Uniform behavior of families of Galois representations on Siegel modular forms

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Abstract

We prove the following uniformity principle: if one of the Galois representations in the family attached to a genus two Siegel cusp form of level 1, weight $k > 3$ and with multiplicity one is reducible (for a prime $p > 4k-5$) then almost all the representations in the family are reducible. The result will be proved more generally for compatible families of geometric, pure and symplectic four-dimensional Galois representations which are "semistable".

1 Introduction

In this article, we will consider a genus two Siegel modular form f of level 1 and weight $k > 3$ (and multiplicity one) and the family of four dimensional symplectic Galois representations attached to it. In [D1], we have given conditions on f to ensure that these Galois representations have generically large image. In particular we have imposed an irreducibility condition on one characteristic polynomial of Frobenius (see [D1], condition (4.8)) to obtain that result. Furthermore, with the same irreducibility condition, we showed in [D2] that for every $p > 4k - 5$ the p-adic representations are irreducible. The only possible reducible case to be considered is the case of two 2-dimensional irreducible components having the same determinant (all other cases can not occur if f is not of Saito-Kurokawa type, cf. [D1],

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[D2]), and from the results of [D2] this case can only happen, for $p > 4k-5$, if all characteristic polynomials are reducible, i.e., the 2-dimensional components will have coefficients in the same field that the 4-dimensional representations: the field E generated by the eigenvalues of f.
One of the consequences of Tate's conjecture on the Siegel threefold is that such a reducibility for the Galois representations attached to f must be a uniform property: if it is verified at one prime, then all the representations in the family are reducible. We will prove the following version of this uniformity principle:

Theorem 1.1 *Let f be a genus 2 level 1 Siegel cuspidal Hecke eigenform of weight $k > 3$, having multiplicity one. Suppose that for some prime $\ell_0 > 4k-5$, $\lambda_0 \mid \ell_0$, the representation ρ_{f,λ_0} is reducible. Then the representations $\rho_{f,\lambda}$ are reducible for almost every λ.*

In fact, it is not necessary to restrict the result to the case of conductor 1 (level 1): we will prove the result more generally for compatible families of geometric, pure and symplectic four-dimensional Galois representations which are "semistable".
With this aim, before starting the proof, in section 2 we will generalize the main results of [D1] and [D2] to the semistable case.
We will use (as in [D2], section 4) as starting point Taylor's recent results on the Fontaine-Mazur conjecture and the meromorphic continuation of L-functions for odd two-dimensional Galois representations (see [T2], [T3] and [T4]). Then, we will combine some of the results and techniques in [D1] (in particular the information about the description of the action of inertia obtained via p-adic Hodge theory) with Ribet's results (see [R]) on two-dimensional semistable Galois representations (slightly generalized to higher weights), and finally Cebotarev density theorem, the fundamental theorem of Galois theory, and some group theory will suffice for the proof.

2 Preliminaries

From now on we will do the following assumption: f is a genus 2 level 1 Siegel cuspidal Hecke eigenform of weight $k > 3$, having multiplicity one, and not of Saito-Kurokawa type (theorem 1.1 is trivial in the Saito-Kurokawa case). Let $E = \mathbb{Q}(\{a_n\})$ be the field generated by its Hecke eigenvalues. Then, there is a compatible family of Galois representations constructed by Taylor [T1] and Weissauer [W2] verifying the following: For any prime number ℓ

and any extension λ of ℓ to E we have a continuous Galois representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \rightarrow \mathrm{GSp}(4, \overline{E}_\lambda)$$

unramified outside ℓ and with characteristic polynomial of $\rho_{f,\lambda}(\text{Frob } p))$ equal to

$$Pol_p(x) = x^4 - a_p x^3 + (a_p^2 - a_{p^2} - p^{2k-4})x^2 - a_p p^{2k-3} x + p^{4k-6}$$

for every $p \neq \ell$. If $\rho_{f,\lambda}$ is absolutely irreducible, then it is defined over E_λ.

