# Operadic Hochschild chain complex and free loop spaces

by

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**Abstract.** We construct, for any algebra $A$ over an operad $\mathcal{O}$, an Hochschild chain complex, $C_*^{\mathfrak{H}}(\mathcal{O}, A)$ which is also an $\mathcal{O}$-algebra. This Hochschild chain complex coincides with the usual one, whenever $A$ is a commutative differential graded algebra. Let $X$ is a simply connected space, $N^*(-)$ be the singular cochain functor, $X^{S^1}$ be the free loop space, $\mathcal{C}_\infty$ be a cofibrant replacement of commutative operad and $M_X$ a $\mathcal{C}_\infty$-cofibrant model of $X$. We prove that The operadic chain complex $C_*^{\mathfrak{H}}(\mathcal{C}_\infty, M_X)$ is quasi-isomorphic to $N^*(X^{S^1})$ as a $\mathcal{C}_\infty$-algebra. In particular, for any prime field of coefficients this identifies the action of the large Steenrod algebra on the Hochschild homomology $H_*^{\mathfrak{H}}(N^*(X))$ with the usual Steenrod operations on $H^*\left(X^{S^1}\right)$.

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**Introduction.**

As illustrated by the fundamental result of Mandell, [20], $E_\infty$-algebras are the good tools for the study of the homotopy theory of topological spaces. Indeed, for a prime field of coefficients the homology of a $E_\infty$-algebra is an unstable algebra over the large Steenrod algebra. This last property originates with the previous works of Dold, [9], May [21] and has been recently extensively studied by the first author and Livernet [8].

In this paper we develop an application of the homotopy theory of $E_\infty$-algebras to the study of the free loop space, $X^{S^1}$ on a space $X$. For this purpose, we consider the almost free operad, denoted $\mathcal{C}_\infty$, which is a cofibrant model of the commutative operad. A quotient of this operad, also denoted $\mathcal{C}_\infty$, has been studied by Kadeisvili, [15], Kontsevich, [16] and Getzler-Jones, [10]. It appears that the category of $\mathcal{C}_\infty$-algebras is a closed model category which is very similar to the category of commutative and associative differential graded algebras.

It follows from Theorem C and Proposition 4.2-2:

**Main Theorem:** *We denote by $N^*(-)$ the normalized singular cochain functor with coefficients in an arbitrary commutative ring $\Bbbk$ and by $H^*(-) =$*

*$H(N^*(-))$ the functor of singular cohomology. Let $X$ be a 1-connected space, if each $H^i(X)$ is finitely generated then there exist natural equivalences of $\mathcal{C}_\infty$-algebras between $N^*(X^{S^1})$ and the operadic Hochschild complex of the $\mathcal{C}_\infty$-algebra $N^*(X)$.*

An associative $\mathcal{C}_\infty$-algebra is a particular case of strongly homotopy algebra (see 3.3). Our result is, in this special case, a substancial improvement of the results proved in [3] and [4] since the structure of unstable algebra over the Steenrod algebra but also all secondary cohomological operations are preserved [8].

All the paper is devoted to the proof of this result. The required knowledge about operads is presented in section 1. In section 2 we define the operad $\mathcal{C}_\infty$ in relation with the Barratt-Eccles operad, extensively studied by Berger and Fresse, [1]. In section 3, we construct for any operad $\mathcal{O}$ in the category of differential graded modules the operadic Hochschild complex of an $\mathcal{O}$-algebra, $A$. When $A$ is supposed to be an associative and commutative differential graded algebra we compare the operadic Hochschild complex of an $\mathcal{O}$-algebra with the usual Hochschild complex. We also study the particular case when $A$ is an almost free $\mathcal{O}$-algebra (Theorem B). In the last section we end the proof of our main result (Theorem C).

## 1. Backgrounds about algebras over an operad.

**1.1 Notation.** We denote by $\Bbbk$-**GM** (resp. $\Bbbk$-**DGM**) the category of graded modules (resp. of differential graded modules). We also consider the forgetfull functor:

$$\# : \Bbbk\text{-}\mathbf{DGM} \to \Bbbk\text{-}\mathbf{GM}\,, \quad (V,d) \mapsto (V,d)_\# = V\,.$$

We are mainly concerned by the following categories :

$\mathbf{\Delta}$, the simplicial category of finite ordered sets with objects $[n] = \{0,1,\dots,n\}$ and non decreassing maps

$\mathbf{C}^{\mathbf{\Delta}}$, the category of cosimplicial objects and cosimplicial maps of $\mathbf{C}$: $\underline{X} = \left(\{\underline{X}^n\}_{n\geq 0}, d^i, s^i\right)$

$\mathbf{C}^{\mathbf{\Delta}^{op}}$, the category of simplicial objects and simplicial maps of $\mathbf{C}$: $\underline{X} = (\{\underline{X}_n\}_{n\geq 0}, d_i, s_i)$

**1.2 Operads.** Recall from [11], [10] and [17], that an operad $\mathcal{O}$ is defined in any symmetric monoidal category $\mathbf{C}$ as a sequence of left $\Bbbk[\Sigma_i]$-modules (where $\Sigma_i$ is the symmetric group) $\mathcal{O}(i)\,, i \geq 0$, with *composition products*

$$\mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \mathcal{O}(i_2) \otimes \cdots \otimes \mathcal{O}(i_n) \to \mathcal{O}(i_1 + i_2 + \cdots + i_n)\,,$$
$$x_0 \otimes x_1 \otimes \cdots \otimes x_n \mapsto x_0(x_1, x_2, \ldots, x_n)$$

which are equivariant, associative and with a unit. Homomorphisms of operads are defined in an obvious way.

The category of operads is a closed model category [2], [12], where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections.

The universal example of operad is the *endomorphim operad*. Let $(V, d_V)$ be a differential graded module, $\mathcal{E}nd_V$ is an operad in $\Bbbk$-**DGM** such that:

$$\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)\,, n \geq 1.$$

**1.3 Algebras over an operad.** Let $\mathcal{O}$ be an operad in $\Bbbk$-**DGM**. An $\mathcal{O}$-*algebra*, $(A, \rho)$, is a differential graded module, $A = (\{A_i\}_{i\in\mathbb{Z}}\,, d_A : A_i \to A_{i-1})$, with an operadic representation

$$\rho : \mathcal{O} \to \mathcal{E}nd_A\,.$$

determined by a sequence of maps differential graded $\Bbbk$-modules, called the *evaluation product*:

$$\tilde{\rho}_n : \mathcal{O}(n) \otimes A^{\otimes n} \to A\,, \quad \tilde{\rho}_n(x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \rho_n(x)(a_1 \otimes a_2 \otimes \cdots \otimes a_n).$$

invariant under the action of $\Sigma_n$ and compatible with the composition product of $\mathcal{O}$.

