Factorization of Polynomials With Estimates of Norms

Alexander Volberg

0. Introduction

We want to give here an overview of what is known about the factorization of polynomials with estimates of norms, and about the relations of this problem with other problems of function theory and operator theory. This subject starts with relatively simple problems, but then, very naturally and very fast, hits on some very difficult ones.

We are interested in the following 2 questions.

Q.1 How to prove that a polynomial $Q(z) = Q_0 + \cdots + Q_n z^n$ of (even) degree n can be represented as

$$Q(z) = \Phi(z)^T \Psi(z), \tag{1}$$

where $\Phi(z) = \Phi_0 + \cdots + \Phi_{n_0} z^{n_0},\ \Psi(z) = \Psi_0 + \cdots + \Psi_{n_1} z^{n_1}$, and where $\Phi,\ \Psi$ are polynomials with vector (columns) coefficients

1) n_0 and $n_1 = n/2$, (or, at worst, $\leq An,\ A$ is absolute).

2) $\Phi_i,\ \Psi_j \in \mathbb{C}^M$. We do not care about M, but we give information about it if available.

3) The main thing we care in (1) is

$$\|\Phi\|_2 \cdot \|\Psi\|_2 \leq A \|Q\|_1, \tag{2}$$

where

$$\|\Phi\|_2 = \left(\sum_{i=0}^{n_0} \|\Phi_i\|^2_{\mathbb{C}^M} \right)^{1/2}, \qquad \|\Psi\|_2 = \left(\sum_{j=0}^{n_1} \|\Psi_j\|^2_{\mathbb{C}^M} \right)^{1/2},$$

and $\| \ \|_{\mathbb{C}^M}$ is a usual Hilbert norm.

Notice that we can introduce scalar polynomials

$$\varphi_m = \sum_{i=0}^{n_0} \varphi_{im} z^i, \qquad \psi_m = \sum_{j=0}^{n_1} \psi_{jm} z^j,$$

and then (1), (2) give us weak factorization of $Q(z)$ in the sense

$$\begin{aligned} Q(z) &= \sum_{m=1}^{M} \varphi_m(z)\psi_m(z), \\ \sum_{m=1}^{M} \|\psi_m\|_2 \|\psi_m\|_2 &\leq A\|Q\|_1. \end{aligned} \tag{3}$$

Notice that the previous estimate of norms is stronger than this one.

We can immediately formulate a

Problem I. Is (3) possible with $M = 1$?

Q.2 How to prove that a polynomial ($z = (z_1, z_2)$) $Q(z) = \sum_{k,\ell=0}^{N} Q_{k,\ell} z_1^k z_2^\ell$ with frequencies in $[0, N] \times [0, N]$ with some (even) N can be represented as

$$Q(z) = \Phi(z)^T \Psi(z),$$

where Φ and Ψ are polynomials with frequencies in $\left[0, \frac{N}{2}\right] \times \left[0, \frac{N}{2}\right]$ (or at worst in $[0, AN] \times [0, AN]$, with absolute constant A), columns $\Phi_{k,\ell}, \Psi_{k,\ell}$ are in $\mathbb{C}^M$ and

$$\left(\sum_{k,\ell} \|\Phi_{k,\ell}\|_{\mathbb{C}^M}^2\right)^{1/2} \left(\sum_{k,\ell} \|\Psi_{k,\ell}\|_{\mathbb{C}^M}^2\right)^{1/2} \leq A\|Q\|_{L^1(\mathbb{T}^2)}. \tag{4}$$

Question 2 has a (probably) weaker version as a question of weak factorization of Q: does there exist a family of polynomials $\varphi_m(z)$, $\psi_m(z)$ with frequencies in $\left[0, \frac{N}{2}\right] \times \left[0, \frac{N}{2}\right]$ such that

$$Q(z) = \sum_{m=1}^{M} \varphi_m(z)\psi_m(z), \qquad \sum_{m=1}^{M} \|\varphi_m\|_{L^2(\mathbb{T}^2)} \, \|\psi_m\|_{L^2(\mathbb{T}^2)} \leq A\|Q\|_1\,. \tag{5}$$

Remark. We will see that in one variable case (3) (weak factorization) and (1), (2) (factorization through vector polynomials) constitute the same problem. However, the best constants we can get in (3) and (2) are different and the one in (2) is of course bigger. In two variable cases we do not know whether (4) holds in general, and that (5) holds may be deduced from [6]. We are making an attempt here to get an approach to (4) and (5) simpler than the one in [6]. We do not succeed, but maybe our approach reveals connections with some other problems.

We start with one variable case, where everything will be easy. We indicate what we would need to change in a one variable case to have two variable cases done.

1 One variable case

We will give three proofs: a) operator proof, b) duality proof, c) function theory proof.

1.1 Operator proof

This one is the best, but it will not give exactly what we need ((1), (2)). If it would give (1), (2) we would be able to carry out the same proof to more than one variable. So this "proof" is the most promising even being incomplete.

Let us consider $k_n := z^{n/2}\ K_{\frac{n}{2}+1}$ - a shifted Fejér kernel, with Fourier coefficients zero outside $[0, n]$, and "triangle"-like over $[0, n]$ (n is even here). Also let us consider $v_{\alpha,n}$ - a shifted de la Vallée-Poussin kernel. Fix α in $(0, 1/4)$, consider Fejér kernel K_m with $m = \frac{n}{2} + 1 - [\alpha n]$, then denote $k_{m,n} = z^{n/2} K_m$. Put $c_\alpha := \frac{2}{n+2}[\alpha n]$, and $v_{\alpha,n} := \frac{k_n - k_{m,n}}{c_\alpha} + k_{m,n}$. Notice $\hat{v}_{\alpha,n}$ is supported by $[0, n]$ and is trapezoid-like over $[0, n]$. In particular $\hat{v}_{\alpha,n}$ is identically 1 on interval of integers $([\alpha n], n - [\alpha n])$.

Notice that $\|k_n\|_1 = 1,\ \|v_{\alpha,n}\|_1 \le A(\alpha) \le \dfrac{A}{\alpha}$.

Lemma 1.1.1 (1) Given $Q = Q_0 + \cdots + Q_n z^n$ one can find $\Phi(z) = \Phi_0 + \cdots + \Phi_{n/2} z^{n/2}$, $\Psi(z) = \Psi_0 + \cdots + \Psi_{n/2} z^{n/2}$ such that $\Phi_i, \Psi_j \in \mathbb{C}^{n/2+1}$, and $Q * k_n = \Phi^T \Psi$ and (10) holds.

