On Strongly Flat Modules over Matlis Domains

L. Fuchs, L. Salce* and J. Trlifaj†

Let R be an integral domain and Q its field of quotients. An R-module M is called *strongly flat* if $\mathrm{Ext}^1_R(Q, C) = 0$ implies $\mathrm{Ext}^1_R(M, C) = 0$ for any $C \in \text{Mod-}R$. The first characterization of strongly flat modules was given by Trlifaj [T2], and they were thoroughly investigated by Bazzoni and Salce [BS3].

The strongly flat modules seem to behave over Matlis domains R particularly nicely; the reason for that may be sought in the fact that they are all of projective dimension 1, and — as has been demonstrated in [FS] — such modules often display features resembling the Dedekind domain case, so they are more tractable. Another reason goes back to an old result of Matlis [M] which — in modern terminology — asserts that Matlis domains can be characterized by the property that strongly flat modules form a resolving class. Flat modules always form a resolving class, and the coincidence of the two classes characterizes an important subclass of the Matlis domains: the almost perfect ones, cf. [BS2-3].

In the Matlis domain case, we can characterize the strongly flat R-modules as torsion-free modules M that satisfy $\text{p.d.}_R(M \otimes_R K) \leq 1$, where K stands for the factor module Q/R; see (1.2). This characterization, along with a theorem by Lee [L] on divisible modules of projective dimension 1 over Matlis domains, leads to a more structural description: they admit continuous well-ordered ascending chains whose factors are strongly flat modules of ranks of bounded cardinality, a bound being the minimal cardinality of generating sets of Q; cf. (2.1).

By making use of these results, we can prove that over Matlis valuation domains the strongly flat modules are exactly the extensions of free modules by divisible torsion-free modules; see (3.3). This result has been established over arbitrary valuation domains in [BS3] only for modules of rank up to the continuum.

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More can be said in the general case of Matlis domains provided we are willing to restrict our considerations to the constructible universe. We show that in this case there is a universal test module to check the strong flatness of an arbitrary module of projective dimension ≤ 1; cf. (5.4).

This result motivates the search for a universal test module for projective dimension ≤ 1. We are able to find such a test module, under the assumption V= L, for Matlis domains of global dimension 2; cf. (5.5). Combining the two test modules, we obtain, in case of V=L, a universal test module for all strongly flat modules over Matlis domains of global dimension 2; see (5.6).

The situation with strongly flat modules over Matlis domains strongly resembles the theory of Baer modules. For Baer modules one can establish the existence of a continuous well-ordered ascending chain with factors of bounded number of generators ($\aleph_0$ suffices as a bound, see Eklof-Fuchs-Shelah [EFS]), but a strong set-theoretical hypothesis (e.g. V = L) is needed to ascertain the existence of a universal test module; see e.g. [FS, XVI.8].

Over certain domains R, the existence of a universal test module for strong flatness or for projective dimension ≤ 1 can be verified without any additional hypothesis, or in a different way. For instance, if R is an almost perfect domain (i.e. all proper factor rings are perfect), then all flat modules are strongly flat (see [BS2] and [BS4]), and there exists a universal test module for flatness. On the other hand, if R is an IC-domain (see [BS1]), then — under the hypothesis V = L — K is a universal test module for projective dimension ≤ 1. The relevant result (6.3) in the present paper shows that, assuming the Uniformization Principle, there is no universal test module for strong flatness over a Matlis domain whenever it is not almost perfect.

The conclusion is that, over a Matlis domain of global dimension 2 that is not almost perfect, the existence of a universal test module for strong flatness is independent of ZFC+GCH; cf. (6.4).

1. Preliminaries.

In what follows R denotes a commutative domain with 1, and $Q \neq R$ its field of quotients. We shall use the notation $K = Q/R$. R is a *Matlis domain* if the projective dimension $\mathrm{p.d.}_R\, Q = 1$.

The R-completion of a module M will be denoted by $\widetilde{M}$, and $\mathrm{gen}\, M$ will denote the minimal cardinality for generating sets of M.

An R-module C is *weakly cotorsion* or *Matlis cotorsion* if $\mathrm{Ext}^1_R(Q, C) = 0$. An R-module M is *strongly flat* if $\mathrm{Ext}^1_R(M, C) = 0$ holds for all weakly cotorsion R-modules C. Evidently, strongly flat modules are flat, and hence torsion-free.

Lemma 1.1. (Trlifaj [T2], Bazzoni-Salce [BS]) *For a reduced torsion-free module M over a Matlis domain R the following are equivalent:*

(i) *M is strongly flat;*

(ii) *M is a summand of an R-module N fitting in an exact sequence*

$$0 \to F \to N \to D \to 0,$$

where F is a free and D is a torsion-free divisible R-module;

(iii) *the R-completion $\widetilde{M}$ of M is a summand of the completion of a free R-module;*

(iv) *$K \otimes M$ is a summand of a direct sum of copies of K.* ∎

The strongly flat modules M over a Matlis domain R can easily be characterized in terms of the tensor product $K \otimes_R M$. This characterization is crucial in the sequel.

