Descending Chains of Modules and Jordan-Hölder Theorem

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1 Introduction

It is clear that there is a relation between the notion of unique factorization domain (a commutative domain in which every non-zero non-invertible element is a product of irreducible elements, and if $a_1 a_2 \dots a_n = a'_1 a'_2 \dots a'_m$ are any two such factorizations, then $n = m$ and there exists a permutation σ such that a_i and $a'_{\sigma(i)}$ are associates for every $i = 1, 2, \dots, n$), the Krull-Schmidt Theorem (every module of finite length is a direct sum of indecomposable modules, and if $A_1 \oplus A_2 \oplus \dots \oplus A_n = A'_1 \oplus A'_2 \oplus \dots \oplus A'_m$ are any two such decompositions, then $n = m$ and there exists a permutation σ such that $A_i \cong A'_{\sigma(i)}$ for every $i = 1, 2, \dots, n$), and the Jordan-Hölder Theorem (every module A of finite length has a composition series, and if $A = A_0 \geq A_1 \geq \dots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \dots \geq A'_m = 0$ are any two composition series, then $n = m$ and there exists a permutation σ such that $A_{i-1}/A_i \cong A'_{\sigma(i)-1}/A'_{\sigma(i)}$ for every $i = 1, 2, \dots, n$).

The relation between these three contexts is that what we state is equivalent to saying that some commutative monoid is free in all these three cases. An integral domain R is a unique factorization domain if and only if the commutative monoid $R^*/U(R)$ is free, where we have denoted by R^* the multiplicative monoid $R \setminus \{0\}$ and by $U(R)$ the group of invertible elements of R. In this case, a free set of generators of $R^*/U(R)$ is

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given by the cosets of all irreducible elements modulo $U(R)$. The Krull-Schmidt Theorem says that if $\mathcal{C}$ is the class of all right modules of finite length over a ring R and $\langle A_R \rangle$ denotes the isomorphism class of a module $A_R \in \mathcal{C}$, that is, the class of all R-modules isomorphic to A_R, then $V(\mathcal{C}) = \{ \langle A_R \rangle \mid A_R \in \mathcal{C} \}$ is a free commutative monoid with respect to the addition defined by $\langle A_R \rangle + \langle B_R \rangle = \langle A_R \oplus B_R \rangle$ for every $A_R, B_R \in \mathcal{C}$. A free set of generators of $V(\mathcal{C})$ is given by the isomorphism classes of the modules indecomposable in $\mathcal{C}$. The Jordan-Hölder Theorem says that if $\sim$ is the congruence relation on the monoid $V(\mathcal{C})$ generated by all the pairs $(\langle B \rangle, \langle A \rangle + \langle C \rangle)$ for which $A, B, C \in \mathcal{C}$ and there exists an exact sequence $0 \to A \to B \to C \to 0$, then the quotient monoid $V(\mathcal{C})/\sim$ is free. A free set of generators of $V(\mathcal{C})/\sim$ is given by the isomorphism classes of all simple R-modules. However, as we shall see in this paper, the relation between existence of descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules, uniqueness up to a permutation of the factors A_{i-1}/A_i, refinements of descending series, validity of the Jordan-Hölder type theorems or the Schreier type theorems that can be found in the mathematical literature, and freeness of the corresponding quotient monoid $V(\mathcal{C})/\sim$ is not immediate. Our paper is born from an attempt to give a general framework to these notions. Since a number of results have been obtained recently as far as Krull-Schmidt type theorems are concerned (cf. [3], [5], [6], [7], [8], [16], [19], [20]), we hoped that we could obtain similar results for Jordan-Hölder type theorems, but the situation turned out to be more complicate than we hoped for.

The Jordan-Hölder theorem and the Schreier theorem concern partially ordered set, and in fact most of the variations on this theme that can be found in the literature pass sooner or later through the Jordan-Hölder theorem and the Schreier theorem for modular lattices [18, Proposition III.3.1 and Corollary III.3.2]. For instance, both the Jordan-Hölder theorem and the Schreier theorem hold in abelian categories because the class $\mathcal{L}(A)$ of all subobjects of an object A of an abelian category is a "modular lattice" (here we write "modular lattice" in inverted commas because it is not necessarily a set). In a number of examples we have found, however, abelian categories do not appear immediately for at least two reasons. Namely, on the one hand only particular descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules are considered in some cases, for example series of pure submodules or divisible submodules or submodules with critical quotients. On the other hand, equivalences $\equiv$ weaker than isomorphism of composition factors are considered in some other cases, for instance being in the same monogeny class or in the same epigeny class.

Our input data are a class $\mathcal{C}$ of right modules over a fixed ring R, a class $\mathcal{R}$ of short exact sequences in $\mathcal{C}$, and a congruence $\equiv$ on the monoid $V(\mathcal{C})$. More precisely, suppose that we have an arbitrary class $\mathcal{C}$ of right R-modules closed under isomorphism and finite direct sums and with only a set of isomorphism classes. Then it is possible to define the monoid $V(\mathcal{C})$, which completely describes the behavior of the class as far as uniqueness of direct sum decompositions is concerned. If we fix a class $\mathcal{R}$ of exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, we can construct the quotient monoid $V(\mathcal{C})/\sim_{\mathcal{R}}$, where $\sim_{\mathcal{R}}$ is the congruence relation on $V(\mathcal{C})$ generated by all pairs $(\langle B\rangle, \langle A\rangle + \langle C\rangle)$ with $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. If $A, B \in \mathcal{C}$ and $A \leq B$, we write $A \leq_{\mathcal{R}} B$ if the canonical exact sequence $0 \to A \to B \to B/A \to 0$ belongs to $\mathcal{R}$. Now let $\equiv$ be an arbitrary congruence on $V(\mathcal{C})$. Our aim is to study the descending series $A_0 \geq A_1 \geq \cdots \geq A_n = 0$, with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, up to the congruence $\equiv$, that is, we identify two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ if $n = m$ and there exists a permutation σ such that $\langle A_{i-1}/A_i\rangle \equiv \langle A'_{\sigma(i)-1}/A'_{\sigma(i)}\rangle$ for every $i = 1, 2, \ldots, n$. In this case, we say that the two descending series are *equivalent modulo* $\equiv$. Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\equiv$ and $\sim_R$. If $A, B \in \mathcal{C}$ and there exist a descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A with $A_i \leq_{\mathcal{R}} A_{i-1}$ for every i, a descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ of submodules of B with $B_i \leq_{\mathcal{R}} B_{i-1}$ for every i and a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i\rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)}\rangle$ for every $i = 1, 2, \ldots, n$, then $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$. We study the correspondence between the existence of such descending series (*descending series in* $\mathcal{R}$) and the quotient monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$.

In the transition from the class $\mathcal{C}$ to the commutative monoid $V(\mathcal{C})$, no information about direct sum decompositions in $\mathcal{C}$ is lost (Krull-Schmidt type theorems). We show that, unluckily, the situation is not so good in the transition from descending series of submodules to the quotient monoids $V(\mathcal{C})/\sim_{\mathcal{R}}$ or $V(\mathcal{C})/\equiv_{\mathcal{R}}$ (Jordan-Hölder type theorems). The paper is organized as follows. In Section 2, we study the relation between classes $\mathcal{R}$ of short exact sequences in $\mathcal{C}$ and the corresponding congruences $\sim_{\mathcal{R}}$ on $V(\mathcal{C})$. In Section 3, we fix an arbitrary congruence $\equiv$ on $V(\mathcal{C})$, construct the congruence $\equiv_{\mathcal{R}}$ generated by the two congruences $\equiv$ and $\sim_R$, and consider the relation between the congruence $\equiv_{\mathcal{R}}$ and the existence of descending series up to equivalence modulo $\equiv$. In Section 4, we determine the conditions on the class $\mathcal{R}$ that allows us to have a reasonably good behavior of descending series in $\mathcal{R}$ as far as taking submodules and quotient modules is concerned. In Section 5, we see how these notions link up to give us information about the existence of refinements (Schreier type theorems) and the uniqueness of composition series (Jordan-Hölder type theorems).

Finally, in Section 6, we analyze some of the many examples of Jordan-Hölder type theorems existing in the mathematical literature from the point of view we have introduced. We also recall an example (critical composition series, Example 12) that falls only partially within our theory, but that we think to be very interesting.

In the literature, there are already other attempts of rationalization of the Jordan-Hölder theory, different from ours. For instance, we mention that due to Hughes [12], concerning a lattice of subsystems of an algebraic system. We must remark that the construction of Grothendieck groups, in which abelian *groups* are considered instead of our monoids $V(\mathcal{C})$, $V(\mathcal{C})/{\sim_{\mathcal{R}}}$ and $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$, cannot be applied in our setting, because in the construction of Grothendieck groups all information concerning cancellation from direct sums and its pathologies is lost.

All rings we consider are associative rings R with identity $1_R \neq 0_R$. Modules are right unital modules. All monoids are commutative additive monoids, that is, commutative additive semigroups with an identity 0_R.

2 Classes of exact sequences and congruences in $V(\mathcal{C})$

Let R be a fixed ring and $\mathcal{C}$ be a class of right R-modules. For every $A_R \in \mathcal{C}$, let $\langle A_R \rangle$ denote the *isomorphism class* of A_R, that is, the class of all right R-modules isomorphic to A_R. We say that $\mathcal{C}$ is *small* if the class $V(\mathcal{C}) = \{\, \langle A_R \rangle \mid A_R \in \mathcal{C}\,\}$ is a set. Suppose that $\mathcal{C}$ is a small class of right R-modules closed under isomorphism (that is, $\langle A_R \rangle \subseteq \mathcal{C}$ for every $A_R \in \mathcal{C}$) and closed under finite direct sums (equivalently, zero modules belong to $\mathcal{C}$ and $A_R \oplus B_R \in \mathcal{C}$ whenever $A_R, B_R \in \mathcal{C}$). The *Krull-Schmidt monoid* of $\mathcal{C}$ is the set $V(\mathcal{C})$ endowed with the addition defined by $\langle A_R \rangle + \langle B_R \rangle = \langle A_R \oplus B_R \rangle$ for every $A_R, B_R \in \mathcal{C}$. It is a commutative monoid.

Let $\mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$. If $\sim$ is a congruence on the monoid $V(\mathcal{C})$, we can construct the subclass $\mathcal{S}_\sim$ of $\mathrm{Ses}(\mathcal{C})$ whose elements are all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $\langle B \rangle \sim \langle A \rangle + \langle C \rangle$.

Conversely, if $\mathcal{R}$ is a subclass of $\mathrm{Ses}(\mathcal{C})$, we may consider the congruence $\sim_{\mathcal{R}}$ on $V(\mathcal{C})$ generated by all pairs $(\langle B \rangle, \langle A \rangle + \langle C \rangle)$ with $0 \to A \to B \to C \to 0$ in $\mathcal{R}$. We shall call $\sim_{\mathcal{R}}$ *the congruence associated to* $\mathcal{R}$.

Thus we have defined two correspondences $\Phi \colon \sim \mapsto \mathcal{S}_\sim$ and $\Psi \colon \mathcal{R} \mapsto \sim_{\mathcal{R}}$ between congruences on the monoid $V(\mathcal{C})$ and subclasses of $\mathrm{Ses}(\mathcal{C})$. Let $\mathcal{L}(V(\mathcal{C}))$ be the lattice of all congruences on $V(\mathcal{C})$. The partial order on $\mathcal{L}(V(\mathcal{C}))$ is defined by $\sim \ \leq \ \sim'$ if $\langle A_R \rangle \sim \langle B_R \rangle$ implies $\langle A_R \rangle \sim' \langle B_R \rangle$ for every $\langle A_R\rangle, \langle B_R\rangle \in V(\mathcal{C})$. Similarly, the class of all subclasses of $\mathrm{Ses}(\mathcal{C})$ is partially ordered by class inclusion $\subseteq$, and the correspondences Φ and Ψ preserve these partial orders, in the sense that $\sim\ \leq\ \sim'$ implies $\mathcal{S}_\sim \subseteq \mathcal{S}_{\sim'}$, and $\mathcal{R} \subseteq \mathcal{R}'$ implies $\sim_\mathcal{R}\ \leq\ \sim_{\mathcal{R}'}$. Moreover, $\Psi\Phi(\sim) \leq\ \sim$ for every congruence $\sim$ on $V(\mathcal{C})$, and $\mathcal{R} \subseteq \Phi\Psi(\mathcal{R})$ for every subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$. From these elementary properties, it immediately follows that $\Phi\Psi\Phi = \Phi$ and $\Psi\Phi\Psi = \Psi$, so that Φ and Ψ induce order preserving bijections, one inverse to the other, between the images of Φ and Ψ. That is, if we call *complete* the subclasses $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ of the type $\mathcal{R} = \mathcal{S}_\sim$ for some congruence $\sim$ on $V(\mathcal{C})$ (equivalently, such that $\mathcal{R} = \Phi\Psi(\mathcal{R})$), *cocomplete* the congruences $\sim$ on $V(\mathcal{C})$ of the type $\sim_\mathcal{R}$ for some subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ (equivalently, such that $\sim\ = \Psi\Phi(\sim)$), and denote by $\mathrm{Cl}(\mathcal{C})$ the set of all complete subclasses of $\mathrm{Ses}(\mathcal{C})$ and by $\mathrm{Cong}(\mathcal{C})$ the set of all cocomplete congruences on $V(\mathcal{C})$, then the partially ordered set $\mathrm{Cl}(\mathcal{C})$ is isomorphic to the partially ordered set $\mathrm{Cong}(\mathcal{C})$ via the restrictions of Φ and Ψ.

