HIGHER ČECH THEORY

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Introduction

The definition and properties of Čech cohomology of topological spaces with coefficients in a (pre)sheaf are well-known, including the key fact that it is isomorphic to sheaf cohomology (in the sense of derived functors) when the underlying space is paracompact and Hausdorff. The theory has been extended by Grothendieck and his collaborators to the topologies that bear his name. Čech cohomology of sites is isomorphic to their derived functor cohomology, in general, only in dimensions 0 and 1. Verdier has introduced a variant of the notion of cover in such a way that the cohomology groups computed along his 'hypercovers' are isomorphic to derived functor cohomology in all dimensions, for arbitrary sites. The starting point for this research was the following naïve but (to the author) irresistible question: Is there a notion of 'cover of level n' with respect to which Čech covers have level 1, and get the cohomology right up to dimension 1, while Verdier covers have level ∞, and get the cohomology right in all dimensions?

The data for an open cover of a topological space can be assembled into a simplicial sheaf with distinguished properties, and hypercovers are simplicial sheaves by definition. The notion of *n-cover* offered here is cast in a simplicial language, and works over an arbitrary site. In fact, it can be defined intrinsically in terms of the category of sheaves on the site (the *topos* in Grothendieck's sense) and this, together with some sheaf-theoretic technology that was only perfected after the publishing of SGA4, makes the proofs much simpler. We also introduce a notion of *Čech n-cover* that is specific to a site; these are genuine combinatorial objects, standing in the same relation to n-covers as topological open covers do to the simplicial sheaf they give rise to.

One of the several equivalent definitions of abelian sheaf cohomology is

$$H^n(\mathcal{E}, A) = \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, \widetilde{K(A, n)})$$

Here $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ is the category of set-valued sheaves on a site, A an abelian sheaf, $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$ stands for the category of simplicial objects in $\mathcal{E}$ with the local weak equivalences (formally) inverted, $\mathbf{1}$ is the terminal object of

$\mathcal{E}^{\Delta^{\mathrm{op}}}$ and $\widetilde{K(A,n)}$ a simplicial Eilenberg–MacLane sheaf corresponding to A in degree n. Level n Čech cohomology will be defined as a filtered colimit of simplicial homotopy classes whose source is an n-cover; from the viewpoint of simplicial homotopy theory, the target could be an arbitrary simplicial sheaf (so perhaps 'level n Čech homotopy' would be a better name) and the main result becomes the following

Theorem. Write $[\mathsf{cov}^n_{\mathcal{E}}]$ for the category whose objects are n-covers of $\mathcal{E}$ and morphisms simplicial homotopy classes of maps. $[\mathsf{cov}^n_{\mathcal{E}}]$ is a cofiltered category. Let X be a simplicial object in $\mathcal{E}$. Writing $[-,X]$ for the contravariant 'simplicial homotopy classes' functor $\mathcal{E}^{\Delta^{\mathrm{op}}} \to \mathit{Set}$, one has a natural map (of sets)

$$\operatorname*{colim}_{[\mathsf{cov}^n_{\mathcal{E}}]} [-,X] \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1},X)$$

which is a bijection if X is an *exact n-type*.

There's an analogous statement for Čech n-covers. An exact n-type is, in turn, a particularly good simplicial model for a (local, in the case of sheaves) weak homotopy type with vanishing local homotopy groups above dimension n. For example, of all simplicial sets of the weak homotopy type of an Eilenberg–MacLane space $K(G,1)$, the nerve of G is also exact. Any homotopical n-type, taken in the ordinary sense, possesses models that are exact n-types. The concept, which is due to Duskin, will be discussed in detail.

The proof of the main theorem proceeds purely by local manipulations on simplicial objects, such as Postnikov sections, and employs no additive tools (e.g. chain complexes or spectral sequences). Setting $n=1$ and $X = K(G,1)$ (here G being a sheaf of groups) recovers Grothendieck's theorem on (not necessarily abelian) H^1 and its computability by 'level 1', that is ordinary Čech cohomology. The preservation of algebraic structure on homotopy classes — for example, in case X is a group or abelian group object up to homotopy — follows by naturality.

0, 1 and 2-covers can be economically reindexed as groupoids (with extra structure), and it might be thought that n-covers, as defined here, could shed light on the elusive notion of 'n-groupoids' or 'weak n-categories'. That is not so, as one can see, already for $n=2$. The reason is that n-covers are defined (in the best tradition of simplicial homotopy) via horn- and simplex-filling conditions, i.e. purely existential conditions, while for groupoids and similar algebraic structures one expects *operations* and identities — such as functional or functorial choices of fillers. For our purposes, the simplicial description seems to be the best in all dimensions.

A few words of warning might be in order regarding the usefulness of the main theorem. First, we do not purport to give an exhaustive treatment of the 'coefficient spaces' X. It is not discussed, for example, what 'small models' of n-types are — that is a question that still awaits a convincing resolution, even in the case of simplicial sets (with no Grothendieck topology present). Secondly, while one adds a certain homotopical touch to the subject by extending the possible coefficients from (sheaves of) Eilenberg–MacLane spaces to (sheaves of) n-types, this in practice is of little value. After all, one is typically interested in generalized cohomology groups of some site, that is, in global homotopy classes with values in (a sheaf of) spectra, and the representing spaces for interesting cohomology theories tend to have infinitely many non-zero homotopy groups. The method of proof, however, lends itself to generalizations to other 'sheafifiable homotopy theories'.

Outline. This introduction is followed by a recollection of classical material on simplicial objects and (perhaps less classical) facts about local arguments, sheaves and sites. The proof of the main theorem relies on the results of the author's [1], but this article ought to be readable separately. We include examples and amplification.

Acknowledgements. It will be abundantly clear to those familiar with [4] how much the work of J. Duskin has influenced my thinking; in addition, I am indebted to him for several email messages and generous simplicial guidance.

Recollections on the homotopy theory of simplicial sheaves. A *topos* is a category abstractly equivalent to the category of *Set*-valued sheaves on some Grothendieck site. To minimize set-theoretical issues, we assume that our toposes can always be written as the category of sheaves on a small site — under this assumption, a topos will possess a proper class of objects and small hom-sets in the usual sense, and there won't be a need to adjoin extra set-theoretical universes.

As general background on sites, toposes and geometric morphisms, we recommend the first three chapters of MacLane–Moerdijk [9].

Write $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$ for $\mathcal{E}^{\Delta^{\mathrm{op}}}[\mathsf{W}^{-1}]$, the category of simplicial sheaves with the class of local weak equivalences inverted. This is a locally small category (i.e. between any two objects there is a hom-*set* rather than proper class of morphisms); this can proved on purely set-theoretic grounds, or follows from W being part of a Quillen model category structure on $\mathcal{E}^{\Delta^{\mathrm{op}}}$. We will not use any model-theoretic arguments, though.

"For logical reasons". Let us illustrate the point with an example that is needed in this paper.

Proposition 0.1. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & B \end{array}$$

be a pullback diagram in SSet, with p a Kan fibration. Assume g is an n-equivalence, that is, it induces isomorphisms on π_i for $0 \leqslant i \leqslant n$, and in addition it induces an epi on π_{n+1} (for all basepoints). Then f is an n-equivalence.

Proposition 0.2. *Let $\mathcal{E}$ be a topos and*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & B \end{array}$$

a pullback diagram in $\mathcal{E}^{\Delta^{op}}$, with p a local Kan fibration. Assume g is a local n-equivalence, that is, it induces isomorphisms on the sheaves π_i for $0 \leqslant i \leqslant n$, and in addition it induces an epi on π_{n+1} (for all local basepoints). Then f is a local n-equivalence.

