A CLASS OF INTERPOLATING BLASCHKE PRODUCTS AND BEST APPROXIMATION IN L^p FOR $p < 1$

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ABSTRACT. We describe the inner functions Θ such that $\|1+\Theta f\|^p_{H^p} \geq 1 - |\Theta(0)|^2$ for all $p > 0$ and $f \in H^p$. We prove that each such inner function Θ satisfying $\Theta(0) \neq 0$ is an interpolating Blaschke product. Moreover, we study the inner functions such that $\|1 + \Theta f\|^p_{H^p} \geq 1 - |\Theta(0)|^2$ for all $p > 0$ and for all $f \in H^p$ for which $1 + \Theta f$ does not vanish in the unit disk.

0. Introduction

Let H^p denote the Hardy space in the unit disk $\mathbb{D} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}$, where $0 < p < +\infty$. Denote by $\boldsymbol{m}$ normalized Lebesgue measure on the unit circle $\mathbb{T} \overset{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}$. As usual, we identify H^p, in the natural way, with a closed subspace of $L^p = L^p(\mathbb{T})$. Put $\|f\|_p \overset{\text{def}}{=} \big(\int_{\mathbb{T}} |f|^p \, d\boldsymbol{m}\big)^{\frac{1}{p}}$ for $f \in L^p(\mathbb{T})$.

Let Θ be an inner function. Clearly, $\operatorname{dist}_{L^p}(\overline{\Theta}, H^p) = \operatorname{dist}_{H^p}(1, \Theta H^p)$. Put

$$\epsilon_p(\Theta) \overset{\text{def}}{=} \operatorname{dist}^p_{H^p}(1, \Theta H^p) \overset{\text{def}}{=} \inf\{\|f\|^p_p : f \in 1 + \Theta H^p\}. \tag{0.1}$$

It is easy to see that the infimum is attained. Substituting $f = \big(1-\overline{\Theta(0)}\Theta\big)^{\frac{2}{p}}$ in (0.1) we get $\epsilon_p(\Theta) \leq 1 - |\Theta(0)|^2$ for all $p > 0$. It is well known and it is easy to prove (see §1) that $\epsilon_p(\Theta) = 1 - |\Theta(0)|^2$ for $p \geq 1$. Moreover, $\epsilon_p(\Theta) = 1$ for all $p > 0$ if $\Theta(0) = 0$. But if $\Theta(0) \neq 0$ and $p < 1$, then $\epsilon_p(\Theta)$ can be less than $1 - |\Theta(0)|^2$. We are going to describe and investigate the nonconstant inner functions Θ such that $\Theta(0) \neq 0$ and $\epsilon_p(\Theta) = 1 - |\Theta(0)|^2$

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for all $p > 0$. It turns out that such an inner function Θ should be an interpolating Blaschke product of a special form.

In Section 1 we state elementary properties of the functional $\epsilon_p(\Theta)$. In Section 2 we describe the Blaschke products B such that $\epsilon_p(B) = 1 - |B(0)|^2$ for all $p > 0$. In Section 3 we prove that for every sequence $\{r_n\}_{n=1}^N$ in $(0,1)$ satisfying the Blaschke condition there exists a Blaschke product $B(z) = \prod_{n=1}^{N} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a}_n z}$ such that $\epsilon_p(B) = 1 - |B(0)|^2$ for all $p > 0$ and $|a_n| = r_n$ for all n. In Section 4 we investigate the functional $\epsilon_p^0(\Theta)$ defined as follows

$$\epsilon_p^0(\Theta) \stackrel{\text{def}}{=} \inf\{\|f\|_p^p : f \in 1 + \Theta H^p, f \neq 0 \text{ in } \mathbb{D}\}.$$