(2) Also fixing $\alpha = \dfrac{m}{n} \in (0, \dfrac{1}{4})$ one can find Φ, Ψ of the same kind such that $Q * v_{\alpha,n} = \Phi^T \Psi$ and (10) holds with constant depending only on α in a way that it is $\le \dfrac{A}{\alpha}$.

Remark. This lemma presents an operator proof of (1) and (10) factorization. Unfortunately, one has to modify $\widehat{Q}$ by multiplying it on $\hat{k}_n$ or $\hat{v}_{\alpha,n}$. To get rid of this difficulty would mean to have a simple factorization proof in 2 variables.

Proof. Consider the following $\left(\dfrac{n}{2}+1\right) \times \left(\dfrac{n}{2}+1\right)$ matrix.

$$\frac{1}{\frac{n}{2}+1}\begin{pmatrix} Q_0 & Q_1 & Q_2 & \dots & Q_{n/2} \\ Q_1 & Q_2 & \dots & Q_{n/2} & Q_{n/2+1} \\ Q_2 & \dots & Q_{n/2} & Q_{n/2+1} & Q_{n/2+2} \\ & \dots\dots\dots\dots\dots\dots & & & \\ Q_{n/2} & \dots & Qn-2 & Q_{n-1} & Q_n \end{pmatrix}$$

Below S_1 and S_2 denote trace class and Hilbert-Schmidt classes respectively.

Note that its trace norm

$$\|A\|_{S_1} \le B\|Q\|_1. \tag{6}$$

This is Peller's lemma, see [8]. Le us recall the proof. Fix $t \in [0, 2\pi)$ and consider two vectors $x_t = (1, e^{-it}, \dots, e^{-i\frac{n}{2}t})^T$, $y_t = \overline{Q(e^{it})}\ \overline{x}_t$. Consider rank one operator $a_t := (\cdot, y_t)x_t$ with matrix coefficients $a_t^{k,\ell} = e^{-ikt}e^{-i\ell t}\, Q(e^{it}) = e^{-i(k+\ell)t}\, Q(e^{it})$. Then ($m(t)$ is normalized Lebesgue measure on $[0, 2\pi)$) we have

1) $A = \dfrac{1}{\frac{n}{2}+1}\int a_t\ dm(t)$,

2) $\|a_t\|_{S_1} \le |Q(e^{it})|\ \|x_t\|_2^2 = \left(1 + \dfrac{n}{2}\right)|Q(e^{it})|$.

Both claims are obvious. Their combination gives

$$\|A\|_{S_1} \le \int |Q(e^{it})|\ dm(t) = \|Q\|_{L^1(\mathbb{T})},$$

and (6) is proved.

Having proved (6) let us use the polar decomposition of $A = V(A^*A)^{\frac14} \cdot (A^*A)^{\frac14}$. Consider $\Phi := V(A^*A)^{\frac14}$, $\Psi = (A^*A)^{\frac14}$. These are two matrices of $\left(\dfrac{n}{2}+1\right) \times \left(\dfrac{n}{2}+1\right)$ size, and we denote columns of Ψ counted from left to right as $\Psi_0, \dots, \Psi_{n/2}$. We denote the rows of Φ counted from top to bottom as $\Phi_0, \dots, \Phi_{n/2}$. Then

$$\sum_{\substack{0\le k',\ \ell'\le\frac{n}{2},\\ k'+\ell'=k+\ell}} \langle\Phi_{k'}, \Psi_{\ell'}\rangle_{\mathbb{C}^{\frac{n}{2}+1}} = \frac{k+\ell+1}{\frac{n}{2}+1}\ Q_{k+\ell}, \qquad k+\ell \le \frac{n}{2}, \tag{7}$$

$$\sum_{\substack{0\le k',\ \ell'\le\frac{n}{2},\\ k'+\ell'=k+\ell}} \langle\Phi_{k'}, \Psi_{\ell'}\rangle_{\mathbb{C}^{\frac{n}{2}+1}} = \frac{n+1-(k+\ell)}{\frac{n}{2}+1}\ Q_{k+\ell}, \qquad \frac{n}{2} \le k+\ell \le n, \tag{8}$$

But numbers in the right hand side of (7), (8) are $\widehat{Q * k_n}(k+\ell)$.
In other words, if we denote $\Phi(z) = \sum_{i=0}^{\frac{n}{2}} \Phi_i^T z^i$, $\Psi(z) = \sum_{i=0}^{\frac{n}{2}} \Psi_i z^i$, then (7), (8) give

$$\Phi^T(z)\Psi(z) = Q(z) * k_n\,. \tag{9}$$

We also have estimate

$$\begin{aligned}\Big(\sum \|\Phi_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\Big)^{1/2}\ \Big(\sum \|\Psi_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\Big)^{1/2} &= \\ = \|\Phi\|_{S_2}\ \|\Psi\|_{S_2} = \|A\|_{S_1} \le \|Q\|_{L^1(\mathbb{T})}.\end{aligned} \tag{10}$$

For the second part of the lemma we want to make the same trick with $Q * v_{\alpha,n}$ instead of $Q * k_n$. This is easy. Fix $m = \frac{n}{2} + 1 - [\alpha n]$. Consider the middle square of size $m \times m$ of the matrix A. Its entries are $Q_{[\alpha n]}, \dots, Q_{n/2}, \dots, Q_{n-[\alpha n]}$ staying on corresponding diagonals. Consider a new matrix A_m of size $\left(\frac{n}{2}+1\right) \times \left(\frac{n}{2}+1-\right)$ which has this square in its middle and zeros everywhere else. This matrix has the same S_1-norm estimate as our A in the first part of the proof (we just consider a square "cut-off" of A, which corresponds to considering PAP with orthogonal projection P, such operation may only diminish S_1 norm):

$$\|A_m\|_{S_1} \le \|Q\|_{L^1(\mathbb{T})}\,. \tag{11}$$

And so $A_m = \Phi_m \Psi_m$, where Φ_m, Ψ_m are $\left(\frac{n}{2}+1\right) \times \left(\frac{n}{2}+1\right)$ matrices with this estimate:

$$\|\Phi_m\|_{S_2} \|\Psi_m\|_{S_2} \le \|Q\|_{L^1(\mathbb{T})}. \tag{12}$$

For the product of corresponding vector polynomials we have

$$\Phi_m^T(z) \cdot \Psi_m(z) = Q(z) * k_{n,m}, \tag{13}$$

where $\widehat{k}_{n,m}$ has been introduces above. But $v_{\alpha,n} = \frac{k_n - k_{m,n}}{c_\alpha} + k_{m,n}$.
So, using (9), (13)

$$Q * v_{\alpha,n} = \left(\frac{1}{c_\alpha}(k_n - k_{n,m}) + k_{n,m}\right) * Q =$$
$$= \frac{1}{c_\alpha}\left(\Phi^T(z)\Psi(z) - \Phi_m^T(z)\Psi_m(z)\right) + \Phi_m^T(z)\Psi_m(z) =: \tilde{\Phi}^T(z)\tilde{\Psi}(z)$$

with estimate

$$\left(\sum_{i=0}^{n/2} \|\tilde{\Phi}_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\right)^{1/2} \left(\sum_{i=0}^{n/2} \|\tilde{\Psi}_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\right)^{1/2} =$$
$$= \frac{2}{c_\alpha}\left(\|\Phi\|^2_{S_2} + \|\Phi_m\|^2_{S_2}\right)^{1/2} \left(\|\Psi\|^2_{S_2} + \|\Psi_m\|^2_{S_2}\right)^{1/2}. \tag{14}$$

But, our construction gives more than (10). It gives $\|\Phi\|_{\mathcal{H}_2} = \|\Psi\|_{\mathcal{H}_2} = \|A\|^{1/2}_{\mathcal{H}_1} \le \|Q\|_1^{1/2}$. Having this in mind and, that $c_\alpha \approx \alpha$, we continue (14):

$$\left(\sum_{i=0}^{n/2} \|\tilde{\Phi}_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\right)^{1/2} \left(\sum_{i=0}^{n/2} \|\tilde{\Psi}_i\|^2_{\mathbb{C}^{\frac{n}{2}+1}}\right)^{1/2} = \frac{A}{\alpha}\, \|Q\|_1, \tag{15}$$

and Lemma 1.1.1 is completely proved. ■

Lemma 1.1.2 Let $\beta > 0$ be given. Every polynomial $P(z) = P_{\frac{n}{2}} z^{N/2} + \cdots + P_n z^N$ can be represented as $\Phi^T(z)\Psi(z)$, with $\Phi_i, \Psi_j \in \mathbb{C}^M,\ \Phi, \Psi$ being polynomials of degrees $\dfrac{N}{2} + \beta N$ in such a way that

$$\|\Phi\|_2 \|\Psi\|_2 \leq A(\beta)\|P\|_1. \tag{16}$$

■

Proof. This is obvious from Lemma 1.1. In fact, we can consider P as a polynomial Q of degree $(1+2\beta)N$. Then

$$Q * v_{\alpha,(1+2\beta)N} = P.$$

It is enough to take $\alpha = \dfrac{2\beta}{1+2\beta}$. Then (16) follows with (15) and $A(\beta) \leq \dfrac{A}{\beta}$. ■

Remark. This β in the denominator is our enemy. Suppose we can prove Lemma 1.1.2 with $\beta = 0$ and $A(0)$ finite. Then we would decompose Q in Lemma 1.1, instead of $Q * v_{\alpha,n}$.

In fact, consider any polynomial $Q(z) = Q_0 + \cdots + Q_n z^n$. Consider $\tilde{Q}(z) = z^{2n} Q(\frac{1}{z}) = Q_n z^n + \cdots + Q_0 z^{2n}$. Apply Lemma 1.1.2 with $N = 2n$ and $\beta \geq 0$ (only $\beta = 0$ will work correctly, precisely the one which is <u>not</u> given by Lemma 1.1.2 as it is now).

Then

$$Q_n z^n + \cdots + Q_0 z^{2n} = \Phi^T(z)\Psi(z), \tag{17}$$

Φ, Ψ are polynomials of degree $\dfrac{N}{2} + \beta N = n + 2\beta n = (1+2\beta)n \quad (\beta \geq 0)$ with estimate (16).

Write (17) as follows:

$$z^{2n}\left(Q_n \frac{1}{z^n} + \cdots + Q_0 \frac{1}{z^{2n}}\right) = z^{2n}\Phi^T\left(\frac{1}{z}\right)\Psi\left(\frac{1}{z}\right) = $$
$$= \left(z^n \Phi\left(\frac{1}{z}\right)\right)^T z^n \Psi\left(\frac{1}{z}\right). \tag{18}$$

Then we can denote $z^n\Phi\left(\dfrac{1}{z}\right)$ by $\Phi_1(z),\ z^n\Psi\left(\dfrac{1}{z}\right)$ by $\Psi_1(z)$. Notice that frequencies of Φ_1, Ψ_1 lie in $[-2\beta n, n]$. Therefore, if $\beta = 0$ would be usable, we would get from (18)

$$Q(z) = \Psi_1^T(z)\Psi_1(z), \tag{19}$$

where Φ_1, Ψ_1 are analytic polynomials (with vector coefficient) of degree n (as Q). And with estimate

$$\|\Phi_1\|_2 \|\Psi_1\|_2 \leq A\|Q\|_1. \tag{20}$$

This would give us (1) and (10)– only polynomials with vector coefficients factorizing Q are of degree $n = \deg Q$, not $n/2$ as we originally wanted. But this is good enough.

Unfortunately, we do not know how to prove Lemma 1.1.2 with $\beta = 0$ and estimate

$$\|\Phi\|_2 \|\Psi\|_2 \leq A\|P\|_1. \tag{21}$$

Remark. Let us say this again: would we know how to do that, then, by virtue of operator proof we would be able to repeat this in 2 variables and then continue the same way to get (19), (20) in $z = (z_1, z_2)$ case.

So, we are ready to formulate the problem:

Problem II. $z = (z_1, z_2)$. $P(z)$ is a polynomial with frequencies in $\left[\frac{N}{2}, N\right] \times \left[\frac{N}{2}, N\right]$.

Can it be written as

$$P(z) = \sum_m \varphi_m(z)\psi_m(z), \tag{22}$$

where φ_m, ψ_m are polynomials with frequencies in $\left[0, \frac{N}{2}\right] \times \left[0, \frac{N}{2}\right]$ and such that the estimate

$$\sum_m \|\varphi_m\|_2 \|\psi_m\|_2 \leq A\|P\|_{L^1(\mathbb{T}^2)} \tag{23}$$

holds.