In general, we can not guarantee that the field of definition is E_λ, but the residual representation $\bar{\rho}_{f,\lambda}$ can be formally defined in any case (see [D1]) as a representation defined over the residue field of λ, $\mathbb{F}_\lambda$. Nevertheless, not knowing the field of definition of the representations that we will study is not a serious problem, we can work instead with the "field of coefficients" (@), i.e., the field generated by the coefficients of the characteristic polynomials $Pol_p(x)$, this field contains all the information we need.

The representations $\rho_{f,\lambda}$ are known to have the following properties (cf [W1], [W2], [D1]): they are pure (Ramanujan conjecture is satisfied), they have conductor 1 and they are crystalline with Hodge-Tate weights $\{0, k-2, k-1, 2k-3\}$. This last property makes possible, via Fontaine-Laffaille theory, to obtain a precise description of the action of the inertia group at ℓ on the residual representation $\overline{\rho}_{f,\lambda}$: it acts through fundamental characters of level one or two, with exponents equal to the Hodge-Tate weights (see [D1] for more details).

Let us change to the more general setting of a family of four-dimensional symplectic Galois representations $\{\rho_\lambda\}$ with coefficients in a number field E (not necessarily defined over E_λ, see (@)), $\det \rho_\lambda = \chi^{4k-6}$, which are pure, and such that there exists a finite set S with, for every $\ell \notin S$, ρ_λ unramified outside $\{\ell\} \cup S$, crystalline at ℓ with Hodge-Tate weights as above, and "semistable" at primes in S, i.e., verifying the following: ρ_λ restricted to I_q is a unipotent group for every $q \in S$.
For every $p \notin S$ we still denote $Pol_p(x)$ the characteristic polynomial of the image of Frob p and a_p the trace of this image. The representations being symplectic, we have the standard factorization

$$Pol_p(x) = (x^2 - (a_p/2 + \sqrt{d_p})x + p^{2k-3})(x^2 - (a_p/2 - \sqrt{d_p})x + p^{2k-3}). \quad (2.1)$$

The results of generically large image and irreducibility proved in previous articles (see [D1], theorem 4.2, and [D2], theorems 2.1 and 4.1) hold also in this generality:

Theorem 2.1 *Let $\{\rho_\lambda\}$ be a family of Galois representations verifying the above properties, with $k > 3$. Assume that there is a prime $p \notin S$ such that*

$$\sqrt{d_p} \notin E \tag{2.2}$$

where d_p is defined by formula (2.1). Then for all but finitely many primes verifying

$$d_p \notin (\mathbb{F}_\lambda)^2$$

and, more generally, for all primes λ in E except at most for a set of Dirichlet density 0, the image of ρ_λ is

$$A_\lambda^k = \{g \in \mathrm{GSp}(4, \mathcal{O}_{E_\lambda}) : \det(g) \in (\mathbb{Q}_\ell^*)^{4k-6}\},$$

where $\mathcal{O}_{E_\lambda}$ denotes the ring of integers of E_λ.
Keeping condition (2.2) we also have: for every prime $\ell \geq 4k-5, \ell \notin S$, $\lambda \mid \ell$, the representation ρ_λ is absolutely irreducible.

Differences with the conductor 1 case:
The proof of the above results given in [D1] and [D2] extends automatically to the semistable case: recall that the determination of the images is done by considering the image of the residual mod λ representations and eliminating all non-maximal proper subgroups of $\mathrm{GSp}(4, \mathbb{F}_\lambda)$. When considering reducible cases (cf. [D1], sections 4.1 and 4.2) if we allow arbitrary ramification at a finite set S then we have to allow the character appearing as one-dimensional component or determinant of a two-dimensional component of a reducible $\bar{\rho}_\lambda$ to ramify at S, but in the semistable case it is easy to see that this character will not ramify at primes in S. The same applies to the case of image equal to a group G having a reducible index 2 normal subgroup M (cf. [D1], section 4.4), the quadratic Galois character G/M can not ramify at primes of S if we assume semistability. Up to these easy remarks, all the proof translates word by word to the semistable case.

Remark 1: Recall that condition (2.2) was introduced (cf. [D1]) specifically to deal with the case where the image of $\bar{\rho}_\lambda$ is reducible, with two 2-dimensional irreducible components of the same determinant. All other cases of non-maximal image can be discarded, for almost every prime, without using condition (2.2).