Corollary 1.2. *Let R be a Matlis domain. A torsion-free module M is strongly flat if and only if*

$$\underset{R}{\text{p.d.}}(K \otimes_R M) \leq 1.$$

Proof. This follows at once from the fact that (1.1)(iv) holds if and only if $\text{p.d.}_R(K \otimes_R M) \leq 1$; see [FS, VII.2.4]. ∎

Observe that $\text{p.d.}\, N \leq \max\{\text{p.d.}\, F, \text{p.d.}\, D\} = 1$ for Matlis domains, consequently,

Corollary 1.3. *A domain R is a Matlis domain if and only if strongly flat R-modules have projective dimension ≤ 1.* ∎

Recall that a submodule N of a module M of projective dimension k is said to be a *tight* submodule if $\text{p.d.}\, M/N \leq k$; then $\text{p.d.}\, N \leq k$ likewise. We now have the simple fact:

Lemma 1.4. *If R is a Matlis domain, then tight submodules of strongly flat R-modules are again strongly flat.*

Proof. Consider the exact sequence $0 \to N \to M \to M/N \to 0$, where M is strongly flat, so $\text{p.d.}\, M \leq 1$ by (1.3). For any weakly cotorsion R-module C we have the induced exact sequence

$$\text{Ext}^1_R(M, C) \to \text{Ext}^1_R(N, C) \to \text{Ext}^2_R(M/N, C).$$

The two ends vanish, since M is strongly flat and $\text{p.d.}\, M/N \leq 1$, respectively, so the middle term is necessarily 0, proving the assertion. ∎

The following corollary is immediate.

Corollary 1.5. *Pure submodules of countably generated strongly flat modules over a Matlis domain are again strongly flat.*

Proof. If N is a pure submodule in the countably generated strongly flat module M, then M/N is countably generated and flat, so it has projective dimension ≤ 1 (see e.g. [FS, VI.9.8]). An appeal to (1.4) completes the proof. ■

2. Chains with strongly flat factors of bounded size.

Assume from now on that R is a Matlis domain with $\operatorname{gen} Q = \mu$, an infinite (regular) cardinal. This cardinal seems to be a most relevant invariant of R. (If $\mu = \aleph_0$, then this is actually a stronger hypothesis than R being a Matlis domain, because $\operatorname{gen} Q \leq \aleph_0$ implies p.d. $Q \leq 1$.) Furthermore, suppose that M is a reduced strongly flat R-module (*reduced* means that it has no divisible submodule $\neq 0$).

We form the exact sequence

$$0 \to M \to D \to T = K \otimes_R M \to 0, \tag{1}$$

where D stands for the injective hull of M; it is the direct sum of copies of Q. Let ϕ denote the canonical map $D \to T$.

We will need the following useful device. By an *$H(\kappa)$-family* $\mathcal{F}$ of submodules of a module M we mean a collection of submodules of M such that

H1. $0, M \in \mathcal{F}$;

H2. $\mathcal{F}$ is closed under sums of submodules;

H3. if $A \in \mathcal{F}$ and X is a subset of M of cardinality $\leq \kappa$, then there is a $B \in \mathcal{F}$ such that $A \cup X \subseteq B$ and $\operatorname{gen} B/A \leq \kappa$.

Select an $H(\mu)$-family $\mathcal{D}$ of summands of D. Owing to (1.2), p.d. $T \leq 1$, so by a theorem of Lee (see [L], or [FS, VII.2.9]) T is a direct sum of countably generated submodules. Therefore, we can select an ($H(\aleph_0)$-, and hence an) $H(\mu)$-family $\mathcal{T}$ of summands of T.

We now extract, by transfinite induction, from these $H(\mu)$-families continuous well-ordered ascending chains

$$0 = D_0 < D_1 < \cdots < D_\sigma < D_{\sigma+1} < \cdots$$
$$0 = T_0 < T_1 < \cdots < T_\sigma < T_{\sigma+1} < \cdots$$

in $\mathcal{D}$ and $\mathcal{T}$, respectively, with $\sigma < \tau$ for a suitable ordinal τ such that

(i) $D = \bigcup_{\sigma<\tau} D_\sigma$ and $T = \bigcup_{\sigma<\tau} T_\sigma$;

(ii) D_σ is a summand of D, and $\phi(D_\sigma) = T_\sigma$ is a summand of T for all $\sigma < \tau$;

(iii) $\operatorname{gen} D_{\sigma+1}/D_\sigma \le \mu$ and $\operatorname{gen} T_{\sigma+1}/T_\sigma \le \mu$ for all $\sigma + 1 < \tau$.

The critical condition is $\phi(D_\sigma) = T_\sigma$. Suppose that we have constructed chains of D_σ and T_σ for all σ up to some ordinal ρ such that all the conditions are satisfied. If ρ is a limit ordinal, then we have to set $D_\rho = \bigcup_{\sigma<\rho} D_\sigma$ and $T_\rho = \bigcup_{\sigma<\rho} T_\sigma$. If ρ is a successor ordinal, then we form a chain

$$D_{\rho-1} = D^0 < D^1 < \cdots < D^n < \cdots \qquad (n < \omega)$$

with links in $\mathcal{D}$, and a chain

$$T_{\rho-1} = T^0 < T^1 < \cdots < T^n < \cdots \quad (n < \omega)$$

with links in $\mathcal{T}$, with at most μ-generated factors, such that

$$T_{\rho-1} = \phi(D^0) < T^1 \le \phi(D^1) \le \cdots \le T^n \le \phi(D^n) < \cdots \quad (n < \omega).$$

Letting $D_\rho = \bigcup_{n<\omega} D^n$ and $T_\rho = \bigcup_{n<\omega} T^n$, $\phi(D_\rho) = T_\rho$ will hold. It is easy to arrange the selections such that (i) will be satisfied.