Remark. We can do this with $\left[0, \frac{N}{2} + \beta N\right] \times \left[0, \frac{N}{2} + \beta N\right]$ frequency support for φ_m, ψ_m. And $A(\beta)$ in (23) $\nearrow \infty$ when $\beta \to 0$. But this is not good enough if one wants the same weak factorization of polynomials of two variables as we are going to get now for one variable.

Here our operator proof stops.

We consider next the duality proof(s). It gives the correct factorization of Q of one variable to $\tilde{\Phi}^T \Psi$ of degree $\frac{\deg Q}{2}$ with estimate (10). However, if we generalize it to $z = (z_1, z_2)$ case, we have to prove a certain extension theorem for Toeplitz matrices. In the case of one variable we prove it here. But the two variable case brings the question of the extension of block Toeplitz matrix of Toeplitz matrices. Let us see how it happens.

1.2 One variable: duality proof

Consider two norms on $(n+1)$ dimensional complex space (n is even here). Given $\overrightarrow{Q} = (Q_0, \dots, Q_n)$ in our space we put $\|\overrightarrow{Q}\|_{(1)} = \|Q_0 + \dots + Q_n z^n\|_{L^1(\mathbb{T})}$.

As $\|\cdot\|_{(2)}$ we introduce $\inf\{\sum |\lambda_i| : Q(z) = \sum_i \lambda_i p_i(z) q_i(z)$, where p_i, q_i are polynomials of degree $\leq n/2$ and $\|p_i\|_2 = 1,\ \|q_i\|_2 = 1\}$.

We want to see that this norms are equivalent. Of course,

$$\|Q\|_{(1)} \leq \|Q\|_{(2)}\,.$$

Consider the dual space consisting of $\Lambda = (\lambda_0, \dots, \lambda_n)$ with two corresponding norms.

One is $|\Lambda|_1 = \inf\{\|f\|_\infty : \hat{f}(m) = \lambda_m,\ m = 0, \dots, n\}$. Another is ($\Lambda(z) := \lambda_0 + \dots + \lambda_n z^n$)

$$|\Lambda|_2 = \sup_{\substack{\|p\|_2 = 1 = \|q\|_2,\\ \deg p = \deg q = n/2}} (\Lambda(z), p(z)q(z)),$$

duality being the usual $\int_{\mathbb{T}} x \cdot \bar{y}\, dm$.

Obviously one can have another description of $|\cdot|_2$ as follows. Consider a finite Hankel matrix

$$\Gamma_\Lambda = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & 0 & \cdots & 0 \\ \lambda_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \lambda_n & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Consider its middle square

$$S\Gamma_\Lambda = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{n/2} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n/2+1} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n/2} & \lambda_{n/2+1} & \cdots & \lambda_n \end{pmatrix}$$

Then $|\Lambda|_2 = \|S\Gamma_\Lambda\|$, with operator norm $\|\cdot\|$. Notice that rotating $S\Gamma_\Lambda$ by ninety degrees we can think of it as a finite Toeplitz matrix. Notice that $|\Lambda|_1$ is infimum of norms of all infinite Toeplitz matrices having $n+1$ fixed diagonals and other diagonals being set free.
And $|\Lambda|_2$ is the norm of one of its squares. Thus, of course $|\Lambda|_2 \le |\Lambda|_1$.

Theorem 1.2.1. $|\Lambda|_1 \le 3|\Lambda|_2$.

Proof. We have to prove that given $m \times m$ ($m = \frac{n}{2} + 1$ in our case) Toeplitz matrix

$$ST_{\mathcal{M}} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{m-1} \\ \mu_{-1} & \ddots & \ddots & \vdots \\ \cdots & \ddots & \ddots & \mu_1 \\ \mu_{-(m-1)} & \mu_{-1} & \cdots & \mu_0 \end{pmatrix}$$

with operator norm $\|ST_{\mathcal{M}}\| \le 1$ one can find an infinite matrix T_M, Toeplitz, and of norm ≤ 3. ■

Remarks. 1. This result can "almost" be found in M. Bakonyi, D. Timotin's paper [3]. The difference is that in [3] all rectangular cut-offs of a band Toeplitz matrix are a priori bounded rather than just one square cut-off. It is interesting to remark that their proof of their extension result uses Peller's lemma, the one we have used above in our operator approach to factorization problem. As we already said, the factorization of polynomials and the extension of Toeplitz matrices constitute two dual problems, so one would expect that a "dual" claim to Peller's lemma should appear. This, strangely, does not happen. Peller's lemma itself appears in two formally totally dual proofs.
2. Constant 3 is probably not sharp but cannot be made 1. I am grateful to Peter Yuditski who showed me a counterexample. In particular the extension problem is not simply a lifting problem.
3. We are going to give two proves of this theorem. It should have been in the textbook literature, but we did not find it there. For our first method to extend a finite Toeplitz matrix to an infinite one with not so much larger norm, we use the lemma of Arveson [1]. This method of extension using Arveson's lemma was presented to us by N.K. Nikolski. We will also give another proof. The idea of the second proof came from [2]. Böttcher and Grudsky prove that boundedness of the norm of finite Toeplitz matrix by 1 implies that Cezaro average of its symbol is bounded by $\sqrt{3\pi}$. From this, it is easy to get the result. But we will do this without using this estimate of

Böttcher and Grudsky. (Actually, recently, they found a proof with much better constant.) Let us also mention here that consideration very similar to our *second* proof can be found in recent preprint of Y. Farforovskaya, L. Nikolskaya [7].

Lemma (Arveson). Given triangular matrix $\begin{pmatrix} \mu_{ij} \diagdown 0 \end{pmatrix}$ one can complement this triangular matrix to a full matrix with the minimal operator norm which can be calculated as follows $\min_x \left\| \begin{pmatrix} \mu_{ij} \diagdown x_{ij} \end{pmatrix} \right\| = \max \left\| \begin{pmatrix} 0 & 0 \\ \mu_{ij} & 0 \end{pmatrix} \right\|$, where the maximum is taken over all rectangular "cut-offs" of matrix (μ_{ij}).

Use it in our case like that. Consider an extension of $ST_{\mathcal{M}}$ to the infinite Toeplitz matrix just by extending diagonals of $ST_{\mathcal{M}}$ to infinity.

