Asymptotic behaviour for occupation times of certain self–similar processes

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Abstract

We give an almost sure asymptotic behaviour for occupation times of some self–similar processes by means of the fractional derivatives of their local times. Fractional Brownian motion and stable Lévy processes are given as examples. Related processes are also discussed.

1 Introduction and main result

Let $\{X_t, t \geq 0\}$ be a real valued stochastic process with stationary increments (SI for brevity). It is called self similar with exponent τ if the two processes $\{X_{ct}, t \geq 0\}$ and $\{c^\tau X_t, t \geq 0\}$ have the same finite dimensional distribution for all $c > 0$. (τ–SS for brevity).

Let us first assume that we can associate a family $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ of stochastic processes to a τ–SS process with SI such that

$$\sup_{x\in\mathbb{R}} \|L(t,x) - L(s,x)\|_p \leq c_1(\tau)(p!)^{\frac{1}{p}} |t-s|^{1-\tau} \qquad \textbf{(H.1)}$$

$$\|L(t,x) - L(s,x) - L(t,y) + L(s,y)\|_p \leq$$

$$\leq c_2(\tau)(p!)^{\frac{1}{p}} |t-s|^{1-\tau(1+\delta)} |x-y|^{\delta}, \qquad \textbf{(H.2)}$$

where $\|\cdot\|_p$ stands for the norm in $L^p(\Omega)$ and $0 < \delta < \gamma_0 =: \min(1, \frac{1}{2\tau} - \frac{1}{2})$.

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If $L(t,x)$ satisfies the so called occupation density formula

$$\int_0^t f(X_s)ds = \int_{\mathbb{R}} f(x)L(t,x)dx, \qquad \textbf{(H.3)}$$

for any bounded or non–negative Borel function f, then the two processes $\{L(ct, c^\tau x), t \geq 0\}$ and $\{c^{1-\tau}L(t,x), t \geq 0\}$ have the same finite dimensional distribution for every $c > 0$.

The family $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ is called as usual local times at the level x and time t of the process $\{X_t, t \geq 0\}$.

By Kolmogorov criterion there exist two random variable C_1 and C_2 such that for any sufficiently small $\varepsilon > 0$, and all $s,t,x,y \in [0,1]$

$$|L(t,x) - L(s,x)| \leq C_1 |t-s|^{1-\tau-\varepsilon} \quad a.s. \tag{1.1}$$

$$|L(t,x) - L(s,x) - L(t,y) + L(s,y)| \leq$$

$$\leq C_2 |t-s|^{1-\tau(1+\gamma_0)-\varepsilon} |x-y|^{\gamma_0-\varepsilon} \quad a.s. \tag{1.2}$$

This allows us to define the fractional derivative of order γ for $L(t,\cdot)$ for all $0 < \gamma < \gamma_0$.

Definition 1.1 *Let $0 < \delta < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that belongs to $\mathcal{C}^\delta(\mathbb{R}) \cap L^1(\mathbb{R})$ where $\mathcal{C}^\delta(\mathbb{R})$ is the space of locally δ–Hölder continuous functions on $\mathbb{R}$. For $0 < \gamma < \delta$ we define $D_\pm^\gamma f$ by (see e.g. Samko* et al. *(1993))*

$$D_\pm^\gamma f(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{f(x) - f(x \mp y)}{y^{1+\gamma}} dy.$$

The operators D_+^γ and D_-^γ are called respectively, right–handed and left–handed Marchaud fractional derivatives of order γ.

They satisfy the (*Switchnig identity*)

$$\int_{\mathbb{R}} f(x) D_-^\gamma g(x) dx = \int_{\mathbb{R}} g(x) D_+^\gamma f(x) dx, \tag{1.3}$$

for any f, $g \in \mathcal{C}^\delta \cap L^1(\mathbb{R})$, and $0 < \gamma < \delta$.

We put $D^\gamma := D_+^\gamma - D_-^\gamma$.

It is known that $D_\pm^\gamma f$ is $(\delta - \gamma)$–Hölder continuous whenever f is δ–Hölder continuous for any $0 < \gamma < \delta$.