Proof of 0.1. Consider the geometric realization functor with values in the category of compactly generated Hausdorff spaces, and apply it to the above diagram. It preserves finite limits by a result of Gabriel–Zisman [6], and takes Kan fibrations to Serre fibrations by a result of Quillen [10]. In the topological world, the claim follows from the five lemma applied to the homomorphism induced between the homotopy long exact sequences of p and q; and that implies the conclusion for the simplicial (or 'combinatorial') homotopy groups too. □

Prop. 0.2 is a formal consequence of Prop. 0.1. There are three strategies for seeing that:

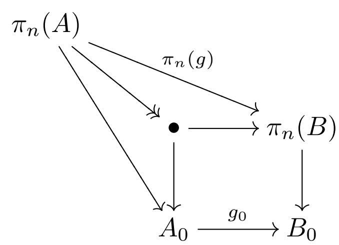
(1) Both the hypotheses and the desired conclusion of Prop. 0.1 can be phrased purely in the language of sets, membership, ordered tuples and projections, and unions and intersections (allowing countable unions as well), in the syntax known as *geometric logic*; see the textbook of Moerdijk–MacLane [9]. There is an interpretation of the language of sets in any topos (akin to the formalism of 'virtual elements' and 'diagram chases via virtual elements' in arbitrary abelian categories, cf. MacLane [8]). This interpretation, at the same time, defines precisely the meaning of *local* and ensures — thanks to a theorem of Joyal, Deligne and others, and one of Makkai and Reyes, respectively — that theorems whose hypotheses and conclusions

can be phrased in finitary resp. countable geometric logic, stay valid in an arbitrary topos.

(2) Both the hypotheses and the conclusion of Prop. 0.2 can be formulated in the language of diagrams, finite limits and countable colimits. For example, to say that $Y \xrightarrow{p} B$ is a local Kan fibration in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ means that the canonical maps $Y_n \twoheadrightarrow \Lambda^k_n(Y) \times_{\Lambda^k_n(B)} B_n$ from the object of n-simplices of Y to the matching object of (n,k)-horns in Y above n-simplices of B, are epimorphisms. To say that a map in $\mathcal{E}^{\Delta^{\mathrm{op}}}$

$$A \xrightarrow{g} B$$

induces an epi on π_n for all local basepoints means that the corner map in the diagram



is an epimorphism. Here A_0, B_0 are the degree 0 parts of the simplicial objects A, B ("all the local basepoints"); $\pi_n(A)$, $\pi_n(B)$ are the bundle of homotopy groups above them (i.e. group objects in $\mathcal{E}/A_0$, $\mathcal{E}/B_0$), and $\bullet$ stands for a pullback. $\pi_n(A)$, in turn, is a certain subquotient of the degree n part of $\mathrm{Ex}_\infty(A)$, which is a countable colimit of $\mathrm{Ex}_k(A)$, each of which, and the connecting maps as well, arises as collections of matching tuples and projection maps...

The data thus packaged — a set of diagrams, together with requirements that given subdiagrams be limits resp. colimits — go by the name of *sketch*; see Borceux [2] for a full and careful definition. Props. 0.1 and 0.2 are identical in this form; they both state that if certain subdiagrams of a given sketch are assumed to be limits resp. colimits, then certain other(s) will be limits (colimits) too. The fundamental theorem of sketches states that if a theorem sketchable via finite limits and finite colimits holds in the category *Set*, then it will hold in any topos; and same holds if the sketch contains countable colimits, but the total cardinality of all diagrams used is countable. (In simplicial homotopy theory, one often needs this second version because of Ex_∞.)

(3) Any Grothendieck topos $\mathcal{E}$ has a Boolean cover, that is to say, a complete Boolean algebra $\mathcal{B}$ and a surjective geometric morphism $\mathrm{Sh}(\mathcal{B}) \xrightarrow{f} \mathcal{E}$. f^* preserves and reflects statements that can be formalized in geometric

logic. Sh($\mathcal{B}$) provides a Boolean-valued model for the axioms of Zermelo–Frankel set theory (including the axiom of choice). Hence if a theorem of geometric logic is provable in ZFC, it holds in the internal logic of every Grothendieck topos.

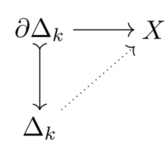
Terminology. The phrase 'for logical reasons', when used in a proof, signifies that one can appeal to any one of the strategies (1), (2) or (3) to verify a claim in *Set*, in order to conclude its validity in every topos.

As a final convention, the symbol **1** will always stand for a terminal object, in the category that's clear from the context.

Recollections on the coskeleton functor. Let $\Delta[0,n]$ denote the full subcategory of Δ (which is the category of finite ordinals and monotone maps) with objects 0, 1, …, n. The truncation functor $SSet = Set^{\Delta^{\mathrm{op}}} \xrightarrow{\mathrm{tr}_n} Set^{\Delta[0,n]^{\mathrm{op}}}$ has a right adjoint (a Kan extension) which we will denote cosk_n.

Proposition 0.3. *For $X \in SSet$, the following are equivalent:*

(1) *X is isomorphic to an object in the image of* cosk_n.
(2) *The canonical morphism $X \to \mathrm{cosk}_n \circ \mathrm{tr}_n(X)$ is an isomorphism.*
(3) *Write $\partial\Delta_k(X)$ for the set of $(k+1)$-tuples of $(k-1)$-simplices of X that are compatible so as to form the boundary of a standard k-simplex. The canonical map $X_k \xrightarrow{b_k} \partial\Delta_k(X)$ (whose coordinates are the boundary mappings) is a bijection for all $k > n$.*
(4) *The canonical map $X_k \to \hom(\mathrm{tr}_n(\Delta_k), \mathrm{tr}_n(X))$ is a bijection for all $k > n$. (This map sends a k-simplex of X, thought of as a simplicial map $\Delta_k \to X$, to its n-truncation.)*
(5) *For any $k > n$ and any diagram in the shape of the solid arrows*



precisely one lift exists that makes it commutative. (Here $\partial\Delta_k$ stands for the $k-1$-skeleton, that is "boundary", of the standard k-simplex Δ_k.)

If X satisfies (any, hence all) these conditions, it is said to be *n-coskeletal*.

The first two versions of this definition can be repeated verbatim for simplicial objects in any category with finite limits (in particular, a topos); and also the third, if one uses iterated pullbacks to assemble the 'boundary object' $\partial\Delta_k(X)$. (1) through (3) remain equivalent, and will serve as the definition of coskeletal objects in general. (4) is probably the most easily visualizable description of the coskeleton functor for simplicial sets, and property (5) is included for completeness and comparison; cf. the notion of exact fibration below.

1. n-COVERS

Given $X \in \mathcal{E}^{\Delta^{\mathrm{op}}}$, a *simplicial torsor over* X (in the broadest sense) is a map $T \to X$ such that $T \to \mathbf{1}$ is a weak equivalence — i.e. T has locally the weak homotopy type of a 'point', the terminal object of $\mathcal{E}^{\Delta^{\mathrm{op}}}$. Defining a map of torsors to be a simplicial map over X, they form a full subcategory $\mathsf{ST}(X)$ of the overcategory $\mathcal{E}^{\Delta^{\mathrm{op}}}/X$. Write $\pi_0\mathcal{C}$ for the class of connected components of a category $\mathcal{C}$. The main theorem of [1] reads

Theorem 1.1. *There is a natural map* $\pi_0\mathsf{ST}(X) \to \mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(\mathbf{1}, X)$ *which is a bijection if* X *is locally Kan.*

See [1] for a proof.

In the 70's, in related work, Duskin defined a certain class of simplicial objects X for which, as it turns out, a much smaller collection of simplicial torsors T suffices to calculate homotopy classes. When one thinks of these torsors as lying over the terminal object instead, they amount to the notion of *cover* that is the subject matter of this article.

Recall that a map $X \xrightarrow{f} Y$ of simplicial sets is a Kan fibration, or satisfies Kan's lifting condition in dimension n if in every commutative square of the form

$$(\bigstar) \qquad \begin{array}{ccc} \Lambda^k_n & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta_n & \longrightarrow & Y \end{array}$$

a lift, as indicated by the dotted arrow, exists. (Λ^k_n is the simplicial set obtained by omitting the non-degenerate simplex of dimension n and the face opposite the vertex k, $0 \leqslant k \leqslant n$, from Δ_n.) The following definition and proposition are due to Duskin [4].

Definition 1.2.

- A simplicial map $X \xrightarrow{f} Y$ satisfies the *exact Kan condition in dimension* n if in any commutative diagram of the type ($\bigstar$), precisely *one* dotted lift exists.
- $X \in SSet$ is an *exact n-type* if $X \to \mathbf{1}$ is a Kan fibration which is exact in all dimensions above n.
- For a topos $\mathcal{E}$, $X \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ is an *exact n-type* if $X \to \mathbf{1}$ is a local Kan fibration which is exact above dimension n.