This extension can have a $\log m \|ST_{\mathcal{M}}\|$ norm. This is too much. We want an estimate independent of m. Consider "corner" matrices $C = \begin{pmatrix} \mu_{m-1} & 0 \ddots & 0 \\ \vdots & \ddots & \ddots \\ \mu_2 & \ddots & \ddots \\ \mu_1 & \mu_2 \dots & \mu_{m-1} \end{pmatrix}$ and use Arveson's lemma to replace 0 by some x_{ij}. The smallest norm we get is the maximum of rectangular cut-offs of C with upper right corner on the diagonal of C.

But this maximum is exactly maximum of rectangular cut-offs of ST_μ with lower left corner on the diagonal of ST_μ. This is just because of Toeplitz structure of ST_μ and because of our trivial extension along diagonals. Now, the maximum of rectangular cut-offs of ST_μ with lower left corner on the diagonal of ST_μ is of course bounded by $\|ST_\mu\| \leq 1$. So we can fill upper triangles by $\{x_{ij}\}$ to have the norm of new $\tilde{C}$ bounded by 1.

Do the same for matrices below main block-diagonal consisting of repeated ST_μ. All other entries will be zeros. As a result we get an infinite matrix of norm at most 3. It is not Toeplitz. But its averaging along the direction of the main diagonal gives the Toeplitz matrix of the same (or smaller) norm. ■

We proved $|\Lambda|_1 \leq 3|\Lambda|_2$.

So the same is true for $\| \; \|_{(1)}$ and $\| \; \|_{(2)}$

$$\frac{1}{3}\|Q\|_{(2)} \leq \|Q\|_{(1)} \leq \|Q\|_{(2)}.$$

We finished our dual approach.

Now 2 variables.

Theorem 1.2.2. Given $(N \times N) \times (N \times N)$ Toeplitz tensor $t_{\alpha,\beta;i,j} = T_{\alpha-i,\beta-j}$ acting on $\ell^2_N \otimes_2 \ell^2_N$ with norm ≤ 1. It can be extended to infinite Toeplitz tensor of bounded norm.

We do not want to do this here, but Theorem 1.2.2 can be deduced from the result of Ferguson and Lacey [6].

Question: How to get an elementary proof?

Actually we want to give another dual approach proof, it is based on Nehari's theorem instead of Arveson's lemma. Consider finite Toeplitz matrix $T_a = \begin{pmatrix} a_0 & & a_n \\ & \ddots & \\ a_{-n} & & a_0 \end{pmatrix}$, $\|T_a\| = 1$. We want to extend it to an infinite Toeplitz matrix with norm bounded by an absolute constant A.

Of course, T_a is the sum of two "Hankel" matrices $R_a = \begin{pmatrix} a_0 & & a_n \\ & \ddots & \\ 0 & & a_0 \end{pmatrix}$ and $L_a = \begin{pmatrix} a_{-1} & & 0 \\ & \ddots & \\ a_{-n} & & a_{-1} \end{pmatrix}$. Unfortunately their norms can be as large as $A_1 \log n$. Otherwise we would use Nehari's theorem, extend L_a, R_a to $\widetilde{L}_a, \widetilde{R}_a$ and then the extension of T_a would be given by $\widetilde{L}_a + \widetilde{R}_a$.

We, instead, split T_a to 3 matrices $T_a = c_a + \ell_a + r_a$. To construct c_a consider $\zeta \in \mathbb{T}$ and $D_\zeta = \text{diag}\,(1, \dots, \zeta^n)$. Consider $D_\zeta T_a D_\zeta^*$, $(D_\zeta T_a D_\zeta^*)_{jk} = a_{j-k}\zeta^{j-k}$.

Consider special convex combinations of $\{D_\zeta T_a D_\zeta^*\}_{\zeta \in \mathbb{T}}$. Namely, let $K_m(\zeta)$ denote Fejér kernel of order $m, m \leq n$.

Consider

$$T_a^m := \int_{\mathbb{T}} K_m(\zeta) D_\zeta T_a D_\zeta^* \, dm(\zeta),$$

this is a Fejér "cut-off" of our matrix, it leaves the main diagonal untouched, diagonals from m to n become zero, the rest has changed accordingly: j-th diagonal is multiplied by $\widehat{K}_m(j)$.

Notice that

$$\|T_a^m\| \leq \|T_a\| = 1. \tag{24}$$

The same can be repeated with $(m_2 < m_1 \leq n)$ $CK_{m_1} - K_{m_2} =: V_{m_1,m_2}$, which is de la Vallée-Poussin type kernels. Here C is chosen to have $\widehat{V}_{m_1,m_2}(j) = 1$, $|j| \leq m_2$. Notice that if $m_2 = \alpha n$, $m_1 = \beta n$ $(\alpha < \beta)$, then $C = C(\alpha, \beta)$ does not depend on n. Put

$$T_a^{\alpha,\beta} = \int_{\mathbb{T}} V_{\beta n, \alpha n}(\zeta) D_\zeta T_a D_\zeta^* \, dm(\zeta).$$

We can now write the analog of (24)

$$\|T_a^{\alpha,\beta}\| \le C(\alpha,\beta) < \infty \tag{25}$$

Now choose $\alpha = \dfrac{3}{8}$, $\beta = \dfrac{1}{2}$ (n can not be divisible by 8, but this is not important, we can always think that it is, or we can slightly change α, β accordingly).

$$c_a := T_a^{3/8,1/2}.$$

Then $r_a + \ell_a = T_a - c_a$, and r_a is a "right upper" corner, ℓ_a is a "left lower" corner.

By (25)

$$\|r_a + \ell_a\| = \|c_a\| + 1 \le A := C\big(\frac{3}{8}, \frac{1}{2}\big) + 1. \tag{26}$$

But r_a and ℓ_a are independent, meaning, that we can find two orthogonal projections P_1, P_2 in such a way that $r_a = P_1(r_a + l_a)P_2$. In fact, let $e_0, ...e_n$ be orthonormal system in which our T_a has Toeplitz structure and let (without loss of generality) think that n is divisible by 8, then one can take P_1 to be the projection on $e_{\frac{3}{8}n}, ..., e_n$ and P_2 be the projection on $e_0, ..., e_{\frac{5}{8}n}$. So (26) implies

$$\|r_a\| \le A, \qquad \|\ell_a\| \le A. \tag{27}$$

Let us apply now Nehari's theorem to get $\tilde{r}_a$, the extension of r_a beyond the upper right corner in the north-east direction. Also $\tilde{\ell}_a$ is the extension of ℓ_a in the south-west direction. Then

$$\|\tilde{r}_a\|, \|\tilde{\ell}_a\| \le \|r_a\|, \|\ell_a\| \le A. \tag{28}$$

To get the extension of T_a we are left to see why c_a gives rise to a bounded Toeplitz operator. Of course, c_a itself is bounded all right by (26). But we want to extend its diagonals to infinite ones, extend this infinite band matrix by zeros, consider the new infinite matrix $\tilde{c}_a$, and prove that its norm is still bounded.

