Periodic Travelling Waves in Nonlinear Reaction–Diffusion Equations via multiple Hopf Bifurcation*

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Abstract

We keep track of periodic wave trains for some classes of one dimensional nonlinear reaction–diffusion partial differential equations. This periodic wave trains can be seen as limit cycles of some planar differential systems. We use standard techniques within the multiple Hopf bifurcation framework.

Keyboards: Reaction–diffusion equations, periodic travelling waves, wave trains, Hopf bifurcation of limit cycles, Kukles' and Liénard's equations.

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1 Introduction

In this paper we study the existence of *periodic travelling wave solutions* for some non-degenerate one dimensional reaction–diffusion equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + f(u, u_x), \tag{1}$$

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where the diffusion function is such that $D(u) > 0$, for all $u \geq 0$. This type of equation arises in many mathematical models for the study of biological and chemical systems, as well as in heat transfer (see [12, 17, 24, 30], and [34] and references therein for example).

A *travelling wave* of equation (1) is a solution of the form $u(x,t) = \phi(x - ct)$. A *periodic wave train* (PWT, from now on) is a travelling wave which corresponds to a periodic function ϕ. The existence of these type of solutions can be determined by studying the main features of a system of ordinary differential equations called *travelling wave system.* This system is obtained as follows: Set $u(x,t) = \phi(\xi)$ (where $\xi = x - ct$), by direct substitution of this expression in equation (1), we obtain

$$D(\phi)\ddot{\phi} + c\dot{\phi} + D'(\phi)(\dot{\phi})^2 + f(\phi, \dot{\phi}) = 0,$$

where $\dot{\{\}} := d/d\xi$. Taking $v := \dot{\phi}$, we have

$$\begin{cases} \dot{\phi} & = v, \\ D(\phi)\dot{v} & = -cv - f(\phi, v) - D'(\phi)v^2. \end{cases}$$

After the time reparametrization given by $d\xi/d\tau = D(\phi)$, we have

$$\begin{cases} \phi' & = D(\phi)\, v, \\ v' & = -cv - f(\phi, v) - D'(\phi)v^2. \end{cases} \tag{2}$$

where $\{'\} := d/d\tau$.

In the case that a singular point of system (2) is a centre (i.e. there is a neighbourhood of the singular point (called the *period annulus*) foliated of periodic orbits) there is a continuum of periodic wave trains unfolding the homogeneous steady-state solution which corresponds to this singular point. On the other hand, the presence of limit cycles (that is, *isolated* periodic orbits) in system (2), leads to the existence of *isolated periodic travelling waves* or *isolated periodic wave trains* (IPWT, from now on).

Therefore, the main idea of the paper is the following: by keeping track of the periodic orbits that can appear in the travelling wave system, we are able to find conditions for the existence of periodic wave train solutions of equation (1). The bifurcation of PWT for two dimensional reaction–diffusion equations via *generic* Hopf bifurcation have been deeply studied previously (see [12, 13, 16, 17, 11] or [24] and references therein). However in this paper we mainly focus on the bifurcation of *small amplitude isolated periodic wave trains* (SAIPWT), via *multiple* Hopf bifurcation.

All the results of the paper rely on known theoretical developments of planar ordinary differential equations. Except on the ones concerning the worked example in Section 4.1.1, no new results concerning planar differential systems are presented. Our contribution consists in the intensive use of this results in the context of the search of PWT for one dimensional reaction–diffusion PDE. Our aim is also to point out that the use of the main techniques for the study of limit cycles of planar differential system can contribute to the theory of travelling waves.

The paper is structured as follows: In Section 3 a generic Hopf bifurcation (AH bifurcation, from now on) of SAIPWT result is presented for equations of type (1) under some additional conditions (which are inspired in the ones studied by Sánchez–Garduño and Maini in [30, 31, 32] and [33]) that lead to the existence of a singular point of weak focus type for system (2).

Section 4 is devoted to study the Hopf bifurcation of SAIPWT for PDE of type (1) with constant diffusion and cubic reaction term. This solutions can be seen as limit cycles of a Kukles' system, which constitutes a well studied family of differential systems.

In Section 5, we use the Corduneanu's idea ([7]), to we present a multiple Hopf bifurcation result for PDE of type $u_t = [D(u)u_x]_x + f(u, u_x)$, where $f(u, u_x) = h(u)u_x + g(u)$. These PWT can be seen as limit cycles of a Liénard's system. We also make review on the known results on limit cycles in Liénard's equations that give rise to IPWT (not necessarily coming from Hopf bifurcation) for this class of reaction–diffusion equations.

For the sake of completeness we present a brief description of the AH limit cycles' bifurcation framework, and the Lyapunov–Poincaré centre problem in Section 2.

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2 Lyapunov–Poincaré centre problem and the multiple Hopf Bifurcation

An isolated singular point of a non–linear planar analytic differential system is said a *weak focus*, if it has a pair of purely imaginary eigenvalues. When the singular point is at the origin the system may be written as

$$\begin{cases} \dot{x} &= -y + P(x,y), \\ \dot{y} &= x + Q(x,y), \end{cases} \tag{3}$$

where P and Q are analytic in a neighbourhood of the origin and with terms of order at least 2. In this case, the origin can be either a centre or a focus. The problem of distinguishing between these two types of singular points is known as the *Lyapunov–Poincaré centre problem*, and it is still an open problem in the sense that, we only know the characterization of the centres for few families of planar differential systems.

For such a system we consider the *Poincaré's first return map*, which is analytic in a neighbourhood of the origin, given by $\Pi : [0,\alpha) \longrightarrow \mathbb{R}$, where the intervals. At any point $(x,0) \in \mathbb{R}^2$, $0 < x < \alpha$, this map gives the first return $(\Pi(x),0)$ to the semi-axis OX^+ of the orbit passing through $(x,0)$. Obviously $\Pi(0) = 0$. The Taylor series of the return map is given by $\Pi(x) = x + \sum_{n\geq 2} \left(\Pi^{(n)}(0)/n!\right) x^n$. It is well-known (see [1]; Chapter IX) that the first non-vanishing nonlinear term appearing in the return map, if there is one, will have odd degree. The constant $V_{2k+1} = \Pi^{(2k+1)}(0)/(2k+1)!$, is called the *k-th Lyapunov constant*.

The role of the Lyapunov constants in the centre problem is the following: if for some $k \geq 1$ we have $V_3 = V_5 = \cdots = V_{2k-1} = 0$ and $V_{2k+1} \neq 0$ then the origin is a *weak focus* of multiplicity k, while if all the Lyapunov constants are zero then the first return map is the identity map and the origin is a centre. If system (3) is represented by a parametric family, then it can be proved that the Lyapunov constants are polynomials in the coefficients of P and Q. One can therefore obtain necessary conditions for the origin to be a centre by imposing the vanishing of the Lyapunov constants on these coefficients. This means that if P and Q are polynomials of a fixed degree, by the Hilbert's basis Theorem it is only necessary to compute a finite number of Lyapunov constants to obtain the characterization of the family. However, in general, one does not know how to determine if a large enough number of constants have been computed to guarantee that the system has a centre at the origin. There is not known algorithmic approach to answer this question.

The Lyapunov constants also play a role in the study the bifurcation of periodic orbits from a weak focus, the so called *small amplitude limit cycles* (this phenomenon is the very well known Hopf bifurcation). The mechanism of the Hopf bifurcation is the following: Given a parametric family of planar differential systems

$$\begin{cases} \dot{x} &= P_{\Lambda_0}(x,y), \\ \dot{y} &= Q_{\Lambda_0}(x,y), \end{cases} \tag{4}$$

where $\Lambda_0 \in \mathbb{R}^m$ is a parameter's vector, and such that $V_1 := \exp\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$ $(0,0) = 1$, $V_j = 0$ for $j = 2, \ldots, 2k$, and $V_{2k+1} \neq 0$, if we take Λ close enough to Λ_0 and such that

$$|V_{2l-1}| << |V_{2l+1}|, \quad \text{and} \quad V_{2l-1} \cdot V_{2l+1} < 0,$$

for $l = 1, \ldots, k$. At each step, the origin reverses stability and a limit cycle bifurcates in a small region of the singular point. Hence we have bifurcate exactly k small amplitude limit cycles. This mechanism is illustrated constructively in the proof of Theorem 4 of Section 4.1.1.

Using the Hopf bifurcation theory we can obtain "at most type" existence results for SAIPWT. Of course, this type of results can be reinterpreted as "at least type" existence results for IPWT.

3 Bifurcation of SAIPWT from steady states

System (2) defines a one-parameter family of rotated vector fields with parameter c. Recall that a one-parameter family of planar differential system given by

$$\begin{cases} \dot{x} &= P(x,y,c), \\ \dot{y} &= Q(x,y,c), \end{cases} \tag{5}$$

is said to define a one-parameter family of rotated vector fields (or it is a rotated system)

$$X(x,y,c) = P(x,y,c)\frac{\partial}{\partial x} + Q(x,y,c)\frac{\partial}{\partial y}, \tag{6}$$

if the singular points of system (5) are isolated, fixed for c, and for all the regular points we have that

$$\begin{vmatrix} P & Q \\ P_c & Q_c \end{vmatrix} > 0, \tag{7}$$

(respectively < 0). Observe that if we set $\Theta := \arctan(Q,P)$, condition (7) implies that the vector field (6) rotates in positive (respectively negative) sense as c increases.