$\mathcal{O}$-algebras and homomorphisms of $\mathcal{O}$-algebras is a category, denoted $\mathcal{O}$-**ALG**. If $\mathcal{O}'$ is another operad in $\Bbbk$-**DGM**, a homomorphism of operads $f : \mathcal{O} \to \mathcal{O}'$ induces a natural functor $f^* : \mathcal{O}'$-**ALG** $\to \mathcal{O}$-**ALG**.

The *free $\mathcal{O}$-algebra* generated by a differential graded module $V$ is the differential graded module

$$F(\mathcal{O}, V) = \bigoplus_{k=0}^{\infty} \mathcal{O}(k) \otimes_{\Sigma_k} V^{\otimes k}\,,$$

with evaluation products $\tilde{\rho}_n : \mathcal{O}(n) \otimes F(\mathcal{O}, V)^{\otimes n} \to F(\mathcal{O}, V)$ induced by the composition products of $\mathcal{O}$. Any homomophism $f : V \to W$ of graded modules extends uniquely in homomorphism of graded modules $F(\mathcal{O}, f) : F(\mathcal{O}, V) \to F(\mathcal{O}, W)$. The functor $F(\mathcal{O}, -) : \Bbbk$-**DGM** $\to \mathcal{O}$-**ALG** is a left adjoint to the forgetful functor $\mathcal{O}$-**ALG** $\to \Bbbk$-**DGM**.

**1.4 The associative operad and the commutative operad.** For each $n \geq 0$, the canonical augmentation $\epsilon_n : \Bbbk[\Sigma_n] \to \Bbbk$ of the group ring $\Bbbk[\Sigma_n]$ defines a homomorphism of operads in $\Bbbk$-**GM**

$$\epsilon : \mathcal{A} \to \mathcal{C}$$

from the *associative operad* $\mathcal{A}$ to the *commutative operad* $\mathcal{C}$ such that $\mathcal{A}(n) = \Bbbk[\Sigma_n]$ and $\mathcal{C}(n) = \Bbbk$ with composition products given respectively by composition of permutations and multiplication in $\Bbbk$. An $\mathcal{A}$-algebra is an associative differential graded algebra while a $\mathcal{C}$-algebra is a commutative differential graded algebra which is also, associative. Indeed, the representations $\rho : \mathcal{A} \to \mathcal{E}nd_A$ and $\mathcal{C} \to \mathcal{E}nd_A$ are the iterated products:

$$\Bbbk[\Sigma_n] \otimes_{\Sigma_n} A^{\otimes n} = A^{\otimes n} \to A \text{ and } \Bbbk \otimes_{\Sigma_n} (A^{\otimes n}) = \left(A^{\otimes n}\right)_{\Sigma_n} \to A\,.$$

The free $\mathcal{C}$-algebra (resp. a free $\mathcal{A}$-algebra) generated by the differential graded module $V$ is the differential graded module $F(\mathcal{C}, V) = \oplus_{n=0}^{\infty} \Bbbk \otimes_{\Sigma_n} V^{\otimes n} = \oplus_{n=0}^{\infty} (V^{\otimes n})_{\Sigma_n} = S(V)$ with graded commutative mutiplication of the elements of $V$ (resp. $F(\mathcal{A}, V) = \oplus_{n=0}^{\infty} \Bbbk[\Sigma_n] \otimes_{\Sigma_n} V^{\otimes n} = \oplus_{n=0}^{\infty} V^{\otimes n} = T(V)$ the usual tensor algebra on $V$).

**1.5** Let $A$ be an $\mathcal{O}$-algebra. Then $A$ is called *almost free* if $A_{\#} = F(\mathcal{O}_{\#}, V)$ for a graded module $V$. If the category of $\mathcal{O}$-algebras is a closed model category (this is the case whenever $\mathcal{O}$ is a cofibrant operad [2]) then any $\mathcal{O}$-algebra admits a cofibrant model. An $\mathcal{O}$-algebra is cofibrant if and only if it is a retract of an almost free $\mathcal{O}$-algebra.

## 2. The operads $\mathcal{BE}$ and $\mathcal{C}_\infty$.

**2.1** Let us denote $\underline{A} = (\{A_n\}_{n\in\mathbb{N}}, d_i, s_i)$ a simplicial differential graded module with internal differential $d_A : A_{n,q} \to A_{n,q-1}$. Then the *total complex* of the bicomplex

$$\underline{A}_{p-1,q} \overset{\sum(-1)^i d_i}{\longleftarrow} \underline{A}_{p,q} \overset{d_A}{\to} \underline{A}_{p,q+1}$$

is denoted by $\mathrm{Tot}_*(\underline{A})$:

$$\mathrm{Tot}(\underline{A}) = (\{\mathrm{Tot}_n(\underline{A})\}_{n\in\mathbb{Z}}, d) \quad d : \mathrm{Tot}_n(\underline{A}) \to \{\mathrm{Tot}_{n-1}(\underline{A})$$

$$\mathrm{Tot}_n(\underline{A}) = \bigoplus_{p+q=n} \underline{A}_{p,q} \qquad dx = d_A x + (-1)^p \textstyle\sum(-1)^i d_i x\,, x \in \underline{A}_{p,q}\,.$$

Let $D_n(\underline{A})$ be the subcomplex generated by degeneracies in $\underline{A}_n$. The quotient complex $\mathrm{Tot}_*(\underline{A})/D_*(\underline{A}) := N_*(\underline{A})$ is the *normalized differential* *graded module*. The quotient map $\mathrm{Tot}(\underline{A}) \twoheadrightarrow N_*(\underline{A})$ is a chain equivalence, [19]-Theorem 6.1.

The singular chain complex of $X$ with coefficients in $\Bbbk$ is $C_*(X;\Bbbk) := C_*(S_*(X;\Bbbk))$ and the normalized chain complex is $N_*(X;\Bbbk) := N_*(S_*(X))$. The subcomplex of normalized cochain complex $N^*(X,\Bbbk) \subset \mathrm{Hom}\,(C_*(X,\mathbb{Z}),\Bbbk)$ is, in this paper, simply denoted $N^*(X)$. It follows from, [19]-theorem 9.1 that $N^*$ is a contravariant functor from the category of pointed topological spaces to the category of augmented associative differential graded algebras.