Such a "natural" extension would fail with T_a, but works fine with c_a. Indeed, $\tilde{c}_a$ is a block matrix with a 3 diagonals of blocks.

Call the blocks in one row I, II, III, from left to right. Block II is c_a itself. It is easy to notice that each of the blocks I and III is equal to $P_i c_a P_j$, $i, j, = 1, 2$, where P_i, $i = 1, 2$ are natural orthogonal projections in $\mathcal{H} = \mathrm{span}\{e_0, ..., e_n\}$. This observation is due to the choice of constants $1/2, 3/8$ in the definition of c_a, of corse only the fact that $\beta \le 1/2$ is important here. Each such "cut-off" is bounded by $\|c_a\| \le A$. So the norm of 3-diagonal block matrix $\tilde{c}_a$ is bounded:

$$\|\tilde{c}_a\| \le 3A. \tag{29}$$

Collecting (2.20), (2.19) and $T_a = \ell_a + c_a + r_a$ we get that $\tilde{\ell}_a + \tilde{c}_a + \tilde{r}_a$ extends T_a and has a uniformly bounded norm.

1.3. Function theory proof

Theorem 1.3.1. $Q = Q_0 + \cdots + Q_n z^n$ can be written as $\sum_{i=1}^{4} p_i q_i$, $\deg p_i$, $\deg q_i \leq \frac{n}{2}$,

$$\|p_i\|_2^2, \|q_i\|_2^2 \leq A\|Q\|_1 \,. \tag{30}$$

Remark. I heard this statement from Sergei Shimorin, but not the proof, and then it was easy to restore the proof.

Remark. The operator proof in Section 1.1 based on Peller's lemma (see [8]) easily allowed for the factorization of $z_1^N z_2^N Q(z_1, z_2)$.

Proof of Theorem 1.3.1. Start with Q and consider $Q_1(z){=}z^{-n/2}Q(z)$, $Q_2(z) = \text{Re } Q_1(z)$, $Q_3 = \text{Im } Q_1$. Notice that their L^1 norms are controlled by $\|Q\|_1$.

Lemma 1.3.2 Given a real trigonometric polynomial $q = a_{-m}z^{-m} + \cdots + a_0 + \cdots + a_m z^m$, there exists a polynomial p of degree sm ($s \in (0,1]$) such that

1) $p > |q|$ on $\mathbb{T}$;

2) $\int_{\mathbb{T}} p \leq e^2 \int_{\mathbb{T}} |q|$.

Proof. Consider analytic polynomial $h(z) = z^m q(z)$.

Consider it in the disc $\left(1 + \dfrac{1}{m+1}\right)\mathbb{D}$.

$$\int_{(1+\frac{1}{m+1})\mathbb{T}} |h| \leq e^2 \int_{\mathbb{T}} |h|. \tag{31}$$

In fact, $h(z)/z^{2m}$ is analytic outside of the unit disc. So just by mean value theorem

$$\int_{(1+\frac{1}{m+1})\mathbb{T}} |h(z)/z^{2m}| \leq \int_{\mathbb{T}} |h(z)/z^{2m}| = \int_{\mathbb{T}} |h(z)|$$

But $\min\limits_{|z|=1+\frac{1}{m+1}} \dfrac{1}{|z|^{2m}} \geq e^{-2}$. So (3.10) is proved.

Using (31), consider $H(z) =$ harmonic extension of $|h|\Big|1+\frac{1}{m+1}\mathbb{T}$ into $\left(1+\frac{1}{m+1}\right)\mathbb{D}$. As $|h(z)|$ is subharmonic there, we get

$$|h(z)| \leq H(z),\ z \in \mathbb{T}. \tag{32}$$

But by (31)

$$\int_{\mathbb{T}} H(z)dm \leq \int_{(1+\frac{1}{m+1})\mathbb{T}} |h| \leq e^2 \int_{\mathbb{T}} |h|. \tag{33}$$

So $H \mid \mathbb{T}$ is what we wanted in 1) and 2) except that $H \mid \mathbb{T}$ is not a polynomial. This is easy to amend. Consider

$$p = K_{sm} * H,$$

where K_{sm} is a Fejér kernel of order sm. Then, from (33) one obtains (recall that $h = (q(z)z^m)$)

$$\int_{\mathbb{T}} p \leq e^2 \int |h| = e^2 \int |q|. \tag{34}$$

On the other hand, p is the averaging of H with Fejér kernel which is $\geq \frac{a}{m}\chi_{[-\frac{1}{m},\frac{1}{m}]}$. And on each interval of length $\frac{2}{m}$ on the circle $\mathbb{T}$ our harmonic function H is not changing too much:

$$\exists\ A \text{ (abs. const.) } : \frac{\max_I H}{\min_I H} \leq A \tag{35}$$

for any I, $|I| = \frac{2}{m}$, $I \subset \mathbb{T}$. This is because H is a positive harmonic function in $\left(1+\frac{1}{m+1}\right)\mathbb{D}$ and we consider it $\frac{1}{m}$ distance to the boundary. Now (35), (32) and $p = K_{sm} * H$ give

$$p(z) \geq a_0 |h(z)|. \tag{36}$$

And (34) and (36) prove the lemma. ■

Recall that $Q_1(z) = z^{-n/2}Q(z),\ Q_2(z) = \text{Re}\ Q_1(z),\ Q_3 = \text{Im}\ Q_1$. Use the lemma for $q = Q_2$, and then for $q = Q_3$. Then $(s = 1,\ m = \frac{n}{2})$

$$\begin{array}{ccccc} Q_2 & = & p_2 - (p_2 - Q_2), & & Q_3 = p_3 - (p_3 - Q_3) \\ & & \| & & \| \\ & & p_2 - p_4 & & p_3 - p_5 \end{array}$$