The singular points of system (2) are isolated, and setting $P(\phi, v, c) := D(\phi)v$ and $Q(\phi, v, c) := -cv - f(\phi, v) - D'(\phi)v^2$, we have that

$$\begin{vmatrix} P & Q \\ P_c & Q_c \end{vmatrix} = -D(\phi)v^2 < 0.$$

Hence system (2) defines a one-parameter family of rotated vector fields.

Duff, in [6], proved that for a rotated system of the form $\dot{\mathbf{x}} = A(c)\mathbf{x} + \mathbf{F}(\mathbf{x}, c)$, such that $\mathbf{F} \in \mathcal{C}^2$ in a neighbourhood of the origin, starting with terms of order at least two, and such that for $c = c_*$ $\mathrm{Spec}(A(c_*)) \in i\,\mathbb{R}$, and the origin is not a centre; the origin absorbs or generates exactly one limit cycle at the bifurcation value $c = c_*$. Therefore we have the following bifurcation result.

Proposition 1. *If $c = c_*$, $P_{\phi_*}(\phi_*, 0)$ is a weak focus of system (2) and it is not a centre, then a unique SAIPWT is absorbed or generated by the steady state $u(x,t) \equiv \alpha$ at the bifurcation value $c = c_*$.*

Example: In [30, 31, 32] and [33], Sánchez–Garduño and Maini have intensively studied the existence of travelling wave solutions of front, pulse and sharp type, as well as the existence of continuum of PWT (in the hamiltonian) case for PDE of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u). \tag{8}$$

In particular, Sánchez-Garduño and Maini in [33], studied the existence of travelling wave solutions of the mentioned type for PDE of the form (8), under the following assumptions

Assumptions (SGM):

(a) $g(0) = g(1) = 0$, and $g(\alpha) = 0$ for a number $\alpha \in (0,1)$, and $g(u) \neq 0$ otherwise.

(b) $g(u) \in \mathcal{C}^2([0,1])$. $g'(0) < 0$, $g'(\alpha) > 0$, and $g'(1) < 0$.

(c) $D \in \mathcal{C}^2([0,1])$; $D(0) = 0$, $D(\phi) > 0$ for all $\phi \in (0,1]$. $D'(u) > 0$ for all $\phi \in [0,1]$, and $D''(0) > 0$.

Conditions (SGMa) and (SGMb) give rise to the existence of the homogeneous steady state solution of (8) $u(x,t) \equiv \alpha$, corresponds to the singular point $P_\alpha(\alpha, 0)$ of the associated travelling wave system (2). If $c = c_* := 0$, then P_α is a Hamiltonian centre of system (2) (giving rise to a continuum of PWT), if $c \neq 0$ and $c^2 < D(\alpha)g'(\alpha)$, then P_α is a robust focus of the planar travelling wave differential system, hence no IPWT bifurcates from this steady state for $c = 0$.

Consider now the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u) + h(u, u_x), \tag{9}$$

satisfying **Assumptions (A):**

(a) $g(0) = g(1) = 0$, and $g(\alpha) = 0$ for a number $\alpha \in (0,1)$, and $g(u) \neq 0$ otherwise.

(b) $g(u) \in \mathcal{C}^2([0,1])$. $g'(0) < 0$, $g'(\alpha) > 0$, and $g'(1) < 0$.

(c) $D \in \mathcal{C}^2([0,1])$; $D(\phi) > 0$ for all $\phi \in (0,1]$.

(d) There exists $\rho > 0$ such that $h(u, u_x) \in \mathcal{C}^2(D_\rho(P_\alpha))$, $h(u,0) = 0$, $v\, h(u,v) \geq 0$ in $D_\rho(P_\alpha) \setminus P_\alpha$, and $h_v(\alpha, 0) = 0$.

Using Duff's Theorem we will see that for $c \lesssim 0$, a unique SAIPWT bifurcates from the steady state $u(x,t) \equiv \alpha$. Indeed, the associated travelling–wave system is

$$\begin{cases} \phi' &= D(\phi)\, v, \\ v' &= -c\, v - g(\phi) - h(\phi, v) - D'(\phi) v^2. \end{cases} \tag{10}$$

Easy computations show that

$$V(\phi, v) = \frac{1}{2}\left(D(\phi)\, v\right)^2 + \int_0^\phi D(s) g(s)\, ds \; - \int_0^\alpha D(s) g(s)\, ds,$$

has a local minimum at P_α. Hence we can use it as Lyapunov Function, to determine the stability of P_α (this functions was introduced in [30] and [33] as a hamiltonian for the travelling wave system associated to the PDE (8) for $c = 0$). The orbital derivative of $V(\phi, v)$ under the flow of system (10) is

$$\begin{aligned} \frac{dV}{d\tau} &= D(\phi) v \left(D'(\phi) v \dot{\phi} + D(\phi) \dot{v} \right) + D(\phi) g(\phi) \dot{\phi} = \cdots \\ &= -c\, D^2(\phi) v^2 - D(\phi) v h(\phi, v). \end{aligned}$$

Using conditions (A) we have that $(dV/d\tau)(\phi, v) < 0$ for all $(\phi, v) \in D_\rho(P_\alpha)$ if $c \geq 0$, hence P_α is an asymptotically stable weak focus for $c = c_* = -h_v(\alpha, 0) = 0$. On the other hand P_α is a robust stable (respectively unstable) focus if $c > c_*$ (respectively if $c < c_*$). Duff's theorem implies that P_α generates a unique limit cycle which grows monotonically when c decreases. This limit cycle corresponds to an IPWT of equation (9) bifurcating from $u(x,t) \equiv \alpha$. □

Consider now that equation (1), satisfying **Assumptions (B):**

(a) $f(0,0) = f(1,0) = 0$, and $f(\alpha, 0) = 0$ for a number $\alpha \in (0,1)$, and $f(u, 0) \neq 0$ otherwise.

(b) $f(\phi, v) \in \mathcal{C}^\omega(\mathbb{R}^2)$. $f_\phi(0,0) < 0, f_\phi(\alpha, 0) > 0$, and $f_\phi(1,0) < 0$.

(c) $D(u) \in \mathcal{C}^\omega(\mathbb{R})$; $D(u) > 0$ for all $u \geq 0$.

Under assumptions (B) (inspired in those of [33]), the steady state $u(x,t) \equiv \alpha$ can be associated to a weak focus of the travelling wave system, which is a possible source of SAIPWT.

Assumptions (Ba) and (Bc), imply that $P_0(0,0)$, $P_1(1,0)$ and $P_\alpha(\alpha,0)$ (where $\alpha \in (0,1)$) are the unique singular points of system (2), which correspond to steady–state solutions of equation (1). First part of Assumptions (Bb) and (Bc) have the following motivation: if both $D(u)$ and $f(u,u_x)$ are analytic functions, then system (2) has analytic right-hand sides, hence if P_α is a weak focus, it can be either a centre or a focus, excluding the possibility of this singular points to be an accumulation of isolated periodic orbits. This is a consequence of the results of finiteness of limit cycles of Ecalle and Il'Yashenko (see [14] for instance).

Let $DX(\phi_*,0)$ denote the differential matrix associated to the singular point $(\phi_*,0)$. Then

$$\mathrm{Spec}\,(DX(\phi_*,0)) = \left\{ \frac{\gamma_{\phi_*} \pm \sqrt{\Delta_{\phi_*}}}{2} \right\},$$

where $\gamma_{\phi_*} := -(c + f_v(\phi_*,0))$, and $\Delta_{\phi_*} := (c + f_v(\phi_*,0)^2 - 4D(\phi_*)f_\phi(\phi_*,0)$.

Taking into account conditions (Bb) and (Bc), is easy to see that P_0 and P_1 are hyperbolic saddles of system (2). Also observe that if $\Delta_\alpha \geq 0$, then P_α is a node, and if $\Delta_\alpha < 0$ and $\gamma_\alpha \neq 0$ then P_α is a focus. In both cases the stability of P_α is given by $\mathrm{sign}(\gamma_\alpha)$. If $\gamma_\alpha = 0$ (that is, when $c = -f_v(\alpha,0)$) then $\Delta_\alpha < 0$, so $\mathrm{Spec}\,(DX(P_\alpha)) \in i\mathbb{R}$. In this case $P(\alpha)$ is a *weak focus.*

Consider equation (1) together with assumptions (B). Set $D(u) = \sum_{i\geq 0} d_i u^i$ and $f(u,u_x) = \sum_{i+j\geq 1} f_{i,j} u^i u_x^j$. An straightforward computation shows that P_α is a weak focus of system (2), if and only if

$$c = c_* := -f_v(\alpha,0).$$

The singular point P_α is a (robust) stable focus if $c - c_* > 0$ (respectively unstable $c - c_* < 0$).