Another way to phrase the condition of X being an exact n-type is: the canonical map $X_k \to \Lambda^i_k(X)$ from the object of k-simplices to the object of (k,i)-horns is an epi for $k \leqslant n$ and an isomorphism for $k > n$. For logical reasons, this formulation is valid in any topos.

Proposition 1.3. *Let $\mathcal{E}$ be a topos, $T \in \mathcal{E}^{\Delta^{op}}$.*

(i) *If T satisfies the exact Kan condition above dimension n, then T must be $n+1$-coskeletal.*
(ii) *If T is n-coskeletal, then it satisfies the exact Kan condition above dimension $n+1$.*
(iii) *If T is locally Kan and n-coskeletal, then it has vanishing local homotopy groups in dimension n and above.*
(iv) *An exact n-type has vanishing local homotopy groups above dimension n.*

Proof. For logical reasons, it suffices to prove these for $\mathcal{E} = Set$.

(i) Suppose one has $n+3$ matching $n+1$-simplices that together can be assembled into an $n+2$-boundary in T, i.e. a map $\partial\Delta_{n+2} \xrightarrow{b} T$. Omitting (say) the 0^{th} of these simplices, one obtains a horn $\Lambda^0_{n+2} \hookrightarrow \partial\Delta_{n+2} \to T$. By assumption, this horn has a filler $\Delta_{n+2} \xrightarrow{f} T$. Consider the two $n+1$-simplices: $d_0\Delta_{n+2} \hookrightarrow \Delta_{n+2} \xrightarrow{f} T$ (the 0^{th} face of f) and the 0^{th} simplex in the matching set $\partial\Delta_{n+2} \xrightarrow{b} T$. They have the same boundary; since T satisfies the exact Kan condition above dimension n, they must coincide. So f is a filler for b. Again by exactness in dimension 2, this filler must be unique.

(ii) By adjunction, $\operatorname{cosk}_n(X)$ satisfies the exact Kan condition with respect to $\Lambda^k_m \rightarrowtail \Delta_m$ iff X satisfies the unique lifting condition with respect to $\operatorname{tr}_n(\Lambda^k_m \rightarrowtail \Delta_m)$. That will certainly happen for $m > n+1$, for $\operatorname{tr}_n(\Lambda^k_m \rightarrowtail \Delta_m)$ is an isomorphism then. (Note that Λ^k_m leaves out cells of dimension $m-1$ and higher from Δ_m.)

(iii) A 0-coskeletal simplicial set is just the nerve of a groupoid that is equivalent (as a category) to the terminal category; so it is automatically Kan, and (if non-empty) simplicially contractible.

If $n > 0$, choose any 0-simplex $x \in T_0$ and any k-simplex $y \in T_k$, $k \geqslant n$, all of whose faces are (the degeneracies of) x. y, together with $k+1$ copies of the (unique) k-dimensional degeneracy of x, makes up a compatible boundary of a $k+1$-simplex. Since T is n-coskeletal, this boundary has a filler. Thence y is null-homotopic modulo its boundary. Since T was assumed to be Kan, this means its homotopy groups are trivial in dimension k, any $k \geqslant n$.

(iv) By definition, an exact n-type is a Kan complex; now use (i) and (iii). □

Proposition 1.4. *Let $n \in \mathbb{N}$ and A be an abelian group (just a group if $n = 1$, or a set if $n = 0$). The standard simplicial model of $K(A, n)$ is an exact n-type.*

By the *standard model* we mean the constant simplicial set A if $n = 0$, the nerve of A if $n = 1$, and the de-normalization of the appropriate chain complex concentrated in degree n if $n > 1$. Recall that this $K(A, n)$ is n-reduced, i.e. has a singleton in dimensions below n, A itself in dimension n, suitable tuples of elements of A in dimension $n + 1$, and is coskeletal above that. Duskin's observation, appearing in [4], that for any n and A, the standard $K(A, n)$ is an exact n-type model of an Eilenberg–MacLane space must have been motivational in his definition of 'exact'.

Remarks. The converse to Prop. 1.3(iv) holds as well, i.e. every (topological) n-type has simplicial models that are exact n-types. In fact, we will see in Prop. 1.5 below that there is a model for the Postnikov section functor that takes values in exact n-types.

• Note that an exact n-type is also an exact m-type for any $m > n$. It is far from the case that exact n-types are *minimal*, or in some sense economical models of homotopical n-types; as we will see, there are infinitely many exact n-type models of the point(!), and they can be quite complicated.

• Interestingly, the exact Kan condition above dimension n places no restriction on the homotopy type of X *in the absence of the Kan condition in lower dimensions*. For example, the nerve of any small category is 2-coskeletal, hence satisfies the exact Kan filling condition in dimensions greater than 3. Nonetheless, any homotopy type can be realized as the nerve of a small category. By way of contrast, if the nerve of a category is a Kan complex (which means, therefore, extra conditions only in dimensions 1 through 3), then that category is a groupoid, and its nerve has the homotopy type of a disjoint union of Eilenberg–MacLane spaces.

• Duskin [4] introduces the term *n-hypergroupoid* for what is called *exact n-type* above, while Glenn [7] uses 'n-hypergroupoid' to mean any simplicial set satisfying the exact Kan condition above dimension n (not requiring it to be a Kan complex in all dimensions). I confess to replacing Duskin's 'n-hypergroupoid' phrase by 'exact n-type' (prompted, of course, by his definition of 'exact Kan fibration', Def. 1.2) so as to emphasize its homotopical meaning (valid for all n) and de-emphasize its link to groupoids (known to be valid for small n).

Functorial Postnikov sections. Duskin has also defined a particularly good model for Postnikov sections (different from that of Moore). A concise version of his results appears in Glenn [7]; we only summarize what we need.

Proposition 1.5. *For each $n \in \mathbb{N}$, there exists a functor $\mathcal{P}_n$ from the full subcategory of simplicial sets whose objects are Kan complexes to its full subcategory with objects the exact n-types, as well as a natural transformation p_n from the identity to $\mathcal{P}_n$ with the following properties:*

- *$\mathcal{P}_n$ preserves Kan fibrations*
- *p_n induces isomorphisms on π_i for $i \leqslant n$*
- *$\mathcal{P}_n$ and p_n are definable in terms of finite limits and countable colimits.*

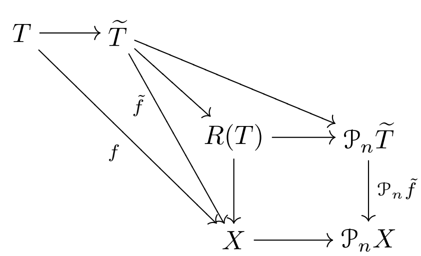
We can now state the main definition and theorem of this section.

Definition 1.6. An *n-cover* of a topos $\mathcal{E}$ is an exact n-type $T \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ such that $T \to \mathbf{1}$ is a local weak equivalence.

Theorem 1.7. *Let $\mathcal{E}$ be a topos and $X \in \mathcal{E}^{\Delta^{op}}$ an exact n-type. Write $\mathsf{cov}^n_{\mathcal{E}}(X)$ for the full subcategory of $\mathsf{ST}(X)$ with objects $T \to X$ such that T is an n-cover of $\mathcal{E}$. Then there is a canonical bijection $\pi_0\mathsf{cov}^n_{\mathcal{E}}(X) \to \mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(\mathbf{1}, X)$.*

Proof. The bijection is $\pi_0\mathsf{cov}^n_{\mathcal{E}}(X) \to \pi_0\mathsf{ST}(X) \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$, where the first map is induced by $\mathsf{cov}^n_{\mathcal{E}}(X) \hookrightarrow \mathsf{ST}(X)$ and the second map is that of Thm. 1.1. So it suffices to prove the first one to be a bijection.