Definition 1.2 *Let $p \geq 1$, we define for any f in $L^p(\mathbb{R})$, $D_\pm^0$ as*

$$D_\pm^0 f(x) := \frac{1}{\pi} \int_0^\infty \frac{f(x)\mathbf{1}_{]0,1]}(y) - f(x \mp y)}{y} dy.$$

Set $D^0 = D^0_+ - D^0_-$. It is known that the operator D^0 maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover for any $f \in L^p(\mathbb{R})$, $p > 1$

$$\left\| D^0 f \right\|_{L^p(\mathbb{R})} \leq c_p \left\| f \right\|_{L^p(\mathbb{R})},$$

(see e.g. Titchmarsh (1948) Chap. V).

Therefore for any $0 \leq \gamma < 1$ we have

$$D^\gamma f(x) = \kappa(\gamma) \int_0^\infty \frac{f(x+y) - f(x-y)}{y^{1+\gamma}} dy,$$

where $\kappa(\gamma) = \frac{\gamma}{\Gamma(1-\gamma)}$ for $0 < \gamma < 1$ and $\kappa(0) = \frac{1}{\pi}$.

The local time and its fractional derivatives always appeared in limit theorems of occupation time of certain self similar processes such as fractional Brownian motion (see Shieh (1996)) and stable Lévy processes (see Fitzsimmons and Getoor (1992)). See also Yamada (1986,1996).

In this article we derive asymptotic behaviour of occupation times of some self similar processes with stationary increments by means of the fractional derivative of their local times.

1.1 Strong approximation

Let $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ be a family of stochastic processes, associated to a τ–SS process with SI $\{X_t, t \geq 0\}$.

Set $\overline{f} = \int_{\mathbb{R}} f(x)dx$ and assume that $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ satisfies **(H.1)–(H.3).**

To simplify notations we set $Y^\gamma_\pm(t) := D^\gamma_\pm L(t,\cdot)(0)$ and $Y^\gamma(t) := D^\gamma L(t,\cdot)(0)$.

The main results are the following

Theorem 1.1 *Let $0 \leq \gamma < \delta < \gamma_0$ and f be in Domain of $D^\gamma_\pm$ such that $\overline{f} \neq 0$ and $\int_{\mathbb{R}} |f(x)|(1 + |x|^\delta)dx$ is finite. Then almost surely for all sufficiently small $\varepsilon > 0$ when t goes to 0*

$$\int_0^t D^\gamma_\mp f(X_s)ds = \overline{f}\; Y^\gamma_\pm(t) + o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

Let f be in $L^1(\mathbb{R})$ such that $\overline{D^0 f} \neq 0$. Then almost surely for all sufficiently small $\varepsilon > 0$ when t goes to 0

$$\int_0^t f(X_s)ds = \overline{D^0 f}\; Y^0(t) + o(t^{1-\tau-\varepsilon}).$$

Theorem 1.2 *Let f be in $L^1(\mathbb{R})$ such that $\overline{f}=0$. Assume that for $0<\gamma<\gamma_0$, $|x|^{1+\gamma}f(x)$ is bounded and*

$$\lim_{x\to+\infty(-\infty)}|x|^{1+\gamma}f(x)=f_+(f_-).$$

Then almost surely for all sufficiently small $\varepsilon>0$ when t goes to 0

$$\int_0^t f(X_s)ds=f_-\ Y_+^\gamma(t)-f_+\ Y_-^\gamma(t)+o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

Remark 1 *Let f be in $\mathcal{C}^\delta(\mathbb{R})\cap L^1(\mathbb{R})$ such that $\overline{f}\neq 0$ and $\{S_t,t\geq 0\}$ be a semimartingale of Itô type. Then almost surely for all sufficiently small $\varepsilon>0$ when t goes to 0*

$$\int_0^t D_\mp^\gamma f(S_s)d\left\langle S\right\rangle_s=\overline{f}\ Y_\pm^\gamma(t)+o(t^{\frac{1-\gamma}{2}-\varepsilon}).$$

Let f be in $L^1(\mathbb{R})$ such that $\overline{D^0 f}\neq 0$. Then almost surely when t goes to 0

$$\int_0^t f(S_s)d\left\langle S\right\rangle_s=\overline{D^0 f}\ Y^0(t)+o(t^{\frac{1}{2}-\varepsilon}).$$