Tedious computations, not reproduced here, show that after the translation of P_α to the origin, and the scaling and reparametrization, given by: $\bar{\phi} = \phi - \alpha$, $\varphi = \bar{\phi}/a$, and $d\tau/ds = b$, where

$$a = -D(\alpha)/\sqrt{D(\alpha)f_\phi(\alpha,0)}, \quad \text{and} \qquad b = 1/\sqrt{D(\alpha)f_\phi(\alpha,0)},$$

system (2), is transformed into system

$$\begin{cases} \dfrac{d\varphi}{ds} &= \sum\limits_{k\geq 0} \delta_k \varphi^k v, \\ \dfrac{dv}{ds} &= -\gamma_{0,1} v + \sum\limits_{k+j\geq 1} \gamma_{k,j} \varphi^k v^j + \sum\limits_{k\geq 0} \mu_k \varphi^k v^2, \end{cases} \tag{11}$$

where

$$\delta_k = b\, a^{k-1} \left[\sum_{n\geq k} d_n \binom{n}{k} \alpha^{n-k} \right]$$

$$\gamma_{k,j} = -b\, a^k \left[\sum_{n\geq k} f_{n,j} \binom{n}{k} \alpha^{n-k} \right]$$

$$\mu_k = -b\, a^k \left[\sum_{n\geq k} n\, d_n \binom{n-1}{k} \alpha^{n-k-1} \right].$$

Observe that system (11) is a system of type (3), since $\delta_0 = -1$, and $\gamma_{1,0} = 1$. From the result in [1], p. 252, the first Lyapunov constant of this system is

$$V_3 = -\frac{\pi}{4} \left(-3\gamma_{0,3} - \gamma_{2,1} + \mu_0 \gamma_{1,1} + \gamma_{0,2}\gamma_{1,1} + \gamma_{2,0}\gamma_{1,1} \right) \tag{12}$$

Hence we obtain the following result:

Proposition 2 (Generic Hopf bifurcation). *Consider equation (1) satisfying assumptions (B). At most one SAIPWT can bifurcate from the steady state solution $u(x,t) \equiv \alpha$, when $c = c_*$ and $V_3 \neq 0$. One SAIPWT bifurcates from the steady state for a small enough perturbation of the coefficients such that $(c - c_*) \cdot V_3 > 0$.*

Example: One SAIPWT bifurcates from the steady state solution $u(x,t) \equiv 1/3$, of equation

$$u_t = \frac{\partial}{\partial x} \left[(9u^2 - 6u + 2)\, u_x \right] - u + u_x + 4u^2 - 3u^3 + u^2 u_x + u_x^3, \tag{13}$$

for $c \lesssim -10/9$.

Using the above computations it is easy to see that the travelling wave system associated to equation (13) has a weak focus at $P_{1/3}$ if and only if

$c = c_* := -10/9$. In this case the particular form of the associated system (11) is

$$\begin{cases} \dfrac{d\varphi}{ds} &= -v - \frac{27}{2}\varphi^2 v, \\ \dfrac{dv}{ds} &= \varphi - \frac{3\sqrt{6}}{4}\varphi^2 + \varphi v - \frac{27}{4}\varphi^3 - \frac{3\sqrt{6}}{4}\varphi^2 v + 27\varphi v^2 - \frac{\sqrt{6}}{2}v^3. \end{cases} \tag{14}$$

From the expression (12), we have $V_3 = -\dfrac{3}{8}\pi\sqrt{6}$, hence a generic Hopf bifurcation of a limit cycle γ_c occurs when $c \lesssim -10/9$. □

We want to remark that in [8], the expressions for the Lyapunov constants V_3, V_5 and V_7, for a general system of type (3) are given. Thus in principle, we are able to present a more general result on multiple bifurcations of SAIPWT. At this stage we have decided not to translate its expression into the coefficients of the travelling wave system because they are extremely large. In summary, to make progress we must consider particular cases of equation (1). This is done in the following sections.

4 PWT in equations with constant diffusion and cubic reaction term

The aim of this Section is to show some results for a more concrete family of PDE than the one presented in Section 3. We are interested in characterizing the existence of periodic wave trains solutions of the particular case of equation (1) with constant diffusion $D(\phi) := d$ and cubic reaction term $f(u, u_x)$:

$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} + \sum_{i+j=1}^{3} f_{ij} u^i u_x^j, \tag{15}$$

Equation (15) obviously satisfies equation (Bc), the particular form of assumptions (Ba) and (Bb) are the following: (Ba) $f_{30} = f_{10}/\alpha$, $f_{20} = -f_{10}(\alpha+1)/\alpha$, and (Bb) $f_{10} < 0$.

After some changes of variables an a time scaling, the associated travelling wave system can be turned into a special kind of cubic planar differential systems known as Kukles systems. This systems are the class of planar differential systems of the form

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \end{cases} \tag{16}$$

and have been intensively studied in the last years.

The particular form of system (2) for equation (15) (which is:

$$\begin{cases} \phi' &= d\,v, \\ v' &= -cv - f(\phi, v). \end{cases} \tag{17}$$

where $\{'\} := d/d\tau$) can be transformed into a Kukles system.

Indeed, assumptions (Ba) and (Bc) imply that $P_1(1,0)$ and $P_\alpha(\alpha, 0)$ (where $\alpha \in (0,1)$) are the only singular points of system (17). Assumption (Ab) implies that $f_\phi(0,0) < 0, f_\phi(\alpha, 0) > 0$, and $f_\phi(1,0) < 0$. Hence we are under the framework of Section 3. Therefore $P_0(0,0)$ and P_1 are hyperbolic saddles of system (17).

Let $DX(P_\alpha)$ denote the differential matrix associated to the singular point P_α.

$$\text{Spec}\,(DX(P_\alpha)) = \left\{ \frac{\gamma_\alpha \pm \sqrt{\Delta_\alpha}}{2} \right\}$$

where $\gamma_\alpha := -(c + f_{01} + f_{11}\alpha + f_{21}\alpha^2)$, and

$$\begin{aligned} \Delta_\alpha := \quad & c^2 + 2cf_{01} + 2cf_{11}\alpha + 2cf_{21}\alpha^2 + f_{01}^2 + 2f_{01}f_{11}\alpha + 2f_{01}f_{21}\alpha^2 \\ & +f_{11}^2\alpha^2 + 2f_{11}f_{21}\alpha^3 + f_{21}^2\alpha^4 - 4df_{10}\alpha + 4df_{10}. \end{aligned}$$

If $\Delta_\alpha \geq 0$, P_α is a node. If $\Delta_\alpha < 0$ and $\gamma_\alpha \neq 0$ then P_α is a focus. In both cases the stability of P_α is given by $\text{sign}(\gamma_\alpha)$. The eigenvalues of $DX(P_\alpha)$ are pure imaginary numbers when $\gamma_\alpha = 0$. Indeed, $\gamma_\alpha = 0$ if and only if $c_* := -(f_{01} + f_{11}\alpha + f_{21}\alpha^2)$. In this case $\Delta_\alpha = -4d\,f_{10}(\alpha - 1) < 0$. Hence

$$\text{Spec}\,(DX_{c_*}(P_\alpha)) = \left\{ \pm\, i\,\sqrt{d(\alpha-1)f_{10}} \right\}.$$

We translate P_α to the origin, and perform the scaling and reparametrization given by: $\bar{\phi} = \phi - \alpha$, $\varphi = \bar{\phi}/a$, and $d\tau/ds = b$, where

$$a = d/\sqrt{d\,f_{10}(\alpha-1)}, \quad \text{and} \qquad b = 1/\sqrt{d\,f_{10}(\alpha-1)},$$

obtaining that system (17) is transformed into a Kukles' system of the form:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \end{cases}$$

where

$$\left.\begin{array}{ll} a_1 := \dfrac{d(2\alpha-1)}{(\alpha-1)\alpha}\eta, & a_2 := -\dfrac{(2f_{21}\alpha+f_{11})}{f_{10}(\alpha-1)}, \\ a_3 := \eta(f_{12}\alpha+f_{02}), & a_4 := -\dfrac{d}{f_{10}(\alpha-1)^2\alpha}, \\ a_5 := \dfrac{f_{21}d}{f_{10}(\alpha-1)}\eta, & a_6 := -\dfrac{f_{21}d}{f_{10}(\alpha-1)}, \\ a_7 := f_{03}\eta. & \text{where } \eta := -1/\sqrt{d\,f_{10}(\alpha-1)}\;. \end{array}\right\} \tag{18}$$

4.1 SAIPWT for equation (15)

In [20] Lloyd and Pearson, and in [28] Sadovskiĭ, proved that under some appropriate conditions, the origin of a Kukles system of type (6) can be a weak focus of order six, and six limit cycles can bifurcate from the origin. Hence there are PDE of type (15) with at least six IPWT. On the other hand Christopher and Lloyd (in [3]; Theorem 3.3 and Corollary 3.5), proved that in the case that $a_7 = 0$ (that is $f_{03} = 0$) the origin of system (16) cannot be a weak focus of order greater than five. They also characterized when the origin of (16) is a weak focus of order five, and obtained a system of five small amplitude limit cycles. Hence we have the following corollary of their results:

Proposition 3. *Consider equation (15) satisfying $f_{10} < 0$, $f_{30} = f_{10}/\alpha$, and $f_{20} = -f_{10}(\alpha+1)/\alpha$.*

(i) At least six SAIPWT can bifurcate from the steady state $u(x,t) \equiv \alpha$, for some particular cases of PDE of this type.

(ii) Assume $f_{03} = 0$. At most five SAIPWT can bifurcate from the steady state $u(x,t) \equiv \alpha$. Furthermore there are PDE of this type with at least five IPWT.

4.1.1 Worked example: a 5–parametric travelling wave system

In the following we present the following example of 4-parametric equation of type (15), to illustrate the mechanism of bifurcation of SAIPWT:

$$\begin{cases} u_t = & d\,u_{xx} - \frac{1}{2}u + f_{01}u_x + \frac{3}{2}u^2 + f_{11}u\,u_x + u_x^2 - u^3 \\ & \qquad +f_{21}u^2\,u_x + uu_x^2 + u_x^3, \end{cases} \tag{19}$$

The set of parameters of the PDE is $\{d, f_{01}, f_{11}, f_{21}\}$. We want to stress that this reaction term is a generalization of the reaction term $f(u) = u(1-u)(u-1/2)$, presented in some examples in [33].