Next define $M_\sigma = M \cap D_\sigma$ ($\sigma < \tau$). Then we get a continuous well-ordered ascending chain

$$0 = M_0 < M_1 < \cdots < M_\sigma < M_{\sigma+1} < \cdots \tag{2}$$

of submodules in M such that

(iv) for every $\sigma < \tau$, $0 \to M_\sigma \to D_\sigma \to T_\sigma \to 0$ is an exact sequence.

After all this preparation we can state:

Theorem 2.1. *Let R be a Matlis domain satisfying* $\operatorname{gen} Q \le \mu$*. Every reduced strongly flat R-module M admits a continuous well-ordered ascending chain* (2) *of submodules M_σ such that all the factors $M_{\sigma+1}/M_\sigma$ are strongly flat modules of rank at most μ.*

Proof. In view of the preceding argument, M has a continuous well-ordered ascending chain (2) of submodules satisfying (iv). Then $M_{\sigma+1}/M_\sigma \cong [D_\sigma + (M \cap D_{\sigma+1})]/D_\sigma$ ($\le D_{\sigma+1}/D_\sigma$) is torsion-free and

$$(M_{\sigma+1}/M_\sigma) \otimes K \cong (M_{\sigma+1} \otimes K)/(M_\sigma \otimes K) \cong T_{\sigma+1}/T_\sigma$$

holds for every $\sigma < \tau$, where we have used the isomorphism $M_\sigma \otimes K \cong T_\sigma$ which is obtained by tensoring the exact sequence in (iv) above with K (recall that $D_\sigma \otimes K = 0 = \operatorname{Tor}_1^R(D_\sigma, K)$ and $\operatorname{Tor}_1^R(T_\sigma, K) \cong T_\sigma$). The last factor module is isomorphic to a summand of T, so it certainly has projective dimension ≤ 1. The factor modules $M_{\sigma+1}/M_\sigma$ are evidently of rank at most μ. It remains to appeal to (1.2) to complete the proof. ∎

The following immediate consequence shows that actually we have a large supply of strongly flat submodules contained in any strongly flat module over a Matlis domain.

Corollary 2.2. *Assuming the hypothesis of* (2.1)*, every strongly flat module has an* $H(\mu)$*-family of strongly flat submodules.*

Proof. Starting from (2.1) use [FS, XVI.8.11] to obtain the stated conclusion. ■

3. The class of projective by divisible modules.

The class $\mathcal{C}$ of R-modules that are extensions of projective modules by divisible torsion-free modules is evidently contained in the class of strongly flat R-modules. As the module N in (1.1) belongs to the class $\mathcal{C}$, it is clear that the class of strongly flat modules coincides with the class of summands of modules in $\mathcal{C}$.

In general, the class $\mathcal{C}$ is not closed under taking direct summands. An easy counterexample has been provided by Bazzoni-Salce: the additive group J_p of the p-adic integers is a summand of the $\mathbb{Z}$-completion of $\mathbb{Z}$, but it is not a member of the class $\mathcal{C}$. However, we wish to show that for Matlis valuation domains R the answer is in the affirmative.

Bazzoni and Salce [BS3, Thm 3.15] show that if R is a valuation domain, then the answer is 'yes' up to cardinality $\aleph_1$, but have no answer for larger cardinalities. We are now going to use a completely different approach that avoids any need for transfinite induction, so we do not have to refer to a sort of singular compactness theorem in our proof. Recall that if R is a valuation domain, then $\text{p.d.}_R\, Q = 1$ implies $\text{gen}\, Q = \aleph_0$.

The next lemmas are important steps in the proof of (3.3).

Lemma 3.1. (Bazzoni-Salce [BS3, Lemma 3.14]) *Let R be a valuation domain. Suppose M is a torsion-free R-module which is, for some cardinal κ, the union of a continuous well-ordered ascending chain*

$$0 = M_0 < M_1 < \cdots < M_\sigma < \cdots \quad (\sigma < \kappa)$$

of submodules M_σ. If all the factors $M_{\sigma+1}/M_\sigma$ belong to the class $\mathcal{C}$, then so does M as well. ■

We quote a basic result from [BS3].

Lemma 3.2. (Bazzoni-Salce [BS3, Lemma 3.13]) *If R is a valuation domain, then a strongly flat R-module of countable rank has a free dense basic submodule.* ■

We can now derive the following conclusion.

Theorem 3.3. *Let R be a Matlis valuation domain and M a strongly flat reduced R-module of any rank. Then M has a free dense basic submodule, i.e. M is the extension of a free module by a divisible module.*

Proof. By (2.1), M has a continuous well-ordered ascending chain (2) with countable rank strongly flat factors $M_{\sigma+1}/M_\sigma$. Hence (3.2) shows that all these factors have free dense basic submodules. It only remains to invoke (3.1) to complete the proof. ∎

4. A test module for strong flatness.

The class $\mathcal{SF}$ of strongly flat modules over a fixed domain R and the class $\mathcal{WC}$ of weakly cotorsion R-modules are, respectively, the cotorsion-free and the cotorsion classes of the cotorsion theory

$$\mathcal{C}_Q = (\mathcal{SF}, \mathcal{WC})$$

cogenerated by Q; see Salce [S] or Trlifaj [T3]. We are wondering whether or not the cotorsion theory $\mathcal{C}_Q$ can also be generated by a single module. In other words, we are looking for a weakly cotorsion *test module* to recognize strong flatness, i.e. for a weakly cotorsion C such that M is strongly flat exactly if $\mathrm{Ext}^1_R(M, C) = 0$ holds. If this happens, then we will have

$$\mathcal{C}_Q = ({}^\perp C, ({}^\perp C)^\perp),$$

where ${}^\perp C = \{M \mid \mathrm{Ext}^1_R(M, C) = 0\}$ and $({}^\perp C)^\perp = \{N \mid \mathrm{Ext}^1_R(M, N) = 0$ for all $M \in {}^\perp C\}$.