Now quite generally, if one has two categories $\mathcal{C}, \mathcal{D}$ and functors $\mathcal{C} \xrightarrow{L} \mathcal{D}$, $\mathcal{C} \xleftarrow{R} \mathcal{D}$ such that every object U of $\mathcal{C}$ is connected to $RL(U)$ via a zig-zag of arrows in $\mathcal{C}$, and every object T of $\mathcal{D}$ is connected to $LR(T)$ via a zig-zag of arrows in $\mathcal{D}$ (these zig-zags are not required to be functorial, or even be of some fixed shape) then $\pi_0 L$ and $\pi_0 R$ are inverse bijections between $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$. It follows that if L is onto on hom-sets (for example, it is the inclusion of a full subcategory) and for any T in $\mathcal{D}$ there exists a map $T \to LR(T)$, then L and R induce bijections on connected components. Apply this to the full inclusion $\mathsf{cov}^n_{\mathcal{E}}(X) \xrightarrow{L} \mathsf{ST}(X)$, R being defined by the diagram:



Starting with a torsor $T \xrightarrow{f} X$, factor it (functorially) as a weak equivalence $T \to \widetilde{T}$ followed by a local fibration $\widetilde{T} \xrightarrow{\tilde{f}} X$. Apply $\mathcal{P}_n$ to $\tilde{f}$, $\widetilde{T} \to \mathcal{P}_n\widetilde{T}$ and $X \to \mathcal{P}_n X$ coming from the natural transformation p_n of Prop. 1.5, and let $R(T)$ be the indicated pullback, the (natural) map $T \to R(T) = LR(T)$ (over X) being the above composite.

First, $\mathcal{P}_n\tilde{f}$ is a local fibration, and so is its pullback $R(T) \to X$; since X is locally Kan, so is $R(T)$. Second, $R(T)$, being a pullback of three objects each of which satisfies the exact Kan condition above dimension n, itself satisfies it; so it is an exact n-type, and has vanishing local homotopy groups above n. Finally, $\mathcal{P}_n\widetilde{T}$, being a Postnikov section of an object that is weakly equivalent to $\mathbf{1}$, is itself weakly equivalent to $\mathbf{1}$. $R(T) \to \mathcal{P}_n\widetilde{T}$ is an n-equivalence by virtue of Prop. 0.2; so $R(T)$ has trivial local homotopy groups up to dimension n too. Therefore $R(T) \to X$ belongs to $\mathsf{cov}^n_{\mathcal{E}}(X)$, and we have a functor and natural transformation as required. □

As in any simplicial category with finite limits and colimits, there is a notion of *simplicial homotopy* in $\mathcal{E}^{\Delta^{\mathrm{op}}}$: the role of the interval is played by the constant simplicial sheaf $\Delta[1]$ with its two global sections corresponding to $\Delta[0] \xrightarrow{i_0,\, i_1} \Delta[1]$. Post- and pre-composition of maps respect simplicial homotopies. Write $[X, Y]$ for the hom-set $\hom_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(X, Y)$ modulo the equivalence relation generated by simplicial homotopy. Let $\mathsf{cov}^n_{\mathcal{E}}$ be the full subcategory of $\mathcal{E}^{\Delta^{\mathrm{op}}}$ whose objects are n-covers, and write $[\mathsf{cov}^n_{\mathcal{E}}]$ for the category with the same objects, but morphisms from T_1 to T_2 being $[T_1, T_2]$.

Proposition 1.8. $[\mathsf{cov}^n_{\mathcal{E}}]$ *is a large, cofiltered category, possessing (non-canonical) small cofinal subcategories.*

In the case of $n = 1$, a bit more can be said. Recall that a *preorder* is a reflexive, transitive relation; it can also be thought of as a category with at most a single arrow from any object to any other.

Proposition 1.9. $[\mathsf{cov}^1_{\mathcal{E}}]$ *is a preorder.*

Proof. Quite generally in $\mathcal{E}^{\Delta^{\mathrm{op}}}$, any two maps $f, g\colon X \to \mathrm{cosk}_0(Y)$ into a 0-coskeletal object are simplicially homotopic: the homotopy $\Delta[1] \times X \xrightarrow{h} \mathrm{cosk}_0(Y)$ comes via adjunction from its 0-truncation, which is $X_0 \sqcup X_0 \xrightarrow{f_0 \sqcup g_0} Y$. □

The next proposition should bring Thm. 1.7 a step closer to resembling classical statements.

Proposition 1.10. *Write* $\hom(-,X)$ *resp.* $[-,X]$ *for the contravariant hom- resp. hom- modulo simplicial homotopy functors* $\mathcal{E}^{\Delta^{op}} \to Set$. *Each of*

$$\operatorname*{colim}_{\mathsf{cov}^n_{\mathcal{E}}} \hom(-,X)$$

$$\operatorname*{colim}_{\mathsf{cov}^n_{\mathcal{E}}} [-,X]$$

$$\operatorname*{colim}_{[\mathsf{cov}^n_{\mathcal{E}}]} [-,X]$$

bijects canonically with $\pi_0 \mathsf{ST}(X)$*; thence, if* X *is an exact* n*-type, with* $\mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(\mathbf{1},X)$.

For the sake of completeness, let us include the classical case of '$n=\infty$' here:

Proposition 1.11. *A* Verdier cover *or* hypercover *of a topos* $\mathcal{E}$ *is a locally Kan simplicial object,* $U \in \mathcal{E}^{\Delta^{op}}$*, such that* $U \to \mathbf{1}$ *is a local weak homotopy equivalence. Write* $\mathsf{cov}_{\mathcal{E}}$ *for this category (thought of as a full subcategory of* $\mathcal{E}^{\Delta^{op}}$*) and* $[\mathsf{cov}_{\mathcal{E}}]$ *for the category with the same objects but as morphisms, simplicial homotopy classes of maps between covers.* $[\mathsf{cov}_{\mathcal{E}}]$ *is a cofiltered category, possessing (non-canonical) small cofinal subcategories. For any* $X \in \mathcal{E}^{\Delta^{op}}$*, each of*

$$\operatorname*{colim}_{\mathsf{cov}_{\mathcal{E}}} \hom(-,X)$$

$$\operatorname*{colim}_{\mathsf{cov}_{\mathcal{E}}} [-,X]$$

$$\operatorname*{colim}_{[\mathsf{cov}_{\mathcal{E}}]} [-,X]$$

bijects canonically with $\pi_0 \mathsf{ST}(X)$*; thence, if* X *is locally Kan, with* $\mathsf{ho}_{\mathcal{E}^{\Delta^{op}}}(\mathbf{1},X)$.

Finitary description of exact *n*-types and *n*-covers. The notions of exact fibration, exact n-type and n-cover are preserved by inverse image parts of topos morphisms, in particular, by stalk functors. In the case of toposes with enough points, these notions are also reflected by stalk functors (for the usual logical reasons), e.g. $T \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ will be an n-cover iff all stalks of T are n-covers in *SSet*.

Def. 1.6, after a little unwrapping, turns out to encode structures well known in low dimensions. We will describe these first.

0-covers. An exact 0-type in *SSet* is a simplicial set that is constant, i.e. all of whose structure maps are the identity. A topos $\mathcal{E}$ has (up to isomorphism) a single 0-cover, the constant simplicial object on the terminal object of $\mathcal{E}$. $\mathsf{cov}^0_{\mathcal{E}}$ is equivalent to the trivial (unique object, only the identity morphism) category.

1-covers. An exact 1-type $T \in SSet$ turns out to be precisely the same as the nerve of a groupoid — see Duskin [4]. The graph underlying this groupoid is just the 1-truncation $T_1 \rightrightarrows T_0$ of T. Such a simplicial set T has the homotopy type of a point iff the corresponding groupoid has one component and trivial vertex groups, which happens iff T is 0-coskeletal and nonempty. Correspondingly, a 1-cover of a topos $\mathcal{E}$ is a 0-coskeletal simplicial object $X \Leftarrow X^2 \Lleftarrow X^3 \ldots$ such that $X \to \mathbf{1}$ is an epimorphism. A simplicial morphism between 0-coskeletal objects is necessarily induced from an ordinary map at level 0; we thus have an adjoint equivalence

$$\mathsf{cov}^1_{\mathcal{E}} \underset{\mathrm{tr}_0}{\overset{\mathrm{cosk}_0}{\leftrightarrows}} \{X \in \mathcal{E} \mid X \twoheadrightarrow \mathbf{1}\}$$

between 1-covers of $\mathcal{E}$ and the full subcategory of $\mathcal{E}$ of *objects with global support*, that is, ones that map to the terminal object via an epimorphism.