For this equation there are three unique steady state solutions given by $u(x,t) \equiv k$, where $k \in \{0, 1/2, 1\}$.

Set $c_* := -f_{01} - \frac{1}{2}f_{11} - \frac{1}{4}f_{21}$, $f_{11}^* := \frac{1}{2} + \frac{1}{3}f_{21}(2d-3)$, and

$$f_{21}^{\pm}(d) := \frac{243 + 24d - 576d^2 \pm \sqrt{(243 + 24d - 576d^2)^2 - 128d^2(756d-63)}}{64d^2},$$

for all d such that $(243 + 24d - 576d^2)^2 - 128d^2(756d - 63) > 0$ (that is $d \notin I_d := (d_1, d_2)$ where $d_1 \approx 0.5120160849$, and $d_2 \approx 0.9830783266$). And finally let d_* be the only positive root of

$$\begin{aligned} P(d) = &24772608d^8 - 5603328d^7 + 9289728d^6 - 5462016d^5 - 77153280d^4 \\ &+ 33258448d^3 + 40557018d^2 - 5960187d - 4516155, \end{aligned}$$

($d_* \approx 0.4117675$). The main result of this section is the following:

Theorem 4. *Consider equation (19) satisfying $f_{10} < 0$, $f_{30} = f_{10}/\alpha$, and $f_{20} = -f_{10}(\alpha+1)/\alpha$. The following statements hold:*

(a) At most one SAIPWT can bifurcate from the steady state $u(x,t) \equiv 1/2$, when $c = c_*$ and $f_{11} \neq f_{11}^*$. Furthermore, in this case, one SAIPWT bifurcates from $u(x,t) \equiv 1/2$, for a small enough perturbation of the parameters such that $(f_{11} - f_{11}^*)(c - c_*) > 0$.

(b) At most two SAIPWT can bifurcate from $u(x,t) \equiv 1/2$, when $c = c_*$, $f_{11} = f_{11}^*$, and $32d^2 f_{21}^2 + (576d^2 - 24d - 243) f_{21} + 756d - 63 \neq 0$, (that is $f_{21} \neq f_{21}^{\pm}(d)$). Furthermore, in this case, two SAIPWT bifurcate from $u(x,t) \equiv 1/2$, for a small enough perturbation of the parameters such that either:

(i) $c > c_*$, $f_{11} > f_{11}^*$ and $f_{21} \in (f_{21}^-(d), f_{21}^+(d))$, or

(ii) $c < c_*$, $f_{11} < f_{11}^*$ and $f_{21} \notin (f_{21}^-(d), f_{21}^+(d))$.

(c) At most three SAIPWT can bifurcate from $u(x,t) \equiv 1/2$, when $c = c_*$, $f_{11} = f_{11}^*$, $f_{21} = f_{21}^{\pm}(d)$, and $d \neq d_*$. Moreover, in this situation at least three SAIPWT bifurcate from $u(x,t) \equiv 1/2$, for a small enough perturbation of the parameters such that either:

(i) $c < c_*$, $f_{11} < f_{11}^*$, $f_{21} < f_{21}^-(d)$ with $d \in (d_*, +\infty) \setminus I_d$, or

(ii) $c < c_*$, $f_{11} < f_{11}^*$, and $f_{21} > f_{21}^+(d)$, or

(iii) $c > c_*$, $f_{11} > f_{11}^*$, $f_{21} \in (f_{21}^-(d), f_{21}^+(d))$ *and* $d \in (0, d_*)$.

(d) At most four SAIPWT can bifurcate from $u(x,t) \equiv 1/2$ *when* $c = c_*$, $f_{11} = f_{11}^*$, $f_{21} = f_{21}^-(d)$, *and* $d = d_*$. *Furthermore four SAIPWT bifurcate from* $u(x,t) \equiv 1/2$, *for a small enough perturbation of the parameters such that* $c < c_*$, $f_{11} < f_{11}^*$, $f_{21} < f_{21}^-(d)$, *and* $d > d_*$.

(e) There is not a choice of the parameters such that there is a continuum of periodic wave train solutions of equation (19) unfolding the steady state solution $u(x,t) \equiv 1/2$.

The SAIPWT bifurcating from the steady state solution $u(x,t) \equiv 1/2$, can be seen as small amplitude limit cycles arising from the singular point $P_{1/2}$ of the associated travelling–wave system, which is the following 5-parameter system:

$$\begin{cases} \dot{\phi} = d\,v, \\ \dot{v} = \frac{1}{2}\phi - (f_{01}+c)v - \frac{3}{2}\phi^2 - f_{11}\phi v - v^2 + \phi^3 - f_{21}\phi^2 v - \phi v^2 - v^3. \end{cases} \tag{20}$$

It is easy to see that $\Delta_{1/2} = (c + f_{01} + \frac{1}{2}f_{11} + \frac{1}{4}f_{21})^2 - d$, and $\gamma_{1/2} = -c - f_{01} - \frac{1}{2}f_{11} - \frac{1}{4}f_{21}$. Hence $P_{1/2}$ is a singular point of weak focus type for system (20), if and only if $c = c_* := -f_{01} - \frac{1}{2}f_{11} - \frac{1}{4}f_{21}$, (in this case $\mathrm{Spec}\left(DX_{c_*}(P_{1/2})\right) = \{\pm i\frac{\sqrt{d}}{2}\}$). On the other hand $P_{1/2}$ is a stable focus if $c > c_*$ and $d > (c-c_*)^2$, (unstable if $c < c_*$ and $d > (c-c_*)^2$).

At this point, the main steps of the proof are the following

(1) Transform system (20) into the canonical form (3).

(2) Compute the Lyapunov constants of the origin of the transformed system.

(3) Characterize all the kind of weak focus that appear in the family and determine its order.

(4) Starting with each type of weak focus, find a sequence of perturbations such that each of them reverses the stability of the origin.

After the translation and the reparametrization given by $\bar{\phi} = \phi - \alpha$, $\varphi = -2\sqrt{d}\bar{\phi}$, and $d\tau/ds = 2\sqrt{d}/d$, system (20) is written in canonical form:

$$\begin{cases} \varphi' &= -v, \\ v' &= \varphi + \frac{\sqrt{d}}{d}\Big[4\sqrt{d}(f_{11}+f_{21})\varphi v - 3v^2 - 16d\sqrt{d}\varphi^3 \\ & \qquad -f_{21}d\varphi^2 v + 4\sqrt{d}\varphi v^2 - 2v^3\Big]. \end{cases} \tag{21}$$

To compute the Lyapunov constants of the origin of (21), we have computed the first four constants for a general cubic system. The computations of the Lyapunov constants have been done using an implementation on MAPLE V release 4, of the algorithm developed by A. Gasull and J. Torregrosa (see [10]). The total time of CPU was 38.2s on a Pentium III 846MHz. The result obtained is summarized in the following result (Step (2) of the proof of Theorem 4):

Lemma 5. *the first Lyapunov constants of system (21) are given by*

(i) $V_3 = \frac{1}{2}\pi\frac{\sqrt{d}}{d}\,(6f_{11} + 6f_{21} - 3 - 4f_{21}d)$.

(ii) $V_5 = \frac{2}{27}\pi\frac{\sqrt{d}}{d}\left(32d^2\, f_{21}^2 + (576d^2 - 24d - 243)\, f_{21} + 756d - 63\right)$.

(iii) $V_7 = \frac{1}{3d}\pi\frac{\sqrt{d}}{d}\left(1536d^4 f_{21} + 2608d^2 f_{21} - 1701f_{21} - 78f_{21}d - 441 + 6102d + 288d^2\right)$.

(iv) V_9 *is a positive constant whose value is* $V_9 \approx 28582.24584$.

The previous result leads to the following one, which characterizes the weak focus and gives its stability and order, (Step (3) of the proof of Theorem 4).

Proposition 6. *The following statements hold.*

(a) $P_{1/2}$ *is a stable weak focus of order 1 of system (20), if* $c = c_*$, *and* $f_{11} < f_{11}^*$ *(respectively unstable if* $f_{11} > f_{11}^*$*).*

(b) $P_{1/2}$ *is a stable weak focus of order 2 of system (20), if* $c = c_*$, $f_{11} = f_{11}^*$, *and*

$$32d^2\, f_{21}^2 + (576d^2 - 24d - 243)\, f_{21} + 756d - 63 < 0,$$

in other words if and only if $f_{21} \in (f_{21}^-(d), f_{21}^+(d))$. *(respectively unstable if* $f_{21} \notin (f_{21}^-(d), f_{21}^+(d))$*).*

(c) $P_{1/2}$ *is a stable weak focus of order 3 of system (20), if* $c = c_*$, $f_{11} = f_{11}^*$ *and either* $f_{21} = f_{21}^+(d)$ *(with* $d \notin I_d$*), or* $f_{21} = f_{21}^-(d)$ *for* $d \in (d_*, +\infty) \setminus I_d$ *(respectively unstable if* $f_{21} = f_{21}^-(d)$ *for* $d \in (0, d_*)$*).*

(d) $P_{1/2}$ *is an unstable weak focus of order 4 of system (20), if* $c = c_*$, $f_{11} = f_{11}^*$, $f_{21} = f_{21}^-(d)$ *and* $d = d_*$.

Proof. (a) Trivial from Lemma 5 (i).