Note that such a module C always exists in the case when $\mathcal{SF}$ coincides with the class of all flat modules (that is, in case R is almost perfect, see [BS2]). Indeed, by the Flat Test Lemma, a module M is flat if and only if $\mathrm{Tor}^R_1(M, R/I) = 0$ for all ideals I of R if and only if $\mathrm{Ext}^1_R(M, C) = 0$ where $C = \prod_{I \leq R} \mathrm{Hom}_{\mathbb{Z}}(R/I, \mathbb{Q}/\mathbb{Z})$.

We first show that, for every module M over any domain there is a single weakly cotorsion test module C_M for strong flatness, and then for Matlis domains we establish the existence of a single test module C_κ for all modules M of cardinality $\leq \kappa$.

For the first part of the next result, see Bazzoni-Salce [BS3], proof of Theorem 2.1. The second part was observed by S. Bazzoni.

Lemma 4.1. *For every torsion-free R-module M, there exists an exact sequence $0 \to C_M \to N \to M \to 0$, where N is strongly flat and C_M is weakly cotorsion.*

M is strongly flat if and only if $\mathrm{Ext}^1_R(M, C_M) = 0$.

Proof. Imitating the construction [BS3, Theorem 2.1], we have the pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & H & \to & F & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \widetilde{H} & \to & N & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & D & = & D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact rows and columns, where F is a free R-module of rank $\mathrm{gen}\, M$, D is a torsion-free divisible module, and N is strongly flat in view of (1). Since the kernel $\widetilde{H} = C_M$ is R-complete, and so weakly cotorsion, the exact sequence splits whenever M is strongly flat. Conversely, if the sequence splits, then M is a summand of the strongly flat module N, so itself strongly flat. ∎

We need an upper estimate for the number of generators of H where $C_M = \widetilde{H}$. F can be chosen with $\kappa = \mathrm{gen}\, M$ generators, so $\mathrm{gen}\, H \leq \kappa |R|$.

Let X_κ henceforth denote a free R-module of rank κ for an infinite cardinal κ. For convenience, we will assume that $\kappa \geq |R|$. By Matlis [M], for a reduced torsion-free module the properties of 'weakly cotorsion' and 'R-complete' are equivalent.

We continue with a simple lemma.

Lemma 4.2. *Let R be a Matlis domain. Any torsion-free weakly cotorsion R-module C that is the completion of a module B with at most κ generators is an epic image of the complete R-module $\widetilde{X}_\kappa$.*

Proof. Evidently, there is an epimorphism $\phi : X_\kappa \to B$. As C is the completion B, ϕ extends to a map $\widetilde{f} : \widetilde{X}_\kappa \to C$. Over Matlis domains, the image of an R-complete torsion-free module is again complete, so $\mathrm{Im}\, \widetilde{f} = C$. ∎

We are now ready to prove the existence of a single test module for strong flatness for all modules with at most κ generators, provided R is a Matlis domain.

Proposition 4.3. *Suppose R is a Matlis domain, and $\kappa \geq |R|$ is an infinite cardinal. A torsion-free R-module M of projective dimension ≤ 1 with* $\operatorname{gen} M \leq \kappa$ *is strongly flat if and only if*

$$\operatorname{Ext}^1_R(M, \widetilde{X}_\kappa) = 0.$$

Proof. Evidently, it is enough to verify that the condition is sufficient, so assume that the indicated Ext vanishes for M. By the proof of (4.1) and its remark it is enough to show that $\operatorname{Ext}^1_R(M, \widetilde{H}) = 0$ for every torsion-free module H with gen $H \leq \kappa$. By (4.2) there exists an exact sequence $0 \to N \to \widetilde{X}_\kappa \to \widetilde{H} \to 0$. This induces the exact sequence

$$\operatorname{Ext}^1_R(M, \widetilde{X}_\kappa) \to \operatorname{Ext}^1_R(M, \widetilde{H}) \to \operatorname{Ext}^2_R(M, N),$$

where the first term is 0 by assumption, while the last term is 0 since $\text{p.d.}\, M \leq 1$. Hence the middle term also vanishes. ■

We shall need an upper estimate of the cardinality of $\widetilde{X}_\mu$.

Proposition 4.4. *If $\lambda = |R|$, then for every cardinal $\mu \geq \lambda$*

$$|\widetilde{X}_\mu| \leq \mu^\lambda.$$

Proof. Clearly, $|X_\mu| \leq \mu\lambda$. Recall that for a torsion-free module X_μ we have $\widetilde{X}_\mu \cong \operatorname{Hom}_R(K, K \otimes_R X_\mu)$, thus $|\widetilde{X}_\mu| \leq |X_\mu|^{|K|} \leq (\lambda\mu)^\lambda = \mu^\lambda$. ■

5. Strongly flat modules in the constructible universe.

We continue with Matlis domains, so in this section R stands for a Matlis domain. We wish to establish the existence of a universal test module for strong flatness, i.e. one which would work for any module, no matter how large it is.