In Section 5 we describe the finite Blaschke products B such that $\epsilon_p^0(B) = 1 - |B(0)|^2$ for all $p > 0$. In Section 6 we prove that $\epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2$ for all $p > 0$ if Θ is an inner function of the form $\Theta(z) = \exp\left(\tau_1 \frac{z+1}{z-1} + \tau_2 \frac{z-1}{z+1}\right) \prod_n \frac{t_n}{|t_n|} \frac{t_n - z}{1 - t_n z}$, where $\tau_1, \tau_2 \in [0, +\infty)$ and $t_n \in (-1, 1)$. In Section 7 we investigate $\epsilon_p^0(B)$ in the case where B is a Blaschke product of degree two. We obtain an explicit formula for the number $p_B^0 \in [0,1)$ such that $\epsilon_p^0(B) = 1 - |B(0)|^2$ if and only if $p \geq p_B^0$, where $p \in (0, +\infty)$. Finally, Section 8 contains several open problems.

We conclude this section with the list of notation.

$\mathbb{Z}$ is the set of all integers.

$\mathbb{Z}_+$ is the set of all nonnegative integers.

$\mathbb{N}$ is the set of all positive integers.

$\mathbb{N}_N \stackrel{\text{def}}{=} \{n \in \mathbb{N} : n \leq N\}$.

$\mathbb{D}$ is the open unit disk.

$\mathbb{T}$ is the unit circle.

$\boldsymbol{m}$ is normalized Lebesgue measure on $\mathbb{T}$.

δ_a is the δ-measure at a point a.

∂G is the boundary of a set G.

$M(\mathbb{T})$ denotes the set of all complex Borel measures μ on $\mathbb{T}$.

For $\mu \in M(\mathbb{T})$, we put $\mu_+(z) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}$, where $z \in \mathbb{D}$.

Put

$p_\Theta \stackrel{\text{def}}{=} \inf\{p \in (0, +\infty) : \epsilon_p(\Theta) = 1 - |\Theta(0)|^2\}$,

$p_\Theta^0 \stackrel{\text{def}}{=} \inf\{p \in (0, +\infty) : \epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2\}$.

$\mathfrak{I}$ denotes the set of all nonconstant inner functions,

$\mathfrak{A}_+ \stackrel{\text{def}}{=} \{\Theta \in \mathfrak{I} : p_\Theta = 0\}$,

$\mathfrak{A} \stackrel{\text{def}}{=} \{\Theta \in \mathfrak{I} : p_\Theta^0 = 0\}$.

1. Properties of $\epsilon_p(\Theta)$

Theorem 1.1. *Let Θ be an inner function and let $p \in (0,+\infty)$. Then the following statements are equivalent:*

(i) $\epsilon_p(\Theta) = 1 - |\Theta(0)|^2$;

(ii) $\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^p - 1)\,d\boldsymbol{m} \geq 0$ *for all $g \in H^p$.*

Proof. Note that

$$\{(1-\overline{\Theta(0)}\Theta)^{\frac{2}{p}}(1+\Theta g) : g \in H^p\} = 1 + \Theta H^p.$$

Hence, the statement (ii) means that $\epsilon_p(\Theta) \geq 1 - |\Theta(0)|^2$. It remains to recall that $\epsilon_p(\Theta) \leq 1 - |\Theta(0)|^2$ for all $p > 0$ (see §0). ■

Corollary 1.2. *Let Θ be an inner function. Suppose that $\epsilon_q(\Theta) = 1 - |\Theta(0)|^2$ for some $q \in (0,+\infty)$. Then $\epsilon_p(\Theta) = 1-|\Theta(0)|^2$ for all $p \in (q,+\infty)$.*

Proof. It suffices to note that

$$\frac{1}{q}\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^q - 1)\,d\boldsymbol{m} \leq \frac{1}{p}\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^p - 1)\,d\boldsymbol{m} \qquad (1.2)$$

since the function $t \mapsto \frac{a^t-1}{t}$ is increasing for any $a \in (0,+\infty)$. ■