(b) Set $f_{11} = f_{11}^*$, thus $V_3 = 0$. From Lemma 5 (ii) we have that

$$\mathrm{sign}(V_5) = \mathrm{sign}\left(32d^2 f_{21}^2 + (576d^2 - 24d - 243) f_{21} + 756d - 63\right),$$

and $V_5 = 0$ if and only if $f_{21} := f_{21}^{\pm}(d)$. It is easy to check that $V_5 < 0$ if and only if $f_{21} \in (f_{21}^-(d), f_{21}^+(d))$, and conversely $V_5 > 0$ if $f_{21} \notin (f_{21}^-(d), f_{21}^+(d))$

(c) Set $c = c_*$, $f_{11} = f_{11}^*$ and $f_{21} = f_{21}^+(d)$ (thus $V_3 = V_5 = 0$). Some tedious computations shows that $V_7|_{\{c=c_*, f_{11}=f_{11}^*, f_{21}=f_{21}^+(d)\}} < 0$, for all $d \in \mathbb{R}^+ \setminus I_d$.

On the other hand a large computation shows that

$$\mathrm{sign}\left(V_7|_{\{c=c_*, f_{11}=f_{11}^*, f_{21}=f_{21}^-(d)\}}\right) = \begin{cases} +1 & \text{if and only if } d \in (0, d_*), \\ -1 & \text{if and only if } d \in (d_*, +\infty) \setminus I_d, \end{cases}$$

where d_* is the only positive root of

$$\begin{aligned} P(d) =& 24772608d^8 - 5603328d^7 + 9289728d^6 - 5462016d^5 - 77153280d^4 \\ &+ 33258448d^3 + 40557018d^2 - 5960187d - 4516155. \end{aligned}$$

(d) Trivial from Lemma 5 (iv). ∎

By the classical multiple Hopf bifurcation theory at most k limit cycles bifurcate from a weak focus of order k, hence to end the proof we only have to proof that the proposed perturbations of the coefficients give rise to the prescribed bifurcation of limit cycles (Step (4) of the proof of Theorem 4).

We will proof that the proposed perturbations stated in Theorem 4 (a) and (d) produce a bifurcation of one and four small amplitude limit cycles respectively. The proof of statements (b) and (c) are analogous.

Proof of Theorem 4. (a) If $c = c_*$ and $f_{11} \neq f_{11}^*$, then $V_3 \neq 0$, hence $P_{1/2}$ is a weak focus of order 1. Assume that $f_{11} < f_{11}^*$, so $P_{1/2}$ is stable. Taking a small enough perturbation of c such that $c < c_*$, $P_{1/2}$ turns to be an unstable focus. Hence an Hopf bifurcation of a stable limit cycle occurs. Conversely if $f_{11} > f_{11}^*$ and c is taken so that $c > c_*$, a bifurcation of an unstable limit cycle takes place.

(d) *Step i:* If $c = c_*$, $f_{11} = f_{11}^*$, $f_{21} = f_{21}^{\pm}(d)$, and $d = d_*$, then $V_3 = V_5 = V_7 = 0$ and $V_9 > 0$, hence $P_{1/2}$ is an unstable focus of order 4.

Step ii: Taking $d > d_*$ ($d \notin I_d$), we obtain $V_7 < 0$ and then $P_{1/2}$ turns to be a stable focus so that an Hopf bifurcation of an unstable limit cycle takes place.

Step iii: If we take $f_{21} < f_{21}^-(d)$, we have that $V_5 > 0$, the singular point turns to be an unstable focus and a new bifurcation of a (stable) limit cycle takes place. By continuity the unstable limit cycle persists.

Step iv: If we now consider $f_{11} < f_{11}^*$, then $V_3 < 0$. Therefore a new unstable limit cycle has bifurcated from $P_{1/2}$.

Step v: Finally, taking $c < c_*$ ($f_{11} > f_{11}^*$, $f_{21} < f_{21}^-(d)$, and $d \in (0, d_*)$), we produce a new stable limit cycle bifurcating from the singular point. ∎

4.1.2 General double Hopf bifurcation for equations of type (15)

The expressions of the Lyapunov constants for a general system of type (16) are extremely large, hence it is not worth to present a result showing the explicit conditions for multiple bifurcations of SAIPWT. With the aim of illustrating this, we present the following result which gives explicitly the first conditions for a double bifurcation of SAIPWT. Set

$$\begin{aligned}V_3 :=& \frac{\pi}{4f_{10}\alpha(\alpha-1)^2}\sqrt{\frac{1}{df_{10}(\alpha-1)}}\\ &(-3\alpha f_{03}f_{10} + 6\alpha^2 f_{03}f_{10} - f_{21}d\alpha + 2f_{11}\alpha d - f_{11}\alpha f_{02} + f_{11}\alpha^3 f_{12}\\ &- f_{11}d - 2f_{21}\alpha^3 f_{12} + 2f_{21}\alpha^3 f_{02} + 3f_{21}\alpha^2 d - 2f_{21}\alpha^2 f_{02}\\ &+ 2f_{21}\alpha^4 f_{12} - f_{11}f_{12}\alpha^2 + f_{11}\alpha^2 f_{02} - 3\alpha^3 f_{03}f_{10}),\end{aligned}$$

and

$$\begin{aligned}V_5 :=& -\frac{\pi}{24df_{10}^3(\alpha-1)^5\alpha^3\sqrt{df_{10}(\alpha-1)}}\\ &(72df_{21}^2\alpha^6 f_{11}f_{12} + 72df_{21}^2\alpha^5 f_{11}f_{02} - 264d\alpha^6 f_{21}^2 f_{03}f_{10}\\ &+ 88d\alpha^5 f_{21}^2 f_{03}f_{10} - 168d^2\alpha^5 f_{21}^2 f_{11} - 88d\alpha^7 f_{21}f_{11}f_{03}f_{10}\\ &+ 120d^2\alpha^6 f_{21}^2 f_{11} - 88d\alpha^8 f_{21}^2 f_{03}f_{10} + 72d^2\alpha^7 f_{21}^3 - 22d\alpha^6 f_{11}^2 f_{03}f_{10}\\ &+ 66d^2\alpha^5 f_{11}^2 f_{21} + 387f_{10}\alpha^5 f_{12}f_{02}^2 f_{11} + 129f_{10}\alpha^7 f_{12}^3 f_{11}\\ &- 258f_{10}\alpha^9 f_{12}^3 f_{21} - 129f_{10}\alpha^8 f_{12}^3 f_{11} + 264d\alpha^6 f_{21}f_{11}f_{03}f_{10}\\ &- 264d\alpha^5 f_{21}f_{11}f_{03}f_{10} + 48d^2\alpha^4 f_{21}^2 f_{11} + 264d\alpha^7 f_{21}^2 f_{03}f_{10}\\ &+ 66d\alpha^5 f_{11}^2 f_{03}f_{10} - 66d\alpha^4 f_{11}^2 f_{03}f_{10} - 96d^2\alpha^4 f_{11}^2 f_{21} + 30d^2\alpha^3 f_{11}^2 f_{21}\\ &+ 22d\alpha^3 f_{11}^2 f_{03}f_{10} + 387f_{10}\alpha^6 f_{12}^2 f_{02}f_{11} + 774f_{10}\alpha^6 f_{12}f_{02}^2 f_{21}\\ &+ 258f_{10}\alpha^8 f_{12}f_{02}^2 f_{21} + 129f_{10}\alpha^7 f_{12}f_{02}^2 f_{11} - 774f_{10}\alpha^7 f_{12}f_{02}^2 f_{21}\\ &+ 57f_{10}\alpha^5 f_{12}d^2 f_{11} + 86f_{10}\alpha^1 0 f_{12}^3 f_{21} + 21f_{10}d^3\alpha^2 f_{11} + 76f_{10}\alpha^3 f_{12}d^2 f_{11}\\ &+ 774f_{10}\alpha^7 f_{12}^2 f_{02}f_{21} + 235f_{10}\alpha^6 f_{12}f_{02}df_{11} - 64f_{10}\alpha^4 f_{12}^2 df_{11}\\ &+ 258f_{10}\alpha^8 f_{12}^3 f_{21} - 258f_{10}\alpha^5 f_{12}f_{02}^2 f_{21} - 129f_{10}\alpha^4 f_{12}f_{02}^2 f_{11}\\ &- 591f_{10}\alpha^5 f_{12}f_{02}df_{11} + 477f_{10}\alpha^4 f_{12}f_{02}df_{11} - 258f_{10}\alpha^6 f_{12}^2 f_{02}f_{21}\end{aligned}$$