Example 1.12. Let G be a discrete group, and $\mathcal{E}$ the topos of G-sets; an abelian object A in $\mathcal{E}$ amounts to a G-module, and $H^n(\mathcal{E}, A)$ is the same as $\mathrm{Ext}_{\mathbb{Z}G}(\mathbb{Z}, A)$. A 1-cover of $\mathcal{E}$ is simply a non-empty G-set. $[\mathsf{cov}^1_{\mathcal{E}}]$, in this case, happens to be equivalent to a small category, and it has an initial object, represented by G acting on itself by multiplication (or equivalently, by any non-empty set with a free G-action). Prop. 1.10 thus specializes to the claim that the canonical map

$$[\mathrm{cosk}_0(G), K(A,n)]_G \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, K(A,n)) = H^n(G,A)$$

from G-equivariant simplicial homotopy classes of maps to global homotopy classes, is a bijection for $n = 0, 1$.

Surely the bar resolution does much better: it is a bijection for *all* n. This is an instance of Cartan's theorem.

Example 1.13. More generally, let $\mathcal{E} = \mathrm{Pre}(\mathcal{C})$ be a category of presheaves. A 1-cover of E is a presheaf ϕ such that $\phi(C)$ is non-empty for all objects C of $\mathcal{C}$. $[\mathsf{cov}^1_{\mathcal{E}}]$ is no longer equivalent to a small category in general, but will always have an initial object. Such an object can be taken to be $\coprod_{C \in \mathrm{ob}(\mathcal{C})} y(C)$, the coproduct of the representable presheaves — or more generally, any 'non-empty, free presheaf'. Here 'free' means that it is (isomorphic to) the left Kan extension of a functor defined on the objects of $\mathcal{C}$ only:

$$\mathrm{Pre}(\mathcal{C}) \underset{L_K}{\overset{f}{\rightleftarrows}} Set^{\mathrm{ob}(\mathcal{C})}$$

(where $\mathrm{ob}(\mathcal{C})$ stands for the set of objects of $\mathcal{C}$, thought of as a discrete category, f is the forgetful functor, and L_K is its left adjoint) and 'non-empty' means 'non-empty at each object of $\mathcal{C}$'. Let $\mathcal{U}$ be such a presheaf, and let X be a locally Kan simplicial presheaf on $\mathcal{C}$; this amounts to $X(C)$ being a Kan complex at every object C of $\mathcal{C}$. Then the canonical map

$$[\operatorname{cosk}_0(\mathcal{U}), X] \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$$

(where $[-,-]$ stands for simplicial homotopy classes in $\mathrm{Pre}(\mathcal{C})$) is a bijection if X is an exact 1-type.

Again, one can do better: there is a Quillen model structure on $\mathrm{Pre}(\mathcal{C})^{\Delta^{\mathrm{op}}}$, due to Bousfield–Kan [3], with respect to which any locally Kan X is fibrant; so $[A, X]$ will biject with $\mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$ if the source A is a cofibrant replacement for $\mathbf{1}$ (unlike $\operatorname{cosk}_0(\mathcal{U})$). This Quillen model structure however, with *local* sheaf-theoretic fibrations serving the role of axiomatic fibrations, presupposes the existence of a set of projective generators, which will not hold in a typical topos. Prop. 1.10, just as Verdier's theorem on hypercovers, can be thought of as 'best approximations' of global homotopy classes by simplicial homotopy classes that work uniformly in every topos.

Remark 1.14. It is interesting to note that when the terminal object of $\mathcal{E} = \mathrm{Pre}(\mathcal{C})$ is not representable (i.e. the category $\mathcal{C}$ has no terminal object) one cannot talk of covering the terminal object and computing its Čech cohomology in the classical sense. The extra flexibility provided by thinking of covers as sheaves —- as objects of the topos satisfying some properties, rather than as data assembled from the site — comes handy here. (See below for Čech covers.)

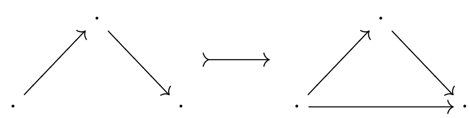
2-covers. The reader sceptical of the intricacies of 'higher dimensional diagram chases' is invited to work out a finitary algebraic description of exact 2-types. Duskin's [5] solution is what he calls a *bigroupoid*, which is a bicategory in which all 1-arrows are equivalences and all 2-arrows are isomorphisms. A bicategory itself (not needed in this paper) is a higher categorical structure made up of objects, 1-arrows and 2-arrows, such that the composition of 1-arrows is *not* associative; its failure to be associative is measured by a (functorially assigned) 2-isomorphism, which in turn is subject to a coherence condition; this coherence condition can be visualized as a 4-simplex, or alternatively as a MacLane–Stasheff pentagon. Identities are likewise relaxed. Duskin assigns a simplicial set, a 'nerve', to any bicategory — this being a rather more involved affair than the nerves of strict higher categories — and proves that the nerve is an exact 2-type iff it arises from a bigroupoid.

What of 2-covers, that is, exact 2-types weakly equivalent to a point? It follows from general principles — see Prop. 1.17 below — that a 2-cover, as a simplicial object, must be 1-coskeletal. Let us take $\mathcal{E} = \mathit{Set}$, and think of the truncation of a simplicial set to levels 0 and 1 as a graph $T_1 \underset{t}{\overset{s}{\rightrightarrows}} T_0$;

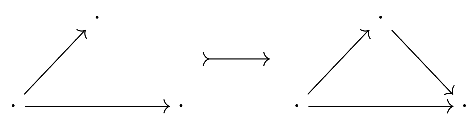
'graph' will mean 'directed graph equipped with unit arrows at each vertex' throughout, but we will omit notation for units for the sake of legibility.

Since π_0 of the 1-coskeleton of this graph must be trivial, T_0 must be non-empty and the graph must be connected. The Kan fibrancy condition on its 1-coskeleton implies that our graph satisfies the right lifting condition (or 'injectivity condition'), in the category of graphs, with respect to the following three inclusions, which should be thought of as the edge truncations of the three Kan horn conditions in dimension 2:

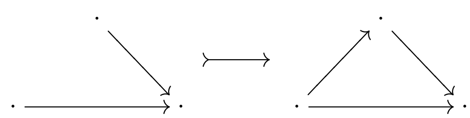
(1.1)



(1.2)



(1.3)



But given that the graph is connected, these imply that for every pair u, v of vertices, there is some edge starting at u and ending at v. Conversely, this latter condition implies that $\operatorname{cosk}_1(T_1 \underset{t}{\overset{s}{\rightrightarrows}} T_0)$ satisfies the Kan condition in dimension 2 (as well as, tautologously, in dimension 1); from being 1-coskeletal, it satisfies the Kan condition exactly above dimension 2; it is connected, and its π_n vanishes for $n \geqslant 1$, by virtue of being Kan and 1-coskeletal. To conclude, a 2-cover in *SSet* is the same data as a graph $T_1 \underset{t}{\overset{s}{\rightrightarrows}} T_0$ such that T_0, the set of vertices, is non-empty, and $T_1 \overset{s\times t}{\twoheadrightarrow} T_0 \times T_0$ is surjective. The 2-cover associated to this data is just the 1-coskeleton of the truncated simplicial object $T_1 \underset{t}{\overset{s}{\rightrightarrows}} T_0$.

For logical reasons, one actually has

Corollary 1.15. *There is an adjoint equivalence*

$$\mathsf{cov}^2_{\mathcal{E}} \underset{\mathrm{tr}_1}{\overset{\operatorname{cosk}_1}{\leftrightarrows}} \{\textit{diagrams } T_1 \underset{t}{\overset{s}{\rightrightarrows}} T_0 \textit{ in } \mathcal{E} \textit{ such that } (0) \textit{ and } (1)\}$$

between 2-covers of $\mathcal{E}$ *and the full subcategory of graph objects in* $\mathcal{E}$ *satisfying*

(0) *the canonical* $T_0 \twoheadrightarrow \mathbf{1}$ *is an epimorphism*

(1) $T_1 \overset{s\times t}{\twoheadrightarrow} T_0 \times T_0$ *is an epimorphism.*

This is also a special case of Prop. 1.18 below.