We assume that we are working in the constructible universe in order to be able to use arguments that require additional set theoretical hypotheses.

We will imitate the discussion in Fuchs-Salce [FS, XVI.10] where Whitehead modules were dealt with. First we fix the cardinal μ as the smallest regular cardinal with $|R|^{|R|} \leq \mu$, and concentrate on R-modules M satisfying

$$\operatorname{Ext}^1_R(M, \widetilde{X}_\mu) = 0,$$

where $\widetilde{X}_\mu$ denotes — as above — the test module for strong flatness for all R-modules with $\leq \mu$ generators. Let us denote by ${}^\perp\widetilde{X}_\mu$ the class of these modules M. It is clear that the class $\mathcal{SF}$ of strongly flat modules is contained in the intersection $\mathcal{P}_1 \cap^\perp \widetilde{X}_\mu$, where $\mathcal{P}_1$ stands for the class of modules of projective dimension ≤ 1. We are going to show that if V = L, then the equality $\mathcal{SF} = \mathcal{P}_1 \cap^\perp \widetilde{X}_\mu$ holds.

Next we formulate results needed for the proof of the main theorem (5.3) infra. They follow the presentation by Becker-Fuchs-Shelah [BFS]; for their proofs we refer to the original source or to [FS, XVI.10].

The proof of (5.3) is via induction on the minimal cardinality of generating sets of M. In the transfinite induction, the argument for regular cardinals is taken care of by the following lemma.

Lemma 5.1. *Assume $V = L$. Let R be a Matlis domain of cardinality λ, and κ a regular cardinal $\geq \lambda^\lambda$. Suppose the R-module M with* $\operatorname{gen} M = \kappa$ *admits a continuous well-ordered ascending chain*

$$0 = M_0 < M_1 < \cdots < M_\alpha < \cdots \quad (\alpha < \kappa) \tag{3}$$

of submodules such that

(i) $M = \bigcup_{\alpha<\kappa} M_\alpha$;

(ii) $\operatorname{gen} M_\alpha < \kappa$ *for each* $\alpha < \kappa$;

(iii) *each* M_α $(\alpha < \kappa)$ *belongs to* ${}^\perp\widetilde{X}_\mu$.

Then: $M \in^\perp \widetilde{X}_\mu$ *if and only if*

$$E = \{\alpha < \kappa \mid \exists \gamma > \alpha \text{such that} M_\gamma/M_\alpha \notin^\perp \widetilde{X}_\mu\}$$

is not a stationary subset of κ. ■

The singular case requires a form of Shelah's singular compactness theorem. To be ready for that, we need a lemma.

Lemma 5.2. *Let κ be an infinite cardinal with $|R|^{|R|} \leq \kappa$ and assume that the R-module M has a filtration* (3) *with* $\operatorname{gen} M_{\alpha+1}/M_\alpha \leq \kappa$. *Then M admits an $H(\kappa)$-family $\mathcal{C}$ of submodules such that every $C \in \mathcal{C}$ is the union of a continuous well-ordered ascending chain of submodules in $\mathcal{C}$ with factors isomorphic to certain quotients $M_{\alpha+1}/M_\alpha$.* ■

Now we have the main result whose proof makes use of a version of (1.4) for modules of projective dimension ≤ 1 in the class ${}^\perp\widetilde{X}_\mu$.

Theorem 5.3. *Assume $V = L$. Let R be a Matlis domain and μ the smallest regular cardinal such that $|R|^{|R|} \leq \mu$. An R-module M of projective dimension ≤ 1 belongs to the class ${}^\perp\widetilde{X}_\mu$ if and only if it admits a chain* (3) *such that*

(i) $M = \bigcup_{\alpha<\kappa} M_\alpha$;

(ii) $\operatorname{gen} M_{\alpha+1}/M_\alpha \leq \mu$ *for each* $\alpha + 1 < \kappa$;

(iii) *each* $M_{\alpha+1}/M_\alpha$ $(\alpha < \kappa)$ *belongs to* ${}^\perp\widetilde{X}_\mu$. ■

If the cardinal μ has the same meaning as above, then we can state the existence of a universal test module for strong flatness for modules of projective dimension ≤ 1 over any Matlis domain; in fact, (5.3) actually covers the strongly flat modules:

Theorem 5.4. *Assume that $V = L$ and R is a Matlis domain. An R-module M of projective dimension ≤ 1 is strongly flat if and only if it satisfies*

$$\mathrm{Ext}^1_R(M, \widetilde{X}_\mu) = 0.$$

Proof. It suffices to verify the 'if' part, so assume $\mathrm{Ext}^1_R(M, \widetilde{X}_\mu) = 0$. If $\mathrm{gen}\, M \leq \mu$, there is nothing to prove. Suppose $\mathrm{gen}\, M > \mu$, so by (5.3) it has a chain (3) satisfying conditions (i)-(iii). Then all the factors $M_{\alpha+1}/M_\alpha$ are strongly flat modules, whence it follows that M is strongly flat as well (see e.g. [FS, VI.2.5]). ∎

Needless to say, for Matlis domains of global dimension 1 (= Dedekind domains) there is a universal test module for strong flatness available in ZFC, since such domains are almost perfect; see §4. In order to improve on (5.4) for domains of global dimension > 1, we search for a universal test module for projective dimension ≤ 1.