The functor $N(-)$

$$\Bbbk\text{-}\mathbf{DGM}^{\mathbf{\Delta}^{op}} \to \Bbbk\text{-}\mathbf{DGM}$$

is a monoidal functor. The functor $N$ transforms operads $\mathcal{O}$ in $\Bbbk\text{-}\mathbf{DGM}^{\mathbf{\Delta}^{op}}$ (resp. $\mathcal{O}$-algebras in $\Bbbk\text{-}\mathbf{DGM}^{\mathbf{\Delta}^{op}}$ ) into operads in $\Bbbk$-**DGM** (resp. agebras over an operad in $\Bbbk$-**DGM** ), [17]-page 51. For further use we need the slightly more general result.

**2.2 Lemma.** *Let* $\underline{A} = \{A_{p,q}\}_{p\in\mathbb{N},q\in\mathbb{Z}}$ *be a simplicial* $\mathcal{O}$*-algebra. Then the total complex* $\mathrm{Tot}(\underline{A})$ *is an* $\mathcal{O}$*-algebra. Moreover,* $N_*(\underline{A})$ *is an* $\mathcal{O}$*-algebra and the quotient map* $\mathrm{Tot}(\underline{A}) \to N_*(\underline{A})$ *is an equivalence of* $\mathcal{O}$*-algebras.*

**Proof.**

Let $\underline{A}$ and $\underline{B}$ be two simplicial graded modules and consider the *shuffle product*

$$sh : C_p\underline{A}\otimes C_q\underline{B} \to C_{p+q}(\underline{A}\times\underline{B})\,, a\otimes b \mapsto \sum_{\mu,\nu}(-1)^{\epsilon(\mu)} s_\nu a \times s_\mu b\,, \left\{ \begin{array}{ll} a & \in A_p \\ b & \in B_q \end{array}\right.$$

when the sum is taken over the $p+q$ shuffles $\mu_1 < \mu_2 < \cdots < \mu_p$,$\nu_1 < \nu_2 < \cdots < \nu_q$, $\mu_i\,,\nu_j \in \{1,2,\dots,p+q\}$, $\epsilon(\mu)$ is the graded signature of the $(p,q)$-shuffle, $s_\mu = s_{\mu_p}\circ s_{\mu_{p-1}}\circ\cdots\circ s_{\mu_1}$ , $s_\nu = s_{\nu_q}\circ s_{\nu_{q-1}}\circ\cdots\circ s_{\nu_1}$, [19]-Chapter 8. If we assume that $\underline{A}$ and $\underline{B}$ are two simplicial differential graded modules, one easily check that $sh$ commutes with the differentials so that we obtain:

$$sh : \mathrm{Tot}\underline{A}\otimes\mathrm{Tot}\underline{B} \to \mathrm{Tot}(\underline{A}\times\underline{B})\,,\ \text{and} \qquad sh : N_p\underline{A}\otimes N_q\underline{B} \to N_{p+q}(\underline{A}\times\underline{B})\,.$$

Since, when $A = B$, the shuflle product is associative (and commutative) one defines the *iterated shuffle product*

$$sh^0 = id\,, sh^{k+1} = (sh\otimes id)\circ sh^k\,, \quad k \geq 0$$

Let $\tilde{\rho}_{n,k} : \mathcal{O}(k) \otimes (\underline{A}_n)^{\otimes k} \to A$ be an evaluation product of $\underline{A}_n$. We consider $\mathcal{O}$ as a constant simplicial module and we define the map $\hat{\rho}_k : \mathcal{O}(k) \otimes (N_*(\underline{A}))^{\otimes k} \to N_*(\underline{A})$ as the composite

$$\begin{array}{ccc} \mathcal{O}(k) \otimes (N_*(\underline{A}))^{\otimes k} & = & N_*(\mathcal{O}(k)) \otimes N_*(\underline{A}))^{\otimes k} \\ & & \downarrow id \otimes sh^k \\ & & N_*(\mathcal{O}(k)) \otimes N_*\left(\underline{A}^{\otimes k}\right) \\ & & \downarrow sh \\ N_*(\underline{A}) & \stackrel{N_*(\tilde{\rho}_k)}{\longleftarrow} & N_*\left(\mathcal{O}(k)) \otimes \underline{A}^{\otimes k}\right) \end{array} .$$

It is then easy to check that the $\hat{\rho}_n$ are composition products.

□

**2.3 The operad $\mathcal{BE}$.** The operad $\mathcal{BE}$ (also called the Barrat-Eccles operad, [1]-1.1.) is an operad in the category of differential graded $\Bbbk$-modules such that:

$$\mathcal{BE}(n) = N_*\left(\mathrm{W}(\Sigma_n)\right) = \text{ the normalized bar construction on } \Sigma_n.$$

**2.4 Some properties of $\mathcal{BE}$.**

1) The operad $\mathcal{BE}$ is a resolution of the operad $\mathcal{C}$: the homomorphism of operads $\overline{\epsilon} : \mathcal{BE} \stackrel{\sim}{\twoheadrightarrow} \mathcal{C}$ is defined by the augmentations of the bar resolution for each component.

2) $\mathcal{BE}$ is an $E_\infty$-operad. Recall that an operad $\mathcal{O} = \{(\mathcal{O}(n))_i\}_{i\geq 0}$ in $\Bbbk$-**DGM** is an $E_\infty$*-operad* if each $\mathcal{O}(n)$ is an acyclic $\Sigma_n$-free module.

3) The natural map $\epsilon : \mathcal{A} \to \mathcal{C}$ factorises as $\mathcal{A} \to \mathcal{BE} \stackrel{\overline{\epsilon}}{\longrightarrow} \mathcal{C}$. In particular, $\mathcal{BE}$-algebras are associative algebras.

4) The operad $\mathcal{BE}$ is not cofibrant.

5) Berger and Fresse [1] have proved that the normalized singular cochains $N^*(-)$ is a functor from the category of topological spaces to the category of $\mathcal{BE}$-differential algebras.