Corollary 1.3. *Let Θ be an inner function. Then $\epsilon_p(\Theta) = 1 - |\Theta(0)|^2$ for all $p \in [1,+\infty)$. Moreover, if $\|f\|_p^p = 1 - |\Theta(0)|^2$ for some $f \in 1 + \Theta H^p$ with $p \geq 1$, then $f = (1-\overline{\Theta}(0)\Theta)^{\frac{2}{p}}$.*

Proof. First, we consider the case where $p = 1$, i. e., we prove that

$$\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2|1+\Theta g|\,d\boldsymbol{m} \geq 1 - |\Theta(0)|^2$$

for all $g \in H^1$. We have

$$\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2|1+\Theta g|\,d\boldsymbol{m} \geq \operatorname{Re}\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(1+\Theta g)\,d\boldsymbol{m}$$

$$= 1 - |\Theta(0)|^2 + \operatorname{Re}\int\limits_{\mathbb{T}} (1-\overline{\Theta(0)}\Theta)(\Theta - \Theta(0))g\,d\boldsymbol{m} = 1 - |\Theta(0)|^2.$$

Clearly, the equality $\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2|1+\Theta g|\,d\boldsymbol{m} = 1-|\Theta(0)|^2$ is fulfilled for $g=0$ only.

Corollary 1.2 implies that $\epsilon_p(\Theta) = 1-|\Theta(0)|^2$ for all $p \ge 1$. Suppose that

$$\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2|1+\Theta g|^p\,d\boldsymbol{m} = 1-|\Theta(0)|^2$$

for some $g \in H^p$ with $p \ge 1$. Then the inequality (1.2) implies that

$$\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2|1+\Theta g|\,d\boldsymbol{m} \le 1-|\Theta(0)|^2.$$

Consequently, $g=0$. ■

Remark 1. Corollary 1.3 is well known, cf. [G] for generalizations. Note that a generalization of the inequality (ii) of Theorem 1.1 was proved by K. M. Dyakonov [D] provided $p \ge 1$. Thus, Corollary 1.3 can be also reduced to this result of K. M. Dyakonov.

Remark 2. Clearly,

$$\inf\Big\{\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^p-1)\,d\boldsymbol{m} : g \in H^p\Big\}$$
$$= \inf\Big\{\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^p-1)\,d\boldsymbol{m} : g \in H^\infty\Big\}.$$

Hence, in the statement (ii) of Theorem 1.1, we could write " for all $g \in H^\infty$ " instead of " for all $g \in H^p$ ".

Remark 3. If $g \in H^\infty$ and $\|g\|_\infty \le 1$, then $\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2(|1+\Theta g|^p - 1)\,d\boldsymbol{m} \ge 0$ for all inner functions Θ and $p > 0$.

Proof. Consider the function $F \stackrel{\text{def}}{=} (1-\overline{\Theta(0)}\Theta)(1+\Theta g)^{\frac{p}{2}}$. Clearly, $F-1 \in \Theta H^2$. Hence, $\|F\|_2^2 \ge 1-|\Theta(0)|^2$. ■

Theorem 1.4. *Let Θ be an inner function. The following statements are equivalent:*

(i) $\epsilon_p(\Theta) = 1-|\Theta(0)|^2$ *for any* $p > 0$, *i. e.,* $p_\Theta = 0$*;*

(ii) $\int\limits_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2\log|1+\Theta g|\,d\boldsymbol{m} \ge 0$ *for all* $g \in H^\infty$.