Lemma 2.1 *The partially ordered set* $\mathrm{Cl}(\mathcal{C})$ *is a complete lattice.*

PROOF. The join of a subset $\{ \mathcal{R}_\lambda \mid \lambda \in \Lambda \}$ of $\mathrm{Cl}(\mathcal{C})$ is $\Phi\Psi(\bigcup_{\lambda\in\Lambda} \mathcal{R}_\lambda)$. ∎

The class $\mathrm{Ses}(\mathcal{C})$ is the greatest element of the lattice $\mathrm{Cl}(\mathcal{C})$. The smallest element of $\mathrm{Cl}(\mathcal{C})$ is the subclass $\mathcal{S}_=$ of $\mathrm{Ses}(\mathcal{C})$ corresponding to the smallest element $=$ of $\mathrm{Cong}(\mathcal{C})$ ($=$ is the identity on $V(\mathcal{C})$). Thus $\mathcal{S}_=$ is the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$. (Notice that $\mathcal{S}_=$ can contain sequences that are not split. An easy example can be constructed with $\mathcal{C}$ the class of all countable abelian groups and a non-split exact sequence $0 \to \mathbf{Z} \xrightarrow{2} (\mathbf{Z}/2\mathbf{Z})^{(\aleph_0)} \oplus \mathbf{Z} \to (\mathbf{Z}/2\mathbf{Z})^{(\aleph_0)} \to 0$. Particular cases in which all sequences in $\mathcal{S}_=$ are split, that is, classes of modules $\mathcal{C}$ such that if $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$, then every exact sequence $0 \to A \to B \to C \to 0$ is split, were studied in [11] and [15]).

Example 1 Let R be a ring and let $\mathcal{C}$ be the class of all finitely generated projective right R-modules. In this case, all exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ split. Therefore in this case $\mathrm{Cl}(\mathcal{C})$ is the lattice whose unique element is $\mathrm{Ses}(\mathcal{C})$. For instance, if D is a division ring and $\mathcal{C}$ is the class of all right vector spaces of finite dimension over D, then $V(\mathcal{C}) \cong \mathbf{N}$. In this example, the correspondence Φ maps all the congruences on $V(\mathcal{C}) \cong \mathbf{N}$ to the class $\mathrm{Ses}(\mathcal{C})$, and Ψ maps all subclasses of $\mathrm{Ses}(\mathcal{C})$ to the equality $=$ on $V(\mathcal{C})$. ∎

Proposition 2.2 *Let $\mathcal{S}_\sim$ be a complete class of short exact sequences. The following properties hold:*

(a) *Every sequence isomorphic to a sequence in $\mathcal{S}_\sim$ also is in $\mathcal{S}_\sim$, that is, if there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

of right R-modules and module homomorphisms in which the vertical arrows denote module isomorphisms and the sequence $0 \to A \to B \to C \to 0$ belongs to $\mathcal{S}_\sim$, then the sequence $0 \to A' \to B' \to C' \to 0$ belongs to $\mathcal{S}_\sim$ as well.

(b) *Every exact sequence $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ and $B \cong A \oplus C$ is in $\mathcal{S}_\sim$.*

(c) *The direct sum $0 \to A \oplus A' \to B \oplus B' \to C \oplus C' \to 0$ of two sequences $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ belonging to $\mathcal{S}_\sim$ belongs to $\mathcal{S}_\sim$ as well.*

The proof of this proposition is elementary.

3 Descending series

In this section, $\mathcal{C}$ will be a small class of right R-modules closed under isomorphism and finite direct sums and $\mathcal{R}$ will be a class of short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$.

By a *descending series* we mean a finite chain $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of a right module A. We call n the *length* of the series.

Definition A *descending series in $\mathcal{R}$* is a descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A for which all the canonical short exact sequences $0 \to A_i \to A_{i-1} \to A_{i-1}/A_i \to 0$ ($i = 1, 2, \ldots, n$) belong to $\mathcal{R}$.

Obviously, if $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ is a descending series in $\mathcal{R}$, then $A, A_1, \ldots, A_n$ belong to $\mathcal{C}$ and $\langle A \rangle \sim_{\mathcal{R}} \langle A_0/A_1 \rangle + \langle A_1/A_2 \rangle + \cdots + \langle A_{n-2}/A_{n-1} \rangle + \langle A_{n-1} \rangle$, where $\sim_{\mathcal{R}}$ denotes the congruence associated to $\mathcal{R}$.

Let A and B be right R-modules. We shall say that two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A and $B = B_0 \geq$

$B_1 \geq \cdots \geq B_m = 0$ of submodules of B are *isomorphic* if $n = m$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i \rangle \cong \langle B_{\sigma(i)-1}/B_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$. In this case, we shall say that A and B *have isomorphic descending series*. Obviously, if $A, B \in \mathcal{C}$ have two descending series in $\mathcal{R}$ that are isomorphic, then $\langle A \rangle \sim_{\mathcal{R}} \langle B \rangle$.

In many examples, however, it is more useful to consider a condition on descending series weaker than isomorphism. Let $\equiv$ be an arbitrarily fixed congruence on $V(\mathcal{C})$ and let A and B be right R-modules. We shall say that two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $B = B_0 \geq B_1 \geq \cdots \geq B_m = 0$ in $\mathcal{R}$ are *equivalent modulo* $\equiv$ if $n = m$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $\langle A_{i-1}/A_i \rangle \equiv \langle B_{\sigma(i)-1}/B_{\sigma(i)} \rangle$ for every $i = 1, 2, \ldots, n$. In this case, we shall say that A and B *have descending series in* $\mathcal{R}$ *equivalent modulo* $\equiv$. Thus two descending series are isomorphic if and only if they are equivalent modulo $=$.

Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\sim_{\mathcal{R}}$ and $\equiv$. Obviously, if $A, B \in \mathcal{C}$ have descending series in $\mathcal{R}$ equivalent modulo $\equiv$, then $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$. More precisely, the congruence $\equiv_{\mathcal{R}}$ is the transitive closure of the relation "having descending series in $\mathcal{R}$ equivalent modulo $\equiv$", as the next theorem shows.

Theorem 3.1 *The following conditions are equivalent for two modules $A, B \in \mathcal{C}$:*

(a) $\langle A \rangle \equiv_{\mathcal{R}} \langle B \rangle$.

(b) *There exist R-modules $B_0, B_1, B_2, \ldots, B_t \in \mathcal{C}$ with $B_0 = A$, $B_t = B$ and such that B_i, B_{i-1} have descending series in $\mathcal{R}$ equivalent modulo $\equiv$ for every $i = 1, 2, \ldots, t$.*

(c) *There exist R-modules $A_1, A_2, \ldots, A_t, B_0, B_1, B_2, \ldots, B_t, C_1, C_2, \ldots, C_t, A'_1, A'_2, \ldots, A'_t, C'_1, C'_2, \ldots, C'_t \in \mathcal{C}$ with $B_0 = A$, $B_t = B$, $\langle A_i \rangle \equiv \langle A'_i \rangle$ and $\langle C_i \rangle \equiv \langle C'_i \rangle$ for every $i = 1, 2, \ldots, t$ and exact sequences*

$$\begin{array}{ccccccccc}
0 & \to & A_1 & \to & A & \to & C_1 & \to & 0 \\
0 & \to & A'_1 & \to & B_1 & \to & C'_1 & \to & 0 \\
0 & \to & A_2 & \to & B_1 & \to & C_2 & \to & 0 \\
0 & \to & A'_2 & \to & B_2 & \to & C'_2 & \to & 0 \\
 & & & & \vdots & & & & \\
0 & \to & A_t & \to & B_{t-1} & \to & C_t & \to & 0 \\
0 & \to & A'_t & \to & B & \to & C'_t & \to & 0
\end{array}$$

in $\mathcal{R}$.

Proof. Since (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is trivial, it is sufficient to show that (a) $\Rightarrow$ (c).

Write $\langle A\rangle \sim \langle B\rangle$ if A and B satisfy condition (c). It is easily seen that $\sim$ is an equivalence relation in $V(\mathcal{C})$ contained in $\equiv_{\mathcal{R}}$. In order to prove that $\sim$ coincides with $\equiv_{\mathcal{R}}$, it is enough to show that $\sim$ is a congruence, that for every $X, Y, Z \in \mathcal{C}$ and every $0 \to X \to Y \to Z \to 0$ in $\mathcal{R}$ one has $Y \sim X \oplus Z$, and that $\langle A\rangle \equiv \langle B\rangle$ implies $A \sim B$.

If $X \sim Y$ and Z is a module in $\mathcal{C}$, then the direct sums of the exact sequence $0 \to Z \to Z \to 0 \to 0$ of $\mathcal{R}$ and the sequences that link X to Y show that $X \oplus Z \sim Y \oplus Z$. Thus $\sim$ is a congruence in $V(\mathcal{C})$.

If $X, Y, Z \in \mathcal{C}$ and the exact sequence $0 \to X \to Y \to Z \to 0$ belongs to $\mathcal{R}$, then the two sequences

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ 0 & \to & X & \to & X \oplus Z & \to & Z & \to & 0 \end{array}$$

of $\mathcal{R}$ show that $Y \sim X \oplus Z$.

Finally, if $\langle A\rangle \equiv \langle B\rangle$, then the two sequences

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A & \to & 0 & \to & 0 \\ 0 & \to & B & \to & B & \to & 0 & \to & 0 \end{array}$$

of $\mathcal{R}$ show that $A \sim B$. ∎

The following corollary has been suggested to us by a similar result due to Heller [17, pp. 731–732].

Corollary 3.2 *If $A, B \in \mathcal{C}$ and $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$, then there exists a module $C \in \mathcal{C}$ such that $A \oplus C$ and $B \oplus C$ have descending series in $\mathcal{R}$ equivalent modulo $\equiv$.*

Proof. By Theorem 3.1(c), there exist R-modules $A_1, A_2, \dots, A_t$, B_0, $B_1, B_2, \dots, B_t$, $C_1, C_2, \dots, C_t$, $A_1', A_2', \dots, A_t'$, $C_1', C_2', \dots, C_t' \in \mathcal{C}$ with $B_0 = A$, $B_t = B$, $\langle A_i\rangle \equiv \langle A_i'\rangle$ and $\langle C_i\rangle \equiv \langle C_i'\rangle$ for every $i = 1, 2, \dots, t$ and exact sequences

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \to & A & \to & C_1 & \to & 0 \\ 0 & \to & A_1' & \to & B_1 & \to & C_1' & \to & 0 \\ 0 & \to & A_2 & \to & B_1 & \to & C_2 & \to & 0 \\ 0 & \to & A_2' & \to & B_2 & \to & C_2' & \to & 0 \\ & & & & \vdots & & & & \\ 0 & \to & A_t & \to & B_{t-1} & \to & C_t & \to & 0 \\ 0 & \to & A_t' & \to & B & \to & C_t' & \to & 0 \end{array}$$

in $\mathcal{R}$.

Then $A \oplus B_1 \oplus \ldots \oplus B_{t-1}$ has a descending series $A \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq A_1 \oplus B_1 \oplus \ldots \oplus B_{t-1} \geq B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq A_2 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq \ldots \geq B_{t-1} \geq A_t \geq 0$ whose factors are isomorphic to $C_1, A_1, C_2, A_2, \ldots, C_t, A_t$ respectively. Similarly, $B \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1}$ has a descending series $B \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq A'_t \oplus B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq B_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq A'_1 \oplus B_2 \oplus \ldots \oplus B_{t-1} \geq \ldots \geq B_{t-1} \geq A'_{t-1} \geq 0$ whose factors are isomorphic to $C'_t, A'_t, C'_1, A'_1, \ldots, C'_{t-1}, A'_{t-1}$ respectively. Set $C = B_1 \oplus \ldots \oplus B_{t-1} \in \mathcal{C}$. The modules $A \oplus C$ and $B \oplus C$ have descending series in $\mathcal{R}$ equivalent modulo $\equiv$. ■

Every commutative monoid can be realized as $V(\mathcal{C})/\equiv_{\mathcal{R}}$, as the next theorem shows.

Theorem 3.3 *For every commutative monoid M, there exists a small class $\mathcal{C}$ of right R-modules, closed under isomorphism and finite direct sums, a subclass $\mathcal{R}$ of* $\mathrm{Ses}(\mathcal{C})$ *closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and a congruence $\equiv$ on $V(\mathcal{C})$ such that $V(\mathcal{C})/\equiv_{\mathcal{R}} \cong M$.*

Proof. Recall that a commutative additive monoid N is said to be *reduced* if $a + b = 0$ implies $a = b = 0$ for every $a, b \in N$, that is, if no non-zero element a of N has an additive inverse $-a$ in N. An element u of a commutative additive monoid N is an *order-unit* if $u \neq 0$ and for any $a \in N$ there exists an element $b \in N$ and an integer $n \geq 0$ such that $a + b = nu$.

Let M be a commutative monoid and let $\varphi \colon F \to M$ be a surjective monoid homomorphism of a free commutative monoid F onto M. Let $F_{+\infty} = F \cup \{+\infty\}$ be the set obtained by adjoining a further element $+\infty$ to the set F. The addition on F extends to an associative addition on $F_{+\infty}$ by setting $a + (+\infty) = (+\infty) + a = +\infty$ for every $a \in F$. Then $F_{+\infty}$ becomes a reduced commutative monoid with order-unit $+\infty$. By Bergman and Dick's Theorem [8, Theorem 2.1], there exists a ring R with $F_{+\infty} \cong V(R)$ via an isomorphism that sends the element $+\infty$ of $F_{+\infty}$ to the element $\langle R_R \rangle$ of $V(R)$. Let $\mathcal{C}$ be the class of all finitely generated projective right R-modules not isomorphic to R_R and let $\mathcal{R}$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$. Notice that all sequences in $\mathcal{R}$ are split. It is easily seen that $\mathcal{C}$ and $\mathcal{R}$ have the required properties and that $V(\mathcal{C}) \cong F$. Let $\equiv$ be the congruence on $V(\mathcal{C})$ corresponding to the congruence $\ker \varphi$ on F. As $\sim_{\mathcal{R}}$ is the equality on $V(\mathcal{C})$, the congruence $\equiv_{\mathcal{R}}$ coincides with $\equiv$, so that $V(\mathcal{C})/\equiv_{\mathcal{R}} = V(\mathcal{C})/\equiv \;\cong F/\ker \varphi \cong M$. ■

4 Transitive and strongly transitive classes

In this section, $\mathcal{C}$ will be a class of right R-modules closed under isomorphism and finite direct sums and $\mathcal{R}$ will be a class of short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$.