6) If $\Bbbk = \mathbb{F}_p$ is the prime field of characteristic $p$ and $A$ is a $\mathcal{BE}$-algebras then $H(A)$ is an unstable algebra over the big Steenrod algebra ([21], [8]). Indeed, consider the Standard small free resolution $\mathcal{W}$ (resp. $\mathcal{W}'$ of the cyclic group of order $p$; $\pi \subset \Sigma_p$ (resp. of $\Sigma_p$. Thus we obtain the homomorphism $\mathcal{W} \to \mathcal{W}' \to \mathcal{BE}(p)$ and the evaluation product $\tilde{\rho}_p : \mathcal{BE}(p) \otimes A^p \to A$ restricts to the structure map $\mathcal{W} \otimes A^{\otimes p} \to A$ considered by May, [21], in order to define "algebraic Steenrod operations". Recall that the big Steenrod algebra, denoted $B_p$, is such that the quotient $B_p/(P^0 = id)$ is the usual Steenrod algebra, see [20]-theorem 1.4. In particular, Adem relations are satisfied ([21] and [8]).

**2.5 The operad $\mathcal{C}_\infty$.** Let $\mathcal{C}_\infty$ be a cofibrant replacement of $\mathcal{C}$. There exists a quasi-isomorphism of operads

$$\mathcal{C}_\infty \to \mathcal{BE}.$$

In particular, by remarks 5) and 6) above, $N^*(X)$ is a $\mathcal{C}_\infty$-algebra and any quasi-isomorphism of $\mathcal{C}_\infty$-algebras $A \to N^*(X)$ identifies the action of the large Steenrod operations on $H(A)$ to the usual action on $N^*(X)$.

By (1.5) any $\mathcal{C}_\infty$-algebra admits a cofibrant replacement which is an almost free $\mathcal{C}_\infty$-algebra, see [13] and [2]

## 3. The Operadic Hochschild chain complex.

**3.1** Let us recall that the category of $\mathcal{O}$-algebras has all limits and colimits, [10]-Theorem 1.13. In particular, [20]-3, the coproduct of two $\mathcal{O}$-algebras $A$ and $B$ is an $\mathcal{O}$-algebra, denoted $A \coprod B$. Hereafter, we will use the following notation: $A \xrightarrow{l} A \coprod B \xleftarrow{r} B$ for the natural inclusions and $A \coprod A \xrightarrow{\nabla} A$ for the folding map. The symmetric group $\Sigma_n$ acts on $A^{\amalg n} = A \coprod \cdots \coprod A$ ($n$ terms) by permutations of factors. We denote by $\tau_n : A^{\amalg n} \to A^{\amalg n}$ the homomophism corresponding to the cyclic permutation $(n, 1, 2, \ldots, n-1) \in \Sigma_n$.

**3.2** Let $\mathcal{O}$ be an operad in $\Bbbk$-**DGM** and $A$ be an $\mathcal{O}$-algebra. We denote by $\underline{A}$ the simplicial $\mathcal{O}$-algebra

$$\underline{A}_n = A^{\amalg\, n+1}\,, n \geq 0\,, \quad d_i : A^{\amalg\, n+1} \to A^{\amalg\, n}\,, \quad s_i : A^{\amalg\, n} \to A^{\amalg\, n+1}$$

$$d_i = \begin{cases} id^{\amalg\, i} \coprod \nabla \coprod id^{\amalg\, n-i} & \text{if } i = 0, 1, \ldots, n-1 \\ \left(\nabla \coprod id^{\amalg\, n}\right) \circ \tau_n & \text{if } i = n \end{cases},$$

$$s_i = id^{\amalg\, i} \coprod l \coprod id^{\amalg\, n-i}\,.$$

The normalization $N(\underline{A})$, (see 2.2), of the simplicial $\mathcal{O}$-algebra $\underline{A}$ is an $\mathcal{O}$-algebra, denoted $C_*^{\mathfrak{H}}(\mathcal{O}, A)$, and called the *Hochschild chain complex of the $\mathcal{O}$-algebra $A$*. The homology of the $C_*^{\mathfrak{H}}(\mathcal{O}, A)$ is the *operadic Hochschild homology*, denoted $H_*^{\mathfrak{H}}(\mathcal{O}, A)$.

**3.3** If $(A, d_A)$ is any *associative* differential graded algebra supposed unital and augmented we denote by $\overline{A}$ the kernel of the augmentation. The (classical) Hochschild chain complex is defined as follows:

$$\mathfrak{C}_* A = \{\mathfrak{C}_k A\}_{k \geq 0}\,, \quad \mathfrak{C}_k A = A \otimes s\overline{A}^{\otimes k}\,,$$

with $A = \Bbbk \oplus \overline{A}$. A generator of $\mathfrak{C}_k A$ is written $a_0[sa_1|sa_2|\dots|sa_k]$ if $k > 0$ and $a[\,]$ if $k = 0$. We set $\epsilon_i = |a_0| + |a_1| + |sa_2| + \cdots + |sa_{i-1}|$, $i \geq 1$. The differential $d = d^1 + d^2$ is defined by:

$$\begin{aligned} d^1 a_0[a_1|a_2|\dots|a_k] &= da_0[a_1|a_2|\dots|a_k] - \sum_{i=1}^{k} (-1)^{\epsilon_i} a_0[a_1|\dots|da_i|\dots|a_k] \\ d^2 a_0[a_1|a_2|\dots|a_k] &= (-1)^{|a_0|} a_0 a_1[a_2|\dots|a_k] + \sum_{i=2}^{k} (-1)^{\epsilon_i} a_0[a_1|\dots] \\ &\quad |a_{i-1}a_i|\dots|a_k - (-1)^{|sa_k|\epsilon_k} a_k a_0[a_1|\dots|a_{k-1}] \end{aligned}$$

Consider the shuffle map, [18] 4.2.1, $sh : \mathfrak{C}_* A \otimes \mathfrak{C}_* A \longrightarrow \mathfrak{C}_*(A \otimes A)$ defined by:

$$sh(a_0[a_1|a_2|\dots|a_n], b_0[b_1|b_2|\dots|b_m]) =$$

$$(-1)^t \sum_{\sigma \in \Sigma_{n,m}} = (-1)^{\epsilon(\sigma)} a_0 \otimes b_0[c_{\sigma(1)}|\dots|c_{\sigma(m+n)}]$$

where $t = |b_0|(|a_0| + \cdots + |a_n|)$, $c_{\sigma(i)} = \begin{cases} a_{\sigma(i)} \otimes 1, & 1 \leq i \leq n \\ 1 \otimes b_{\sigma(i-n)}, & n+1 \leq i \leq n+m \end{cases}$
and $\epsilon(\sigma) = \sum |c_{\sigma(i)}||c_{\sigma(m+j)}|$, summed over all pairs $(i, m+j)$ such that $\sigma(m+j) < \sigma(i)$. Clearly, $sh$ induces a chain map still denoted $sh$, $\mathfrak{C}_* A \otimes \mathfrak{C}_* A \longrightarrow \mathfrak{C}_*(A \otimes A)$. Let $HH_*(A)$ be the homology of $\mathfrak{C}_* A$.