Of course, $p_2, p_3, p_4, p_5 \geq 0$. So Fejér-Riesz theorem decomposes them

$$p_i(z) = \alpha_i(z)^* \alpha_i(z), \quad i = 2,3,4,5.$$

$$\|\alpha_i\|_2 = \|p_i\|_1^{1/2} \leq A\Big(\int |Q_2|\Big)^{1/2} \leq A\Big(\int_{\mathbb{T}} |Q|\Big)^{1/2}.$$

Degrees of analytic polynomials α_i are at most $m = \frac{n}{2}$.
Finally,

$$\begin{aligned} z^{-n/2} Q(z) &= Q_1(z) = Q_2 + iQ_3 = p_2 - p_4 + ip_3 - ip_5 \\ &= \alpha_2^* \alpha_2 - \alpha_4^* \alpha_4 + i\alpha_3^* \alpha_3 - i\alpha_5^* \alpha_5. \end{aligned}$$

Multiplying α_i^* by $z^{n/2}$ we get new *analytic* polynomials. So Q is decomposed exactly as Theorem 3.1 states. ■

Theorem 2.1 (2 variables) is proved in exactly the same way using operator Fejér-Riesz theorem. Lemma 1.3.2 holds for 2 variables, and the proof is verbatim the same.

2. Two variables

Theorem 2.1. $Q = \sum_{i,j=0}^{N} Q_{ij} z^i z^j$, then $z_2^N Q$ (or $z_1^N Q$) can be written as $\sum_{i=1}^{\infty} p_i q_i$, frequencies of p_i, q_i being in $[0, N] \times [0, N]$, and

$$\sum_{i=1}^{\infty} \|p_i\|_{L^2(\mathbb{T}^2)} \, \|q_i\|_{L^2(\mathbb{T}^2)} \leq A \|Q\|_{L^1(\mathbb{T}^2)}. \tag{37}$$

Remark. One can prove that Q itself (not $z_1^N Q$) can be factorized as above. This follows from Theorem 1.2.2, which in its turn follows from [6]. But we want to prove a "much softer" result by much softer methods.

Proof of Theorem 2.1. We will use Lemma 1.3.2 for two variables (the same proof) combined with an idea from M. Dritschel [4].

Our polynomial $Q(z_1, z_2)$ has all frequencies in $[0, N]^2$. Let us shift it (as in 1-dimensional case, but more). Consider $Q_1(z_1, z_2) = z_1^{-N} z_2^{-N} Q(z_1, z_2)$. Now it has frequencies in $[-N, 0]^2$. Let

$$Q_2 = \text{Re } Q_1, \qquad Q_3 = \text{Im } Q_1.$$

Using Lemma 1.3.2 we can find $p_2, p_4 := p_2 - Q_3,\ p_3,\ p_5 := p_3 - Q_3$ which have frequencies in $[-N,0]^2 \cup [0,N]^2$, which are all nonnegative polynomials and which satisfy

$$\int_{\mathbb{T}^2} |p_i| \leq A \int_{\mathbb{T}^2} |Q|, \quad i = 2,3,4,5. \tag{38}$$

Now it seems to be the time to use operator Fejér-Riesz theorem. But actually it is not. Fix p_i, and call it $\widetilde{P}$. It can be written as block Toeplitz matrix of Toeplitz matrices:

$$\widetilde{P} = \begin{pmatrix} \cdot & \widetilde{P}_{-1} & \widetilde{P}_0 & \widetilde{P}_1 & \cdot & & \\ & \cdot & \widetilde{P}_{-1} & \widetilde{P}_0 & \widetilde{P}_1 & \cdot & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \end{pmatrix}$$

$\widetilde{P}_i$ is the Toeplitz matrix in which coefficient $\widetilde{P}_{i,j}$ of polynomial $\widetilde{P}$ stands on jth diagonal. It is, of course, a nonnegative infinite matrix, as $\widetilde{P}$ is a nonnegative polynomial.

Let us compress each block $\widetilde{P}_i$ to $N \times N$ square block. After that let us divide each block by N. The resulting infinite block Toeplitz matrix whose blocks are $N \times N$ finite Toeplitz matrices is denoted by $P^{(N)}$. Its blocks are $P_i^{(N)}$:

$$P^{(N)} = \begin{pmatrix} \cdot & P_{-1}^{(N)} & P_0^{(N)} & P_1^{(N)} & \cdot & & \\ & \cdot & P_1^{(N)} & P_0^{(N)} & P_1^{(N)} & \cdot & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \end{pmatrix}$$

Consider operator polynomial of one variable

$$P^{(N)}(z) = \sum_{i=-N}^{N} P_i^{(N)} z^i$$

(notice that $\widetilde{P}_i$, and so, $P_i^{(N)} = 0$ if $|i| > N$). It is positive definite operator polynomial. We apply operator version of Fejér-Riesz theorem (M. Rosenblum): (F_k are $N \times N$ matrices)

$$P^{(N)}(z) = \left(\sum_{k=0}^{N} F_k z^k \right)^* \left(\sum_{k=0}^{n} F_k z^k \right) =: F(z)^*\, F(z).$$

Then, of course,

$$P_k^{(N)} = \sum_{i=0}^{N} F_i^* \; F_{k+i},$$

then

$$P_{k;s,t}^{(N)} = \sum_{i=0}^{N} F_{i;s}^* \; F_{k+i;t},$$

where $F_{k+i;t}$ is the t-th column of $N \times N$ matrix F_{k+i} counted from the left. If we are on the j-th diagonal of $P_k^{(N)}$, we have on it $N-j$ times $\frac{1}{N} \; P_{k,j}$ (recall that $P_{k,j}$ are coefficients of original polynomial $\widetilde{P}$, (this polynomial is, say, p_2 (or p_i, i=2,3,4,5)).

Summing up along the j-th diagonal one gets

$$\begin{aligned} \frac{N-j}{N} P_{k,j} &= \sum_{i=0}^{N} \sum_{s=0}^{N-1-|j|} F_{i;s}^* \; F_{k+i;s+j}, \qquad j \geq 0 \\ \frac{N-|j|}{N} P_{k,j} &= \textstyle\sum_{i=0}^{N} \; \sum_{s=|j|}^{N-1} \; F_{i;s}^* \; F_{k+i;s+j}, \quad j < 0 \end{aligned} \tag{39}$$

Consider vector polynomial of two variables (coefficients are s-th columns of i-th matrix F)

$$F(z_1, z_1) = \sum_{i=0}^{N} \sum_{s=0}^{N-1} F_{i;s} z_1^i z_2^s.$$

Now (39) shows that

$$F(z)^* \; F(z) \quad \text{has coefficient} \quad \frac{N-|j|}{N} \; P_{k,j} \quad \text{in front of} \quad z_1^k \; z_2^j. \tag{40}$$

If not for this $\frac{N-|j|}{N}$ coefficient, we would finish the proof of factorization in 2 variables - because the rest would be exactly as in 1 variable.