Remark 2: In [D1], the large images result was proved (for the case of conductor 1) with an additional condition, called "untwisted": this condition was imposed to eliminate the possibility that the projective residual image falls in a smaller symplectic group $\mathrm{PGSp}(4, k')$, k' a proper subfield of k, where k is the field generated by the traces of the residual representation.

We have not included a similar condition in the above theorem because in the following lemma, we will explain that this condition is superfluous, i.e., that the case of smaller projective symplectic group can never happen if we assume semistability. In particular, this applies to level 1 Siegel cusp forms, so the condition "untwisted" can be removed from theorem 4.2 of [D1].

Lemma 2.2 *Let $\{\rho_\lambda\}$ be a compatible families of Galois representations as above (in particular, a semistable family). Then for every prime $q > 2k-2, q \notin S$ and Q a prime in E dividing q, if we call G the image of $\bar{\rho}_Q$ and $P(G)$ its projectivization, $P(G)$ lies in $\mathrm{PGSp}(4,k)$ if and only if G lies in $\mathrm{GSp}(4,k)$, for every subfield k of $\mathbb{F}_Q$.*

Proof: A similar result, for semistable two-dimensional representations, is lemma 2.4 in [R]. The proof given there translates word by word, once we have explained why in our case we also have an element c in the inertia group I_q such that $\chi(c)$ is a generator of $\mathbb{F}_q^*$ and the trace of $\bar{\rho}_Q(c)$ is a non-zero element of $\mathbb{F}_q$ (we know a priori, from the description of the action of I_q, that this trace will be in $\mathbb{F}_q$, what requires a proof is the fact that it is not 0).
We have given in [D1], proposition 3.1, a description of the action of I_q that applies in the current situation, because we are assuming that ρ_Q is symplectic and crystalline with Hodge-Tate weights $\{0, k-2, k-1, 2k-3\}$, and $q > 2k-2$. Let ψ be a level 2 fundamental character, and take $c \in I_q$ such that $\psi(c)$ generates $\mathbb{F}_{q^2}^*$. We have four possibilities for the trace of c, whose values are, after a suitable factorization:

$$(1+\chi(c)^{k-1})(1+\chi(c)^{k-2})$$

$$(\psi(c)^{k-2}+\psi(c)^{(k-2)q})(\psi(c)^{k-1}+\psi(c)^{(k-1)q})$$

$$(1+\psi(c)^{(k-2)+(k-1)q})(1+\psi(c)^{(k-1)+(k-2)q})$$

$$(\psi(c)^{k-2}+\psi(c)^{(k-1)q})(\psi(c)^{k-1}+\psi(c)^{(k-2)q}).$$

In all cases, the inequality $q > 2k-2$ implies that these traces are not 0.

3 Uniformity of reducibility

At this point, we can say that the validity or not of condition (2.2) at some prime $p \notin S$ determines the behavior of the family of representations ρ_λ: If condition (2.2) is satisfied, then we have generically large image and irreducibility for every ℓ sufficiently large compared with the weights.
What happens if condition (2.2) is not satisfied at any prime? This implies

that the factorization (2.1) takes place over E, i.e., that for every $p \notin S$, $Pol_p(x)$ reduces over E. The coefficients of all characteristic polynomials $Pol_p(x)$ generate an order $\mathcal{O}$ of E, and if we restrict to primes λ not dividing the conductor of this order (we are neglecting only finitely many primes), we see that the field generated by the coefficients of the mod λ reduction of all the $Pol_p(x)$ gives the whole $\mathbb{F}_\lambda$. Thus, we see that for almost every prime, the failure of (2.2) implies that $\bar{\rho}_\lambda$ has its image in $\mathrm{GSp}(4, \mathbb{F}_\lambda)$ and not in a smaller symplectic, but all characteristic polynomials reduce over $\mathbb{F}_\lambda$: clearly in this case the image can not be the whole symplectic group, and we know that for almost every prime only one possibility (see remark 1 after theorem 2.1 and lemma 2.2) remains:

Lemma 3.1 *Let $\{\rho_\lambda\}$ be as in the previous section, and assume that for every $p \notin S$, condition (2.2) is not satisfied. Then, for almost every prime λ, the residual representation $\bar{\rho}_\lambda$ is reducible with two 2-dimensional irreducible components of the same determinant.*