Proof. Note that

$$\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|1+\Theta g|\, d\boldsymbol{m}$$

$$= \lim_{p\to 0+} \frac{1}{p}\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 (|1+\Theta g|^p - 1)\, d\boldsymbol{m}$$

$$= \inf_{p>0} \frac{1}{p}\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 (|1+\Theta g|^p - 1)\, d\boldsymbol{m}.$$

It remains to apply Theorem 1.1. ■

Remark 1. Suppose that there exists a function $g\in H^\infty$ such that

$$\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|1+\Theta g|\, d\boldsymbol{m} < 0.$$

Then

$$\inf\left\{ \int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|1+\Theta g|\, d\boldsymbol{m} : g\in H^\infty \right\} = -\infty.$$

Indeed, it suffices to note that

$$\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|(1+\Theta g)^n| = n\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|1+\Theta g|$$

for all positive integers n.

Remark 2. If $g\in H^\infty$ and $\|g\|_\infty \le 1$, then $\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 \log|1+\Theta g|\, d\boldsymbol{m} = 0$ for every inner function Θ.

Note that $\log(1+\Theta g)\in \Theta H^2$ and $\int_{\mathbb{T}} |1-\overline{\Theta(0)}\Theta|^2 h\,\Theta\, d\boldsymbol{m} = 0$ for all $h\in H^2$.

Put $H^p(\Theta) \stackrel{\text{def}}{=} \{f\circ\Theta : f\in H^p\}$ and

$$\varepsilon_p(\Theta) \stackrel{\text{def}}{=} \inf\{\|f\|_p^p : f\in 1+\Theta H^p(\Theta)\}.$$

Clearly, $\epsilon_p(\Theta)\le\varepsilon_p(\Theta)\le 1-|\Theta(0)|^2$. It is interesting to note that $\varepsilon_p(\Theta) = 1-|\Theta(0)|^2$ for all $p>0$ and inner functions Θ.

Theorem 1.5. *Let Θ be an inner function. Then $\varepsilon_p(\Theta) = 1-|\Theta(0)|^2$ for all $p\in(0,+\infty)$. Moreover, the equality $\|f\|_p^p = 1-|\Theta(0)|^2$ is fulfilled for $f\in 1+\Theta H^p(\Theta)$ if and only if $f = \big(1-\overline{\Theta}(0)\Theta\big)^{\frac{2}{p}}$.*

Proof. It is well known that

$$\int\limits_{\mathbb{T}} h(\Theta)\,d\boldsymbol{m} = \int\limits_{\mathbb{T}} h(\zeta)\frac{1-|\Theta(0)|^2}{|1-\overline{\Theta(0)}\zeta|^2}\,d\boldsymbol{m}$$

for all $h \in L^1(\mathbb{T})$ and nonconstant inner functions Θ. Therefore

$$\int\limits_{\mathbb{T}} |1+\Theta g(\Theta)|^p\,dm = \int\limits_{\mathbb{T}} |1+\zeta g(\zeta)|^p \frac{1-|\Theta(0)|^2}{|1-\overline{\Theta(0)}\zeta|^2}\,d\boldsymbol{m}$$

$$= (1-|\Theta(0)|^2)\left\|\frac{1+zg(z)}{(1-\overline{\Theta(0)}z)^{\frac{2}{p}}}\right\|_p^p \ge 1-|\Theta(0)|^2$$

because $|F(0)| \le \|F\|_p$ for all $F \in H^p$ and $p > 0$. It remains to note that $|F(0)| = \|F\|_p$ if and only if $F(z) = F(0)$ for all $z \in \mathbb{D}$. ■

Corollary 1.6. *Let* $\Theta(z) = \tau\dfrac{z-a}{1-\overline{a}z}$ *with* $a \in \mathbb{D}$ *and* $\tau \in \mathbb{T}$. *Then* $\epsilon_p(\Theta) = 1-|\Theta(0)|^2$.

Proof. It suffices to note that $\epsilon_p(\Theta) = \varepsilon_p(\Theta)$ since Θ is one-to-one. ■

Remark. We will show that, in general, the equality $\epsilon_p(\Theta) = 1-|\Theta(0)|^2$ does not hold for $p < 1$.