Remark 1.16. If one prefers the bigroupoid formalism, then one can think of the injectivity condition with respect to (1.1) as expressing the *possibility* of composing graph edges, and (1.2) and (1.3) resp. as permitting 'left' and 'right' division, *without* assuming that such operations have been chosen, or indeed, that they can be chosen with any kind of consistency. The bigroupoid associated to the data $T_1 \rightrightarrows T_0$ in *Set* has T_0 as objects, T_1 as 1-arrows, and exactly one 2-arrow from any 1-arrow to any other with the same source and target. This also fixes the composition of 2-arrows. To define composition of 1-arrows, *choose*, completely arbitrarily, a composite for each composable pair, respecting source and target — this is made possible by (1.1). The association isomorphism, being a 2-arrow between the 1-arrows $(ab)c$ and $a(bc)$, is uniquely defined. The coherence condition on the next level, which requires that the two possible re-association sequences from $(((ab)c)d$ to $a(b(cd)))$, when expressed as 2-arrows, coincide, is satisfied tautologously. In this bicategory, obviously every 1-arrow is an equivalence and every 2-arrow an isomorphism; so it is a bigroupoid.

The correspondence between bigroupoids and exact 2-types breaks down when one passes to sheaves of them, since the requisite choices can be made locally at best. A bigroupoid does carry more information than an exact 2-type; for example, *functional* choices of fillers for certain horns. Equivalently, the category of bigroupoids (as a many-sorted universal algebra) will not be naturally equivalent to the category of exact 2-types (as full subcategory of *SSet*). In this paper, we will employ direct simplicial descriptions throughout.

n-covers. These turn out to be, quite simply, truncated simplicial objects with simplex filling conditions up to the top dimension.

Proposition 1.17. *If T is an n-cover, then it is $(n-1)$-coskeletal.*

Proof. For logical reasons, it is enough to prove this for simplicial sets. By Prop. 0.3, what we have to prove is that the canonical maps $T_k \xrightarrow{b_k} \partial\Delta_k(T)$ are bijections for $k > n-1$.

T is a Kan complex that has the weak homotopy type of a point, which implies that the b_k are surjective for all k (if one prefers, T satisfies the right lifting property w.r.t. all the boundary inclusions $\partial\Delta_k \to \Delta_k$).

By virtue of T being exact Kan above dimension n, b_k is injective for $k > n$ (if any k-horn has precisely one filler, then certainly any k-boundary can have at most one filler). Thence b_k is bijective for $k > n$.

So we are left to prove that b_n is injective. It is easiest to argue topologically. Let $\Delta_n \xrightarrow{s} T$, $\Delta_n \xrightarrow{t} T$ be two n-simplices of T whose boundaries coincide. Since T was assumed to have the weak homotopy type of a point, the geometric realizations of s and t are homotopic modulo the boundary. Since T was assumed to be a Kan complex, s and t are simplicially homotopic modulo the boundary. But two n-simplices of an exact n-type that are simplicially homotopic modulo the boundary, must coincide. □

Proposition 1.18. *Write $\Delta[0, n-1]$ for the full subcategory of Δ (i.e. the category of finite ordinals and monotone maps) whose objects are $0, 1, \ldots, n-1$. There is an adjoint equivalence*

$$\mathsf{cov}^n_{\mathcal{E}} \underset{\mathrm{tr}_{n-1}}{\overset{\mathrm{cosk}_{n-1}}{\leftrightarrows}} \{T \in \mathcal{E}^{\Delta[0,n-1]^{op}} \mid (\bigstar) \text{ holds for } k = 0, 1, 2, \ldots, n-1\}$$

between n-covers of $\mathcal{E}$ and the full subcategory of $n-1$-truncated simplicial objects in $\mathcal{E}$ satisfying

$$(\bigstar) \quad T_k \overset{b_k}{\twoheadrightarrow} \partial\Delta_k(T) \text{ is an epimorphism}$$

where $\partial\Delta_k(T)$ and b_k are defined as in Prop. 0.3(3).

2. Čech n-covers

To begin with, let us describe how Thm. 1.7 specializes, for $n = 1$, to the familiar formalism of Čech cohomology.

Example 2.1. Let X be a topological space. For convenience, we make no notational distinction between spaces étale over X and the sheaves on X they represent, and we talk of 'n-covers of $\mathrm{Sh}(X)$' as 'n-covers of X'.

(1) Let $\{U_i \mid i \in I\}$ be an open cover of X in the usual sense. Each U_i represents, as an étale space over X, a subobject of the terminal object $\mathbf{1}$ of $\mathrm{Sh}(X)$. Set $\mathcal{U}$ to be the coproduct of the U_i considered as objects of $\mathrm{Sh}(X)$ (equivalently, take their disjoint union as spaces over X). $\mathcal{U} \twoheadrightarrow \mathbf{1}$ then, so $\mathrm{cosk}_0(\mathcal{U})$ is a 1-cover of X.

(2) Let $C \xrightarrow{p} X$ be a covering space of X in the usual sense. Then p is certainly étale and surjective over X, so $\mathrm{cosk}_0(C)$ is a 1-cover of X.

(3) One can have a mixture of (1) and (2): a different covering space over each element of an open cover of X. These (and in fact all 1-covers of a space) are dominated by trivializations, thus Čech covers as in (1).

Note that there is a proper class of non-isomorphic 1-covers of X, while there are certainly no more open covers of X than collections of open subsets of X. Nonetheless, open covers are cofinal among 1-covers (having passed to simplicial homotopy classes). This is a general phenomenon, as we will see.

Let now $(\mathcal{C}, J)$ be an arbitrary site. Denote by ϵ the 'canonical functor' $\mathcal{C} \xrightarrow{y} \mathrm{Pre}(\mathcal{C}) \xrightarrow{L} \mathrm{Sh}(\mathcal{C}, J)$, the Yoneda embedding followed by sheafification. Assume $\mathcal{C}$ has a terminal object, which we will denote by $\mathbf{1}$ as well.[1]

Let $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I_1\}$ be an arbitrary set of arrows in $\mathcal{C}$ with common target $\mathbf{1}$. Set $\mathcal{U}_{I_1} = \coprod_{i \in I_1} \epsilon(U_i)$. For another such collection $\{U_k \xrightarrow{u_k} \mathbf{1} \mid k \in I_2\}$, a *refinement* from the latter to the former is a mapping $\Phi : I_2 \to I_1$ together with a factorization $U_k \to U_{\Phi(k)} \to \mathbf{1}$ for each $k \in I_2$. Φ then induces a map $\mathcal{U}_{I_2} \to \mathcal{U}_{I_1}$. Refinements can be composed in the obvious way.

Note that $\mathcal{U}_I = \coprod_{i \in I} \epsilon(U_i) \twoheadrightarrow \mathbf{1}$ in $\mathcal{E}$ iff the $U_i \xrightarrow{u_i} \mathbf{1}$ form a covering family; in that case, $\mathrm{cosk}_0(\mathcal{U}_I)$ is a 1-cover of $\mathcal{E}$. (Here and hereafter, we write *covering family* to mean *a collection of arrows with common codomain generating a covering sieve for the topology J*, to ease somewhat on the multiple uses of the term 'cover'.) Let $\mathsf{cech}^1_{(\mathcal{C},J)}$ be the category whose objects are covering families $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I_\lambda\}$ and whose morphisms are refinements. (It is a small category if $(\mathcal{C}, J)$ is a small site.) In line with the notation of Prop. 1.8, write $[\mathsf{cech}^1_{(\mathcal{C},J)}]$ for the category with the same objects as $\mathsf{cech}^1_{(\mathcal{C},J)}$, but morphisms being (the simplicial maps induced by) refinements modulo simplicial homotopies. Finally let $J(\mathbf{1})$ be the poset of J-sieves covering $\mathbf{1}$, ordered by inclusion.