If $A, B \in \mathcal{C}$, we shall write $A \leq_{\mathcal{R}} B$ if A is a submodule of B and the canonical short exact sequence $0 \to A \to B \to B/A \to 0$ belongs to $\mathcal{R}$. Thus descending chains in $\mathcal{R}$ are exactly the chains $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of submodules of A for which $A_i \leq_{\mathcal{R}} A_{i-1}$ for every $i = 1, 2, \dots, n$. For every $B \in \mathcal{C}$, let $\mathcal{L}_{\mathcal{R}}(B)$ be the set of all submodules A of B with $A \leq_{\mathcal{R}} B$ (so that, in particular, both A and B/A must belong to $\mathcal{C}$). Then $\mathcal{L}_{\mathcal{R}}(B)$, ordered by set inclusion, is a partially ordered subset of the lattice $\mathcal{L}(B)$ of all submodules of B. The submodules 0 and B of B are the smallest element and the greatest element of $\mathcal{L}_{\mathcal{R}}(B)$ respectively.

In general, descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in $\mathcal{R}$ do not coincide with finite descending chains in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$, as the following example shows.

Example 2 Let D be a division ring, let $\mathcal{C}$ be the class of all finite dimensional right vector spaces over D of dimension $\neq 1$, and let $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$. Then $V(\mathcal{C}) = V(\mathcal{C})/\sim_{\mathcal{R}} \;\cong \mathbf{N} \setminus \{1\}$. Let $D^5 > D^3 > D^2 > 0$ be vector spaces of dimension $5, 3, 2, 0$ respectively, each contained in the previous one. Then D^2 and D^3 belong to $\mathcal{L}_{\mathcal{R}}(D^5)$, so that $D^5 > D^3 > D^2 > 0$ is a chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(D^5)$. But $D^5 > D^3 > D^2 > 0$ is not a descending series in $\mathcal{R}$. ∎

Proposition 4.1 *The following conditions are equivalent for a subclass $\mathcal{R}$ of* $\mathrm{Ses}(\mathcal{C})$*:*

(a) *For every $B_0 \in \mathcal{C}$, the set of all descending series $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in $\mathcal{R}$ coincides with the set of all finite descending chains $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ whenever $B \in \mathcal{C}$ and $B_0 \leq_{\mathcal{R}} B$.*

(b) *For every $A', A, B \in \mathcal{C}$ with $A' \leq A$ and $A \leq_{\mathcal{R}} B$, one has that $A' \leq_{\mathcal{R}} A$ if and only if $A' \leq_{\mathcal{R}} B$.*

(c) *For every A, B in $\mathcal{C}$ with $A \leq_{\mathcal{R}} B$, the position $A' \mapsto A'$ for every $A' \in \mathcal{L}_{\mathcal{R}}(A)$ defines an injective mapping of $\mathcal{L}_{\mathcal{R}}(A) \to \mathcal{L}_{\mathcal{R}}(B)$, whose image is the interval $[0, A]$ of $\mathcal{L}_{\mathcal{R}}(B)$.*

Proof. (a) ⇒ (b) Suppose that (a) holds. Let $A' \leq A \leq B$ be modules in $\mathcal{C}$ with $A \leq_{\mathcal{R}} B$. Then $A' \leq_{\mathcal{R}} A$ if and only if $B \geq A \geq A' \geq 0$ is a descending series in $\mathcal{R}$, if and only if $B \geq A \geq A' \geq 0$ is a chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$. This happens if and only if $A' \in \mathcal{L}_{\mathcal{R}}(B)$, i.e., if and only if $A' \leq_{\mathcal{R}} B$.

(b) $\Rightarrow$ (a) Suppose that (b) holds and that $B_0 \in \mathcal{C}$. If $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a descending series in $\mathcal{R}$, $B \in \mathcal{C}$ and $B_0 \leq_{\mathcal{R}} B$, then $B_i \leq_{\mathcal{R}} B_{i-1}$ for every i, so that $B_i \leq_{\mathcal{R}} B$ by (b). Thus $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a finite descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$. Conversely, let $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ be a finite descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ for some $B \in \mathcal{C}$ with $B_0 \leq_{\mathcal{R}} B$. Then $B_i \leq_{\mathcal{R}} B$ and $B_{i-1} \leq_{\mathcal{R}} B$ imply that $B_i \leq_{\mathcal{R}} B_{i-1}$ by (b). Thus $B_0 \geq B_1 \geq \cdots \geq B_n = 0$ is a descending series in $\mathcal{R}$.

(c) is merely a restatement of (b). ∎

We shall say that a subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ is *transitive* if it satisfies the equivalent conditions of Proposition 4.1. In this case, the relation $\leq_{\mathcal{R}}$ is necessarily a transitive relation in the class $\mathcal{C}$.

Example 3 Let $\mathcal{C}$ be the class of all finite dimensional right vector spaces of dimension $\neq 1$ over a division ring D considered in Example 2. We have already seen that, in this case, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ is not a transitive subclass. Nevertheless, it is easily seen that the relation $\leq_{\mathcal{R}}$ is a transitive relation in $\mathcal{C}$. Here is an example that shows that the relation $\leq_{\mathcal{R}}$ can be non-transitive. Let $\mathcal{C}$ be the class of all finitely generated abelian groups, so that $V(\mathcal{C})$ is the free commutative monoid having $\langle \mathbf{Z} \rangle$ and the $\langle \mathbf{Z}/p^n\mathbf{Z} \rangle$'s ($n \geq 1$ and p a prime number) as a free set of generators.

Let $\mathcal{R}$ be the complete subclass of $\mathrm{Ses}(\mathcal{C})$ generated by the canonical exact sequence $0 \to p\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$. Then $\sim_{\mathcal{R}}$ is the congruence on $V(\mathcal{C})$ generated by the pair $(\langle \mathbf{Z} \rangle, \langle \mathbf{Z} \rangle + \langle \mathbf{Z}/p\mathbf{Z} \rangle)$. Then $\langle \mathbf{Z} \rangle \not\sim_{\mathcal{R}} \langle \mathbf{Z} \rangle + \langle \mathbf{Z}/p^2\mathbf{Z} \rangle$, so that no exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z} \to 0$ belongs to the complete class $\mathcal{R}$. Therefore $p^2\mathbf{Z} \leq_{\mathcal{R}} p\mathbf{Z}$ and $p\mathbf{Z} \leq_{\mathcal{R}} \mathbf{Z}$, but $p^2\mathbf{Z} \not\leq_{\mathcal{R}} \mathbf{Z}$. ∎

Proposition 4.2 *The following conditions are equivalent for a transitive subclass $\mathcal{R}$ of* $\mathrm{Ses}(\mathcal{C})$*:*

(a) *For every descending chain $B_0 \geq B_1 \geq \cdots \geq B_t$ of modules of $\mathcal{C}$ with $B_t \leq_{\mathcal{R}} B_0$, one has that $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{R}$ if and only if $B_0 \geq B_1 \geq \cdots \geq B_t$ is a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B_0)$.*

(b) *For every $A \leq C \leq B$ with $A, B \in \mathcal{C}$ and $A \leq_{\mathcal{R}} B$, one has that $C \in \mathcal{C}$ and $C \leq_{\mathcal{R}} B$ if and only if $C/A \in \mathcal{C}$ and $C/A \leq_{\mathcal{R}} B/A$.*

(c) *For every $A, B \in \mathcal{C}$ with $A \leq_{\mathcal{R}} B$, the injective mapping from the interval $[A, B]$ of $\mathcal{L}_{\mathcal{R}}(B)$ to $\mathcal{L}_{\mathcal{R}}(B/A)$, defined by the position $C \mapsto C/A$ for every $C \in [A, B]$, is well defined and surjective.*

PROOF. (a) $\Rightarrow$ (c) Suppose that (a) holds. Let A, B be in $\mathcal{C}$ and $A \leq_{\mathcal{R}} B$. In order to show that the mapping is well defined, notice that if $C \in [A, B]$ (that is, $C \in \mathcal{C}$, $A \leq C \leq B$ and $C \leq_{\mathcal{R}} B$), then $B \geq C \geq A$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B)$, so by (a) $B/A \geq C/A \geq A/A$ is a descending chain in $\mathcal{R}$. In particular, $C/A \leq_{\mathcal{R}} B/A$. This shows that the mapping is well defined.

In order to prove that the mapping is surjective, fix an element C/A of $\mathcal{L}_{\mathcal{R}}(B/A)$. Then $C/A \leq_{\mathcal{R}} B/A$, and so $B/A \geq C/A \geq 0$ is a descending chain in $\mathcal{R}$. By (a), $B \geq C \geq A$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B)$. Thus $C \in [A, B]$.

(b) $\Rightarrow$ (a) Let $B_0 \geq B_1 \geq \cdots \geq B_t$ be a descending chain of modules of $\mathcal{C}$ such that $B_t \leq_{\mathcal{R}} B_0$ and $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{R}$, i.e., a chain of modules B_i/B_t of $\mathcal{C}$ with $B_i/B_t \leq_{\mathcal{R}} B_{i-1}/B_t$ for every $i = 1, 2, \ldots, t$. Apply (b) to the triple $B_t \leq B_i \leq B_{i-1}$ for every $i = 1, 2, \ldots, t$ (this is possible because $B_t \leq_{\mathcal{R}} B_{i-1}$). Thus $B_i \leq_{\mathcal{R}} B_{i-1}$ for every $i = 1, 2, \ldots, t$. By transitivity, one has that $B_i \leq_{\mathcal{R}} B_0$ for every $i = 1, 2, \ldots, t$, and so $B_0 \geq B_1 \geq \cdots \geq B_t$ is a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B_0)$.

Conversely, let $B_0 \geq B_1 \geq \cdots \geq B_t$ be a descending chain in $\mathcal{L}_{\mathcal{R}}(B_0)$, i.e., $B_i \leq_{\mathcal{R}} B_0$ for every $i = 1, 2, \ldots, t$. Apply (b) to the triple $B_t \leq B_i \leq B_0$ for every $i = 1, 2, \ldots, t$ (which is possible because $B_t \leq_{\mathcal{R}} B_0$). Then $B_i/B_t \leq_{\mathcal{R}} B_0/B_t$ for every $i = 1, 2, \ldots, t$, i.e., $B_0/B_t \geq B_1/B_t \geq \cdots \geq B_t/B_t$ is a descending chain in $\mathcal{L}_{\mathcal{R}}(B_0/B_t)$. By transitivity, it is a descending chain in $\mathcal{R}$.

(c) is merely a restatement of (b). ∎

A subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ will be called a *strongly transitive* class if it is transitive and satisfies the equivalent conditions of Proposition 4.2.

Remarks 4.3 (a) Notice that if $\mathcal{R}$ is a transitive subclass of $\mathrm{Ses}(\mathcal{C})$, then the set inclusion $\subseteq$ on the set $\mathcal{L}_{\mathcal{R}}(A)$ and the relation $\leq_{\mathcal{R}}$ on $\mathcal{L}_{\mathcal{R}}(A)$ coincide, and descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$ coincide with finite descending chains in the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$.

(b) If, moreover, $\mathcal{R}$ is strongly transitive, then, for every $A, B \in \mathcal{C}$ with $A \leq_{\mathcal{R}} B$, one has that the partially ordered set $\mathcal{L}_{\mathcal{R}}(B/A)$ is canonically order isomorphic to the interval $[A, B]$ of $\mathcal{L}_{\mathcal{R}}(C)$ for every module $C \in \mathcal{C}$ with $B \leq_{\mathcal{R}} C$.

5 Refinements and composition series

In Section 2 we considered a completely arbitrary congruence $\equiv$ on $V(\mathcal{C})$. This choice had the advantage of a great generality, but it immediately leads to pathologies. Suppose, for instance, that there exist non-zero modules $A \in \mathcal{C}$ with $\langle A \rangle \equiv \langle 0 \rangle$. Then, not only does there exist non-zero modules $A \in \mathcal{C}$ of $\mathcal{C}$ that are zero in $V(\mathcal{C})/\equiv_{\mathcal{R}}$, but also there could be non-zero modules $B \in \mathcal{C}$ that become invertible in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ (consider $\langle B \rangle$ and $\langle A/B \rangle$ for any exact sequence $0 \to B \to A \to A/B \to 0$ in $\mathcal{R}$ with $\langle A \rangle \equiv \langle 0 \rangle$). This would lead us to a theory that could be interesting, but far from the applications and the examples we have in mind. Thus we shall consider only congruences $\equiv$ with the property that $\langle A \rangle \equiv \langle 0 \rangle$ implies $A = 0$ for every $A \in \mathcal{C}$.