$$
\begin{aligned}
&- 129f_{10}\alpha^5 f_{12}^2 f_{02} f_{11} + 86f_{10}\alpha^7 f_{02}^3 f_{21} + 43f_{10}\alpha^6 f_{02}^3 f_{11} - 258f_{10}\alpha^6 f_{02}^3 f_{21}\\
&- 129f_{10}\alpha^5 f_{02}^3 f_{11} - 86f_{10}\alpha^7 f_{12}^3 f_{21} - 43f_{10}\alpha^6 f_{12}^3 f_{11} - 86f_{10}\alpha^4 f_{02}^3 f_{21}\\
&- 43f_{10}\alpha^3 f_{02}^3 f_{11} - 17f_{10}d^3\alpha f_{11} - 14f_{10}d^3\alpha^3 f_{11} + 121f_{10}\alpha^7 f_{12}^2 df_{11}\\
&- 285f_{10}\alpha^4 f_{02}^2 df_{11} - 94f_{10}\alpha^3 f_{02}d^2 f_{11} + 5f_{10}d^3 f_{11} - 121f_{10}\alpha^3 f_{12} f_{02} df_{11}\\
&- 9f_{10}\alpha f_{02}d^2 f_{11} - 57f_{10}\alpha^2 f_{02}^2 df_{11} + 114f_{10}\alpha^5 f_{02}^2 df_{11} + 56f_{10}\alpha^2 f_{02}d^2 f_{11}\\
&+ 47f_{10}\alpha^4 f_{02}d^2 f_{11} - 306f_{10}\alpha^6 f_{12}^2 df_{11} + 249f_{10}\alpha^5 f_{12}^2 df_{11}\\
&+ 258f_{10}\alpha^9 f_{12}^2 f_{02} f_{21} + 129f_{10}\alpha^8 f_{12}^2 f_{02} f_{11} - 119f_{10}\alpha^4 f_{12}d^2 f_{11}\\
&- 387f_{10}\alpha^6 f_{12} f_{02}^2 f_{11} - 774f_{10}\alpha^8 f_{12}^2 f_{02} f_{21} - 387f_{10}\alpha^7 f_{12}^2 f_{02} f_{11}\\
&+ 43f_{10}\alpha^9 f_{12}^3 f_{11} + 258f_{10}\alpha^5 f_{02}^3 f_{21} - 96d^2\alpha^6 f_{21}^3 + 12d^2 f_{11}^3\alpha^4 + 24d^2\alpha^5 f_{21}^3\\
&+ 36df_{11}^2\alpha^6 f_{21} f_{02} + 6d^2 f_{11}^3\alpha^2 - 72df_{11}^2\alpha^5 f_{21} f_{02} + 36df_{11}^2\alpha^5 f_{21} f_{12}\\
&- 144df_{21}^2\alpha^6 f_{11} f_{02} - 144df_{21}^2\alpha^7 f_{11} f_{12} + 153\alpha^4 f_{10}^2 d^2 f_{03} - 11\alpha^4 f_{10} d^3 f_{21}\\
&+ 8\alpha^3 f_{10} d^3 f_{21} - 72df_{11}^2\alpha^6 f_{21} f_{12} + 48df_{21}^3\alpha^8 f_{02} - 501\alpha^7 f_{10} f_{12}^2 f_{21} d\\
&- 258\alpha^8 f_{10}^2 f_{12} f_{02} f_{03} + 396\alpha^7 f_{10} f_{12} f_{02} f_{21} d - 129\alpha^9 f_{10}^2 f_{12}^2 f_{03}\\
&+ 205\alpha^8 f_{10} f_{12}^2 f_{21} d + 1032\alpha^7 f_{10}^2 f_{12} f_{02} f_{03} + 36df_{11}^2\alpha^4 f_{21} f_{02} + 48df_{21}^3\alpha^9 f_{12}\\
&+ 72df_{21}^2\alpha^7 f_{11} f_{02} + 36df_{11}^2\alpha^7 f_{21} f_{12} + 72df_{21}^2\alpha^8 f_{11} f_{12} + 103\alpha^6 f_{10} f_{12} d^2 f_{21}\\
&- 209\alpha^5 f_{10} f_{12} d^2 f_{21} - 558\alpha^5 f_{10}^2 f_{12} df_{03} - 168\alpha^3 f_{10}^2 d^2 f_{03} + 81\alpha^2 f_{10}^2 d^2 f_{03}\\
&- 12\alpha^2 f_{10} d^3 f_{21} + 5\alpha f_{10} d^3 f_{21} - 15\alpha f_{10}^2 d^2 f_{03} + 516\alpha^6 f_{10}^2 f_{12}^2 f_{03}\\
&+ 387\alpha^6 f_{10} f_{12}^2 f_{21} d - 91\alpha^5 f_{10} f_{12}^2 f_{21} d - 129\alpha^5 f_{10}^2 f_{12}^2 f_{03} - 129\alpha^7 f_{10}^2 f_{02}^2 f_{03}\\
&+ 516\alpha^6 f_{10}^2 f_{02}^2 f_{03} + 191\alpha^6 f_{10} f_{02}^2 f_{21} d - 459\alpha^5 f_{10} f_{02}^2 f_{21} d - 774\alpha^7 f_{10}^2 f_{12}^2 f_{03}\\
&+ 516\alpha^8 f_{10}^2 f_{12}^2 f_{03} + 127\alpha^4 f_{10} f_{12} d^2 f_{21} + 312\alpha^4 f_{10}^2 f_{12} df_{03}\\
&- 960\alpha^6 f_{10} f_{12} f_{02} f_{21} d - 168\alpha^4 f_{10} f_{02} d^2 f_{21} - 1548\alpha^6 f_{10}^2 f_{12} f_{02} f_{03}\\
&+ 1032\alpha^5 f_{10}^2 f_{12} f_{02} f_{03} + 732\alpha^5 f_{10} f_{12} f_{02} f_{21} d - 168\alpha^4 f_{10} f_{12} f_{02} f_{21} d\\
&- 258\alpha^4 f_{10}^2 f_{12} f_{02} f_{03} - 774\alpha^5 f_{10}^2 f_{02}^2 f_{03} + 516\alpha^4 f_{10}^2 f_{02}^2 f_{03}\\
&+ 345\alpha^4 f_{10} f_{02}^2 f_{21} d - 77\alpha^3 f_{10} f_{02}^2 f_{21} d - 129\alpha^3 f_{10}^2 f_{02}^2 f_{03} - 51\alpha^5 f_{10}^2 d^2 f_{03}\\
&+ 6df_{11}^3\alpha^6 f_{12} - 14\alpha^2 f_{10} f_{02} d^2 f_{21} - 60\alpha^2 f_{10}^2 f_{02} df_{03}\\
&- 120\alpha^6 f_{10}^2 f_{02} df_{03} + 420\alpha^5 f_{10}^2 f_{02} df_{03} + 86\alpha^5 f_{10} f_{02} d^2 f_{21}\\
&- 540\alpha^4 f_{10}^2 f_{02} df_{03} + 96\alpha^3 f_{10} f_{02} d^2 f_{21} - 18d^2 f_{11}^3\alpha^3 - 12df_{11}^3\alpha^5 f_{12}\\
&- 12df_{11}^3\alpha^4 f_{02} + 6df_{11}^3\alpha^4 f_{12} - 96df_{21}^3\alpha^8 f_{12} - 96df_{21}^3\alpha^7 f_{02}\\
&+ 48df_{21}^3\alpha^7 f_{12} + 6df_{11}^3\alpha^3 f_{02} + 48df_{21}^3\alpha^6 f_{02} + 6df_{11}^3\alpha^5 f_{02}\\
&- 21\alpha^3 f_{10} f_{12} d^2 f_{21} - 63\alpha^3 f_{10}^2 f_{12} df_{03} - 123\alpha^7 f_{10}^2 f_{12} df_{03}\\
&+ 432\alpha^6 f_{10}^2 f_{12} df_{03} + 300\alpha^3 f_{10}^2 f_{02} df_{03} - 14f_{10}\alpha^2 f_{12} d^2 f_{11}\\
&+ 129f_{10}\alpha^4 f_{02}^3 f_{11} + 228f_{10}\alpha^3 f_{02}^2 df_{11} + 88d\alpha^4 f_{21} f_{11} f_{03} f_{10}).
\end{aligned}
$$

We have the following result:

Proposition 7. *Consider equation (15), satisfying* $f_{10} < 0$, $f_{30} = f_{10}/\alpha$, *and* $f_{20} = -f_{10}(\alpha+1)/\alpha$. *The following statements hold:*

(a) At most one SAIPWT can bifurcate from the steady state $u(x,t) \equiv \alpha$, *when* $c = c_*$ *and* $V_3 \neq 0$. *Furthermore, in this case one SAIPWT bifurcates from this steady state for a small enough perturbation of the parameters such that* $(c - c_*) \cdot V_3 > 0$, *and* $|c - c_*| << |V_3|$.

(b) At most two SAIPWT can bifurcate from $u(x,t) \equiv \alpha$, *when* $c = c_*$, $V_3 = 0$, *and* $V_5 \neq 0$. *In this case two SAIPWT bifurcate from the steady state for a small enough perturbation of the parameters such that* $(c - c_*) \cdot V_3 > 0$ *and* $V_3 \cdot V_5 < 0$, *and* $|c - c_*| << |V_3| << |V_5|$.

Proof. Directly from [8], we have that the first Lyapunov constants for the origin of system (18) are V_3 and V_5 given above. The result follows as a direct application of the Hopf bifurcation theory. ∎

Example: Using the above result, it is easy to see that two SAIPWT bifurcate from the steady state $u(x,t) \equiv 1/3$ of the equation

$$u_t = u_{xx} - u + 4u^2 - 3uu_x - 3u^3 + 9u^2u_x + f_{03}u_x^3, \qquad (22)$$

for $c \gtrsim 0$ and $f_{03} \lesssim -9/4$.