Put $p_\Theta \overset{\text{def}}{=} \inf\{p > 0 : \epsilon_p(\Theta) = 1-|\Theta(0)|^2\}$. Corollary 1.3 implies that $0 \le p_\Theta \le 1$. It is easy to see that $\epsilon_{p_\Theta}(\Theta) = 1-|\Theta(0)|^2$ if $p_\Theta > 0$.

Theorem 1.7. *Let* Θ *be a finite Blaschke product. Then* $p_\Theta < 1$.

Proof. Suppose that $p_\Theta = 1$. Then for every $p < 1$ there exists a function $f_p \in H^\infty$ such that

$$\|(1-\overline{\Theta(0)}\Theta)^{\frac{2}{p}} + \Theta f_p\|_p^p < 1-|\Theta(0)|^2.$$

Note that if $f \in H^\infty$ and $\|f\|_\infty \le (1-|\Theta(0)|)^{\frac{2}{p}}$, then

$$\|(1-\overline{\Theta(0)}\Theta)^{\frac{2}{p}} + \Theta f\|_p^p = \left\|\Big((1-\overline{\Theta(0)}\Theta)^{\frac{2}{p}}\big(1+\Theta f(1-\overline{\Theta(0)}\Theta)^{-\frac{2}{p}}\big)\Big)^{\frac{p}{2}}\right\|_2^2$$

$$\ge 1-|\Theta(0)|^2.$$

Hence, $\|f_p\|_\infty > (1-|\Theta(0)|)^{\frac{2}{p}} > (1-|\Theta(0)|)^4$ provided $p > \frac{1}{2}$. Clearly, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $(0,1)$ such that $\lim\limits_{n\to+\infty} p_n = 1$ and the sequence $\{f_{p_n}\}_{n\in\mathbb{N}}$ converges in H^∞. Put $f \overset{\text{def}}{=} \lim\limits_{n\to+\infty} f_{p_n}$. Note that $\|f\|_{H^\infty} \ge (1-|\Theta(0)|)^4$. Moreover, $\|(1-\overline{\Theta(0)}\Theta)^2 + \Theta f\|_{H^1} \le 1-|\Theta(0)|^2$ which contradicts Corollary 1.3. ■

In the same way we can prove the following assertion.

Theorem 1.8. *Let Θ be a finite Blaschke product. Suppose that $p_\Theta > 0$. Then there exists a function $g \in 1 + \Theta H^{p_\Theta}$ such that $g \neq (1 - \overline{\Theta(0)}\Theta)^{\frac{2}{p_\Theta}}$ and $\|g\|_{H^{p_\Theta}}^{p_\Theta} = 1 - |\Theta(0)|^2$.* ■

Remark. The author does not know whether there exists an inner function Θ with $p_\Theta = 1$.

Denote by $\mathfrak{I}$ the set of all nonconstant inner functions. Put $\operatorname{dist}(\Theta_1, \Theta_2) \stackrel{\text{def}}{=} \|\Theta_1 - \Theta_2\|_2$.

Theorem 1.9. *The map $\Theta \mapsto p_\Theta$ is lower semicontinuous.*

Proof. Clearly, the set $\{\Theta \in \mathfrak{I} : p_\Theta > p\}$ is open in $\mathfrak{I}$ for each $p > 0$. Consequently, the set $\{\Theta \in \mathfrak{I} : p_\Theta > 0\}$ is also open in $\mathfrak{I}$. ■

Theorem 1.10. *Let Θ and Ψ be inner functions. Then*

$$\epsilon_p(\Theta(\Psi)) \leq \frac{1 + |\Psi(0)|}{1 - |\Psi(0)|}\epsilon_p(\Theta).$$

In particular, $\epsilon_p(\Theta(\Psi)) \leq \epsilon_p(\Theta)$ if $\Psi(0) = 0$.