Lemma 2.2. *Given $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I_1\}$, let $\{U_k \xrightarrow{u_k} \mathbf{1} \mid k \in I_2\}$ be the sieve it generates. Then $\mathrm{cosk}_0(\mathcal{U}_{I_1})$ and $\mathrm{cosk}_0(\mathcal{U}_{I_2})$ are simplicially homotopy equivalent.*

Proof. By Prop. 1.9, it suffices to exhibit a refinement from I_2 to I_1, and one in the other direction. Since each element of I_2 must factor through an element of I_1 (by definition), a choice of such for each $u_k \in I_2$ gives a refinement from I_2 to I_1. In the other direction one has the inclusion $I_1 \subseteq I_2$. □

Proposition 2.3. *$J(\mathbf{1})$ and $[\mathsf{cech}^1_{(\mathcal{C},J)}]$ are equivalent as categories.*

Proof. By Prop. 1.9, $[\mathsf{cech}^1_{(\mathcal{C},J)}]$ is a preorder. Since each covering sieve is a covering family, and each inclusion of sieves a refinement, one has a functor

[1]For the sake of exposition, we are purely concerned with derived functors of the global section functor, meaning global homotopy classes with source the terminal object; that is the reason for limiting the description to Čech covers of $\mathbf{1}$, though it leads to tautological notation occasionally.

$J(\mathbf{1}) \hookrightarrow [\mathsf{cech}^1_{(\mathcal{C},J)}]$. It is faithful and full, and surjective on isomorphism types of objects by the above lemma. □

Proposition 2.4. *The functor* $[\mathsf{cech}^1_{(\mathcal{C},J)}] \to [\mathsf{cov}^1_{\mathcal{E}}]$ *induced by the functor sending* $\{U_i \xrightarrow{u_i} \mathbf{1}\}$ *to* $\operatorname{cosk}_0\left(\bigsqcup \epsilon(U_i)\right)$ *is cofinal.*

Proof. For any object T_0 of global support, there exists a covering family $\{U_i \xrightarrow{u_i} \mathbf{1}\}$ that allows a map $\bigsqcup \epsilon(U_i) \to T_0$. Any two Čech 1-covers $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I_1\}$ and $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I_2\}$ have a common refinement, namely the intersection of the sieves generated by I_1 and I_2. □

In the next formula, to reduce notational clutter, we do not distinguish between a Čech cover and the simplicial sheaf it gives rise to.

Corollary 2.5. $[\mathsf{cech}^1_{(\mathcal{C},J)}]$ *is a cofiltered category. For any* $X \in \mathcal{E}^{\Delta^{op}}$, *one has a natural bijection*

$$\operatorname*{colim}_{[\mathsf{cech}^1_{(\mathcal{C},J)}]} [-, X] \to \operatorname*{colim}_{[\mathsf{cov}^1_{\mathcal{E}}]} [-, X]$$

Set $X = K(A, n)$ and define

$$\check{H}^n(A) = \operatorname*{colim}_{[\mathsf{cech}^1_{(\mathcal{C},J)}]} [-, K(A, n)]$$

Čech cohomology of the site $(\mathcal{C}, J)$ with coefficients in A.

If iterated products of the objects U_i that make up Čech covers exist in $\mathcal{C}$, then the objects of higher-dimensional simplices of the $\operatorname{cosk}_0\left(\bigsqcup \epsilon(U_i)\right)$ are representable, hence $\check{H}^n(A)$ can be computed by the usual recipe, i.e. as a filtered colimit of cohomology of chain complexes. The map of Cor. 2.5 followed by $\operatorname*{colim}_{[\mathsf{cov}^1_{\mathcal{E}}]} [-, X] \to \mathsf{ho}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}(\mathbf{1}, X)$ specializes to the natural transformation $\check{H}^n(A) \to H^n(A)$ from Čech to derived functor cohomology, and as a corollary of Cor. 2.5 and the main theorem, it is an isomorphism for $n = 0, 1$.

Remark 2.6. Čech covers can only be defined for a *site*, and are (covariantly) functorial for site morphisms; here a site morphism from $(\mathcal{C}, J)$ to $(\mathcal{D}, K)$ (let us assume these have finite limits) is a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ that preserves finite limits and takes K-covers to J-covers. It induces a topos map $\mathrm{Sh}(\mathcal{C}, J) \to \mathrm{Sh}(\mathcal{D}, K)$. Recall that any category of sheaves can be presented on a proper class of different sites, and it may well happen that $\mathrm{Sh}(\mathcal{C}, J)$ and $\mathrm{Sh}(\mathcal{D}, K)$ are equivalent as categories without there being any site map between $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$! (There will exist a zig-zag of site maps, though.) At any rate, it is not at all obvious that $\check{H}^n(A)$ depends purely on the category of sheaves and the object A in it. It seems best to concur with Grothendieck's point of view that algebraic topological invariants are owned by the abstract category of sheaves, and the language of the topos should be used to define them and to establish their basic properties. A good choice of site and translation of the theorems into the language of sieves, presheaves etc. will facilitate calculations.

In the case of an *arbitrary* site, the best that can be said is that 'level 1 Čech homotopy' and global homotopy classes (from the terminal object) coincide when the target is an exact 1-type. There are easy examples to show that in favorable geometric situations one can do much better.

Let us move to Čech 2-covers.

Example 2.7. Continuing with the setting of Ex. 2.1,

(0) set $V_{ij} = U_i \cap U_j$ for each ordered pair i, j, and let $\mathcal{V}$ be the coproduct of the V_{ij}. The inclusions $V_{ij} \hookrightarrow U_i$ resp. $V_{ij} \hookrightarrow U_j$ together with $U_i = V_{ii}$ define a sheaf of (directed, unit arrow-equipped) graphs $\mathcal{V} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{U}$, the 1-coskeleton of which is a 2-cover of X.

But the graph $\mathcal{V} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{U}$ is nothing but the 1-truncation of the simplicial object $\mathrm{cosk}_0(\mathcal{U})$ constructed in Ex. 2.1(1), and $\mathrm{cosk}_1(\mathcal{V} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{U})$ is canonically isomorphic to $\mathrm{cosk}_0(\mathcal{U})$. All this is just an instance of the tautology that an n-cover (considered as a simplicial object) is also an m-cover for any $m > n$.

(1) Suppose, however, that the family $\mathcal{V}$ is only a *refinement* of the collection $\{U_i \cap U_j \mid i, j \in I\}$ in the following sense: for each ordered pair $i, j \in I$ one has an index set K_{ij}, and for each $k \in K_{ij}$ an open subset V_k of $U_i \cap U_j$ such that $\bigcup_{k \in K_{ij}} V_k = U_i \cap U_j$, *and* each original open U_i is included in the collection K_{ii} (this is to ensure that the unit condition can be satisfied). Let $\mathcal{V}$ be the coproduct of the V_k, $k \in K_{ij}$, $i, j \in I$. The inclusions $V_k \hookrightarrow U_i \cap U_j \hookrightarrow U_i$ (resp. U_j) give a sheaf of graphs $\mathcal{V} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{U}$ such that $\mathcal{V} \twoheadrightarrow \mathcal{U} \times \mathcal{U}$, and this is our first non-trivial example of a 2-cover. In short, *in a 2-cover a second refinement occurs on double intersections.*

(2) Let $C \xrightarrow{p} X$ be a covering space, with $D \xrightarrow{d} C \times_X C$ another covering. $C \times_X C$ contains a 'marked' copy of C, thanks to the diagonal $C \xrightarrow{\Delta} C \times_X C$. Assume d has a section above Δ, so the unit condition is satisfied. The composite $D \xrightarrow{d} C \times_X C \rightrightarrows C$, the double arrows being the projections, is then a 2-cover of X. (Even

if not a covering space, D is certainly étale and surjective above C, and that is all that matters sheaf-theoretically.)

(3) Analogously to Ex. 2.1, one can have a mixture of (1) and (2), i.e. a different 2-tier covering space system above each open, but any 2-cover of a space is dominated by one of type (1).