3.1 A reducible member in the family: Residual consequences

From now on, assume that for a prime $q > 4k-5, q \notin S$, $Q \mid q$, the Q-adic representation ρ_Q is reducible, we know (using semistability and purity) that the only possible case is the case of two 2-dimensional irreducible components both with determinant χ^{2k-3}. Thus we have:

$$\rho_Q \cong \sigma_{1,Q} \oplus \sigma_{2,Q}. \tag{3.1}$$

Furthermore, we know from theorem 2.1 that condition (2.2) must fail at every prime, and therefore, that $\sigma_{1,Q}$ and $\sigma_{2,Q}$ will also have coefficients in E and from the lemma above, that for every prime λ in a cofinite set Λ of primes of E, $\bar{\rho}_\lambda$ will verify:

$$\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}$$

where $\pi_{i,\lambda}$ is an irreducible two dimensional representation defined over $\mathbb{F}_\lambda$ having determinant χ^{2k-3}, for $i = 1, 2$ and for every $\lambda \in \Lambda$.
Moreover, we can determine the image of $\bar{\rho}_\lambda$ for almost every prime in Λ:

Lemma 3.2 *Keep the above assumptions. For every prime $\lambda \in \Lambda_2$, a cofinite subset of Λ, the image of $\bar{\rho}_\lambda$ is a subgroup of $\mathrm{GSp}(4, \mathbb{F}_\lambda)$ conjugated to $M_\lambda = \{A \times B \in \mathrm{GL}(2, \mathbb{F}_{1,\lambda}) \times \mathrm{GL}(2, \mathbb{F}_{2,\lambda}) : \det(A) = \det(B) \in \mathbb{F}_\ell^{2k-3}\}$, where $\mathbb{F}_{1,\lambda}, \mathbb{F}_{2,\lambda} \subseteq \mathbb{F}_\lambda$ are the fields of coefficients of $\pi_{1,\lambda}$ and $\pi_{2,\lambda}$.*

Proof: We have assumed that the representations ρ_λ have a finite ramification set S and they are semistable at every prime $q \in S$. A fortiori, the same applies to their residual components $\pi_{i,\lambda}$. Moreover, these two dimensional representations are irreducible for every $\lambda \in \Lambda$. In a similar situation, Ribet has proved a large image result for $\ell \geq 5$, but he assumes that the action of I_ℓ, given by fundamental characters of level 1 or 2, has weights (i.e., exponents of the fundamental characters) 0 and 1. The main point of his proof is to exclude the dihedral case. In our case, using the information on the Hodge-Tate decomposition, we have this extra condition at I_ℓ also verified by the twisted representation $\pi_{i,\lambda} \otimes \chi^{-k+2}$ for, say, $i = 2$ (cf. [D1],[D2]). On the other hand, for $\pi_{1,\lambda}$ Ribet's result still holds if we restrict to primes $\ell > 4k - 5$, because the weights of the action of I_ℓ being 0 and $2k - 3$, the projectivization of the image if I_ℓ gives a cyclic group of order $(\ell \pm 1)/\gcd(\ell \pm 1, 2k - 3) > 2$, and this is all that you need to follow Ribet's argument. We also have a statement as lemma 2.2 for these two dimensional representations, again adapting lemma 2.4 in [R].
We conclude (cf. [R], theorem 2.5 and the remark after) that for ℓ sufficiently large, the images of both irreducible components are conjugated to the subgroup of matrices in $\mathrm{GL}(2, \mathbb{F}_{i,\lambda})$ with determinant in $\mathbb{F}_\ell^{2k-3}$.
Finally, to prove that the image of $\bar{\rho}_\lambda$ is as we want, it remains to show that the Galois fields corresponding to $P(\pi_{1,\lambda})$ and $P(\pi_{2,\lambda})$ are disjoint (P denotes projectivization). These fields having Galois groups isomorphic to the simple groups $\mathrm{PGL}(2, \mathbb{F}_{i,\lambda})$ or $\mathrm{PSL}(2, \mathbb{F}_{i,\lambda})$, they are either disjoint or equal: the second is not possible because the restriction of these two projective representations to I_ℓ are different, and this proves the result.