It was proved in [BS1] that, in case V=L, K is a universal test module for projective dimension ≤ 1 for a valuation IC-domain R. (A valuation domain is an IC-domain if it is a Matlis domain of global dimension 2 that satisfies a topological incompleteness condition.) Over an arbitrary Matlis domain of global dimension 2 we can prove — again in case V=L — that there is a test module, viz. $K^{(\lambda)}$, where $\lambda = 2^{|R|}$.

Theorem 5.5. *Assume $V = L$. Let R be a Matlis domain of global dimension 2, and set $Y = K^{(\lambda)}$ with $\lambda = 2^{|R|}$. Then an R-module M has projective dimension ≤ 1 if and only if it satisfies*

$$\mathrm{Ext}^1_R(M, Y) = 0.$$

Proof. Since the module Y is (h-)divisible, we have $\mathrm{Ext}^1_R(M, Y) = 0$ for any module M of projective dimension ≤ 1; see [FS, VII.2.5(i)].

For the proof of the converse, we can restrict ourselves to torsion divisible modules. Indeed, for an arbitrary module M, there is an exact sequence $0 \to M \to E \to P \to 0$ where E is divisible and P has projective dimension ≤ 1 [FS, VII.1.4]. If M satisfies $\mathrm{Ext}^1_R(M, Y) = 0$, then the exact sequence $0 = \mathrm{Ext}^1_R(P, Y) \to \mathrm{Ext}^1_R(E, Y) \to \mathrm{Ext}^1_R(M, Y) = 0$ implies $\mathrm{Ext}^1_R(E, Y) = 0$, thus every M embeds in a divisible module E satisfying the same condition. If we can verify that E has to be of projective dimension ≤ 1, then the same will follow for M. Note that if E is divisible, then $E = F \oplus T$ where F is a Q-module (so of projective dimension 1) and T is torsion divisible [FS, VII.2.2], thus $\mathrm{Ext}^1_R(E, Y) = 0$ if and only if $\mathrm{Ext}^1_R(T, Y) = 0$.

A torsion divisible R-module M will be called *"free"* provided that p.d. $M \leq 1$; equivalently, M is a direct sum of countably generated torsion divisible modules of projective dimension ≤ 1; cf. [L] or [FS, VII.2.9].

By induction on $\kappa = |M|$, we will prove that if M is a torsion divisible R-module satisfying $\mathrm{Ext}^1_R(M, Y) = 0$, then M is "free".

First, assume $\kappa \leq \lambda$. By [FS, VII.2.10], there is an exact sequence

$$0 \to D \to C \to M \to 0$$

where $C = K^{(\sigma)}$, $\sigma = \mathrm{Hom}_R(K, M)$, and D is torsion divisible. We have $|\sigma| \leq |M^R| \leq \lambda$, so $|D| \leq \lambda$. If p.d. $M = 2$, then p.d. $D = 1$ and another application of [FS, VII.2.10] shows that D is a direct summand of a direct sum of $\leq \lambda$ copies of K. Then D is isomorphic to a direct summand in Y, and the exact sequence above splits by hypothesis, a contradiction. This proves that in the present case p.d. $M \leq 1$.

Next, suppose that κ is a regular uncountable cardinal $> \lambda$, and A is a torsion divisible module of cardinality κ. A will be called "*κ-projective*" if each subset of A of cardinality $< \kappa$ is contained in a "free" submodule of A of cardinality $< \kappa$. By the inductive premise and [FS, VII.2.5(ii)], all torsion divisible submodules of M of cardinality $< \kappa$ are "free", hence M is "κ-projective". Jensen's Diamond Principle then provides for a continuous well-ordered ascending chain $\{M_\alpha \mid \alpha < \kappa\}$ of "free" submodules of M such that $|M_\alpha| < \kappa$ for all $\alpha < \kappa$, $M = \bigcup_{\alpha<\kappa} M_\alpha$, and $\mathrm{Ext}^1_R(M_{\alpha+1}/M_\alpha, Y) = 0$, see e.g. [T1, 3.7 and 3.8]. By the inductive hypothesis, $M_{\alpha+1}/M_\alpha$ has projective dimension ≤ 1, so $M_{\alpha+1} = M_\alpha \oplus F_\alpha$ for a "free" module F_α $(\alpha < \kappa)$, so $M \cong \bigoplus_{\alpha<\kappa} F_\alpha$ is indeed "free".

Finally, assume κ is a singular cardinal $> \lambda$. For each "free" module A and each decomposition $A = \bigoplus_{i \in I} A_i$ into a direct sum of countably generated modules, set

$$\mathcal{F} = \{C = \bigoplus_{i \in X} A_i \mid X \subseteq I\}.$$

and call it a "basis" of A. The notions of "free" and "basis" satisfy conditions (a)-(e) of [EM, IV.3.6], so by Shelah's Singular Compactness Theorem and the inductive premise we can conclude that M is "free", cf. [EM, IV.3.7]. ∎

It is now clear that (5.4) and (5.5) together yield a universal test module for strong flatness for Matlis domains of global dimension 2.

Corollary 5.6. *Assume $V = L$. Let R be a Matlis domain of global dimension 2, and let $\widetilde{X}_\mu$ and Y be as in* (5.4) *and* (5.5)*. Then an R-module M is strongly flat if and only if it satisfies*

$$\mathrm{Ext}^1_R(M, Z) = 0,$$

where $Z = \widetilde{X}_\mu \oplus Y$. ■

6. Strongly flat modules under uniformization.

In this section, we will prove that in a model of ZFC with Shelah's Uniformization Principle adjoined, there are no universal test modules for strong flatness for any Matlis domain that is not almost perfect, in particular, for any Matlis valuation domain of global dimension > 1. Combined with the results of §5, this asserts that the existence of universal test modules for strong flatness for Matlis domains of global dimension 2 which are not almost perfect is independent of ZFC.