In this section, $\mathcal{C}$ will be a small class of right R-modules closed under isomorphism and finite direct sums, $\mathcal{R}$ will be a transitive subclass of $\mathrm{Ses}(\mathcal{C})$*, closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and $\equiv$ will be a congruence on $V(\mathcal{C})$ such that, for every $A \in \mathcal{C}$, one has $\langle A \rangle \equiv \langle 0 \rangle$ if and only if $A = 0$.*

Lemma 5.1 *For every $A \in \mathcal{C}$, one has that $\langle A \rangle \equiv_{\mathcal{R}} \langle 0 \rangle$ if and only if $A = 0$.*

Proof. Suppose that $A \in \mathcal{C}$ and $\langle A \rangle \equiv_{\mathcal{R}} \langle 0 \rangle$. By the hypothesis that for every $B \in \mathcal{C}$ one has $\langle B \rangle \equiv \langle 0 \rangle$ if and only if $B = 0$, the only module that has a descending series equivalent modulo $\equiv$ to a descending series of the zero module is the zero module itself. Therefore $A = 0$ by Theorem 3.1((a) $\Rightarrow$ (b)). ■

Thus, under the hypotheses of this section, the commutative monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is reduced. Conversely, using the idea of the proof of Theorem 3.3, it is easy to see that, for every reduced commutative monoid M, there exists a small class $\mathcal{C}$ of right R-modules, closed under isomorphism and finite direct sums, a subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{C})$ closed for isomorphism and finite direct sums and containing all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and a congruence $\equiv$ on $V(\mathcal{C})$ such that $V(\mathcal{C})/\equiv_{\mathcal{R}} \;\cong M$ (Apply Bergman and Dick's Theorem [8, Theorem 2.1] to the reduced monoid $M \cup \{+\infty\}$ with order-unit $+\infty$, and then let $\mathcal{C}$ be the class of all finitely generated projective R-modules non-isomorphic to R_R, $\mathcal{R} = CalS_=$, and $\equiv$ be the identity on $V(\mathcal{C})$.)

As usual, a *refinement* of a descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ is obtained by possibly inserting further elements in the chain, and a *composition series of A in $\mathcal{R}$* is a properly descending chain in $\mathcal{L}_{\mathcal{R}}(A)$ which has no refinements except by introducing repetitions of the elements of the chain. We say that a module A is *$\mathcal{R}$-simple* if $\mathcal{L}_{\mathcal{R}}(A)$ has exactly two elements (necessarily A and 0).

If M is a commutative monoid, we say that an element $a \in M$ is *indecomposable* (in M) if it is non-zero and, for every $a', a'' \in M$, $a = a' + a''$ implies $a' = 0$ or $a'' = 0$.

Let $\equiv_{\mathcal{R}}$ be the congruence on $V(\mathcal{C})$ generated by the two congruences $\sim_{\mathcal{R}}$ and $\equiv$. The image in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ of an element $\langle A\rangle$ of $V(\mathcal{C})$ will be denoted $\langle A\rangle_{\equiv_{\mathcal{R}}}$.

Lemma 5.2 *Let $\mathcal{R}$ be a transitive subclass of* $\operatorname{Ses}(\mathcal{C})$. *Suppose that for every $A, B \in \mathcal{C}$, if A is $\mathcal{R}$-simple and $\langle A\rangle \equiv \langle B\rangle$, then B is $\mathcal{R}$-simple as well.*

Then:

(1) *For every $A \in \mathcal{C}$, $\langle A\rangle_{\equiv_{\mathcal{R}}}$ is indecomposable in the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ if and only if A is $\mathcal{R}$-simple.*

(2) *For every $A, C \in \mathcal{C}$ with A $\mathcal{R}$-simple, one has $\langle A\rangle \equiv_{\mathcal{R}} \langle C\rangle$ if and only if $\langle A\rangle \equiv \langle C\rangle$.*

Proof. Implication $\Rightarrow$ of (1). If A is not $\mathcal{R}$-simple, there exists $B \in \mathcal{C}$ with $B \leq_{\mathcal{R}} A$ and $0 \neq B \neq A$. Then the canonical exact sequence $0 \to B \to A \to A/B \to 0$ belongs to $\mathcal{R}$. Thus $\langle A\rangle_{\equiv_{\mathcal{R}}} = \langle B\rangle_{\equiv_{\mathcal{R}}} + \langle A/B\rangle_{\equiv_{\mathcal{R}}}$ in $V(\mathcal{C})/\equiv_{\mathcal{R}}$, and these elements of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are not zero by Lemma 5.1.

(2) Suppose that $A \in \mathcal{C}$ is $\mathcal{R}$-simple, $C \in \mathcal{C}$ and $\langle A\rangle \equiv_{\mathcal{R}} \langle C\rangle$. Then there exist R-modules $A_0, A_1, \dots, A_t \in \mathcal{C}$ with $A_0 = A$, $A_t = C$ and such that A_i, A_{i-1} have descending series in $\mathcal{R}$ equivalent modulo $\equiv$ for every $i = 1, 2, \dots, t$ (Theorem 3.1). The module A has a unique properly descending series in $\mathcal{R}$, namely $A > 0$. Thus A_1 has a descending series in $\mathcal{R}$ equivalent to this modulo $\equiv$, that is, A_1 has a descending series of length 1 in $\mathcal{R}$ with its factor equivalent to A modulo $\equiv$. It follows that $\langle A_1\rangle \equiv \langle A\rangle$ and that A_1 is R-simple. By induction on t one proves that $\langle A_i\rangle \equiv \langle A\rangle$ and A_i is R-simple for every i. The proof of (2) follows immediately.

For the implication $\Leftarrow$ of (1), suppose A $\mathcal{R}$-simple and that $\langle A\rangle_{\equiv_{\mathcal{R}}} = \langle A'\rangle_{\equiv_{\mathcal{R}}} + \langle A''\rangle_{\equiv_{\mathcal{R}}}$ for some $A', A'' \in \mathcal{C}$, that is, $\langle A\rangle \equiv_{\mathcal{R}} \langle A' \oplus A''\rangle$. Apply (2) to the modules A and $C = A' \oplus A''$, so that one gets that $\langle A\rangle \equiv \langle C\rangle = \langle A' \oplus A''\rangle$. Thus $A' \oplus A''$ is $\mathcal{R}$-simple. It follows that either $A' = 0$ or $A'' = 0$. ■

Proposition 5.3 *Let $\mathcal{R}$ be a strongly transitive subclass of* $\operatorname{Ses}(\mathcal{C})$. *Then a descending series $B = B_0 > B_1 > \dots > B_n = 0$ in $\mathcal{R}$ is a composition series of B in $\mathcal{R}$ if and only if all the elements $\langle B_{i-1}/B_i\rangle_{\equiv_{\mathcal{R}}}$, $i = 1, 2, \dots, n$, of the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are indecomposable in $V(\mathcal{C})/\equiv_{\mathcal{R}}$.*

Proof. A descending series $B = B_0 > B_1 > \cdots > B_n = 0$ in $\mathcal{R}$, that is, a properly descending chain in the partially ordered set $\mathcal{L}_{\mathcal{R}}(B)$ (Proposition 4.1), is a composition series of B in $\mathcal{R}$ if and only if all the intervals $[B_i, B_{i-1}]$ in $\mathcal{L}_{\mathcal{R}}(B)$ have exactly two elements, or, equivalently, all the sets $\mathcal{L}_{\mathcal{R}}(B_{i-1}/B_i)$ have exactly two elements, i.e., all the quotients B_{i-1}/B_i are $\mathcal{R}$-simple. By Lemma 5.2(1), this holds if and only if every $\langle B_{i-1}/B_i\rangle_{\equiv_{\mathcal{R}}}$ is indecomposable in $V(\mathcal{C})/\equiv_{\mathcal{R}}$. ∎

Thus if $B = B_0 > B_1 > \cdots > B_n = 0$ is a composition series of B in $\mathcal{R}$, then $\langle B\rangle_{\equiv_{\mathcal{R}}} = \langle B_0/B_1\rangle_{\equiv_{\mathcal{R}}} + \langle B_1/B_2\rangle_{\equiv_{\mathcal{R}}} + \cdots + \langle B_{n-2}/B_{n-1}\rangle_{\equiv_{\mathcal{R}}} + \langle B_{n-1}\rangle_{\equiv_{\mathcal{R}}}$ is a decomposition of $\langle B\rangle_{\equiv_{\mathcal{R}}}$ as a sum of indecomposable elements in $V(\mathcal{C})/\equiv_{\mathcal{R}}$.

For any module $A \in \mathcal{C}$, the set $\mathcal{L}_{\mathcal{R}}(A)$ is only a partially ordered subset of the lattice $\mathcal{L}(A)$. If $\mathcal{L}_{\mathcal{R}}(A)$, with the partial order induced from $\mathcal{L}(A)$, turns out to be a lattice, we shall denote by $B \vee C$ and $B \wedge C$ the join and the meet of two elements B, C of $\mathcal{L}_{\mathcal{R}}(A)$.

Recall that a monoid M is said to be a *refinement monoid* if whenever $a+b = c+d$ in M, there exist $x, y, z, t \in M$ such that $a = x+y$, $b = z+t$, $c = x+z$ and $d = y+t$.

Theorem 5.4 *Let $\mathcal{R}$ be a strongly transitive subclass of* $\mathrm{Ses}(\mathcal{C})$. *Suppose that:*

(a) $\mathcal{L}_{\mathcal{R}}(A)$ *is a modular lattice for every $A \in \mathcal{C}$.*

(b) *For every $A \in \mathcal{C}$ and every $B, C \in \mathcal{L}_{\mathcal{R}}(A)$, $\langle B \vee C/B\rangle \equiv \langle C/B \wedge C\rangle$.*

(c) *For every $A, B \in \mathcal{C}$ with $\langle A\rangle \equiv \langle B\rangle$ and every descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ of A in $\mathcal{R}$, there exists a descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ of B in $\mathcal{R}$ equivalent modulo $\equiv$ to the previous one.*

Then the following statements hold:

(1) *For every $A, B \in \mathcal{C}$, $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$ if and only if A and B have descending series in $\mathcal{R}$ equivalent modulo $\equiv$.*

(2) (Schreier Refinement Theorem) *For every $A \in \mathcal{C}$, any two descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ in $\mathcal{R}$ have refinements in $\mathcal{R}$ equivalent modulo $\equiv$.*

(3) *For every $A, B \in \mathcal{C}$, if A is $\mathcal{R}$-simple and $\langle A\rangle \equiv \langle B\rangle$, then B is $\mathcal{R}$-simple as well.*

(4) $V(\mathcal{C})/\equiv_{\mathcal{R}}$ *is a refinement monoid.*

Proof. (3) follows immediately form (c).

(2) By Schreier refinement theorem for modular lattices [18, Proposition III.3.1], the two series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $A = A'_0 \geq A'_1 \geq \cdots \geq A'_m = 0$ in the lattice $\mathcal{L}_{\mathcal{R}}(A)$ have "equivalent" refinements, where

"equivalent" here means that the corresponding intervals of $\mathcal{L}_{\mathcal{R}}(A)$ are projective, that is, in the transitive closure of the relation "being similar". Now two intervals of $\mathcal{L}_{\mathcal{R}}(A)$ are similar if and only if they can be written in the form $[X, X \vee Y]$ and $[X \wedge Y, Y]$ for suitable $X, Y \in \mathcal{L}_{\mathcal{R}}(A)$. By (b), the isomorphism classes of the corresponding quotient modules are equivalent modulo $\equiv$, that is, $\langle X \vee Y/X\rangle \equiv \langle Y/X \wedge Y\rangle$. It follows that the two series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ and $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$ have two refinements, which are descending series in $\mathcal{R}$ equivalent modulo $\equiv$.

(1) By Theorem 3.1, it is sufficient to show that the relation $\approx$, defined for all $\langle A\rangle, \langle B\rangle \in V(\mathcal{C})$ by $\langle A\rangle \approx \langle B\rangle$ if A and B have descending series in $\mathcal{R}$ equivalent modulo $\equiv$, is transitive. Let A, B, C be elements of $\mathcal{C}$, let $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ and $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ be descending series in $\mathcal{R}$ equivalent modulo $\equiv$, and let $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$ and $C = C_0 \geq C_1 \geq \cdots \geq C_m = 0$ be descending series in $\mathcal{R}$ equivalent modulo $\equiv$. Apply (2) to the two descending series $B = B_0 \geq B_1 \geq \cdots \geq B_n = 0$ and $B = B'_0 \geq B'_1 \geq \cdots \geq B'_m = 0$. By (2), these series have two refinements $B = B_0 = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s = B_1 \geq \cdots \geq \overline{B}_t = 0$ and $B = B'_0 = \overline{B'}_0 \geq \overline{B'}_1 \geq \cdots \geq \overline{B'}_r = B'_1 \geq \cdots \geq \overline{B'}_t = 0$, which are descending series in $\mathcal{R}$ equivalent modulo $\equiv$. Assume that $\langle B_0/B_1\rangle \equiv \langle A_{i-1}/A_i\rangle$. As $\mathcal{R}$ is strongly transitive, we can apply hypothesis (c) and find a refinement of the series $A_{i-1} \geq A_i$ corresponding to the refinement $B_0 = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s = B_1$ of the series $B_0 \geq B_1$. In this way, we get a refinement of the descending series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$ equivalent modulo $\equiv$ to the descending series $B = \overline{B}_0 \geq \overline{B}_1 \geq \cdots \geq \overline{B}_s \geq \cdots \geq \overline{B}_t = 0$. Similarly, one constructs a refinement of the descending series $C = C_0 \geq C_1 \geq \cdots \geq C_m = 0$ equivalent modulo $\equiv$ to the descending series $B = \overline{B'}_0 \geq \overline{B'}_1 \geq \cdots \geq \overline{B'}_r \geq \cdots \geq \overline{B'}_t = 0$.