3.2 A reducible member in the family: λ-adic consequences

In the decomposition (3.1) of ρ_Q it is clear that $\sigma_{1,Q}$ has Hodge-Tate weights $\{0, 2k - 3\}$ and $\sigma_{2,Q}$ has Hodge-Tate weights $\{k - 2, k - 1\}$ (or viceversa). Now, we invoke a result of Taylor (see [T2] and [T3], recall that $q > 4k - 5$) asserting that for a representation such as $\sigma_{1,Q}$ it is possible to find a totally real number field F such that it is modular when restricted to this field, and therefore it agrees on F with the Q-adic motivic irreducible Galois representation (constructed by Blasius and Rogawski) attached to a Hilbert modular form h. This implies that $\sigma_{1,Q}$ appears in the cohomology of the restriction of scalars of the motive M_h associated to h, and it can be checked from the fact that the Q-adic representation of the absolute Galois group of F attached to h has descended to a 2-dimensional representation

of $G_{\mathbb{Q}}$, Cebotarev density theorem, and the fact that all modular Galois representations in the family $\{\sigma_{h,\lambda}\}$ attached to h are known to be irreducible, that the whole family descends to a compatible family $\{\sigma_{1,\lambda}\}$ of Galois representations of $G_{\mathbb{Q}}$ containing $\sigma_{1,Q}$. To do this, one has to write the representation $\sigma_{1,Q}$ as in the proof of theorem 6.6 in [T3], and define the representations $\sigma_{1,\lambda}$ formally in the same way using the strongly compatible families associated to the base change of h to each E_i (recall that, for each i, F/E_i is soluble, cf. [T3]). Then, following an idea suggested to us by R. Taylor, one can check that the virtual representations $\sigma_{1,\lambda}$ constructed this way are true Galois representations by applying the arguments of [T4], section 533.
It follows from the main result of [T3] that the family $\{\sigma_{1,\lambda}\}$ is a strongly compatible family (cf. [T3] for the definition) of Galois representations. Strong compatibility proves the last steps of the following:

Proposition 3.3 *Let ρ_Q be as above, reducible as in (3.1), and let $\sigma_{1,Q}$ be its irreducible component having Hodge-Tate weights $\{0, 2k-3\}$. Then, there exists a compatible family of Galois representations $\{\sigma_{1,\lambda}\}$ containing $\sigma_{1,Q}$, such that for every $\ell \notin S$, $\lambda \mid \ell$, the representation $\sigma_{1,\lambda}$ is unramified outside $\{\ell\} \cup S$, is crystalline at ℓ with Hodge-Tate weights $\{0, 2k-3\}$, and is semistable at every prime of S. Of course, these representations are pure because ρ_Q is.*

Recall that, the representation ρ_λ being symplectic, for every $g \in G_{\mathbb{Q}}$ the roots of $\rho_\lambda(g)$ come in reciprocal pairs: $\{\alpha, \chi^{2k-3}(g)/\alpha, \beta, \chi^{2k-3}(g)/\beta\}$. The following lemma is a first approach to compare the representations $\sigma_{1,\lambda}$ and ρ_λ:

Lemma 3.4 *For every $\ell \notin S$, $\lambda \mid \ell$, and every $g \in G_{\mathbb{Q}}$, the roots of $\sigma_{1,\lambda}(g)$ form a pair of reciprocal roots of those of $\rho_\lambda(g)$.*

Proof: From the compatibility of the families $\{\sigma_{1,\lambda}\}$ and $\{\rho_\lambda\}$ and the fact that $\sigma_{1,Q}$ is a component of ρ_Q the lemma is obvious for the dense set of Frobenius elements at unramified places. Then, by continuity and Cebotarev the lemma follows for every element of $G_{\mathbb{Q}}$.