1.2 Law of the iterated logarithm

Theorem 1.3 *Let $0\leq\gamma<\gamma_0$ Then there exist a constant $c_1(\tau,\gamma)>0$ such that*

$$\operatorname*{Limsup}_{h\to 0}\frac{|Y_\pm^\gamma(h)|}{h^{1-\tau(1+\gamma)}(\log\log\frac{1}{h})^{\tau\gamma}}\leq c_1(\tau,\gamma)\quad a.s.$$

$$\operatorname*{Limsup}_{h\to 0}\frac{\sup_{0\leq t\leq 1-h}\sup_{0\leq s\leq h}|Y_\pm^\gamma(t+s)-Y_\pm^\gamma(t)|}{h^{1-\tau(1+\gamma)}(\log\log\frac{1}{h})^{\tau\gamma}}\leq c_1(\tau,\gamma)\quad a.s.$$

Remark 2 *If $\alpha=2$ that is X is a Brownian motion Theorems 1.3 much the results of Csáki* et al. *(2001) (see Theorems 1.1 and 1.2).*

2 Examples

Example 1. Let $B^{\mathrm{H}}=\{B_t^{\mathrm{H}},t\geq 0\}$ be a real valued standard fractional Brownian motion (fBm for brevity) with Hurst parameter $\mathrm{H}\in(0,1)$. The sample paths of B^{H} are a.s. $(\mathrm{H}-\varepsilon)$–Hölder continuous for any $\varepsilon>0$ and it is H–SS with SI.

Let $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ denotes the family of local times associated with the fBm B^{H}. In the sequel we shall omit the superscript H.

Geman and Horowitz (1980) have proved that the local time $L(t,x)$ of B exists and has a.s. Hölder continuous modification of order $\gamma_0 - \varepsilon$ in space ($\gamma_0 = \min(1, \frac{1}{2\mathrm{H}} - \frac{1}{2})$) and of order $1 - \mathrm{H} - \varepsilon$ in time for any $\varepsilon > 0$. More precisely It has been proved by Xiao (1997), Lemma 2.5 p. 137–140 that

$$\sup_x \|L(t,x) - L(s,x)\|_p \leq c_1 (p!)^{\frac{1}{p}} |t-s|^{1-\mathrm{H}}$$

and

$$\|L(t,x) - L(s,x) - L(t,x) + L(s,y)\|_p \leq c_1 (p!)^{\frac{1}{p}} |t-s|^{1-\mathrm{H}(1+\delta)} |x-y|^{\delta}$$

for any $0 < \delta < \gamma_0$.

This means that the family $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ satisfies the hypotheses **(H.1)**–**(H.3)** for $\tau = \mathrm{H}$.

Example 2. Let $\{X_t, t \geq 0\}$ be a symmetric stable process of index $1 < \alpha \leq 2$. That is $X_0 = 0$, X has stationary independent increments with the characteristic

$$\mathbb{E}\exp(izX_t) = \exp(-t|z|^{\alpha}), \quad \text{for any} \quad z \in \mathbb{R}.$$

It is well know that this process is $\frac{1}{\alpha}$–SS with SI and admits a continuous local time process $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ (see Barlow (1985, 1988)) which satisfies also the condition **(H.3)**. Moreover by Marcus and Rosen (1992), Lemma 3.3 p. 1706, we know that $\{L(t,x), t \geq 0, x \in \mathbb{R}\}$ satisfies the hypotheses **(H.1)** and **(H.2)** for $\tau = \frac{1}{\alpha}$.

Remark that in this example the sample paths of $\{X_t, t \geq 0\}$ are just càdlàg a.s. however its local time is known to have Hölder continuous sample paths in both variables.

Example 3. We mention also two important classes of H–SS symmetric α–stable process with SI namely linear fractional stable and harmonizable fractional stable processes, which are defined respectively by

$$\begin{aligned} Z_{a,b}(t) &= \int_{\mathbb{R}} \Big[a\,(t-u)_+^{\mathrm{H}-\frac{1}{\alpha}} - (-u)_+^{\mathrm{H}-\frac{1}{\alpha}} \\ &\qquad + b\,(t-u)_-^{\mathrm{H}-\frac{1}{\alpha}} - (-u)_-^{\mathrm{H}-\frac{1}{\alpha}} \Big] X_\alpha(du), \end{aligned}$$

and

$$\widetilde{Z}_{a,b}(t) = \mathrm{Re}\int_{\mathbb{R}} \left(a u_+^{1-\mathrm{H}-\frac{1}{\alpha}} + b u_-^{1-\mathrm{H}-\frac{1}{\alpha}} \right) \frac{e^{itu}-1}{iu} \widetilde{X}_\alpha(du),$$