Indeed, it is easy to see that for $c = c_* := 0$, $P_{1/3}$ is a weak focus. An straightforward computation shows that

$$V_3 = -\frac{3}{32}\pi\sqrt{6}(9 + 4f_{03}).$$

Hence if $f_{03} = -9/4$ then $V_3 = 0$ and

$$V_5 = -\frac{1215}{256}\pi\sqrt{6} < 0,$$

and therefore $P_{1/3}$ is a weak focus of order two. Taking a small enough perturbation of f_{30} such that $f_{03} \lesssim -9/10$ we obtain that $V_3 \gtrsim 0$. Proposition 7 assures that taking $c \gtrsim 0$ we produce a double Hopf bifurcation giving rise to two SAIPWT. □

4.2 Continuum of periodic wave trains for equation (15)

Necessary and sufficient conditions for the origin of a Kukles' system to be a centre have been intensively studied. See [18, 15, 3, 25] and [29];

The following result (which is an immediate corollary of Pearson, Lloyd and Christopher's Theorems 3.1 and 3.2 of [25]) gives sufficient conditions for equation (16) to have a continuum of PWT unfolding the steady state $u(x,t) \equiv \alpha$, for the PDE (15).

Theorem 8. *Assume that $f_{10} < 0$, $f_{30} = f_{10}/\alpha$, and $f_{20} = -f_{10}(\alpha+1)/\alpha$. Set*

$$\begin{cases} a_1 := \dfrac{d(2\alpha-1)}{(\alpha-1)\alpha}\eta, & a_2 := -\dfrac{(2f_{21}\alpha+f_{11})}{f_{10}(\alpha-1)}, \\ a_3 := \eta(f_{12}\alpha+f_{02}), & a_4 := -\dfrac{d}{f_{10}(\alpha-1)^2\alpha}, \\ a_5 := \dfrac{f_{21}d}{f_{10}(\alpha-1)}\eta, & a_6 := -\dfrac{f_{21}d}{f_{10}(\alpha-1)}, \\ a_7 := f_{03}\eta. & \text{where } \eta := -1/\sqrt{d\,f_{10}(\alpha-1)}\,. \end{cases}$$

If one of the following conditions (i)–(v) hold, there is a continuum of PWT surrounding the steady state solution $u(x,t) \equiv \alpha$ of equation (15), with constant speed $c = c_* := -(f_{01}+f_{11}\alpha+f_{21}\alpha^2)$.

(i) $a_7 = a_5 = a_2 = 0$.

(ii) $a_7 = a_5 = a_3 = a_1 = 0$.

(iii) $a_7 = 0$, $a_4 = a_3(a_1+a_3)$, $a_5 = -a_2(a_1+a_3)$, $a_6(a_1+2a_3) = -a_3^2(a_1+a_3)$.

(iv) $a_1a_2+a_5+\mu = 0$, $a_4a_2^2+a_5\mu = 0$, $(3a_7\mu+\mu^2+a_6a_2^2)a_5-3a_7\mu^2 -a_6a_2^2\mu = 0$, $9a_6a_2^2+2a_2^4+27a_7\mu+9\mu^2 = 0$, *where* $\mu := 3a_7+a_2a_3$.

(v) $81a_1^3a_3-2(18a_2\vartheta-4a_2^4-27a_2^2a_1^2-81a_1^4) = 0$, $9\delta a_4+36a_2\vartheta+8a_2^4+90a_2^2a_1^2+243a_1^4 = 0$, $\delta a_5-27a_1\vartheta+a_2a_1(2a_2^2+9a_1^2) = 0$, $81a_1^2a_6\delta+2a_2(144a_2^2\vartheta+243a_1^2\vartheta-32a_2^5-270a_2^3a_1^2-567a_1^4a_2) = 0$. $3\delta a_7+27a_1\vartheta+a_2(a_3\delta+14a_2^2a_1+72a_1^3) = 0$, *where* $162\vartheta^2 := (2a_2^2+9a_1^2)^3$ *and* $\delta := 16a_2^2+81a_1^2 \neq 0$.

Example: Using Statement (ii) of the Theorem 8, it can be checked that for $d>0$, $\alpha<0$ and for all values of the parameters $\beta,\gamma,\delta \in \mathbb{R}$, there exists a continuum of PWT surrounding the steady state solution $u(x,t) \equiv 1/2$ of the equation

$$u_t = d\,u_{xx}+\alpha u+\beta u_x-3\alpha u^2+\gamma u u_x+\delta u_x^2+2\alpha u^3-2\delta u u_x.$$

All the PWT of the continuum evolve with constant speed $c = c_* = -(\beta+\gamma/2)$. □

5 PWT for equation of type (1) arising from Liénard's limit cycles

Corduneanu, in [7], gave a characterization of a class of PDE of the form

$$\begin{cases} u_t = u_{xx} + f(u, u_x), \\ \text{where } f(u, u_x) = h(u)u_x + g(u) \end{cases}$$

having a unique IPWT. The main idea was to realize that the associated travelling wave system has the form of the well known Liénard ordinary differential equation. Inspired by his paper, in this section we deal with the reaction diffusion PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left[D(u)\dfrac{\partial u}{\partial x}\right] + f(u, u_x), \\ \text{where } f(u, u_x) = h(u)u_x + g(u), \text{ and } D(u) \neq 0, \text{ for all } u. \end{cases} \tag{23}$$

We can also keep track of IPWT for equations of type (23), using a Liénard's–coordinates associated system. Indeed, an easy computation shows that taking $u = \phi(\xi)$ (where $\xi = x - ct$), the associated travelling wave second order equation has the form

$$D(\phi)\ddot{\phi} + D'(\phi)(\dot{\phi})^2 + [c + h(\phi)]\,\dot{\phi} + g(\phi) = 0. \tag{24}$$

Set

$$H(\phi) := \int_0^{\phi} h(s)\,ds.$$

Taking the new variable $w = D(\phi)\dfrac{d\phi}{d\xi} + c\phi + H(\phi)$, equation (24) is transformed into

$$\begin{cases} D(\phi)\dfrac{d\phi}{d\xi} & = -c\phi + w - H(\phi), \\ \dfrac{dw}{d\xi} & = -g(\phi). \end{cases} \tag{25}$$

After the time reparametrization given by $d\xi/d\tau = D(\phi)$, we obtain the classical Liénard system

$$\begin{cases} \phi' & = -c\phi + w - H(\phi), \\ w' & = -D(\phi)\,g(\phi). \end{cases} \tag{26}$$

where $\{'\} := d/d\tau$.

Next result on bifurcation of SAIPWT, is a straightforward application of the result of Blows, Lloyd and Zuppa (see [2] and [36]).

Theorem 9. *Assume that $D(u)$ and $g(u)$ are functions such that $D(u)g(u) = G_1 u$, with $G_1 > 0$, and that $h(u) = \sum_{k\geq 1} h_k u^k$ is an analytic function such that $h_2 = h_4 = \cdots = h_{2m-2} = 0$ and $h_{2m} \neq 0$, then*

(i) At most m SAIPWT can bifurcate from the steady state $u(t,x) \equiv 0$ of equation (23), when $c = 0$. Furthermore m SAIPWT bifurcate from it for a small enough perturbation of the parameters such that $ch_2 < 0$ and $h_{2j}h_{2j+2} < 0$ for $j = 1, \ldots, m-1$.

(ii) There is a continuum of stationary PWT unfolding $u(x,t) \equiv 0$ if and only if $h_{2k} = 0$ for all $k \geq 1$.

Proof. An straightforward computation shows that the origin of system (24) (which corresponds to the homogeneous steady state solution $u(x,t) \equiv 0$, of (23)) is a weak focus if and only if $c = 0$ and $G_1 > 0$. Also, if $c \neq 0$ and $c^2 - 4G_1 < 0$, the origin is a focus with stability given by sign$(-c)$.

Taking the change of variables given by $\varphi = a\phi$ and $d\tau/ds = b$, where $a = -\sqrt{G_1}$, and $b = \sqrt{G_1}/G_1$. We transform system (26) into a system of the form (3):

$$\begin{cases} \dfrac{d\varphi}{ds} & = -\frac{c}{a}\varphi - w + H(\varphi/a) = -w + \sum_{j\geq 2} H_j \varphi^j, \\ \dfrac{dw}{ds} & = \varphi \end{cases} \tag{27}$$

where

$$H_k := \left(\frac{1}{k} \left(-\frac{1}{\sqrt{G_1}} \right)^k \right) h_{k-1}.$$

Consider $c = 0$, so that the origin of system (27) is a weak focus. Following the result in [2] and [36] (see also [5]), we have that if $h_2 = h_4 = \cdots = h_{2m-2} = 0$ and $h_{2m} \neq 0$, then the Lyapunov constants associated to the origin satisfy

$$V_3 = V_5 = \cdots = V_{2m-1} = 0,$$

and

$$V_{2m+1} = -\left[\frac{(2m-1)!!}{(2m+2)!!} 2\pi \left(\frac{1}{\sqrt{G_1}} \right)^{2m+1} \right] h_{2m},$$

where $k!! := k(k-2)\ldots 1$ (or $k!! := k(k-2)\ldots 2$) if k is odd (or k is even respectively). Hence, under the hypothesis of statement (i) of Theorem 9, the origin of system (27) is a weak focus of order m, and the result follows.

If $h_{2k} = 0$ for all $k \geq 1$, system is invariant under the change $(\varphi, w, t) \to (-\varphi, w, -t)$. It is well known that a system of type (3) satisfying this property has a *reversible* centre at the origin, hence (ii) is proved. ∎

Example: Using Statement (i) of Theorem 9 we will see that there exists three SAIPWT unfolding the steady state $u(x,t) \equiv 0$, of equation

$$u_t = \frac{\partial}{\partial x}\left[\frac{u_x}{1+u^2}\right] + (\varepsilon u^2 + \delta u^4 + \mu u^6)u_x + u(1+u^2), \tag{28}$$

for c,ε,δ, and μ small enough and such that $c\varepsilon < 0$, $\varepsilon\delta < 0$ and $\delta\mu < 0$.