Recall that Λ_2 denotes the cofinite set of primes of E where lemma 3.2 is satisfied. We will shrink again this set by eliminating a finite set of primes, namely, those primes where the image of $\sigma_{1,\lambda}$ fails to be maximal: in fact, if we call $E' \subseteq E$ the field of coefficients of this family of representations and $\mathcal{O}'$ its ring of integers, using semistability and again the slight modification of the methods of [R] to higher weights (as we did before to obtain lemma 3.2) we see that for almost every prime $\lambda \in E$ the image of $\sigma_{1,\lambda}$ can be conjugated to the subgroup of $\mathrm{GL}(2, \mathcal{O}'_\lambda)$ of matrices with determinant in

$\mathbb{Z}_\ell^{2k-3}$ (after proving the similar result for the residual representations, we apply a lemma of Serre in [S1] that shows that the λ-adic image is also large).

Remark: Here we need to know that the residual representations $\bar{\sigma}_{1,\lambda}$ are almost all of them irreducible. This follows again from the good properties of the λ-adic family: purity, the fact that they are all crystalline with Hodge-Tate weights $\{0, 2k-3\}$ (and the uniform description of inertia that one gets from this), and semistability.

Thus, we exclude from Λ_2 the finite set of primes where the image of $\sigma_{1,\lambda}$ fails to be maximal, and we obtain a cofinite set Λ_3 where the residual image of ρ_λ is the full M_λ and the image of $\sigma_{1,\lambda}$ is maximal.

We want to extract more information from the relation derived in lemma 3.4. To start with, we work at the level of residual representations. Observe that the same relation proved in lemma 3.4 holds for the roots of the matrices in the image of the residual representations $\bar{\rho}_\lambda$ and $\bar{\sigma}_{1,\lambda}$:

Lemma 3.5 *Let λ be a prime in Λ_3, then in the decomposition $\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}$ we have $\pi_{1,\lambda} \cong \bar{\sigma}_{1,\lambda}$.*

Remark: Of course, we should write the above equality with $\pi_{i,\lambda}$ for $i = 1$ or 2. But to fix notation, we will always call $\pi_{1,\lambda}$ the component of $\bar{\rho}_\lambda$ where the inertia group at ℓ acts with weights 0 and $2k-3$ (as we did in section 3.1), this is a good way to distinguish the two components, and of course this is the only component that deserves being compared to $\bar{\sigma}_{1,\lambda}$.

Proof: Take $\lambda \in \Lambda_3$. Let L be the Galois field corresponding to $\bar{\rho}_\lambda$, thus $\mathrm{Gal}(L/\mathbb{Q}) \cong M_\lambda$, and B the one corresponding to $\bar{\sigma}_{1,\lambda}$, thus if $\mathbb{F}'_\lambda$ is the residue field of $\mathcal{O}'_\lambda$ and $U_\lambda = \{A \in \mathrm{GL}(2, \mathbb{F}'_\lambda) : \det(A) \in \mathbb{F}_\ell^{2k-3}\}$, $\mathrm{Gal}(B/\mathbb{Q}) \cong U_\lambda$. We want to prove that $B \subseteq L$. Let $M = L \cap B$, and consider an element $z \in \mathrm{Gal}(B/M)$. Let $\check{z}$ be a preimage of z in $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$, that we can choose such that $\bar{\rho}_\lambda(\check{z}) = \mathbf{1}_4$ (because it is trivial on $M = L \cap B$). Then, the residual version of lemma 3.4 implies that 1 is a double root of the characteristic polynomial of $\bar{\sigma}_{1,\lambda}(\check{z})$. This implies that the group $\mathrm{Gal}(B/M)$ is unipotent, but this group is a normal subgroup of $\mathrm{Gal}(B/\mathbb{Q}) \cong U_\lambda$, and U_λ has no non-trivial unipotent normal subgroup, thus $B = M$, i.e., $B \subseteq L$. Then, we have a projection: $\phi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(B/\mathbb{Q})$, that is to say, ϕ sends M_λ onto U_λ, and from this we see that $\mathrm{Gal}(B/\mathbb{Q})$ must correspond to one of the irreducible components of $\mathrm{Gal}(L/\mathbb{Q})$, i.e., that $\bar{\sigma}_{1,\lambda} \cong \pi_{1,\lambda}$.