Proof. Clearly,

$$\epsilon_p(\Theta(\Psi)) \stackrel{\text{def}}{=} \inf\{\|1+\Theta(\Psi)f\|_p^p : f \in H^p\} \leq \inf\{\|1+\Theta(\Psi)f(\Psi)\|_p^p : f \in H^p\}.$$

It remains to note that

$$\int_{\mathbb{T}} |1 + \Theta(\Psi)f(\Psi)|^p \, dm = \int_{\mathbb{T}} |1 + \Theta(\zeta)f(\zeta)|^p \frac{1 - |\Psi(0)|^2}{|1 - \overline{\Psi(0)}\zeta|^2} \, d\boldsymbol{m}$$

$$\leq \frac{1 + |\Psi(0)|}{1 - |\Psi(0)|}\|1 + \Theta f\|_p^p. \qquad ■$$

Theorem 1.11. *Let Θ be an inner function. Put $\Theta_n(z) = \Theta(z^n)$ for a positive integer n. Then $\epsilon_p(\Theta_n) \leq \epsilon_p(\Theta) \leq \epsilon_{pn}(\Theta_n)$ for all $p \in (0, +\infty)$.*

Proof. The inequality $\epsilon_p(\Theta_n) \leq \epsilon_p(\Theta)$ follows from Theorem 1.10. Let us prove that $\epsilon_p(\Theta) \leq \epsilon_{pn}(\Theta_n)$. There exists a function $f \in H^{pn}$ such that $\epsilon_{pn}(\Theta_n) = \|1+\Theta_n f\|_{pn}^{pn}$. Consider the function $G \stackrel{\text{def}}{=} \prod_{k=1}^{n}(1+\Theta_n(z)f(\zeta^k z))$, where $\zeta = e^{\frac{2\pi i}{n}}$. Clearly, $G(\zeta z) = G(z)$ for all $z \in \mathbb{D}$ and $G(z) = 1 + \Theta(z^n)F(z^n)$ for some $F \in H^p$. Using Hölder's inequality we obtain

$$\epsilon_p(\Theta) \leq \|1 + \Theta F\|_p^p = \|G\|_p^p \leq \prod_{k=1}^{n} \|1 + \Theta_n(z)f(\zeta^k z)\|_{pn}^p$$

$$= \|1 + \Theta_n f\|_{pn}^{pn} = \epsilon_{pn}(\Theta_n). \qquad ■$$

Corollary 1.12. *Let Θ be an inner function and let n be a positive integer. Then $p_\Theta = 0$ if and only if $p_{\Theta_n} = 0$.* ■

2. Inner functions Θ with $p_\Theta = 0$

Denote by $\mathfrak{A}_+$ the set of all inner functions Θ such that $p_\Theta = 0$ and $\Theta(0) \neq 0$. In this section we obtain a description of the inner functions Θ of the class $\mathfrak{A}_+$.

First, we prove several auxiliary assertions.

Lemma 2.1. *Let $f, g \in H^1$. Suppose that $f = \Theta\overline{g}$ almost everywhere on $\mathbb{T}$. Then*

$$\int_{\mathbb{T}} f|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = f(0) - \Theta(0)\overline{g(0)}.$$

In particular,

$$\int_{\mathbb{T}} f|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = f(0)$$

for all $f \in H^2 \ominus \Theta H^2$.

Proof. We have

$$\int_{\mathbb{T}} f|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = \int_{\mathbb{T}} f(1 - \overline{\Theta(0)}\Theta)(1 - \Theta(0)\overline{\Theta}) \, d\boldsymbol{m}$$

$$= \int_{\mathbb{T}} f(1 - \overline{\Theta(0)}\Theta + |\Theta(0)|^2) \, d\boldsymbol{m} - \Theta(0) \int_{\mathbb{T}} f\overline{\Theta} \, d\boldsymbol{m} = f(0) - \Theta(0)\overline{g(0)}. \quad \blacksquare$$

Lemma 2.2. *Let I be a nonconstant inner function and let $\rho \in L^1(\mathbb{T})$. Suppose that*

$$\int_{\mathbb{T}} (\log|f \circ I|)\, \rho d\boldsymbol{m} \geq 0 \tag{2.1}$$

for all $f \in H^\infty$ such that $f(0) = 1$ and $f'(0) = 0$. Then the inequality (2.1) is fulfilled for all $f \in H^\infty$ such that $f(0) = 1$.