where $0 < \mathrm{H} < 1$, $0 < \alpha < 2$, $\mathrm{H} \neq \frac{1}{\alpha}$, a and b are real numbers such that $a^2 + b^2 > 0$ and $\{X_\alpha(s), s \in \mathbb{R}\}$ and $\{\widetilde{X}_\alpha(s), s \in \mathbb{R}\}$ are respectively real and complex symmetric Lévy α–stable motions. Remark that in the case $\mathrm{H} > \frac{1}{\alpha}$ it is known that sample paths of the $Z_{a,b}(t)$ are continuous, while in case $\mathrm{H} < \frac{1}{\alpha}$ are nowhere bounded and hence everywhere discontinuous. However its local time exists and is $(1 - \mathrm{H} - \varepsilon)$–Hölder continuous in time and $(\gamma_0 - \varepsilon)$–Hölder continuous in space for $0 < \mathrm{H} < 1$ and $1 \leq \alpha < 2$. For related discussions see Kôno and Shieh (1993).

3 Proofs of Theorems 1.1 and 1.2

Before starting the proofs of Theorems 1.1 and 1.2 we proof two technical lemmas which are the keys here.

Lemma 3.1 *Let $0 < \gamma < \gamma_0$ and $D \in \{D_+^\gamma, D_-^\gamma, D^\gamma\}$. Then almost surely for any $0 < \beta < 1 - \tau(1+\gamma)$, there exist a finite random variable $R(\omega) > 0$ such that for every s, t in $\mathbb{R}_+^2$*

$$\sup_x |DL(t,\cdot)(x) - DL(s,\cdot)(x)| \leq R(\omega) \, |t-s|^\beta ,$$

and for any $\varepsilon > 0$, small enough

$$\sup_x \left| D^0 L(t,\cdot)(x) - D^0 L(s,\cdot)(x) \right| \leq R(\omega) \, |t-s|^{1-\tau-\varepsilon} .$$

Corollary 3.1 *Let $0 \leq \gamma < \gamma_0$ and $D \in \{D_+^\gamma, D_-^\gamma, D^\gamma\}$. Then for all $\varepsilon >$, small enough when t goes to 0*

$$\sup_x |DL(t,\cdot)(x)| = o(t^{1-\tau(1+\gamma)-\varepsilon}).$$

and

$$\sup_{t\in[0,T]} \sup_x |x|^{1+\gamma} \, |DL(t,\cdot)(x)| < \infty.$$

Lemma 3.2 *Let $0 \leq \gamma < \gamma_0$ and $D \in \{D_+^\gamma, D_-^\gamma, D^\gamma\}$. Then almost surely for any $0 < \delta < 1 - \tau(1+\gamma)$, there exist a finite random variable $R(\omega) > 0$ such that for every t in $\mathbb{R}_+$ and $\varepsilon > 0$ small enough*

$$|DL(t,\cdot)(x) - DL(t,\cdot)(y)| \leq R(\omega) t^{1-\tau(1+\gamma)-\varepsilon} \left(1 + t^{-\tau\delta} \, |x-y|^\delta \right).$$

Proof of Lemma 3.1: Let us give the proof for D_+^γ the other case can be derived similarly and by linearity.
From the definition of D_+^γ we have fro all integers $p \geq 1$

$$\begin{aligned}
& \left\| D_+^\gamma L(t,\cdot)(x) - D_+^\gamma L(s,\cdot)(x) \right\|_p \\
\leq \; & \kappa(\gamma) \int_0^b \frac{\|L(t,x) - L(t,x-y) - L(s,x) + L(s,x-y)\|_p}{y^{1+\gamma}} dy \\
& + \kappa(\gamma) \int_b^{+\infty} \frac{\|L(t,x) - L(t,x-y) - L(s,x) + L(s,x-y)\|_p}{y^{1+\gamma}} dy \\
= \; & : A + B.
\end{aligned}$$

Using **(H.1)** and **(H.2)** we obtain

$$A \leq c_1(\tau)\kappa(\gamma)(p!)^{\frac{1}{p}} |t-s|^{1-\tau(1+\delta)} b^{\delta-\gamma},$$

and

$$B \leq c_2(\tau)\kappa(\gamma)(p!)^{\frac{1}{p}} |t-s|^{1-\tau} b^{-\gamma}.$$

Choose $b = |t-s|^\tau$ we get

$$\sup_x \left\| D_+^\gamma L(t,\cdot)(x) - D_+^\gamma L(s,\cdot)(x) \right\|_p \leq (p!)^{\frac{1}{p}} c(\tau,\gamma)|t-s|^{1-\tau(1+\gamma)}, \qquad (3.4)$$

which by Kolmogorov criteria finishes the proof.