3.3 Proof of Theorem 1.1

We start by observing that part of the proof of lemma 3.5 can be translated to the λ-adic setting. Take $\lambda \in \Lambda_3$, and call L' the (infinite) Galois field corresponding to ρ_λ and B' the one corresponding to $\sigma_{1,\lambda}$. Recall that $\mathrm{Gal}(B'/\mathbb{Q})$ is isomorphic to the subgroup U'_λ of $\mathrm{GL}(2, \mathcal{O}'_\lambda)$ composed of matrices with determinant in $\mathbb{Z}_\ell^{2k-3}$, and therefore again we have a group with no non-trivial unipotent subgroups, thus we conclude from lemma 3.4 as in the proof of lemma 3.5 that $B' \subseteq L'$ and that we have a projection: $\phi' : \mathrm{Gal}(L'/\mathbb{Q}) \to \mathrm{Gal}(B'/\mathbb{Q})$. We have $\phi' \circ \rho_\lambda = \sigma_{1,\lambda}$. Let us consider the normal subgroup $\mathrm{Gal}(L'/B')$ of $\mathrm{Gal}(L'/\mathbb{Q})$, i.e., we are considering the restriction $\rho_\lambda|_{\ker \phi'}$. The elements in this subgroup fix B', thus by lemma 3.4 we see that the corresponding matrices in $\mathrm{GSp}(4, \mathcal{O}_\lambda)$ will have 1 as a double root.

On the other hand, we know that the residual representation $\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda} \cong \bar{\sigma}_{1,\lambda} \oplus \pi_{2,\lambda}$ has maximal image M_λ (see lemmas 3.2 and 3.5). Moreover, the representation $\sigma_{1,\lambda}$ being a "deformation" of $\pi_{1,\lambda}$ which is disjoint from $\pi_{2,\lambda}$ (in the sense established during the proof of lemma 3.2, i.e., up to the equality of determinants), we see that restricting to $\ker \phi'$ will only shrink the image of $\pi_{2,\lambda}$ by making the determinant trivial, in other words: the residual representation $\overline{\rho_\lambda|_{\ker \phi'}}$ has image

$$\mathrm{SL}(2, \mathbb{F}_{2,\lambda}) \oplus \mathbf{1}_2 \subseteq M_\lambda \tag{3.2}$$

So, what do we know about $\rho_\lambda|_{\ker \phi'}$? We have determined its residual image and we also know that all matrices in its image have 1 as a double root: this last property extends to the Zariski closure of the image, and using the information we have together with the list of possibilities for this Zariski closure given in [T1], we see that the image of $\rho_\lambda|_{\ker \phi'}$ must be contained in $\mathrm{SL}(2, \mathcal{O}_\lambda) \oplus \mathbf{1}_2$. If we call $\mathcal{O}''_\lambda \subseteq \mathcal{O}_\lambda$ the field generated by the traces of the image of $\rho_\lambda|_{\ker \phi'}$, we can apply a lemma of Serre (cf. [S1]) (and Carayol's lemma for the assertion about the field of definition, cf. [C]) and conclude from (3.2) that the image of $\rho_\lambda|_{\ker \phi'}$ must be conjugated to $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$.

Remark: $\ker \phi'$ fixes B' which is an infinite extension of $\mathbb{Q}$, but Serre's lemma can still be applied because $G_\mathbb{Q}$ is compact and the fixer of B' is a closed subgroup.

Thus, we conclude that $\mathrm{Image}(\rho_\lambda) \subseteq \mathrm{GSp}(4, \bar{E}_\lambda)$ contains a normal subgroup isomorphic to $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$, and the quotient by this subgroup gives U'_λ. But it is easy to see that the normalizer of $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$ in $\mathrm{GSp}(4, \bar{E}_\lambda)$ is contained in the reducible group $\mathrm{GL}(2, \bar{E}_\lambda) \oplus \mathrm{GL}(2, \bar{E}_\lambda)$. Thus, ρ_λ is reducible, for every $\lambda \in \Lambda_3$, and $\sigma_{1,\lambda}$ is one of its two-dimensional irreducible components.

4 Final Remarks

The Galois representations attached to a Siegel cusp form f of level greater than one are known to verify the semistability condition when the ramified local components of (the automorphic representation corresponding to) f are of certain particular types (for example, a Steinberg representation), as follows from recent works of Tilouine-Genestier and Genestier. Thus, the results in this article apply to these cases. We thank J. Tilouine for pointing out this fact to us.

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