The proof of (4) is similar to the proof of [1, Proposition 3.8]. ∎

The hypotheses of Theorem 5.4 are not sufficient to assure that the monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ is cancellative, even in the case in which $\mathcal{R}$ is the strongly transitive class $\mathrm{Ses}(\mathcal{C})$ and $\equiv$ is the identity. For instance, the monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ is not cancellative when $\mathcal{C}$ is the class of all right vector spaces of dimension $\leq \aleph_0$ over a division ring D. Notice that, in this example, $\mathcal{L}_{\mathcal{R}}(A) = \mathcal{L}(A)$ for every $A \in \mathcal{C}$. Here, the reason why the monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ is not cancellative is due to the lack of a suitable finiteness condition. We need such a condition for the Jordan-Hölder Theorem to hold.

Recall that a partially ordered set L is said to *have finite length* if there is a natural number n such that every chain in L has at most n elements. Let $\mathcal{R}$ be a transitive subclass of $\mathrm{Ses}(\mathcal{C})$ and $\equiv$ be a congruence on $V(\mathcal{C})$. If we want a Jordan-Hölder type theorem to hold, that is, if we want every properly descending series in $\mathcal{R}$ to have a refinement that is a composition series in $\mathcal{R}$, and any two composition series of A in $\mathcal{R}$ to be equivalent modulo $\equiv$, then the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ must have finite length. Conversely, if the partially ordered set $\mathcal{L}_{\mathcal{R}}(A)$ has finite length, then (1) every $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$, (2) $\mathcal{L}_{\mathcal{R}}(A)$ satisfies the acc and the dcc, and (3) every properly descending series in $\mathcal{R}$ has a refinement that is a composition series in $\mathcal{R}$. The following theorem shows that all these concepts finds their natural setting under the hypotheses of Theorem 5.4.

Theorem 5.5 *Suppose that the hypotheses* (a), (b) *and* (c) *of Theorem* 5.4 *hold. Then the following conditions are equivalent:*

(1) (The Jordan-Hölder Theorem) *Any $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$, and any two composition series of A in $\mathcal{R}$ are equivalent modulo $\equiv$.*

(2) *The commutative monoid $V(\mathcal{C})/\!\equiv_{\mathcal{R}}$ is free.*

(3) *The lattice $\mathcal{L}_{\mathcal{R}}(A)$ has finite length for every $A \in \mathcal{C}$.*

(4) *Any $A \in \mathcal{C}$ has a composition series in $\mathcal{R}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that (1) holds. It suffices to show that the set of all $\langle A\rangle_{\equiv_{\mathcal{R}}}$, where A ranges in the set of all $\mathcal{R}$-simple modules of $\mathcal{C}$, is a free set of generators for $V(\mathcal{C})/\equiv_{\mathcal{R}}$. In view of Lemma 5.2 and Proposition 5.3, it is clearly a set of generators.

If $B_1, \dots, B_m$ are $\mathcal{R}$-simple modules of $\mathcal{C}$ with $\langle B_1\rangle_{\equiv_{\mathcal{R}}}, \dots, \langle B_m\rangle_{\equiv_{\mathcal{R}}}$ distinct elements of $V(\mathcal{C})/\!\equiv_{\mathcal{R}}$, and $s_1, \dots, s_m, t_1, \dots, t_m$ are non-negative integers with $\sum_{i=1}^m s_i\langle B_i\rangle_{\equiv_{\mathcal{R}}} = \sum_{i=1}^m t_i\langle B_i\rangle_{\equiv_{\mathcal{R}}}$, then $\langle \oplus_{i=1}^m B_i^{s_i}\rangle \equiv_{\mathcal{R}} \langle \oplus_{i=1}^m B_i^{t_i}\rangle$, so that the two modules $\oplus_{i=1}^m B_i^{s_i}$ and $\oplus_{i=1}^m B_i^{t_i}$ have descending series $\mathcal{D}_1, \mathcal{D}_2$ in $\mathcal{R}$ equivalent modulo $\equiv$ (Theorem 5.4(1)). Now the series $\mathcal{D}_1'$

$$\oplus_{i=1}^m B_i^{s_i} > B_1^{s_1-1} \oplus \oplus_{i=2}^m B_i^{s_i} > \mathcal{B}_1^{s_1-2} \oplus \oplus_{i=2}^m B_i^{s_i} > dots > \oplus_{i=2}^m B_i^{s_i} > dots > 0$$

is a composition series in $\mathcal{L}_{\mathcal{R}}(\oplus_{i=1}^m B_i^{s_i})$, which has s_1 factors isomorphic to B_1, s_2 factors isomorphic to B_2, ..., s_m factors isomorphic to B_m. Similarly, the module $\oplus_{i=1}^m B_i^{t_i}$ has a composition series $\mathcal{D}_2'$ with t_1 factors isomorphic to B_1, t_2 factors isomorphic to B_2, ..., t_m factors isomorphic to B_m. As we have seen in the proof of Theorem 5.4(1), two descending series in $\mathcal{R}$ of a module always have refinements in $\mathcal{R}$ equivalent modulo $\equiv$, so that the two descending series $\mathcal{D}_2$ and $\mathcal{D}_2'$ have two refinements in $\mathcal{R}$ equivalent modulo $\equiv$. But $\mathcal{D}_2'$ has no proper refinements in $\mathcal{R}$, so that $\mathcal{D}_2$ has a refinement $\mathcal{D}_2''$ in $\mathcal{R}$ with t_1 factors equivalent modulo $\equiv$ to B_1, ..., t_m factors equivalent modulo $\equiv$ to B_m. Since $\mathcal{D}_1$ and $\mathcal{D}_2$ in $\mathcal{R}$ are equivalent, it follows that $\mathcal{D}_1$ has a refinement $\mathcal{D}_1''$ in $\mathcal{R}$ with t_1 factors equivalent to B_1, ..., t_m factors equivalent to B_m. Condition (a), applied to the two composition series $\mathcal{D}_1'$ and $\mathcal{D}_1''$ implies that $s_1 = t_1, \dots, s_m = t_m$.

(2) $\Rightarrow$ (3) Let $A \in \mathcal{C}$. Suppose that $\mathcal{L}_{\mathcal{R}}(A)$ has not finite length. Then for every $n > 0$ there is a chain in $\mathcal{L}_{\mathcal{R}}(A)$ with more than n elements. Recall that the only modules $C \in \mathcal{C}$ with $\langle C\rangle_{\equiv_{\mathcal{R}}} = \langle 0\rangle_{\equiv_{\mathcal{R}}}$ in $V(\mathcal{C})/\equiv_{\mathcal{R}}$ are the modules $C = 0$ (Lemma 5.1). Thus $\langle A\rangle_{\equiv_{\mathcal{R}}}$ can be written as the sum of $\geq n$ non-zero elements of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ for every $n > 0$. This cannot happen in a free commutative monoid.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) follows from Theorem 5.4(2). ■

6 Examples

Example 4 The first obvious example is that of $\mathcal{C}$ the class of all R-modules of finite composition length, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ and $\equiv$ the identity $=$ on $V(\mathcal{C})$. Then $\mathcal{R}$ is a strongly transitive class and $\mathcal{L}_{\mathcal{R}}(A) = \mathcal{L}(A)$ for every $A \in \mathcal{C}$, so that Theorems 5.4 and 5.5 apply. The monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is a free commutative monoid.

More generally, if $\mathcal{A}$ is an arbitrary abelian category and A is an object of $\mathcal{A}$, then the class of all subobjects of A is a "modular lattice" (it can be a proper class, and not necessarily a set, but it satisfies the axioms of a modular lattice), so that the Schreirer refinement Theorem in $\mathcal{A}$ follows from the Schreirer refinement Theorem for modular lattices [18, pp. 67, 91 and 92]. If $\mathcal{F}$ is the full subcategory of all objects of $\mathcal{A}$ of finite length, then the Jordan-Hölder theorem holds in $\mathcal{F}$. ■

Example 5 *Biuniform modules.* For an arbitrary module A, we shall denote by $\dim A$ and $\operatorname{codim} A$ the Goldie dimension and the dual Goldie dimension of A respectively [7]. Recall that an R-module A is said to be *uniform* if $\dim A = 1$, *couniform* (or *hollow*) if $\operatorname{codim} A = 1$, and *biuniform* if it uniform and couniform. Also, recall that if A and B are two R-modules, we say that A and B belong to the same monogeny class, and write $[A]_m = [B]_m$, if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$ [7]. Similarly, we say that A and B belong to the same epigeny class, and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$. Thus, we denote by $[A]_m$ ($[A]_e$, respectively) the class of all the R-modules that belong to the monogeny (epigeny) class of A.

Let $\mathcal{C}$ be the small class of all the R-modules which are direct sums of finitely many biuniform modules. Let $\mathcal{R}$ be the class of all split exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, so that $\sim_R$ coincides with the equality $=$ on $V(\mathcal{C})$, and let $\equiv$ be the congruence on $V(\mathcal{C})$ defined by $\langle A\rangle \equiv \langle B\rangle$ if $[A]_m = [B]_m$.

Proposition 6.1 *Let $\mathcal{C}$ be the class of all the R-modules which are direct sums of finitely many biuniform modules. If $A, B \in \mathcal{C}$ and A is a direct summand of B, then $B/A \in \mathcal{C}$ also.*

PROOF. Induction on $n = \dim B = \operatorname{codim} B$. The cases $n = 0$ and $n = 1$ are trivial. Suppose $n \geq 2$. The case $A = 0$ also is trivial, so that we may assume $A \neq 0$. In this case A has a direct sum decomposition $A = D \oplus A'$ with D biuniform and $A' \in \mathcal{C}$. Moreover, $B = C_1 \oplus \cdots \oplus C_n$ with the C_i's biuniform, and $A \oplus A'' = B$ for a suitable submodule A'' of B. Apply [7, Proposition 9.5] to the direct sum decompositions $D \oplus (A' \oplus A'') = C_1 \oplus \cdots \oplus C_n$. Then there are two distinct indices $i, j = 1, \ldots, n$ and a direct sum decomposition $\overline{D} \oplus E = C_i \oplus C_j$ of $C_i \oplus C_j$ such that $D \cong \overline{D}$ and $A' \oplus A'' \cong E \oplus (\oplus_{k \neq i,j} C_k)$. As dim and codim are additive on direct sums, the module E is biuniform as well. Thus we can apply the inductive hypothesis to the direct summand A' of $A' \oplus A''$, and obtain that $A'' \in \mathcal{C}$. ∎

By this proposition, if $A, B \in \mathcal{C}$, $A \leq_{\mathcal{R}} B$ simply means that A is a direct summand of B. It easily follows that:

Corollary 6.2 *The class $\mathcal{R}$ is strongly transitive.*

The $\mathcal{R}$-simple modules of $\mathcal{C}$ are exactly the biuniform modules, and the composition series of a module $B \in \mathcal{C}$ with $\dim B = \operatorname{codim} B = n$ are exactly the descending series $B = B_0 > B_1 > \cdots > B_n = 0$ of direct summands B_i of B with every B_i in $\mathcal{C}$ and $\dim B_i = \operatorname{codim} B_i = n - i$.

Notice that the conclusions of Theorem 5.4 and the equivalent conditions of Theorem 5.5 hold [5], though $\mathcal{L}_{\mathcal{R}}(A)$ is not a modular lattice in general.

The situation can be dualized, in the sense that all we have said in this example remains true if, instead of defining $\langle A\rangle \equiv \langle B\rangle$ if $[A]_m = [B]_m$, we set $\langle A\rangle \equiv \langle B\rangle$ if $[A]_e = [B]_e$. Also note that if we take as $\mathcal{C}$ the class of R-modules that are direct sums of finitely many uniform modules, as $\mathcal{R}$ the class of all split exact sequences in $\operatorname{Ses}(\mathcal{C})$, and as $\equiv$ the congruence on $V(\mathcal{C})$ defined by $\langle A\rangle \equiv \langle B\rangle$ if $[A]_m = [B]_m$, then $V(\mathcal{C})/\!\equiv_{\mathcal{R}}$ turns out to be a free commutative monoid [5]. Similarly when $\mathcal{C}$ is the class of R-modules that are direct sums of finitely many couniform modules and $\equiv$ is the congruence "belonging to the same epigeny class". ∎

Remark 6.3 (and notations for the rest of this section). We saw in Example 4 that if $\mathcal{A}$ is an arbitrary abelian category and $\mathcal{F}$ is the full subcategory of $\mathcal{A}$ whose objects are all objects of $\mathcal{A}$ of finite length, then the Schreirer Refinement Theorem holds in $\mathcal{A}$ and the Jordan-Hölder Theorem holds in $\mathcal{F}$. It follows that if $\mathcal{C}$ is a class of right R-modules, viewed as a full subcategory of Mod-R, and F is a functor from $\mathcal{C}$ to $\mathcal{A}$ or $\mathcal{F}$, then the information we have about descending chain of subobjects in $\mathcal{A}$ or $\mathcal{F}$ can be lifted to get information about descending chains of subobjects in $\mathcal{C}$.