Proof of Lemma 3.2: Let us give the proof for D_-^γ. We have

$$\begin{aligned}
& \left| D_-^\gamma L(t,\cdot)(x) - D_-^\gamma L(t,\cdot)(y) \right| \\
\leq \; & \kappa(\gamma) \int_0^b \frac{|L(t,x) - L(t,x+z)| + |L(t,y) - L(t,y+z)|}{z^{1+\gamma}} dz \\
& + \kappa(\gamma) \int_b^{+\infty} \frac{|L(t,x) - L(t,y)| + |L(t,x+z) - L(t,y+z)|}{z^{1+\gamma}} dz \\
= \; & : C + D.
\end{aligned}$$

Using inequalities (1.1) and (1.2) we get for any $\varepsilon > 0$ sufficiently small

$$C \leq c_1(\tau,\gamma) t^{1-\tau(1+\delta)-\varepsilon} b^{\delta-\gamma},$$

and

$$D \leq c_2(\tau,\gamma)t^{1-\tau(1+\delta)-\varepsilon} |x-y|^{\delta} b^{-\gamma}.$$

Choosing $b = t^{\tau}$ we get

$$\left|D_{-}^{\gamma}L(t,\cdot)(x) - D_{-}^{\gamma}L(t,\cdot)(y)\right| \leq R(\omega)t^{1-\tau(1+\gamma)-\varepsilon}\left(1 + t^{-\tau\delta}|x-y|^{\delta}\right).$$

The of the lemma is done.

Proof of Theorem 1.1. Set

$$I(t) := \int_0^t D_{\pm}^{\gamma} f(B_s)ds - \overline{f}\, Y_{\mp}^{\gamma}(t)$$

Thanks to the occupation density formula and (1.3) we have

$$I(t) = \int_{\mathbb{R}} f(x)\left\{D_{\mp}^{\gamma}L(t,\cdot)(x) - Y_{\mp}^{\gamma}(t)\right\}dx.$$

By Lemma 3.2,

$$|I(t)| \leq R(\omega)t^{1-\tau(1+\gamma)-\varepsilon}\int_{\mathbb{R}} |f(x)|(1+|x|^{\delta})dx,$$

for all $t \in [0,1]$ and ε sufficiently small. The hypothesis on the function f gives the result.

Proof of Theorem 1.2. Let us set

$$J(t) = \int_0^t f(B_s)ds - f_{-}\, Y_{+}^{\gamma}(t) + f_{+}\, Y_{-}^{\gamma}(t).$$

Using occupation density formula and the fact that $\overline{f} = 0$, we obtain

$$J(t) = \int_{\mathbb{R}} f(y)(L(t,y) - L(t,0))dy - f_{-}\, Y_{+}^{\gamma}(t) + f_{+}\, Y_{-}^{\gamma}(t).$$

Then

$$\begin{aligned} |J(t)| \quad &\leq \quad \left|\int_{\mathbb{R}} y^{1+\gamma} f(y)\frac{L(t,y)-L(t,0)}{y^{1+\gamma}}dy\right| \\ &\quad + |f_{-}|\left|Y_{+}^{\gamma}(t)\right| + |f_{+}|\left|Y_{-}^{\gamma}(t)\right|. \end{aligned}$$

Since $|y|^{1+\gamma}|f(y)|$ is bounded, we get

$$|J(t)| \leq c\left\{\left|Y_{+}^{\gamma}(t)\right| + \left|Y_{-}^{\gamma}(t)\right|\right\}.$$

By Lemma 3.1 and Corollary 3.1

$$J(t) = o(t^{1-\tau(1+\gamma)-\varepsilon}),$$

which gives the desired estimate since ε is arbitrary.

The Proof of Theorem 1.3 is similar to that given by Csáki et *al.* (2001).

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