Following [T3, 2.3], UP_κ will denote Shelah's Uniformization Principle for κ (where κ is a singular cardinal of cofinality ω), and UP will denote "UP_κ for all singular cardinals κ of cofinality ω." Shelah proved that UP is consistent with ZFC + GCH.

More specifically, let κ be a singular cardinal of cofinality ω and E a subset of κ^+ consisting of ordinals of cofinality ω. Then $\{n_\alpha \mid \alpha \in E\}$ is called a *ladder system* provided that for each $\alpha \in E$, $n_\alpha = \{n_\alpha(i) \mid i < \omega\}$ is a strictly increasing sequence consisting of non-limit ordinals in κ^+ such that $\sup_{i<\omega} n_\alpha(i) = \alpha$ holds.

UP_κ says the following:

"There are a stationary subset $E \subseteq \kappa^+$ consisting of ordinals of cofinality ω, and a ladder system $\{n_\alpha \mid \alpha \in E\}$ which have the uniformization property with respect to each cardinal $\lambda < \kappa$. That is, for each $\lambda < \kappa$ and each sequence $\{h_\alpha \mid \alpha \in E\}$ of mappings from ω to λ, there is a mapping $f : \kappa^+ \to \lambda$ such that for each $\alpha \in E$, $f(n_\alpha(i)) = h_\alpha(i)$ for almost all $i < \omega$."

In the balance of this section, R will denote a Matlis domain. We will be interested in the cotorsion theories $(\mathcal{P}_0, \mathrm{Mod}R)$ and $(\mathcal{P}_1, \mathcal{D})$ where $\mathcal{P}_i$ denotes the class of all R-modules of projective dimension $\leq i$, and $\mathcal{D}$ is the class of all (h-)divisible R-modules, cf. [FS, VII.2.5].

If $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion theory, then a module M is a *local splitter* for $\mathcal{C}$ provided that $M^{(\omega)} \in \mathcal{A} \cap \mathcal{B}$ (hence $\mathrm{Ext}^1_R(M, M^{(\omega)}) = 0$), and there is a non-split embedding $\nu : M^{(\omega)} \to M^{(\omega)}$ which is locally split, that is, for each $n < \omega$, there is a submodule $C_n \subseteq M^{(\omega)}$ such that $\nu(M^{(n)}) \oplus C_n = M^{(\omega)}$ and $\nu(M^{(\omega \setminus n)}) \subseteq C_n$. Here, for a subset $A \subseteq \omega$, the symbol $M^{(A)}$ denotes the direct summand of $M^{(\omega)}$ consisting of all $x \in M^{(\omega)}$ such that $\pi_i(x) = 0$ for all $i \in \omega \setminus A$ (where $\pi_i : M^{(\omega)} \to M$ is the ith canonical projection).

Example 6.1. The module $M = R$ is a local splitter for the cotorsion theory $(\mathcal{P}_0, \mathrm{Mod}R)$.

Indeed, take any sequence $\mathcal{S} = \{a_i \mid i < \omega\}$ of elements of R such that $\{Ra_n \cdots a_0 \mid n < \omega\}$ is a strictly decreasing chain of principal ideals of R, and consider the embedding $\nu_{\mathcal{S}} : R^{(\omega)} \to R^{(\omega)}$ defined at the canonical basis of $R^{(\omega)}$ by $\nu_{\mathcal{S}}(1_i) = 1_i - 1_{i+1}a_i$, and $C_n = \bigoplus_{i \geq n} 1_i R$ $(n < \omega)$. By a classical result of Bass, in this case $\mathrm{Coker}\, \nu_{\mathcal{S}}$ is a non-projective flat module. ∎

Example 6.2. If R contains a strictly increasing infinite chain of principal ideals, then $M = K$ is a local splitter for the cotorsion theory $(\mathcal{P}_1, \mathcal{D})$.

In order to see this, let $\{r_i R \mid i < \omega\}$ where $0 \neq r_i \in R$ and $r_i R \subset r_{i+1} R$ for all $i < \omega$. Put $J = \bigcup_{i<\omega} r_i R$. Define $a_i \in R$ by $r_i = r_{i+1} a_i$ $(i < \omega)$. Then $\mathcal{S} = \{Ra_n \cdots a_0 \mid n < \omega\}$ is a strictly decreasing chain of principal ideals of R, and we have the exact sequence

$$0 \to R^{(\omega)} \xrightarrow{\nu_{\mathcal{S}}} R^{(\omega)} \xrightarrow{\pi_{\mathcal{S}}} J \to 0,$$

where $\nu_{\mathcal{S}}$ is as in (6.1), and $\pi_{\mathcal{S}}(1_i) = r_i$ for all $i < \omega$. In particular, J is a flat module of projective dimension 1. Tensoring the last exact sequence with K, we obtain

$$0 \to K^{(\omega)} \xrightarrow{\mu_{\mathcal{S}}} K^{(\omega)} \to K \otimes_R J \to 0.$$

It is easy to see that the embedding $\mu_{\mathcal{S}} = 1_K \otimes_R \nu_{\mathcal{S}}$ splits locally (cf. [T3, Lemma 5]). Since $R \otimes_R J \cong RJ = J$ and $Q \otimes_R J \cong QJ = Q$ canonically, we have $K \otimes_R J \cong Q/J$. Because of $\mathrm{p.d.}\, Q/J = \mathrm{p.d.}\, R/J = \mathrm{p.d.}\, J + 1 = 2$, $\mu_{\mathcal{S}}$ is non-split. This establishes our claim that K is a local splitter for $(\mathcal{P}_1, \mathcal{D})$. ∎

It is worthwhile pointing out that the domains satisfying the hypothesis of Example 6.2 are exactly those domains which are not almost perfect. This fact can easily be checked by recalling the following property that characterizes almost perfect domains R: given any $0 \neq r \in R$, the factor ring R/rR has the descending chain condition on principal ideals.