Let us apply this remark to the case in which $\mathcal{A} = \text{Mod-}R'$ for another suitable ring R'. *Let R, R' be two rings and $\mathcal{C}, \mathcal{C}'$ be small classes of right R-modules and right R'-modules respectively, both closed under isomorphism and finite direct sums. View $\mathcal{C}$ and $\mathcal{C}'$ as full subcategory of* Mod-R *and* Mod-R'. *Let $\mathcal{R}'$ be a class of exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}'$ and let $\equiv'$ be a congruence on $V(\mathcal{C}')$. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be an additive functor with the following two properties:*

(a) *for every exact sequence $0 \to A' \to B' \to C' \to 0$ in $\mathcal{R}'$ there exists an exact sequence $0 \to A \to B \to C \to 0$ in* $\text{Ses}(\mathcal{C})$ *with $F(A) \cong A', F(B) \cong B'$ and $F(C) \cong C'$.*

(b) *for every $A \in \mathcal{C}$, $F(A) = 0$ implies $A = 0$.*

Let $\mathcal{R}$ be the class of all exact sequences $0 \to A \to B \to C \to 0$ belonging to $\text{Ses}(\mathcal{C})$ *such that the corresponding sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact and belongs to $\mathcal{R}'$, and let $\equiv$ be the congruence on $V(\mathcal{C})$ defined, for every $A, B \in \mathcal{C}$, by $\langle A \rangle \equiv \langle B \rangle$ if and only if $\langle F(A) \rangle \equiv' \langle F(B) \rangle$. Then F induces a monoid isomorphism $\widetilde{F}$ of $V(\mathcal{C})/\equiv_{\mathcal{R}}$ onto $V(\mathcal{C}')/\equiv'_{\mathcal{R}'}$.*

Example 6 *Finitely generated modules, polyserial modules and finite-rank torsion-free modules over commutative valuation domains.* A modules is *uniserial* if its submodules form a chain under inclusion. Non-zero uniserial modules are biuniform. A *valuation domain* is a commutative integral domain R with R_R uniserial. A module A_R over a valuation domain R is *polyserial* if it has a series $A = A_0 > A_1 > \cdots > A_n = 0$ of submodules with each A_i pure in A and each A_{i-1}/A_i uniserial [9, p. 403]. For instance, every finitely generated module over a valuation domain is polyserial [9, Lemma I.7.8].

Let R be a valuation domain and $\mathcal{C}$ be the class of all finitely generated R-modules. Let R' be a maximal immediate extension of R [9, pp. 58–60], so that R' is a flat R-algebra. If $\mathcal{C}'$ is the class of all finitely generated R'-modules, then $V(\mathcal{C}')$ is a free commutative monoid, because every finitely generated R'-module is a direct sum of cyclic R'-modules in an essentially unique way (the Krull-Schmidt Theorem holds for finitely generated modules over maximal valuation domains). Let $\mathcal{R}'$ denote the class of all sequences in $\text{Ses}(\mathcal{R}')$ that are split, and let $\equiv'$ be the identity on $V(\mathcal{C}')$. Thus $\equiv_{\mathcal{R}'}$ is the identity on $V(\mathcal{C}')$ as well, and $V(\mathcal{C}')/\equiv_{\mathcal{R}'} \ = V(\mathcal{C}')$.

Let $F\colon \mathcal{C} \to \mathcal{C}'$ be the functor defined by $F(A) = A \otimes_R R'$ for every $A \in \mathcal{C}$, so that F satisfies conditions (a) and (b) above. If $0 \to A \to B \to C \to 0$ is a pure exact sequence with $A, B, C \in \mathcal{C}$, then the sequence $0 \to F(A) = A \otimes_R R' \to F(B) = B \otimes_R R' \to F(C) = C \otimes_R R' \to 0$ is pure as well. Now $C \otimes_R R'$ is a direct sum of uniserial R'-modules, hence it is pure-injective by [9, Theorem XIII.5.2]. Thus the exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ splits. Conversely, let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{Ses}(\mathcal{C})$ such that $0 \to F(A) \to F(B) \to F(C) \to 0$ is a splitting exact sequence of R'-modules. For every R-module M we have a commutative square

$$\begin{array}{ccc} A \otimes_R R \otimes_R M & \to & A \otimes_R R' \otimes_R M \\ \downarrow & & \downarrow \\ B \otimes_R R \otimes_R M & \to & B \otimes_R R' \otimes_R M \end{array}$$

of R-module homomorphisms. The horizontal arrows are injective because the embedding $R \to R'$ is a pure monomorphism of R-modules. As $X \otimes_R R' \otimes_R M \cong X \otimes_R R' \otimes_{R'} R' \otimes_R M = F(X) \otimes_{R'} (R' \otimes_R M)$ for every R-module X and $F(A) \to F(B)$ is a pure monomorphism of R'-modules, the vertical arrow on the right of the commutative square is injective as well. It follows that the vertical arrow on the left is injective, that is, the exact sequence $0 \to A \to B \to C \to 0$ is pure. In the notation above, we have proved that the class $\mathcal{R}$ consists exactly of all pure sequences in $\mathrm{Ses}(\mathcal{C})$.

Let us see what the congruence $\equiv$ on $V(\mathcal{C})$ is in this case. Let A be a finitely generated R-module. By [9, Lemma I.7.8], A has a *pure composition series with cyclic factors*, that is, there exists a descending series $A = A_0 > A_1 > \cdots > A_n = 0$ of submodules with each A_i pure in A and each A_{i-1}/A_i cyclic. We have just seen that $F(A) \cong \oplus_{i=1}^n (A_{i-1}/A_i) \otimes_R R'$. Moreover, two cyclic R-modules are isomorphic if and only if they remain isomorphic when they are tensored with R'. It follows that for every $A, B \in \mathcal{C}$, $\langle F(A) \rangle = \langle F(B) \rangle$ if and only if A and B have isomorphic pure composition series with cyclic factors. Thus the congruence $\equiv$ on $V(\mathcal{C})$ is defined by $\langle A \rangle \equiv \langle B \rangle$ if and only if A and B have isomorphic pure composition series with cyclic factors. Notice that $\sim_{\mathcal{R}}$ is contained in $\equiv$, because if $0 \to A \to B \to C \to 0$ belongs to $\mathcal{R}$, then $0 \to F(A) \to F(B) \to F(C) \to 0$ splits, so that $\langle F(B) \rangle = \langle F(A) \rangle + \langle F(C) \rangle$, that is, $\langle F(B) \rangle \equiv' \langle F(A \oplus C) \rangle$, hence $\langle B \rangle \equiv \langle A \oplus C \rangle$. Therefore the congruences $\equiv$ and $\equiv_{\mathcal{R}}$ coincide, and the monoids $V(\mathcal{C})/\!\equiv_{\mathcal{R}}\, = V(\mathcal{C})/\!\equiv\, \cong V(\mathcal{C}')$ are free commutative monoids. The Jordan-Hölder Theorem holds in $\mathcal{C}$ (Salce and Zanardo, [9, Theorem V.5.5]).

Notice that this Jordan-Hölder Theorem in $\mathcal{C}$ does not follow from the Jordan-Hölder Theorem in the abelian category of all additive functors from finitely presented modules to abelian groups. More precisely, let ${}_R\mathrm{FP}$ denote the full subcategory of R-Mod whose objects are all finitely presented left modules over an arbitrary (not necessarily commutative) ring R. Let $({}_R\mathrm{FP},\mathrm{Ab})$ denote the category of all additive functors of ${}_R\mathrm{FP}$ into the category Ab of abelian groups. Then there is a full embedding of categories $\boldsymbol{\Psi}\colon \text{Mod-}R \to ({}_R\mathrm{FP},\mathrm{Ab})$ defined by $M_R \mapsto M \otimes_R -$ [7, p. 26]. This embedding sends pure subobjects of Mod-R to subobjects of $({}_R\mathrm{FP},\mathrm{Ab})$, and pure-injective objects of Mod-R to injective objects of $({}_R\mathrm{FP},\mathrm{Ab})$. Thus the Jordan-Hölder Theorem in the abelian category $({}_R\mathrm{FP},\mathrm{Ab})$ yields some kind of information on the right modules M_R for which $M \otimes_R -$ is an object of finite length in $({}_R\mathrm{FP},\mathrm{Ab})$. Now if R is a commutative valuation domain and M_R is a cyclic R-module, then $\mathrm{End}(M_R)$ is a valuation ring, but not necessarily a field. Thus the corresponding object $M \otimes_R -$ is not necessarily a simple object in $({}_R\mathrm{FP},\mathrm{Ab})$, because its endomorphism ring is isomorphic to $\mathrm{End}(M_R)$. Thus the Jordan-Hölder Theorem in $\mathcal{C}$ due to Salce and Zanardo does not follow from the Jordan-Hölder Theorem in the abelian category $({}_R\mathrm{FP},\mathrm{Ab})$.

This example generalizes to the case of arbitrary polyserial modules [9, Theorem XII.1.6]. Namely, let $\mathcal{C}, \mathcal{C}'$ be the classes of all polyserial modules over a valuation domain R and over a maximal immediate extension R' of R, respectively. Arguing as in the previous paragraph (R' is a pure R-algebra and uniserial R'-modules are pure-injective), one sees that every polyserial R'-module is a direct sum of uniserial R'-modules in an essentially unique way (the Krull-Schmidt Theorem holds because endomorphism rings of uniserial modules over commutative rings are local). Thus $V(\mathcal{C}')$ is a free commutative monoid having the set of all isomorphism classes $\langle U \rangle$ of uniserial R'-modules U as a free set of generators. Let $\mathcal{R}'$ denote the class of all splitting exact sequences in $\mathrm{Ses}(\mathcal{C}')$, $\equiv'$ be the identity on $V(\mathcal{C}')$, and $F\colon \mathcal{C} \to \mathcal{C}'$ be the functor $F\colon A \mapsto A \otimes_R R'$. As in the case of finitely generated R-modules, the class $\mathcal{R}$ of all exact sequences in $\mathrm{Ses}(\mathcal{C})$ that are mapped to split sequences via F turns out to be the class of all pure exact sequences in $\mathrm{Ses}(\mathcal{C})$. If A is a polyserial R-module and $A = A_0 > A_1 > \cdots > A_n = 0$ is a descending series of submodules with the A_i's pure in A and the A_{i-1}/A_i's uniserial, then $F(A) \cong \oplus_{i=1}^{n} (A_{i-1}/A_i) \otimes_R R'$, so that for every $A, B \in \mathcal{C}$, $\langle F(A) \rangle = \langle F(B) \rangle$ if and only if A and B have pure composition series with uniserial factors equivalent modulo having the same type [9, p. 346]. The monoid $V(\mathcal{C})/\equiv_{\mathcal{R}} \;\cong V(\mathcal{C}')$ is the free commutative monoid having the isomorphism classes of uniserial R'-modules as a free set of generators. This yields a Jordan-Hölder Theorem for $\mathcal{C}$ (cf. [9, Proposition XII.1.6]).

Another possible generalization is that one to finite-rank torsion-free R-modules [9, Theorem XV.1.7]. In this case, $\mathcal{C}$ and $\mathcal{C}'$ are the classes of all finite-rank torsion-free modules over a valuation domain R and over a maximal immediate extension R' of R, respectively. As before, every finite-rank torsion-free R'-module is a direct sum of standard uniserial R'-modules in an essentially unique way. Thus $V(\mathcal{C}')$ is a free commutative monoid having the set of all isomorphism classes $\langle U \rangle$ of standard uniserial R'-modules U as a free set of generators. Let $\mathcal{R}'$ denote the class of all splitting exact sequences in $\mathrm{Ses}(\mathcal{R}')$, $\equiv'$ be the identity on $V(\mathcal{C}')$ and $F\colon \mathcal{C} \to \mathcal{C}'$ be the functor $F\colon A \mapsto A \otimes_R R'$. If $A \in \mathcal{C}$ and $A = A_0 > A_1 > \cdots > A_n = 0$ is a descending series of submodules with the A_i's pure in A and the A_{i-1}/A_i's uniserial, then $F(A) \cong \oplus_{i=1}^{n} (A_{i-1}/A_i) \otimes_R R'$. The monoid $V(\mathcal{C})/{\equiv_{\mathcal{R}}} \cong V(\mathcal{C}')$ is the free commutative monoid having the isomorphism classes of standard uniserial R'-modules as a free set of generators, and a Jordan-Hölder Theorem for $\mathcal{C}$ holds [9, Proposition XV.1.7]). ■

Example 7 *Artinian divisible modules over a commutative, noetherian, local, 1-dimensional, Cohen-Macaulay ring.* This example is taken from [14, Chapter V]. Let R be a commutative, noetherian, local, 1-dimensional, Cohen-Macaulay ring. Recall that an element of R is called *regular* if it is not a zero-divisor. An R-module A is *divisible* if $Ar = A$ for every regular element $r \in A$. Let $\mathcal{C}$ be the class of all artinian divisible R-modules. The class $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$ is strongly transitive, as is easily seen. Let $\equiv$ be the congruence on $V(\mathcal{C})$ defined, for every $A, B \in \mathcal{C}$, by $\langle A \rangle \equiv \langle B \rangle$ if A and B belong to the same epigeny class (terminology as in Example 5). Recall that an R-module A is said to be a *simple divisible module* if it is a non-zero, torsion, divisible module that has no proper non-zero divisible submodules [14, p. 46]. Every $A \in \mathcal{C}$ has a composition series $A = A_0 \geq A_1 \geq \cdots \geq A_n = 0$ in $\mathcal{R}$, and any two composition series of A are equivalent modulo $\equiv$ [14, Theorem 5.10]. Thus $V(\mathcal{C})/{\equiv_{\mathcal{R}}}$ is a free commutative monoid isomorphic to the free commutative monoid freely generated by the set of all epigeny classes $[A]_e$, where A ranges in the class of all simple divisible R-modules [14, Theorem 5.10]. ■

Example 8 *h-divisible torsion modules and complete torsion-free modules.* Let R be a commutative ring, Q its total ring of fractions and $K = Q/R$. An R-module A is *h-divisible* if it is a R-homomorphic image of a Q-module [14]. The *torsion submodule* $t(A)$ of A is the set of all $x \in A$ with $xr = 0$ for some regular $r \in R$. An R-module A is *torsion* if $t(A) = A$, *torsion-free* if $t(A) = 0$, and *complete* if it is Hausdorff and complete with respect to the topology on A defined by taking the submodules of the form Ar, where r is a regular element of R, as a basis of neighborhoods of 0 in A. Let $\mathcal{D}$ be the class of all h-divisible torsion R-modules and let $\mathcal{C}$ be the class of all complete torsion-free R-modules. Equivalently, $\mathcal{C}$ is the class of all torsion-free R-modules X that are *cotorsion*, that is, have the property that $\mathrm{Hom}_R(Q,X)=0$ and $\mathrm{Ext}^1_R(Q,X)=0$. If we view $\mathcal{D}$ and $\mathcal{C}$ as full subcategories of Mod-R, there is a category equivalence between $\mathcal{D}$ and $\mathcal{C}$ given by the functors $\mathrm{Hom}_R(K,-)\colon \mathcal{D}\to\mathcal{C}$ and $K\otimes_R -\colon \mathcal{C}\to\mathcal{D}$ [14, Corollaries 2.3 and 2.4]. Let $\mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0\to X\to Y\to Z\to 0$ with $X,Y,Z\in\mathcal{C}$. This is a complete strongly transitive class, as is easily verified.