If $A$ is commutative (in the graded sense) then the multiplication $\mu_A : A \otimes A \to A$ is a homomorphism of differential graded algebras. Thus the composite $\mathfrak{C}_* \mu_A \circ sh : \mathfrak{C}_* A \otimes \mathfrak{C}_* A \to \mathfrak{C}_* A$ defines a multiplication on $\mathfrak{C}_* A$ which makes it into a commutative differential graded algebra [18] 4.2.2.

If $A$ is an associative (non commutative) differential graded algebra, there is no interesting product on $\mathfrak{C}_*(A)$ while $C_*^{\mathfrak{H}}(\mathcal{A}, A)$ is naturally an associative differential graded algebra. Obviously, $\mathfrak{C}_*(A)_\# \neq C_*^{\mathfrak{H}}(\mathcal{A}, A)_\#$. N. Bitjong and second named author, [3], have proved that for any strongly homotopy commutative $\Bbbk$-algebra $A$, in the sense of [22], there is a well defined cup product on $HH_*(A)$ , ([3]-Theorem 1), which is induced from a non canonical product on $\mathfrak{C}_*(A)$. In the formalism of operads, a strongly homotopy commutative $\Bbbk$-algebra $A$ is an associative $\mathcal{B}_\infty$-algebra, in the sense of [10]- 5.2, (see [3]-Proposition 2). The graded vector space $C_*^{\mathfrak{H}}(\mathcal{B}_\infty, A)$ is not isomorphic to $\mathfrak{C}_*(A)$. An interesting question is: *Let $A$ be a strongly homotopy commutative algebra $A$ does $H_*^{\mathfrak{H}}(\mathcal{B}_\infty, A) \cong HH_*(A)$ as commutative graded algebras?*

**3.4** Let $A$ be an associative (unital) differential graded algebra. There is classically associated to $A$ an other simplicial differential graded algebra, which we denote $\underline{\underline{A}}$ and which is defined as follows (see [14]-Exemple 1.4):

$$\underline{\underline{A}}_n = A^{\otimes n+1}\,, n\geq 0\,, \quad d_i : A^{\otimes n+1}\to A^{\otimes n}\,, \quad s_i : A^{\otimes n}\to A^{\otimes n+1}$$

$$d_i = \begin{cases} id^{\otimes i}\otimes\mu_A\otimes id^{\otimes n-i} & \text{if } i=0,1,\dots,n-1 \\ (\otimes\mu_A\otimes id^{\otimes n})\circ\tau_n & \text{if } i=n \end{cases} ,$$

$$s_i = id^{\otimes i}\otimes 1\otimes id^{\otimes n-i}\,.$$

where $\mu_A$ demotes the multiplication of $A$ and $\tau_n$ the map $a_0\otimes a_1\otimes\cdots\otimes a_n \mapsto (1)^{(|a_0|+\cdots+|a_{n_1}|)|a_n]} \to a_n\otimes a_0\otimes\cdots\otimes a_{n-1}$.

The complex $\tilde{\mathfrak{C}}(A) = N_*\underline{\underline{A}}$ is the *unreduced Hochschild chain complex*. By [19]-Chapter X-Corollary 2.2 and Theorem 9.1, the maps $(id\otimes s^{\otimes n})$ define a quasi-isomorphism

$$(1) \qquad \tilde{\mathfrak{C}}(A)\to\mathfrak{C}_*(A)\,.$$

If we assume that $A$ is commutative, $\tilde{\mathfrak{C}}_*(A)$ is also a differential graded algebra. The multiplication is the shuffle product defined in the same way that the shuffle product on $\mathfrak{C}_*(A)$. Therefore the quasi-isomorphism (1) is a homomorphism of differential graded algebras.

**3.5 Theorem A.** *Assume that $A$ is associative commutative differential graded algebra. There exists a natural quasi-isomorphism of commutative differential graded algebras*

$$C_*^{\mathfrak{H}}(\mathcal{C},A)\to\mathfrak{C}(A)\,.$$

**Remark.** If the $\mathcal{C}$-algebra $A$ is considered as a $\mathcal{C}_\infty$-algebra then, by naturality there is a surjective homomorphism of $\mathcal{C}_\infty$-algebras

$$C_*^{\mathfrak{H}}(\mathcal{C}_\infty,A)\to C_*^{\mathfrak{H}}(\mathcal{C},A)\,.$$

Does this map induces an isomorphism in homology?

**Proof.** Let $B$ be a $\mathcal{C}$-algebra. By universal property, there exists a natural isomorphism $\Phi_{A,B}:A\coprod B\to A\otimes B$ of commutative differential graded algebras such that the following diagrams commute where we put $\Phi_{A,A}=\Phi_1$.

$$\begin{array}{ccc} A\coprod A & \xrightarrow{\nabla} & A \\ \Phi_1\downarrow & & || \\ A\otimes A & \xrightarrow{\mu} & A \end{array} \quad , \quad \begin{array}{ccc} A\coprod A & \xrightarrow{T} & A\coprod A \\ \Phi_1\downarrow & & || \\ A\otimes A & \xrightarrow{T} & A\otimes A \end{array}$$

where, $\mu$ denotes the usual product on $A \otimes A$ and $T$ the usual twisting map. The associativity properties permit iteration so that we obtain for any $n \geq 0$ an isomorphism $\Phi_n : A^{\coprod n_1} \to A^{\otimes n+1}$. These $\Phi_n$'s induce an isomorphism of simplicial differential graded modules $\underline{A} \to \underline{\underline{A}}$ which in turn induces an isomorphism

$$(2) \qquad C_*^{\mathfrak{H}}(A) = N_*(\underline{A}) \to N_*(\underline{\underline{A}}) = \tilde{\mathfrak{C}}_*(A)\,.$$

Composition of the isomorphism (2) with the quasi-isomorphism (1) gives the quasi-isomorphism

$$(3) \qquad C_*^{\mathfrak{H}}(A) \to \mathfrak{C}_*(A)\,.$$

On the other hand, the $\mathcal{C}$-algebra structure on $C_*^{\mathfrak{H}}(A)$ is such that the isomorphism (2) is an isomorphism of differential graded algebras. Thus (3) is a quasi-isomorphism of commutative differential graded algebras.

