1-u^2)^3}{(1-bt^3)^4(1-bu^3)^4}[(t-u)^2(t^2+tu+u^2)]dtdu < 0.
\end{aligned}$$

Since $\beta(0) = 0$, to get statement (3), it is enough to show that

$$I(b,c) \equiv c\int_{-1}^{1} f(t,b)tdt \int_{-1}^{1} f(t,c)dt - b\int_{-1}^{1} f(t,c)tdt \int_{-1}^{1} f(t,b)dt > 0.$$

As above we define

$$h(t,u) = cf(t,b)tf(u,c) - bf(u,c)uf(t,b) = (ct-bu)f(t,b)f(u,c).$$

Then

$$\begin{aligned}
I(b,c) &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}(h(t,u)+h(u,t))dtdu \\
&= \int_{-1}^{1}\int_{-1}^{1}\frac{(c-b)(1-t^2)^3(1-u^2)^3(u+t)}{2\,(1-bt^3)^3(1-cu^3)^3}dtdu.
\end{aligned}$$

In the rectangle $\{-1 \le t \le 1, -1 \le u \le 1\}$ we make a change of coordinates for the points in the half–rectangle $\{u+t \le 0\}$ by means of $u = -t, t = -u$, then $I(b,c)$ is taken over the half–rectangle $\{u+t \ge 0\}$, and the integrant becomes

$$\frac{(c^2-b^2)[(1-t^2)(1-u^2)]^3(u+t)^2(u^2-ut+t^2)(A^2-AB+B^2)}{2(AB)^3} > 0,$$

where $A = (1-bt^3)(1-cu^3) > 0$ and $B = (1+bt^3)(1+cu^3) > 0$. This finishes the proof of the lemma. □

Lemma 7.2 *Let* $\xi(b,\beta) = b(5b^2+2)\beta^2 + (9b^2+3)\beta - b(b^2-2)$. *Then* $\bar{\xi}(b) = \xi(b,\beta(b))$ *has a unique real root* $b_\xi \approx 0.8991$ *for* $b \in (0,1)$, *and* $\bar{\xi}(b)(b-b_\xi) < 0$ *for* $b \in (0,1)\backslash\{b_\xi\}$.

Proof. Easy computations show that $\beta(0) = 0$ and $\beta(1) \approx -0.15267$. From lemma 7.1, we see that $\beta'(b) < 0$. We consider a rectangle $D = \{0 < b < 1, -0.2 < \beta < 0\}$ in (b, β)–plane. It is easy to verify that the equation $\xi(b, \beta) = 0$ defines a curve Γ in D (one more branch is in the complement of D). We prove that the curve $\gamma = \{(b, \beta) : \beta = \beta(b)\}$ crosses Γ at a unique point in D. Eliminating β from $\xi(b, \beta) = 0$ and $\xi'_b(b, \beta) = 0$ we obtain a unique zero point $b_1 = \sqrt{2}/3$. An additional calculation shows that Γ has a unique inflection point at $b = b_2 \approx 0.9733 > b_1$. Thus, from $(0, 0)$ to its unique minimum point $(\sqrt{2}/3, -\sqrt{2}/8)$, the curve Γ is concave, and Γ crosses the line $b = 1$ at $\beta \approx -0.08783$. From these properties of Γ and the fact that γ is decreasing and pseudo–convex (lemma 6.1), we immediately get the conclusion that $\Gamma \cap \gamma$ consists of a unique point $(b_\xi, \beta(b_\xi))$ in D. Numerical computation gives $b_\xi \approx 0.8991$. From the expression of $\xi(b, \beta)$, it is easy to see that for $b = 1$ we have $\bar{\xi}(1) \approx -0.66889$, so we have $\bar{\xi}(b)(b - b_\xi) < 0$ for $b \in (0, 1)\backslash\{b_\xi\}$. □

In a similar way we have the following result.

Lemma 7.3 *Let* $\eta(b, \beta) = b^2(11b^4 - 2b^2 + 5)\beta^4 + 3b(11b^6 + 41b^4 + 9b^2 + 3)\beta^3 + 2(3b^8 + 35b^6 + 118b^4 + 35b^2 + 3)\beta^2 + 3b(3b^6 + 9b^4 + 41b^2 + 11)\beta + b^2(5b^4 - 2b^2 + 11)$. *Then* $\bar{\eta}(b) = \eta(b, \beta(b))$ *has a unique real root* $b_\eta \approx 0.7882$ *for* $b \in (0, 1)$, *and* $\bar{\eta}(b)(b - b_\eta) < 0$ *for* $b \in (0, 1)\backslash\{b_\eta\}$.

Now we are ready to prove that $R_0(h) = r_3(b)h^3 + r_2(b)h^2 + r_1(b)h + r_0(b)$ has no double zero for $h \in (0, 1/6)$, where

$$\begin{aligned}
r_3(b) &= -432b^5(b\beta(b) + 1)(b - \beta(b)),\\
r_2(b) &= -72b^3[b(5b^2 + 2)(\beta(b))^2 + (9b^2 + 3)\beta(b) - b(b^2 - 2)],\\
r_1(b) &= 6b(b^2 + 1)[b(13b^2 + 1)(\beta(b))^2 + (2b^4 + 18b^2 + 4)\beta(b) + (b^2 + 1)b],\\
r_0(b) &= -(b^2 + 1)^2[(5b^2 - 1)\beta(b) + 6b]\beta(b).
\end{aligned}$$

From lemma 7.1 and the proof of lemma 7.2 we see that $-0.16 < \beta(b) < 0$ for $b \in (0, 1)$, hence $r_3(b) < 0$. It is easy to see that $R''_0(0) = -144b^3\bar{\xi}(b)$, where $\bar{\xi}(b)$ is given in lemma 7.2. Hence, by this lemma, if $0 < b < b_\xi$ then $R''_0(0) < 0$. These two facts imply that $R_0(h)$ has no double zero for $h > 0$. Otherwise, a cubic polynomial would have two inflection points. If $b_\xi \le b < 1$ then, by lemma 7.3, we must have $\bar{\eta}(b) < 0$ (note $b_\xi > b_\eta$). Since $R'_0(h) = 3r_3(b)h^2 + 2r_2(b)h + r_1(b)$ and $(r_2(b))^2 - 3r_3(b)r_1(b) = 2592b^6\bar{\eta}(b)$, we conclude that in this case $R_0(h)$ is globally monotone, and (3.20) implies $R_0(h) > 0$ for $h \in [0, 1/6]$.

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