Following [4], we say that a short exact sequence $0\to A\to B\xrightarrow{f} C\to 0$ is *h-exact* if for every homomorphism $g\colon K\to C$ there exists a homomorphism $h\colon K\to B$ such that $g=fh$. If $0\to X\to Y\to Z\to 0$ is a short exact sequence in $\mathrm{Ses}(\mathcal{C})$, the corresponding sequence $0\to K\otimes_R X\to K\otimes_R Y\to K\otimes_R Z\to 0$ is an h-exact sequence that belongs to $\mathrm{Ses}(\mathcal{D})$ (cf. [4], where this is proved for an integral domain, but the same proof holds for an arbitrary commutative ring). Conversely, if $0\to A\to B\to C\to 0$ is an exact sequence in $\mathrm{Ses}(\mathcal{D})$ that is h-exact, than the sequence $0\to \mathrm{Hom}_R(K,A)\to\mathrm{Hom}_R(K,B)\to\mathrm{Hom}_R(K,C)\to 0$ is exact and belongs to $\mathrm{Ses}(\mathcal{C})$. Thus the complete strongly transitive class $\mathrm{Ses}(\mathcal{C})$ corresponds, via the category equivalence $\mathcal{C}\to\mathcal{D}$, to the subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{D})$ consisting of all h-exact sequences.

Proposition 6.4 *The subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{D})$ consisting of all h-exact sequences is a strongly transitive subclass of $\mathrm{Ses}(\mathcal{D})$.*

The proof is straightforward. Let us show, for instance, that if $A\le C\le B$, $A,B,C/A\in\mathcal{D}$, $A\le_{\mathcal{R}} B$ and $C/A\le_{\mathcal{R}} B/A$, then $C\in\mathcal{D}$. Let c be an element of C. As $C/A\in\mathcal{D}$, the natural map $\varphi\colon K\otimes_R\mathrm{Hom}_R(K,C/A)\to C/A$ defined by $\varphi(x\otimes f)=f(x)$ is an isomorphism [14, Corollary 1.2]. Thus there exist $x_1,\dots,x_n\in K$ and $f_1,\dots,f_n\in\mathrm{Hom}_R(K,C/A)$ such that $\sum_{i=1}^n f_i(x_i)=c+A$. Each f_i can be viewed as a map of K into B/A, and the sequence $0\to A\to B\to B/A\to 0$ is h-exact because $A\le_{\mathcal{R}} B$. Thus there exist homomorphisms $g_i\colon K\to B$ such that $\pi g_i=f_i$, where $\pi\colon B\to B/A$ denotes the canonical projection. From $f_i(K)\subseteq C/A$, it follows that $g_i(K)\subseteq C$. Moreover, $c-\sum_{i=1}^n g_i(x_i)\in A$. This implies that there exist $y_1,\dots,y_m\in K$ and $h_1,\dots,h_m\in\mathrm{Hom}_R(K,A)$ such that $\sum_{j=1}^m h_j(y_j)=c-\sum_{i=1}^n g_i(x_i)$. Thus there exists a homomorphism $K^{n+m}\to C$ whose image contains c. This proves that $C\in\mathcal{D}$. ■

Example 9 *Torsion-free abelian groups of finite rank.* Let $\mathcal{F}$ be the class of all torsion-free abelian groups of finite rank and $\mathcal{P} = \mathrm{Ses}(\mathcal{F})$. Then $\mathcal{P}$ is a complete strongly transitive class. A module $A \in \mathcal{F}$ is $\mathcal{P}$-simple if and only if it has torsion-free rank 1. We shall show that the monoid $V(\mathcal{F})/\sim_{\mathcal{P}}$ is not cancellative, and that there exist $A, B \in \mathcal{F}$ that do not have isomorphic descending series in $\mathcal{P}$, but $\langle A\rangle \sim_{\mathcal{P}} \langle B\rangle$.

Let $A' \subseteq \mathbf{Q}$ be a torsion-free abelian group (of rank 1) such that $pA' \neq A'$ for every prime p and A' is not isomorphic to $\mathbf{Z}$. For instance, A' could be the group of all rationals with square-free denominators. By [17, Lemma 7], there exists a group E with exact sequences $0 \to A' \to E \to \mathbf{Q} \to 0$ and $0 \to \mathbf{Z} \to E \to \mathbf{Q} \to 0$. Thus $\langle A'\rangle + \langle \mathbf{Q}\rangle \sim_{\mathcal{P}} \langle E\rangle \sim_{\mathcal{P}} \langle \mathbf{Z}\rangle + \langle \mathbf{Q}\rangle$.

Suppose $\langle A'\rangle \sim_{\mathcal{P}} \langle \mathbf{Z}\rangle$. By Theorem 3.1, there exist $A_0, A_1, \dots, A_t \in \mathcal{F}$ with $A_0 = A'$, $A_t = \mathbf{Z}$ and such that A_i, A_{i-1} have isomorphic descending series in $\mathcal{P}$ for every $i = 1, 2, \dots, t$. As two modules with isomorphic descending series in $\mathcal{P}$ have the same torsion-free rank, all the abelian groups $A_0, A_1, \dots, A_t$ must have torsion-free rank 1. But two groups of rank 1 with isomorphic descending series in $\mathcal{P}$ are isomorphic. It follows that $A' \cong \mathbf{Z}$, a contradiction. This shows that the monoid $V(\mathcal{F})/\sim_{\mathcal{P}}$ is not cancellative.

Let $\varphi\colon A' \oplus \mathbf{Q} \to \mathbf{Q}$ be any group homomorphism. Then $\varphi = \varphi_{a,b}$ for suitable $a, b \in \mathbf{Q}$, where $\varphi_{a,b}\colon (x, y) \mapsto ax + by$. Thus φ is onto and $\ker\varphi \cong A'$ for $b \neq 0$, while $\varphi(A' \oplus \mathbf{Q}) \cong A'$ and $\ker\varphi = 0 \oplus \mathbf{Q}$ for $b = 0$ and $a \neq 0$. This proves that all non-trivial strictly descending series of $A' \oplus \mathbf{Q}$ in $\mathcal{P}$ are isomorphic. Similarly, all non-trivial strictly descending series of $\mathbf{Z} \oplus \mathbf{Q}$ are isomorphic. Therefore $A' \oplus \mathbf{Q}$ and $\mathbf{Z} \oplus \mathbf{Q}$ do not have isomorphic descending series in $\mathcal{P}$. Thus $A = A' \oplus \mathbf{Q}$ and $B = \mathbf{Z} \oplus \mathbf{Q}$ have the required properties.

Notice that $\mathcal{L}_{\mathcal{P}}(A)$ is a lattice of finite length for every $A \in \mathcal{F}$, and that the monoid $V(\mathcal{F})/\sim_{\mathcal{P}}$ is generated by all classes of torsion-free modules of rank 1, which are exactly the indecomposable elements of $V(\mathcal{F})/\sim_{\mathcal{P}}$. Thus all composition series in $\mathcal{P}$ of $A' \oplus \mathbf{Q}$ are isomorphic, and the same holds for $\mathbf{Z} \oplus \mathbf{Q}$, but $\langle A' \oplus \mathbf{Q}\rangle_{\sim_{\mathcal{P}}} = \langle \mathbf{Z} \oplus \mathbf{Q}\rangle_{\sim_{\mathcal{P}}}$ is an element of $V(\mathcal{F})/\sim_{\mathcal{P}}$ that can be written as a sum of two indecomposable elements in infinitely many different ways.

Now fix a prime p, and let J_p be the ring of p-adic integers. Let $\mathcal{F}_p$ be the class of torsion-free J_p-modules of finite rank. If Q_p denotes the field of fractions of J_p, $\mathcal{F}_p$ consists of all J_p-modules isomorphic to a submodule of Q_p^n for some $n \geq 0$. Every module in $\mathcal{F}_p$ is the direct sum of finitely many copies of J_p's and Q_p in an essentially unique way [13, Theorem 12], so that $V(\mathcal{F}_p)$ is the free commutative monoid with two generators $\langle J_p\rangle$ and $\langle Q_p\rangle$. Let $\mathcal{R}'$ be the class of all split exact sequences of $\mathrm{Ses}(\mathcal{F}_p)$ and $\equiv'$ be the identity of $V(\mathcal{F}'_p)$. If $F\colon \mathcal{F} \to \mathcal{F}_p$ is the functor $- \otimes_{\mathbf{Z}} J_p$, then

F satisfies conditions (a) and (b) of Remark 6.3. Let us show that in the notation introduced there, the subclass $\mathcal{R}$ of $\mathrm{Ses}(\mathcal{F})$ coincides with the whole $\mathcal{P} = \mathrm{Ses}(\mathcal{F})$. To this end, let $0 \to A \to B \to C \to 0$ be an element of $\mathrm{Ses}(\mathcal{F})$. Then the sequence $0 \to A \to B \to C \to 0$ is pure, so that applying the functor $F = - \otimes_{\mathbf{Z}} J_p$ we get a pure exact sequence belonging to $\mathrm{Ses}(\mathcal{F}_p)$. Since J_p and Q_p are pure-injective J_p-modules [9, Theorem XIII.4.6], all pure-exact sequences in $\mathrm{Ses}(\mathcal{F}_p)$ split, and therefore belong to $\mathcal{R}'$. Thus $\mathcal{R} = \mathcal{P} = \mathrm{Ses}(\mathcal{F})$. The congruence $\equiv$ on $V(\mathcal{F})$ is defined by setting, for all $A, B \in \mathcal{F}$, $\langle A \rangle \equiv \langle B \rangle$ if $\langle F(A) \rangle \equiv' \langle F(B) \rangle$, that is, if and only if A and B have the same torsion-free rank and the same p-rank. Here the *p-rank* of a torsion-free abelian group A is the number of direct summands of $A \otimes_{\mathbf{Z}} J_p$ isomorphic to J_p, or, equivalently, the number of p-reduced factor groups of any compositions series of A in $\mathcal{P}$ [17, p. 730]. Thus $V(\mathcal{F})/\equiv_{\mathcal{P}} \; = V(\mathcal{F})/\equiv_{\mathcal{R}} \; \cong V(\mathcal{F}_p)$ is the free commutative monoid freely generated by two elements. ■

Example 10 *Noetherian modules.* Let R be an arbitrarily fixed unital ring, $\mathcal{C}$ be the class of all noetherian right R-modules, $\mathcal{R} = \mathrm{Ses}(\mathcal{C})$ be the class of all short exact sequences $0 \to A \to B \to C \to 0$ with $A, B, C \in \mathcal{C}$, and $\equiv$ be the identity on $V(\mathcal{C})$. In this case, the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$ was studied by Brookfield in [1], and, for R right noetherian, in [2]. A commutative monoid M is said to be *strongly separative* if $a + a = a + b$ implies $a = b$ for all $a, b \in M$. This property is weaker than the cancellation property. For an arbitrary ring R, Brookfield proved that $V(\mathcal{C})/\equiv_{\mathcal{R}}$ is strongly separative [1, Theorem 5.1]. Moreover, if $\approx$ is the smallest congruence on $V(\mathcal{C})$ with $V(\mathcal{C})/\approx$ cancellative, so that $\approx$ is defined by $\langle A \rangle \approx \langle B \rangle$ if $\langle A \rangle + \langle C \rangle = \langle B \rangle + \langle C \rangle$ for some $\langle C \rangle \in V(\mathcal{C})$, then $\approx$ is smaller than $\equiv_{\mathcal{R}}$. Thus the canonical projection $V(\mathcal{C}) \to V(\mathcal{C})/\equiv_{\mathcal{R}}$ induces a homomorphism of the cancellative monoid $V(\mathcal{C})/\approx$ onto the monoid $V(\mathcal{C})/\equiv_{\mathcal{R}}$. Notice that, in this case, $\mathcal{R}$ is trivially a strongly transitive class and that the hypotheses of Theorem 5.4 hold. The modules $A \in \mathcal{C}$ for which the lattice $\mathcal{L}_{\mathcal{R}}(A)$ has finite length are the modules of finite composition length. ■

Example 11 *Torsion-free modules in a hereditary torsion theory.* Let $\mathfrak{F}$ be a right Gabriel topology on a ring R [18, p. 146]. For a right R-module A, $\mathbf{Sat}_{\mathfrak{F}}(A)$ will denote the set of all *$\mathfrak{F}$-saturated submodules* of A, that is, the set of all submodules B of A with A/B $\mathfrak{F}$-torsion-free. Let Mod-$(R, \mathfrak{F})$ be the full subcategory of Mod-R whose objects are all $\mathfrak{F}$-closed modules. The partially ordered set $\mathbf{Sat}_{\mathfrak{F}}(A)$ is a complete modular lattice, isomorphic to the lattice of all subobjects of $A_{\mathfrak{F}}$ in the abelian category Mod-$(R, \mathfrak{F})$ [18, Theorem IX.4.1 and Corollary IX.4.4]. Let $\mathcal{C}$ be the class of all $\mathfrak{F}$-torsion-free right R-modules A for which $\mathbf{Sat}_{\mathfrak{F}}(A)$ has finite length.