What we got is the following:

$$Q_1 = z_1^{-N} \; z_2^{-N} Q = Q_2 + iQ_3 = p_2 - p_4 + i(p_3 - p_5),$$

where p_i has frequencies in $[-N, 0]^2 \cup [0, N]^2$, and for $\ell = 2, 3, 4, 5$

$$K_N(z_2) * \; p_\ell = \sum_{m=0}^{N} \left(F^{(m)}(z_1, z_2) \right)^* \; F^{(m)}(z_1, z_2), \tag{41}$$

where $F^{(m)}(z_1, z_2) := \sum_{i=0}^{N} \sum_{s=0}^{N-1} F_{i;s,m}\, z_1^i z_2^s$, where $F_{i;s,m}$ is (m, s)-th element of F_i.

Now just by (41)

$$\sum_{m=0}^{N} \|F^{(m)}\|^2_{L^2(\mathbb{T}^2)} \le \|p_\ell\|_{L^1(\mathbb{T}^2)}. \tag{42}$$

A problem which awaits us is the existence of a "cut" function K_N in (41). Analogous place for one variable did not have any "cut" function (see section 1.3).

Using (41) four times (for $\ell = 2, 3, 4, 5$) we get that

$$K_N(z_2) * (z_1^{-N} z_2^{-N} Q) = \sum \alpha_m^* \alpha_m, \tag{43}$$

where frequencies of α_m lie in $[0, N]^2$ and

$$\sum \|\alpha_m\|_2^2 \le A\|Q\|_1.$$

Multiply (43) by $z_1^N z_2^N$. Let $\beta_m := z_1^N z_2^N \alpha^*$. These are analytic polynomials with frequencies in $[0, N]^2$. Also

$$\sum \|\beta_m\|_2\ \|\alpha_m\|_2 = \sum \|\alpha_m\|^2 \le A\|Q\|_1. \tag{44}$$

And (43) becomes

$$V_0^N(z_2) * Q(z_1, z_2) = \sum_{m=0}^{N} \beta_m(z_1, z_2)\ \alpha_m(z_1, z_2). \tag{45}$$

Here $V_0^N(z_2)*$ multiplies coefficients $Q_{k,j}$ of Q on $|j|/N,\ j = 0, \dots, N$. And (44) is satisfied. The moral is that if we would have this equality without Fejér like cut-off by V we would be done with weak factorization of analytic polynomials of 2 variables. In fact, it looks like the symmetry allows us to get rid of V.

Namely, it looks like we can write a symmetric equality

$$V_N^0(z_2) * Q(z_1, z_2) = \sum_{m=0}^{N} \gamma_m(z_1, z_2)\ \delta_m(z_1, z_2). \tag{46}$$

With (44) being satisfied for γ, δ instead of β, α. Here $V_N^0(z_2)*$ multiplies coefficients $Q_{k,j}$ of Q by $\dfrac{N - |j|}{N},\ j = 0, \dots, N$. Add (45), (46). As $(V_0^N +$

$V_N^0)(z_2)^*$ multiply coefficients by $j/N + (N-j)/N = 1$ we get the desired decomposition of $Q(z_1, z_2)$. Unfortunately, (45) is, of course, correct (it is just (43), and we have proved it already), but (46) needs a correction. Let us see why. Let us try to prove this "symmetric" formula.

Consider $\widetilde{Q}(z_1, z_2) := z_2^N Q\left(z_1, \frac{1}{z_2}\right)$. Apply (45) to $\widetilde{Q}$:

$$V_0^N * z_2^N Q\left(z_1, \frac{1}{z_2}\right) = V_0^N * \widetilde{Q}(z_1, z_2) = \sum_{m=0}^N \tau_m(z_1, z_2)\ \eta_m(z_1, z_2)$$

with frequencies of $\tau_m,\ \eta_m$ in $[0, N]^2$ and with (44) satisfied. This is of course true.

But now we want polynomial Q. Notice that $Q = \widetilde{\widetilde{Q}}$ and

$$\widetilde{V_0^N(z_2) * \widetilde{Q}} = V_N^0(z_2) * Q \widetilde{\left(\sum_{m=0}^N \tau_m\ \eta_m\right)} = z_2^N \sum_{m=0}^N \tau_m \left(z_1, \frac{1}{z_2}\right) \eta_m \left(z_1, \frac{1}{z_2}\right)$$

$$= z_2^{-N} \sum_{m=0}^N \left(z_2^N \tau_m \left(z_1, \frac{1}{z_2}\right)\right) \left(z_2^N \eta_m \left(z_1, \frac{1}{z_2}\right)\right).$$

Denote $\gamma_m = z_2^N \tau_m \left(z_1, \frac{1}{z_2}\right)$, $\delta_m = z_2^N \eta_m \left(z_1, \frac{1}{z_2}\right)$. Then γ_m, δ_m are analytic polynomials with frequencies in $[0, N]^2$ and

$$V_N^0(z_2) * Q = z_2^{-N} \sum_{m=0}^N \gamma_m(z_1, z_2)\ \delta_m(z_1, z_2). \tag{47}$$

This is what we get instead of (46).

Adding this to (45), one gets

$$Q = \sum\ \alpha_m \beta_m + z_2^{-N} \sum\ \gamma_m \delta_m.$$

The estimates on norms are okay (see (44)):

$$\sum\ \|\alpha_m\|_2\ \|\beta\|_2 + \sum\ \|\gamma_m\|_2\ \|\delta_m\|_2 \leq A \|Q\|_1. \tag{48}$$

But we get only factorization of $z_2^N Q(z_1, z_2)$, as we see. Theorem 2.1 is proved. ■

Question: Is it possible to modify the proof to avoid the shift? The paper by J. Geronimo and H. Woerdeman [5] may give some hope.

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Address: Department of Mathematics
Michigan State University
East Lansing, MI 48824
USA