□

**3.6** An operad $\mathcal{C}_\infty$ is a Hopf operad "up to homotopy" (for the notion of Hopf operad we refer to [10]-5.3), that is to say it has a diagonal wich is not coassociative but only up to homotopy. In this case, the tensor product of two $\mathcal{C}_\infty$-algebras $A$ and $B$ is a $\mathcal{C}_\infty$-algebra [7] with underlying differential graded module being the tensor product of the underlying differential graded modules, denoted $A \otimes B$. Indeed, Hinich [13] has proved that if the $\mathcal{C}_\infty$-algebras, $A$ and $B$ are cofibrant there exists a natural quasi-isomorphism

$$\Phi_{A,B} : A \coprod B \to A \otimes B\,.$$

Let $f : A \to A'$ and $g : B \to B'$ be two homomorphisms of $\mathcal{C}_\infty$-algebras. Then, by naturallity of $\Phi_{A,B}$, we obtain the commutative diagram

$$\begin{array}{ccc} A \coprod B & \stackrel{f \coprod g}{\to} & A' \coprod B' \\ \Phi_{A,B} \downarrow & & \downarrow \Phi_{A',B'} \\ A \otimes B & \stackrel{f \otimes g}{\to} & A' \otimes B' \end{array}$$

If we assume that $A$ and $A'$ are cofibrant and that $f$ and $g$ are quasi-isomorphisms, then by [20]-Theorem 3.2, the homomorphism $f \coprod g$ is a quasi-isomorphism. Therefore, $f \otimes g$ is also a quasi-isomorphism.

**3.7** Assume that $A = F(\mathcal{O}, V)$ is an almost free $\mathcal{O}$-algebra. Thus, we have the sequence of direct summands

$$\underline{A}_n \supset \mathcal{O}(k) \otimes_{\Sigma_k} (V^{\oplus n+1})^{\otimes k} \supset \mathcal{O}(k) \otimes_{\Sigma_k} \left(V^{\otimes k_0} \otimes V^{\otimes k_1} \otimes \cdots \otimes V^{\otimes k_n}\right)\,,$$

with $k = k_0 + k_1 + \cdots + k_n$. Therefore, an element of $\underline{A}_n$ is finite sum of elements of the form $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k$ with $x \in \mathcal{O}(k)$ and $v_{k_0+\cdots+k_{i-1}+1} \dots v_{k_0+\cdots+k_i} \in V^{\otimes k_i}$ with usual convention $V^{\otimes 0} = \Bbbk$. With this notation we obtain explicit formulas for the map $r$, $l$ and $\nabla$ defined in 3.1:

$$\begin{aligned}
r : \underline{A}_0 \to \underline{A}_1 \quad & r(x \otimes v_1 \dots v_{k_0} = x \otimes v_1 \dots v_{k_0} \otimes 1 \\
l : \underline{A}_0 \to \underline{A}_1 \quad & l(x \otimes v_1 \dots v_{k_0} = x \otimes 1 \otimes v_1 \dots v_{k_0} \\
\nabla : \underline{A}_1 \to \underline{A}_0 \quad & \nabla(x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1}) \\
& \qquad = x \otimes v_1 \dots v_{k_0} v_{k_0+1} \dots v_{k_0+k_1} .
\end{aligned}$$

Therefore,

$$\begin{aligned}
& d_i : \underline{A}_{n+1} \to \underline{A}_n , \\
& d_i \left( x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k \right) = \\
& \begin{cases}
x \otimes v_1 \dots v_{k_0} \otimes \cdots \otimes v_{k_0+k_i+1} \dots v_{k_0+\cdots+k_{i+2}} \otimes \dots \\
\qquad \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k \qquad \text{if } i = 0, 1, \dots, n-1 \\
= x \otimes v_{k_0+\cdots+k_n+1} \dots v_k v_1 \dots v_{k_0} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-2}+1} \dots \\
\qquad v_{k_0+\cdots+k_{n-1}} \qquad \text{if } i = n
\end{cases} \\
& s_i : \underline{A}_n \to \underline{A}_{n+1} , \\
& s_i \left( x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k \right) = \\
& \qquad x \otimes v_1 \dots v_{k_0} \otimes \cdots \otimes 1 \otimes \cdots \otimes v_{k_0+\cdots+k_i+1} \dots v_{k_0+\cdots+k_{i+1}} \otimes \dots \\
& \qquad \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k .
\end{aligned}$$

Let us denote by $\underline{A}_n^+$ the submodule of the $\Bbbk$-module $\underline{A}_n$ generated by the elements of the form $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k$ with $x \in \mathcal{O}(k)$ and $v_{k_0+\cdots+k_{i-1}+1} \dots v_{k_0+\cdots+k_i} \in V^{\otimes k_i}$ such that each $k_i > 0$. The above fomulas for $d_i$ and $s_i$ show that

a) the graded module $\underline{A}_n^+$ is stable for the $d_i$'s but not stable for the $s_i$'s.

b) the submodule $D\underline{A}_n$ of $\underline{A}_n$ generated by all degenerate elements ($D\underline{A}_0 = 0$) is exactly the submodule generated by the elements $x \otimes v_1 \dots v_{k_0} \otimes v_{k_0+1} \dots v_{k_0+k_1} \otimes \cdots \otimes v_{k_0+\cdots+k_{n-1}+1} \dots v_k$ such that at least one $k_i = 0$.

Since $C_*^{\mathfrak{H}}(\mathcal{O}, A) = N(\underline{A}) = \mathrm{Tot}(\underline{A})/D(\underline{A})$ and since $d_A(\underline{A}_n^+) \subset \underline{A}_n^+$ we have proved the first part of the next result.