Lemma 6.5 *The injective envelope $E(A)$ of any module $A \in \mathcal{C}$ is an R-module of finite Goldie dimension. In particular, the class $\mathcal{C}$ is small.*

Proof. Let $A \in \mathcal{C}$ be an R-module with $E(A)$ of infinite Goldie dimension. Then A has a family of non-zero R-submodules B_i, $i \geq 0$, such that $A \supseteq \oplus_{i=0}^{\infty} B_i$. It follows that $E(A)$ has an ascending chain of direct summands $E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$ with $E_n = E(\oplus_{i=0}^{n} B_i)$ and $E_n/E_{n-1} \cong E(B_n)$. Consider the ascending chain of submodules

$$A \cap E_0 \subseteq A \cap E_1 \subseteq A \cap E_2 \subseteq \cdots \tag{1}$$

of A. The inclusion $A \to E(A)$ induces an embedding $A/A \cap E_n \to E(A)/E_n$, and $E(A)/E_n$ is isomorphic to a direct summand of $E(A)$. Since the class of $\mathfrak{F}$-torsion-free R-modules is closed for injective envelopes and submodules, is follows that $A/A \cap E_n$ is $\mathfrak{F}$-torsion-free. Thus (1) is an ascending chain in $\mathbf{Sat}_{\mathfrak{F}}(A)$. Moreover, $A \cap E_n \supseteq B_n$ and $A \cap E_{n-1} \cap B_n = 0$, which shows that (1) is a strictly ascending chain, a contradiction to the fact that $\mathbf{Sat}_{\mathfrak{F}}(A)$ has finite length.

In particular $\mathcal{C}$ is small, because every module in $\mathcal{C}$ is isomorphic to a submodule of the injective envelope of a direct sum of finitely many cyclic R-modules. ∎

Since the functor Mod-$R \to$ Mod-$(R, \mathfrak{F})$, $A \mapsto A_{\mathfrak{F}}$, is additive, it is clear that $\mathcal{C}$ is closed for finite direct sums and isomorphism. Let $\mathcal{F}$ be the full subcategory of Mod-$(R, \mathfrak{F})$ whose objects are all objects of finite length of Mod-$(R, \mathfrak{F})$ and $F \colon \mathcal{C} \to \mathcal{F}$ be the functor localization defined by $F(A) = A_{\mathfrak{F}}$ for every $A \in \mathcal{C}$, so that F satisfies conditions (a) and (b) of Remark 6.3. Let $\mathcal{R}'$ be Ses$(\mathcal{F})$ and $\equiv'$ be the equality on $V(\mathcal{F})$. Then $\mathcal{R} =$ Ses$(\mathcal{C})$, as the following proposition shows, and $\equiv$ is the congruence on $V(\mathcal{C})$ defined, for all $A, B \in \mathcal{C}$, by $\langle A \rangle \equiv \langle B \rangle$ if $A_{\mathfrak{F}} \cong B_{\mathfrak{F}}$.

Proposition 6.6 *If $A, B, C \in \mathcal{C}$ and $0 \to A \to B \xrightarrow{\varphi} C \to 0$ is an exact sequence of R-modules, then the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} C_{\mathfrak{F}} \to 0$ is exact in the category* Mod-$(R, \mathfrak{F})$.

Proof. Let $0 \to A \to B \xrightarrow{\varphi} C \to 0$ be an exact sequence of R-modules with $A, B, C \in \mathcal{C}$. As the functor localization F is left exact, the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} C_{\mathfrak{F}}$ is an exact sequence of $R_{\mathfrak{F}}$-modules. Thus we only have to prove that the morphism $\varphi_{\mathfrak{F}} \colon B_{\mathfrak{F}} \to C_{\mathfrak{F}}$ is an epimorphism in the category Mod-$(R,\mathfrak{F})$. To this end, it suffices to show that if $\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ denotes the image of the mapping $\varphi_{\mathfrak{F}}$ (that is, the image of $\varphi_{\mathfrak{F}}$ in the category Mod-$R_{\mathfrak{F}}$), then $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is an $\mathfrak{F}$-torsion R-module. Let D be the R-submodule of $C_{\mathfrak{F}}$ such that $\varphi_{\mathfrak{F}}(B_{\mathfrak{F}}) \subseteq D$ and $D/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is the $\mathfrak{F}$-torsion submodule of $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$. Then the sequence $0 \to A_{\mathfrak{F}} \to B_{\mathfrak{F}} \xrightarrow{\varphi_{\mathfrak{F}}} D \to 0$ is exact in Mod-$(R,\mathfrak{F})$. Thus if $L(X)$ denotes the length of an object $X \in \mathcal{F}$, then $L(D) = L(B_{\mathfrak{F}}) - L(A_{\mathfrak{F}})$. Since the lattice $\mathbf{Sat}_{\mathfrak{F}}(Y)$ and the lattice of all subobjects of $Y_{\mathfrak{F}}$ in the abelian category Mod-$(R,\mathfrak{F})$ are isomorphic, it follows that $L(D) = \ell(B) - \ell(A)$, where we have denoted by $\ell(Y)$ the length of the lattice $\mathbf{Sat}_{\mathfrak{F}}(Y)$ for an arbitrary $Y \in \mathcal{F}$. These are modular lattices, $\mathbf{Sat}_{\mathfrak{F}}(A)$ is isomorphic to the interval $[0, A]$ of the lattice $\mathbf{Sat}_{\mathfrak{F}}(B)$, and $\mathbf{Sat}_{\mathfrak{F}}(C)$ is isomorphic to the interval $[A, B]$ of the lattice $\mathbf{Sat}_{\mathfrak{F}}(B)$. It follows that $L(D) = \ell(C) = L(C_{\mathfrak{F}})$. Thus $C_{\mathfrak{F}}$ and its subobject D have the same length in the category Mod-$(R,\mathfrak{F})$, so that $D = C_{\mathfrak{F}}$ and $C_{\mathfrak{F}}/\varphi_{\mathfrak{F}}(B_{\mathfrak{F}})$ is $\mathfrak{F}$-torsion. ■

In particular, $V(\mathcal{C})/\sim_{\mathcal{R}}$ is a free commutative monoid. ■

Example 12 *Critical composition series.* We conclude with an example that is beyond the theory we have developed so far, but that we think interesting. Let R be an arbitrary ring and $\mathcal{C}$ be the full subcategory of Mod-R whose objects are all noetherian right R-modules. We shall denote by K.dim(A) the Krull dimension of a module A. For an ordinal $\alpha \geq 0$, recall that a module A is *α-critical* if K.dim$(A) = \alpha$ and $K(A/B) < \alpha$ for all non-zero submodules B of A. A module is *critical* if it is α-critical for some ordinal α. A *critical composition series* of a noetherian module A is a chain $A = A_0 \geq A_1 \geq \ldots \geq A_n = 0$ of submodules of A such that each of the factors A_{i-1}/A_i is critical and such that K.dim$(A_{i-1}/A_i) \geq$ K.dim(A_i/A_{i+1}) for all $i = 1, 2, \ldots, n$ [10, p. 229].

Let $\mathcal{R}$ be the class of all exact sequences $0 \to A \to B \to C \to 0$ in Ses$(\mathcal{C})$ with either (1) K.dim$(A) \leq$ K.dim(C) and C critical, or (2) $A = 0$, or (3) $C = 0$. Notice that this class $\mathcal{R}$ does not contain all split exact sequences.

Lemma 6.7 (a) *Let $A, B \in \mathcal{C}$ with $0 < A < B$. Then $A \leq_{\mathcal{R}} B$ if and only if* K.dim$(A) \leq$ K.dim(B/A) *and B/A is critical, if and only if* K.dim$(B) =$ K.dim(B/A) *and B/A is critical.*

(b) *A module $A \in \mathcal{C}$ is $\mathcal{R}$-simple if and only if it is critical.*

(c) *A chain $A = A_0 \geq A_1 \geq \ldots \geq A_n = 0$ of submodules of a module $A \in \mathcal{C}$ is a critical composition series of A if and only if it is a composition series of A in $\mathcal{R}$.*

Proof. (a) is clear, because $\mathrm{K.dim}(B) = \max\{\mathrm{K.dim}(A), \mathrm{K.dim}(C)\}$ for every exact sequence $0 \to A \to B \to C \to 0$.

(b) Let $A \in \mathcal{C}$ be an $\mathcal{R}$-simple module and let α be its Krull dimension. By [10, Exercise 13G], A has a proper submodule B with A/B α-critical. Thus $0 \to B \to A \to A/B \to 0$ belongs to $\mathcal{R}$. As A is $\mathcal{R}$-simple, it follows that $B = 0$, so that A is critical. Conversely, if A is critical, then $\mathrm{K.dim}(A/B) < \mathrm{K.dim}(A)$ for every non-zero submodule B of A, and thus the exact sequence $0 \to B \to A \to A/B \to 0$ does not belong to $\mathcal{R}$. This shows that A is $\mathcal{R}$-simple.

(c) Let $A = A_0 \geq A_1 \geq \ldots \geq A_n = 0$ be a critical composition series of a module $A \in \mathcal{C}$. First of all, we shall show that this is a descending chain in $\mathcal{R}$, that is, that $A_i \leq_{\mathcal{R}} A_{i-1}$ for all i. Induction on $n-i$. The case $i = n$ is trivial. Assume that $A_{i+1} \leq_{\mathcal{R}} A_i$. Then $\mathrm{K.dim}(A_i) = \mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$, so that $A_i \leq_{\mathcal{R}} A_{i-1}$. This proves that the chain is a descending chain in $\mathcal{R}$.

In order to prove that it is a composition series in $\mathcal{R}$, suppose that $C \in \mathcal{C}$, $A_i < C < A_{i-1}$ and $A_i \leq_{\mathcal{R}} C$. As C/A_i is a non-zero submodule of the critical module A_{i-1}/A_i, we have that $\mathrm{K.dim}(A_{i-1}/C) < \mathrm{K.dim}(A_{i-1}/A_i)$, from which $\mathrm{K.dim}(A_{i-1}/C) < \mathrm{K.dim}(C/A_i)$. Now $A_i \leq_{\mathcal{R}} C$ yields $\mathrm{K.dim}(C) = \mathrm{K.dim}(C/A_i)$, so that $\mathrm{K.dim}(C) > \mathrm{K.dim}(A_{i-1}/C)$. Thus $C \not\leq_{\mathcal{R}} A_{i-1}$.

Conversely, if $A = A_0 > A_1 > \ldots > A_n = 0$ is a composition series of A in $\mathcal{R}$, then $0 \to A_i \to A_{i-1} \to A_{i-1}/A_i \to 0$ belongs to $\mathcal{R}$ for every $i = 1, \ldots, n$, so that A_{i-1}/A_i is critical for every $i = 1, \ldots, n-1$. Moreover, the term A_{n-1} of the composition series is always $\mathcal{R}$-simple, hence critical by (b). To conclude, we must prove that $\mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$ for every $i = 1, \ldots, n-1$. Now $A_i \neq 0$, so that from $A_i \leq_{\mathcal{R}} A_{i-1}$, it follows that $\mathrm{K.dim}(A_{i-1}) = \mathrm{K.dim}(A_{i-1}/A_i)$. Similarly, $A_{i+1} \leq_{\mathcal{R}} A_i$ implies $\mathrm{K.dim}(A_i) = \mathrm{K.dim}(A_i/A_{i+1})$ if $A_{i+1} \neq 0$, but the same equality holds trivially in the case $A_{i+1} = 0$ also. Now $A_i \leq A_{i-1}$ implies that $\mathrm{K.dim}(A_i) \leq \mathrm{K.dim}(A_{i-1})$. It follows that $\mathrm{K.dim}(A_i/A_{i+1}) \leq \mathrm{K.dim}(A_{i-1}/A_i)$. ■

Let $\equiv$ be the congruence on $V(\mathcal{C})$ generated by all the pairs $(\langle A \rangle, \langle B \rangle)$ with $A, B \in \mathcal{C}$ critical and with isomorphic injective envelopes $E(A) \cong E(B)$. Every module $A \in \mathcal{C}$ has a critical composition series and any two critical composition series of A are equivalent modulo $\equiv$ (Jategaonkar, Gordon [10, Theorem 13.9]). Let F be the free commutative monoid freely generated by the set of the isomorphism classes $\langle E(A) \rangle$ of the injective envelopes of all critical modules $A \in \mathcal{C}$. There is a monoid homomorphism $\varphi \colon V(\mathcal{C}) \to F$ defined as follows. For every $A \in \mathcal{C}$, there is, as we have already said, a critical composition series $A = A_0 > A_1 > \ldots > A_n = 0$, unique up to the congruence $\equiv$. Set $\varphi(\langle A \rangle) = \sum_{i=1}^{n} \langle E(A_{i-1}/A_i) \rangle$. The uniqueness up to equivalence of the critical composition series says that this mapping is well defined. Let us prove that the congruence $\equiv_{\mathcal{R}}$ on $V(\mathcal{C})$ generated by $\equiv$ and $\sim_{\mathcal{R}}$ is the kernel $\ker\varphi$ of the homomorphism φ. If $A, B \in \mathcal{C}$ and $(\langle A\rangle, \langle B\rangle) \in \ker\varphi$, then A and B have critical composition series equivalent modulo $\equiv$, i.e., they have composition series in $\mathcal{R}$ equivalent modulo $\equiv$. It follows that $\langle A\rangle \equiv_{\mathcal{R}} \langle B\rangle$. Conversely, in order to prove that $\equiv_{\mathcal{R}}$ is contained in the kernel, it suffices to show that $\equiv$ and $\sim_{\mathcal{R}}$ are both contained in the kernel, and both these facts are easily verified. Thus $V(\mathcal{C})/\equiv_{\mathcal{R}} \cong F$ is a free commutative monoid. ∎

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