For the case of a general site $(\mathcal{C}, J)$ with terminal object $\mathbf{1}$, this suggests

Definition 2.8. A *Čech 2-cover* (of the terminal object) is a covering family $\{U_i \xrightarrow{u_i} \mathbf{1} \mid i \in I\}$ together with covering families $\{V_k \xrightarrow{v_k} U_i \times U_j \mid k \in K_{ij}\,;\, i,j \in I\}$ such that K_{ii}, for each $i \in I$, contains $U_i \times U_i \xrightarrow{\text{id}} U_i \times U_i$. The 2-cover associated to this data is $\operatorname{cosk}_1(\mathcal{V}_K \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{U}_I)$ where $\mathcal{V}_K = \coprod_{\substack{k \in K_{ij} \\ i,j \in I}} V_k$ and $\mathcal{U}_I = \coprod_{i \in I} U_i$; s has components $V_k \xrightarrow{v_k} U_i \times U_j \xrightarrow{\text{pr}_1} U_i \to \mathcal{U}_I$, with the other projection for t, and a splitting $\mathcal{U}_I \to \mathcal{V}_K$ as assumed. A *refinement* of Čech 2-covers $\mathcal{V}_{K_2} \rightrightarrows \mathcal{U}_{I_2} \rightsquigarrow \mathcal{V}_{K_1} \rightrightarrows \mathcal{U}_{I_1}$ is given by a function $\Phi : I_2 \to I_1$ together with factorizations $U_i \to U_{\Phi(i)} \to \mathbf{1}$ for $i \in I_2$, and a family of functions $\Psi_{ij} : K_{ij} \to K_{\Phi(i)\Phi(j)}$ together with maps $V_k \to V_{\Psi_{ij}(k)}$ such that

$$\begin{array}{ccc} V_k & \longrightarrow & V_{\Psi_{ij}(k)} \\ {\scriptstyle v_k}\downarrow & & \downarrow{\scriptstyle v_{\Psi_{ij}(k)}} \\ U_i \times U_j & \longrightarrow & U_{\Phi(i)} \times U_{\Phi(j)} \end{array}$$

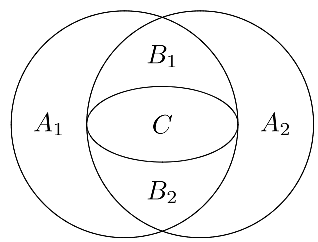
commutes. This will induce a morphism of graphs from $\mathcal{V}_{K_2} \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{U}_{I_2}$ to $\mathcal{V}_{K_1} \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{U}_{I_1}$, hence a simplicial map between their 1-coskeleta. Write $\mathsf{cech}^2_{(\mathcal{C},J)}$ for the category whose objects are Čech 2-covers, morphisms being refinements, and $[\mathsf{cech}^2_{(\mathcal{C},J)}]$ for the category with the same objects, but morphisms the simplicial maps induced by refinement modulo (the equivalence relation generated by) simplicial homotopy. (Here again, we make no notational distinction between a Čech 2-cover and the simplicial object, i.e. 2-cover it gives rise to.)

Remark 2.9. To ease notational congestion, we have assumed above that the site $\mathcal{C}$ has finite products. One can get around that: instead of asking for a covering family $\{V_k \xrightarrow{v_k} U_1 \times U_2 \mid k \in K\}$, ask for a set of pairs of maps $\{V_k \xrightarrow{v_{1,k}} U_1,\, V_k \xrightarrow{v_{2,k}} U_2 \mid k \in K\}$ such that $\coprod_{k \in K} \epsilon(V_k) \xrightarrow{\epsilon(v_{1,k}) \times \epsilon(v_{2,k})} \epsilon(U_1) \times \epsilon(U_2)$ is epi in $\text{Sh}(\mathcal{C}, J)$, etc.

The cheap analogue of Prop. 1.9 fails for n-coskeletal objects with $n > 0$, and there seems to be no way to construe a canonical *poset* (or even preorder) of Čech 2-covers on an arbitrary site. The analogues of 2.4 and 2.5 do survive:

Proposition 2.10. $[\mathsf{cech}^2_{(\mathcal{C},J)}]$ *is a cofiltered category. The inclusion* $[\mathsf{cech}^2_{(\mathcal{C},J)}] \to [\mathsf{cov}^2_{\mathcal{E}}]$ *is cofinal.*

It is easy enough to concoct examples of sites, in fact finite topological spaces, whose Čech covers are not cofinal among their Čech 2-covers (when both are thought of as subcategories of simplicial sheaves). Let $X = A_1 \sqcup A_2 \sqcup B_1 \sqcup B_2 \sqcup C$ be the disjoint union of five non-empty sets. Put a topology on X by declaring the following to be its non-trivial opens: C, $B_1 \cup C$, $B_2 \cup C$, $B_1 \cup B_2 \cup C$, $A_1 \cup B_1 \cup B_2 \cup C$, $A_2 \cup B_1 \cup B_2 \cup C$.



Note that this space is compact, but never Hausdorff (though it can be sober). It has a 2-cover defined by $\operatorname{cosk}_1(\mathcal{V} \rightrightarrows \mathfrak{U})$ where $\mathcal{V}$ is (the sheaf represented by) $(B_1 \cup C) \sqcup (B_2 \cup C) \sqcup (A_1 \cup B_1 \cup B_2 \cup C) \sqcup (A_2 \cup B_1 \cup B_2 \cup C)$ and $\mathfrak{U}$ stands for $(A_1 \cup B_1 \cup B_2 \cup C) \sqcup (A_2 \cup B_1 \cup B_2 \cup C)$. There is no Čech 1-cover $\operatorname{cosk}_0(\mathcal{W})$ that permits a map $\operatorname{cosk}_0(\mathcal{W}) \to \operatorname{cosk}_1(\mathcal{V} \rightrightarrows \mathfrak{U})$. Indeed, such a cover $\mathcal{W}$ would have to contain both of the opens $A_1 \cup B_1 \cup B_2 \cup C$ and $A_2 \cup B_1 \cup B_2 \cup C$ in degree 0, hence it would have to contain the intersection $B_1 \cup B_2 \cup C$ in simplicial degree 1. But $B_1 \cup B_2 \cup C$ has nowhere to map to in $\operatorname{cosk}_1(\mathcal{V} \rightrightarrows \mathfrak{U})$.

The pattern is now quite clear. A *Čech 3-cover* is the data for a Čech 2-cover together with a refinement (i.e. epimorphism onto) the sheaf of 3-simplices of the associated simplicial sheaf. Precisely, a Čech 3-cover of (the terminal object of) $(\mathcal{C}, J)$ means a covering family $\{U_\alpha \xrightarrow{u_\alpha} \mathbf{1} \mid \alpha \in I\}$ together with covering families $\{V_k \xrightarrow{v_k} U_\alpha \times U_\beta \mid k \in K_{\alpha\beta}\,;\, \alpha, \beta \in I\}$, and the following. Consider the set of triples of indices of the form $\{\langle k_1, k_2, k_3\rangle \mid k_1 \in K_{\alpha\beta}, k_2 \in K_{\alpha\gamma}, k_3 \in K_{\beta\gamma};\, \alpha, \beta, \gamma \in I\}$. $T_{k_1k_2k_3}$ is defined to be the subobject of $V_{k_1} \times V_{k_2} \times V_{k_3}$ that is the intersection of three equalizers; the first of these is the equalizer of $V_{k_1} \times V_{k_2} \times V_{k_3} \xrightarrow{\text{pr}} V_{k_1} \to U_\alpha \times U_\beta \xrightarrow{\text{pr}} U_\alpha$ and $V_{k_1} \times V_{k_2} \times V_{k_3} \xrightarrow{\text{pr}} V_{k_2} \to U_\alpha \times U_\gamma \xrightarrow{\text{pr}} U_\alpha$, and the other two are re-labelled analogues. (What this means is that $T_{k_1k_2k_3}$, as the indices k_i take all possible values, gives the 'object of triangles' of the graph associated to the Cech 2-cover: three edges compatible at the three corners. That is the same as the object of 3-simplices of the 2-coskeleton.) Now give a covering family $\{W_i \xrightarrow{w_i} T_{k_1 k_2 k_3} \,|\, i \in L_{k_1 k_2 k_3}\}$ for each such triple $\langle k_1, k_2, k_3 \rangle$. (The data are subject to the degeneracy conditions in the case of repeated indices.)

One associates a 3-truncated simplicial sheaf to this data (via $\mathcal{C} \xrightarrow{y} \mathrm{Pre}(\mathcal{C}) \xrightarrow{L} \mathrm{Sh}(\mathcal{C}, J) = \mathcal{E}$), and hence a 3-cover of $\mathcal{E}$. The notion of refinement of Čech 3-covers, the associated cofiltered diagram (after modding out by simplicial homotopies) and its properties can be formulated as expected — but at this stage, such an approach hardly seems more elegant than that offered by n-covers.

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