**Theorem B.** *Assume that $A = F(\mathcal{O}, V)$ is an almost free $\mathcal{O}$-algebra. Then the restriction of the natural chain equivalence* $\mathrm{Tot}\underline{A} \to N_*\underline{A}$ *to the total complex,* $\mathrm{Tot}(\underline{A}^+)$ *is an isomorphism of differential graded modules*

$$\mathrm{Tot}(\underline{A}^+) \to C_*^{\mathfrak{H}}(\mathcal{O}, A) .$$

*Moreover, this homomorphism is an isomorphism of $\mathcal{O}$-algebras.*

**End of proof.** Let us precise first that the evaluation products of the simplicial algebra $\underline{A}$ are determined by the maps

$$\mathcal{O}(k)_q \otimes \underline{A}^+_{p_1,q_1} \otimes \underline{A}^+_{p_2,q_2} \otimes \cdots \otimes \underline{A}^+_{p_k,q_k} \to \underline{A}^+_{p_1+p_2+\cdots+p_k,q+q_1+\ldots q_1}$$

which are explicitely given by the shuffle products and the evaluation product of $\mathcal{O}$ (see proof of lemma 2.2). This implies that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(n)_q \otimes \underline{A}^+_{p_1,q_1} \otimes \cdots \otimes \underline{A}^+_{p_n,q_n} & \hookrightarrow & \mathcal{O}(n)_q \otimes \underline{A}_{p_1,q_1} \otimes \cdots \otimes \underline{A}_{p_n,q_n} \\ \downarrow & & \downarrow \\ \underline{A}^+_{p_1+\cdots+p_n,q+q_1+\cdots+q_n} & \hookrightarrow & \underline{A}_{p_1+\cdots+p_n,q+q_1+\cdots+q_n} \end{array}$$

Therefore, each $\underline{A}^+_n$ is a sub $\mathcal{O}$-algebra of $\underline{A}_n$.

□

It results from the formula above that $d_0 = d_1 : \underline{A}_1 \to \underline{A}_0$ and that $d_i \left( \bigoplus_{p>0,q\in\mathbb{Z}} \underline{A}_{p,q} \right) \subset \bigoplus_{p>0,q\in\mathbb{Z}} \underline{A}_{p,q}$. Thus we obtain:

**Proposition.** *Assume that $A = F(\mathcal{O}, V)$ is an almost free $\mathcal{O}$-algebra. Then we have the natural splitting of $\mathcal{O}$-algebras*

$$\mathrm{Tot}(\underline{A}^+) = (A, 0) \oplus \left( \bigoplus_{p>0} \underline{A}^+_{p,q}, d \right) .$$

## 4. Free loop space.

**4.1** Write **Top** (resp. **Costop**) for the category of topological spaces (resp. of cosimplicial topological spaces). The geometric realization of a cosimplicial set is the covariant functor

$$||.|| : \mathbf{Costop} \to \mathbf{Top}\,, \quad \underline{Z} \mapsto ||\underline{Z}|| = \mathbf{Costop}(\underline{\triangle}, \underline{Z}) \subset \prod_{n\geq 0} \mathbf{Top}(\triangle^n, \underline{Z}(n))\,,$$

where $||\underline{Z}||$ is equipped with the topology induced by this inclusion. Here $\underline{\triangle}$ denotes the cosimplicial space defined by $\underline{\triangle}(n) = \triangle^n$ with the usual coface and codegeneracy maps. If $\underline{Z}$ is any cosimplicial topological space, then

$N^*\underline{Z}$ is a simplicial cochain complex with total complex $\mathrm{Tot}(N^*\underline{Z})$:

$$(\mathrm{Tot}_n(N^*\underline{Z}) = \oplus_{p-q=n} N^q \underline{Z}(p)\,,$$
$$Dx = \sum_{i=1}^{p} (-1)^i C^*(d_i) + (-1)^p \delta x\,, \quad x \in N^*\underline{Z}(p)\,.$$

The $d_i$ are the coface operators of $\underline{Z}$ and $\delta$ is the internal differential of $N^*(\underline{Z}(p))$. Recall that in general the natural map $\mathrm{Tot}(N^*\underline{Z}) \to N^*(||\underline{Z}||)$ is not a weak equivalence, [5]. It results from lemma 2.2 and 2.3-5 that the total complex $\mathrm{Tot}(N^*(\underline{Z}))$ is naturally a $\mathcal{C}_\infty$-algebra and $\mathrm{Tot}(N^*(\underline{Z})) \to N^*(||\underline{Z}||)$ a homomorphism of $\mathcal{C}_\infty$-algebras.

Hereafter we denote $N^*\left(N^*(\underline{Z})\right)$ the normalization of $\mathrm{Tot}\left(N^*(\underline{Z})\right)$.

**4.2** One of the interest for considering cosimplicial spaces is the following result, [5]-Proposition 5.1, (see also [23]-Corollary 1): *If $\underline{L}$ is a simplicial set and $T$ a topological space then the cosimplicial space $T^{\underline{L}}$ is such that there is a homeomorphism:*

$$||T^{\underline{L}}|| := \mathbf{Costop}(\underline{\triangle}, T^{\underline{L}}) \cong \mathbf{Top}(|\underline{L}|, T) = T^{|\underline{L}|}\,.$$

In particular, if we consider the simplicial set $K$ defined as follows: $K(n) = \mathbb{Z}/(n+1)\mathbb{Z}$, and, if $\overline{k}^n$ denotes an element in $\mathbb{Z}/n\mathbb{Z}$, the face maps $d_i : K(n) \to K(n-1)$ with $0 \le i \le n-1$ and the degeneracy maps $s_j : K(n) \to K(n+1)$ with $0 \le j \le n$ are:

$$d_i \overline{k}^{n+1} = \begin{cases} \overline{k}^n & \text{if } k \le i \\ \overline{k-1}^n & \text{if } k > i \end{cases} \qquad s_j \overline{k}^{n+1} = \begin{cases} \overline{k}^{n+2} & \text{if } k \le j \\ \overline{k+1}^{n+2} & \text{if } k > j. \end{cases}$$

and $d_n \overline{k}^{n+1} = \overline{k}^n$. The geometric realization of $K$, [6] (proposition 1.4), $|K|$ is homeomorphic to the circle $S^1$. Therefore, the cosimplicial model, $\underline{X}$, for the free loop space, used by Jones, [14],

$$\begin{array}{ll} \underline{X}(n) & = Map(K(n), X) = \underbrace{X \times \cdots \times X}_{(n+1)\text{-times}} \\ d_i(x_0, x_1, \ldots, x_n) & = (x_0, x_1, \ldots, x_i, x_i, \ldots x_n)\,, 0 \le i \le n \\ d_{n+1}(x_0, x_1, \ldots, x_n) & = (x_0, x_1, \ldots, x_n, x_0) \\ s_j(x_0, x_1, \ldots, x_n) & = (x_0, x_1, \ldots x_j, x_{j+2}, \ldots, x_n)\,, 0 \le j \le n. \end{array}$$

is such that $||\underline{X}|| \cong \mathbf{Top}(|\underline{K}|, X) = X^{S^1}$, ($\cong$ means homeomorphism). From lemma 2.2, we deduce then:

**Proposition.** *If $X$ is simply connected, the natural map* $\mathrm{Tot}(N^*\underline{X}) \to N^*(X^{S1})$ *is a quasi-isomomorphism of $\mathcal{C}_\infty$-algebras.*

**4.3 Theorem C.** *Let $X$ be a simply connected space such that each $H^i(X)$ is finitely generated. Given an almost free model of the space $X$*

$$\varphi_X : M_X = (F(\mathcal{C}, V), d) \to N^*(X)$$

*there exists a natural quasi-isomorphism of $\mathcal{C}_\infty$-algebras*

$$C_*^{\mathfrak{H}}(\mathcal{C}_\infty, M_X) \longrightarrow N^*\left(N^*(\underline{X})\right) .$$

**Proof.** Let $\varphi_X : M_X = (F(\mathcal{C}, V), d) \to N^*(X)$ (resp. $\varphi_Y : M_Y = (F(\mathcal{C}, V), d) \to N^*(Y))$ be a almost free model for the space $X$ (resp. for the space $Y$). By universal property, we obtain the commutative diagram

$$\begin{array}{ccccc} M_X & \xrightarrow{l} & M_X \coprod M_Y & \xleftarrow{r} & M_Y \\ \sim\downarrow \varphi_X & & \psi_{X,Y}\downarrow & & \sim\downarrow \varphi_Y \\ N^*(X) & \xrightarrow{N^*(pr_X)} & N^*(X\times Y) & \xleftarrow{N^*(pr_Y)} & N^*(Y) \end{array}$$

where $pr_X$ and $pr_Y$ (resp. $i_X$ and $i_Y$) are the natural projections (resp. inclusions). We have also the following commutative diagrams

$$\begin{array}{ccc} M_X \coprod M_X & \xrightarrow{\nabla} & M_X \\ \psi_{X,X}\downarrow & & \sim\downarrow \varphi_X \\ N^*(X\times X) & \xrightarrow{N^*(\Delta_X)} & N^*(X) \end{array} \qquad \begin{array}{ccc} M_X \coprod M_X & \xrightarrow{T^{\coprod}} & M_X \coprod M_X \\ \psi_{X,X}\downarrow & & \downarrow \psi_{X,X} \\ N^*(X\times Y) & \xrightarrow{N^*(T_X)} & N^*(X) \end{array}$$

where $\Delta_X$ is the diagonal and $T_X$ the topological interchange map.

By [20]-Lemma 5.2, we know that if each $H^i(X)$ is finitely generated then $\psi_{X,X}$ is a weak equivalence of differential graded module. It is now easy to prove that iteration furnishes a quasi-isomorphism of simplicial $\mathcal{C}_\infty$-algebras $\underline{M}_X \to N^*(\underline{X})$.

□

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# References

[1] C. Berger and B. Fresse, *Combinatorial operad action on cochains*, to appear in J. of Cambridge Math. Phil. Soc.

[2] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operad.* Preprint Univ. Nice **651** (2002).

[3] N. Bitjong and J-C. Thomas, *On the cohomology algebra of free loop spaces*, Topology **41** (2002) 85–106.

[4] N. Bitjong and J-C. Thomas, *Steenrod operations and Hochschild homology*, Preprint Université d'Angers (2000).

[5] R. Bott and G. Segal, *The cohomology of the vector fields on a manifold*, Topology **16** (1977) 285–298.

[6] D. Burghelea and Z. Fiedorowicz, *Cyclic homology and algebraic K-theory of spaces -II*, Topology **25** (1986) 303–318.

[7] D. Chataur, *Formes diffÚrentielles sur une opérade et modèle algÚbrique d'une fibration.* Algebraic and Geometric Topology **2** (2002) 51–93.

[8] D. Chataur and M. Livernet *Operadic description of Steenrod operations.* Preprint C.R.M. **507** (2002).

[9] A. Dold, *Über die Steenrodschen Kohomologieoperationen*, Annals of Mathematics **73** (1961) 258–294.

[10] E. Getzler and J.D.S. Jones, Operads, homotopy algebras, and iterated integrals for double loop spaces Preprint 1994 http://arXiv.org/abs/hep-th:9403055.

[11] V. Ginsburg and M. M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1994) 203–288.

[12] V. A. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997) 3291–3323.

[13] V. A. Hinich, *Virtual operad algebras and realization of homotopy types.* J. Pure and Applied Algebra **159** (2000) 173–185.

[14] J. D.S. Jones, *Cyclic homology and equivariant homology.* Invent. math. **87** (1987) 403–423.

[15] T. Kadeishvili, *$A_\infty$-algebra structure in cohomology and the rational homotopy type*, Forschungsschwerpunkt Geometrie, Universität Heidelberg, Mathematishes Institut **37** (1988) 1–64.

[16] M. Kontsevich, *Formal (non)-commutative symplectic geometry*, in "The Gelfand mathematics seminar, 1990–1992" eds L.Coerwin, I. Gelfand, J. Lepowsky, Birkhauser, Boston-Basel-Berlin, 1993.

[17] I. Kriz and P. May, *Operads, algebras, modules and motives*, AstÚrisque **223** 1995.

[18] J-L. Loday, *Cyclic homology*, Grundleren der mathematischen Wissenschaften **301** Springer-Verlag 1991.

[19] S. Mac Lane, *Homology*, Grundleren der mathematischen Wissenschaften Springer-Verlag **114** 1963.

[20] M.A. Mandell, *$E_\infty$-algebras and p-adic homotopy theory*, Topology **40** (2001) 43-94.

[21] J.P. May, *A general algebraic approach to Steenrod operations*, Springer Lecture Notes in Math, **168** (1970) 153-231.

[22] H. J. Munkholm, *The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps*, Jour. of Pure and Appl. Alg. **5** (1974) 1-50.

[23] F. Patras and J.-C. Thomas, *Cochain algebras of mapping spaces and finite group actions.* To appear in Topology and its Applications (2003)

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