Any almost perfect domain is a Matlis domain ([M], [BS2]); consequently, there exist Matlis domains of global dimension > 1 possessing universal test modules in ZFC, cf. §4 (observe that a valuation domain is almost perfect exactly if it is a DVR, i.e. it has global dimension 1). Our next result shows that it is consistent with ZFC + GCH that, among Matlis domains, only the almost perfect ones admit universal test modules for strong flatness.

Theorem 6.3. *Assume UP, and let R be a Matlis domain that is not almost perfect* (*e.g., a Matlis valuation domain of global dimension* > 1). *For every R-module C, there exists a flat, but not strongly flat R-module F of projective dimension* 1 *such that*

$$\mathrm{Ext}^1_R(F, C) = 0.$$

Proof. We use the notations of (6.2). Take a singular cardinal κ of cofinality ω such that $\kappa > |\mathrm{End}(Q/J)|, |R|, |C|$. UP_κ provides for a stationary subset $E \subseteq \kappa^+$ and a ladder system $\{n_\alpha \mid \alpha \in E\}$ which has the uniformization property with respect to all cardinals $< \kappa$.

Let $\{P_\alpha \mid \alpha < \kappa^+\}$ be a sequence of free modules defined as follows: $P_\alpha = R^{(\omega)}$ for $\alpha \in E$, and $P_\alpha = R$ otherwise. For $\alpha \notin E$, denote by 1_α the canonical generator of P_α. For $\alpha \in E$, let $\{1_{\alpha,i} \mid i < \omega\}$ be the canonical basis of P_α. Define $P = \bigoplus_{\alpha<\kappa^+} P_\alpha$.

For $\alpha \in E$ and $i < \omega$, let $g_{\alpha i} = 1_{n_\alpha(i)} - 1_{\alpha,i} + 1_{\alpha,i+1}a_i \in P$. Setting $L_\alpha = \bigoplus_{i<\omega} g_{\alpha,i}R \subseteq P$ and $L = \bigoplus_{\alpha\in E} L_\alpha \subseteq P$, let $F = P/L$. Note that F coincides with the module constructed in [T3, p.319] (for $K = R$, $\nu = \nu_{\mathcal{S}}$ and $N = J$). Also, F is the same as the module M constructed in [T1, 2.1].

Since $\kappa > |C|$, the uniformization property yields $\mathrm{Ext}^1_R(F, C) = 0$, see [T1, 2.4]. Applying [T3, 1.7] (for $\mathcal{C} = (\mathcal{P}_0, \mathrm{Mod}R)$, $K = R$ and $N = J$) we have $\mathrm{p.d.}\, F = 1$. Moreover, $F = \bigcup_{\alpha<\kappa^+} F_\alpha$ where $\{F_\alpha \mid \alpha < \kappa^+\}$ is a continuous chain of submodules of F such that either $F_{\alpha+1}/F_\alpha \cong R$, or else $F_{\alpha+1}/F_\alpha \cong J$, for all $\alpha < \kappa^+$. It follows that F is flat.

It remains to prove that F is not strongly flat. By (1.2), it suffices to show that $\mathrm{p.d.}(K \otimes_R F) > 1$. Tensoring the exact sequence $0 \to R^{(\omega)} \to R^{(\omega)} \to J \to 0$ with K, we get the exact sequence $0 \to K^{(\omega)} \to K^{(\omega)} \to Q/J \to 0$, see (6.2). Similarly, the exact sequence $0 \to L \to P \to F \to 0$ tensored with K yields the exact sequence

$$0 \to \bigoplus_{\alpha\in E} K \otimes_R L_\alpha \to \bigoplus_{\alpha<\kappa^+} K \otimes_R P_\alpha \to K \otimes_R F \to 0.$$

Another application of [T3, 1.7] (this time to $\mathcal{C} = (\mathcal{P}_1, \mathcal{D})$ and $N = Q/J$) yields $K \otimes F \notin \mathcal{P}_1$, as we wished to prove. ■

We have just completed the proof of an independence result:

Corollary 6.4. *Let R be a Matlis domain of global dimension* 2 *that is not almost perfect. Then the assertion "There is a universal test module for strong flatness in* Mod-R*" is independent of ZFC + GCH.* ■

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Department of Mathematics
Tulane University
New Orleans, Louisiana 70118, U.S.A.
e-mail: fuchs@tulane.edu

Dipartimento di Matematica Pura e Applicata
Via Belzoni 7
35131 Padova, Italy
e-mail: salce@math.unipd.it

Katedra Algebry, MFF UK
Sokolovská 83
186 75 Praha 8, Czech Republic
e-mail:trlifaj@karlin.mff.cuni.cz