For the particular equation (28), the associated Liénard system (27), writes

$$\begin{cases} \frac{d\varphi}{ds} &= -c\varphi - w + \frac{\varepsilon}{3}\varphi^3 - \frac{\delta}{5}\varphi^5 - \frac{\mu}{7}\varphi^7, \\ \frac{dw}{ds} &= \varphi. \end{cases} \tag{29}$$

If $c \in (-2,2) \setminus \{0\}$ the origin of system (29) is robust focus which stability is given by $\text{sign}(-c)$.

If $c = \varepsilon = \delta = 0$, and $\mu \neq 0$, by Theorem 9 (i) the origin of system (29) is a weak focus of order three, which stability is given by $\text{sign}(V_7) = \text{sign}(-\mu)$. If we take δ small enough and such that $\delta\mu < 0$, and since $\text{sign}(V_5) = \text{sign}(-\delta)$, the origin of system (29) turns to be a weak focus of order two, and its stability have been reversed. Hence an Hopf bifurcation occurs. Taking ε and c small enough such that $c\varepsilon < 0$ and $\varepsilon\delta < 0$ we complete the sequence of multiple AH bifurcation of three limit cycles giving rise to the three SAIPWT.

In Figure 1, it is shown (by numerical integration using MATLAB) the three limit cycles associated to system (29) obtained by multiple Hopf bifurcation, and the corresponding SAIPWT for equation (28), for the values $c = -0.05$, $\epsilon = 15$, $\delta = -250$ and $\mu = 700$. □

Example: As a direct consequence of Statement (ii) of Theorem 9, for all values of h_{2k+1}, $k \geq 0$, we have that

$$u_t = \frac{\partial}{\partial x}\left[\frac{u_x}{1+u^2}\right] + \sum_{k\geq 0} h_{2k+1}u^{2k+1}u_x + u(1+u^2),$$

has a continuum of stationary PWT unfolding the steady state $u(x,t) \equiv 0$. □

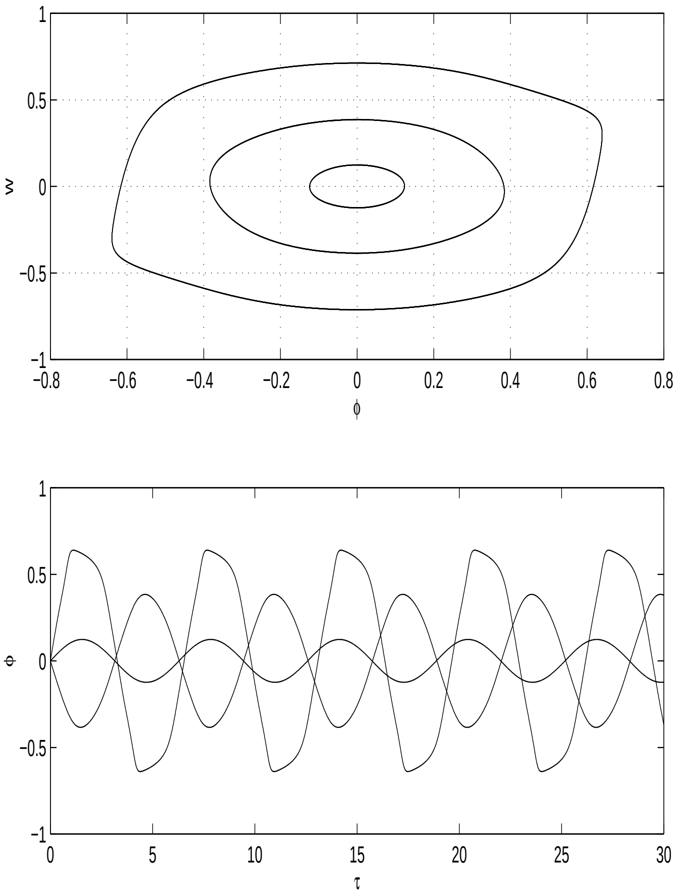

Figure 1: SAIPWT obtained by multiple Hopf bifurcation for Equation (28), for $c = -0.05$, $\epsilon = 15$, $\delta = -250$ and $\mu = 700$.

Since the early paper of Liénard in 1928, a deep research have been done in order to determine conditions for the existence of periodic solutions of Liénard's systems, and to determine the number of limit cycles. A review on the known results can also be found in [23, 27] or [35]. Also in the last years a lot of research have been done in order to determine the maximum number of limit cycles that can appear in a Liénard equation of the type (26), when both $H(u)$ and $D(u)g(u)$ are polynomials (see [2, 4, 19, 22, 23] and [9], for example). The maximum number of small amplitude limit cycles for a system of type (26), in the polynomial case is known only for some cases shown in Table 1 below (reproduced from [23]). Of course it gives the maximum number of SAIPWT for equations of type (23).

	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$	48	49	50	n
1	0	1	1	2	2	3	3	4	4	5	5	6	6	$\cdots$	24	24	$\rightarrow$	
2	1	1	2	3	3	4	5	5	6	7	7	8	9	$\cdots$	32	33	$\rightarrow$	
3	1	2	2	4	4	6	6	6	8	8	8	10	10	$\cdots$	36	38	38	
4	2	3	4	4	6	7	8	9	9									
5	2	3	4	6	6													
6	3	4	6	7														
7	3	5	6	8														
8	4	5	6	9														
9	4	6	8	9														
10	5	7	8															
11	5	7	8															
12	6	8	10															
13	6	9	10															
$\vdots$	$\vdots$	$\vdots$	$\vdots$															
48	24	32	36															
49	24	33	38															
50	$\downarrow$	$\downarrow$	38															
m																		

Table 1: Maximum number of SAIPWT for equation (23) when $h(u)$ and $D(u)g(u)$ are polynomials of degree n and m respectively (reproduced from [23]).

Just as an example of the type of results concerning (global) IPWT that can be obtained using this approach, we rewrite the classical results of Liénard and Zhang in terms of solutions of the PDE (23) (see [27] or [35], to obtain more information about other criteria of existence of limit cycles in Liénard equations).

Theorem 10 (Liénard–Corduneanu). *Suppose that*

(i) Both $c + H(u)$ and $D(u)g(u)$ are odd functions and $u\,D(u)\,g(u) > 0$ for all $u \neq 0$.

(ii) $h(u)$ and $D(u)g(u)$ are continuous for all u, and $D(u)g(u)$ satisfies Lipschitz condition for all u.

(iii) $c + H(u)$ has a single positive zero at $u = a > 0$, and $cu + H(u)$ increases monotonically to infinity for $u \geq a$ as $u \to +\infty$.

Then equation (23) has a unique IPWT with speed c.

Example: As a direct application of the above result we have that there exist a unique IPWT for each speed $c < 1$ for both equations

(i) $u_t = [(1+u^2)u_x]_x + u - u_x + 3u^2u_x$.

(ii) $u_t = \dfrac{\partial}{\partial x}\left[\dfrac{u_x}{1+u^2}\right] + u - u_x + u^3 + 3u^2u_x$. □

Theorem 11. *Assume that $a < 0 < b$, and*

(i) Both $H(u)$ and $D(u)g(u)$ are $\mathcal{C}^1(a,b)$ functions and $u\,D(u)\,g(u) > 0$ for all $u \neq 0$.

(ii) If $a = -\infty$, then $\lim_{u\to a}\int_0^u D(s)g(s)\,ds = \infty$, and if $b = \infty$, then $\lim_{u\to b}\int_0^u D(s)g(s)\,ds = \infty$,

(iii) $h(u)/(D(u)g(u))$ increases monotonically on $(a,0)\cup(0,b)$, and is not constant in a neighbourhood of 0.

Then equation (23) has at most one IPWT of the form $u(x,t) = \phi(x-ct)$, where ϕ is a periodic function such that $\phi(x-ct) \in (a,b)$.

6 Final consideration

In Section 3, we have seen that system (2) generates a one-parameter family of rotated systems. By virtue of Perko's Planar Termination Principle ([26] and [27]), and since the rotated family of systems (2) is defined for $c \in (-\infty,\infty)$, any family of limit cycles Γ_c of (2), is either unbounded, or terminates on a graphic of system, or terminates on a singular point of (2). As limit cycles of rotated systems expands or contracts monotonically with the rotation parameter, a limit cycle family Γ_c arising from a singular point of system (2), is either unbounded or terminates in a graphic. This last case has very significative implications for the study of other type of travelling waves solutions for equation (1), since the presence of the heteroclinic or homoclinic connections of the graphic correspond to the presence of travelling wave solutions of front or pulse type for the corresponding boundary value problem. For example, if we consider equation (1) under assumptions (A), the SAIPWT corresponding to a limit cycle generated at $c = c_*$ evolves (as c varies) from the steady state solution $u(x,t) \equiv \alpha$ to some of the following types of travelling wave solutions of equation (1) that only occur for a unique value of $c = \bar{c}$:

(a) If Γ_c terminates in a heteroclinic graphic connecting P_0 and P_1, then the family of IPWT bifurcates at $c = \bar{c}$ into two travelling wave solutions of equation (1) of front type (see [30, 31] and [33] for example, for a definition and more details).

(b) If Γ_c terminates in a homoclinic connection of P_0, or a homoclinic connection of P_1, then the family of IPWT bifurcates at $c = \bar{c}$ into a travelling wave solutions of pulse type.

Further research must be done in order to determine an classify the exact terminal behaviour of the families of IPWT, an their possible terminal bifurcation into a pulse or front type travelling wave solution.

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