Proof. Assume that $f \in H^\infty$ and $f(0) = 1$. Consider the function $g(z) \stackrel{\text{def}}{=} f(z)^n(1 - nf'(0)z)$, where n is a positive integer. Clearly, $g \in H^\infty$, $g(0) = 1$ and $g'(0) = 0$. Hence, g satisfies the inequality (2.1), i.e.,

$$n \int_{\mathbb{T}} (\log|f \circ I|)\, \rho d\boldsymbol{m} + \int_{\mathbb{T}} \log|1 - nf'(0)I(z)|\, \rho(z) d\boldsymbol{m}(z) \geq 0$$

for all $n \in \mathbb{N}$. It remains to note that

$$\int_{\mathbb{T}} \log|1 - nf'(0)I(z)|\, \rho(z) d\boldsymbol{m}(z) = o(n) \quad \text{as} \quad n \to +\infty. \quad \blacksquare$$

Lemma 2.3. *Let Θ be an inner function with $\Theta(0) \neq 0$. Suppose that there exists an inner function I such that I is a proper divisor of Θ and Θ is a divisor of I^2. Then $p_\Theta > 0$.*

Proof. Suppose that $p_\Theta = 0$. By Theorem 1.4, we have

$$\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta g| \, d\boldsymbol{m} \geq 0$$

for all $g \in H^\infty$. Hence,

$$\int_{\mathbb{T}} (\log|f \circ I|)|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} \geq 0 \tag{2.2}$$

for all $f \in H^\infty$ such that $f(0) = 1$ and $f'(0) = 0$. Lemma 2.2 implies that the inequality (2.2) is fulfilled for all $f \in H^\infty$ such that $f(0) = 1$. In particular, this inequality is fulfilled for $f(z) = 1 - \overline{I(0)}z$. Note that

$$\int_{\mathbb{T}} (\log(1 - \overline{I(0)}I))|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = -\int_{\mathbb{T}} \overline{I(0)}I|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m}$$

since $\log(1 - \overline{I(0)}I) + \overline{I(0)}I \in I^2 H^\infty \subset \Theta H^\infty$. We have

$$\int_{\mathbb{T}} \overline{I(0)}I|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = |I(0)|^2 - |\Theta(0)|^2$$

by Lemma 2.1. Hence,

$$\int_{\mathbb{T}} (\log|1 - \overline{I(0)}I|)|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m}$$

$$= \operatorname{Re} \int_{\mathbb{T}} \log(1 - \overline{I(0)}I)|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = |\Theta(0)|^2 - |I(0)|^2 < 0,$$

and we get a contradiction. ∎

Theorem 2.4. *Let $\Theta \in \mathfrak{A}_+$. Then Θ is a Blaschke product with simple zeros.*

Proof. We may represent the function Θ in the form $\Theta = SB$, where S is a singular inner function and B is a Blaschke product. Suppose that $S \neq \text{const}$. Applying Lemma 2.3 for $I = S^{\frac{1}{2}}B$, we get a contradiction. Hence, $\Theta = B$ is a Blaschke product. Suppose that there exists a zero a of Θ such that $\Theta'(a) = 0$. Applying Lemma 2.3 for $I(z) \stackrel{\text{def}}{=} \Theta(z)\frac{1-\overline{a}z}{z-a}$, we again get a contradiction. ∎

Lemma 2.5. *Let B be a Blaschke product with simple zeros such that $B(0) \neq 0$ and $p_B = 0$. Let g be a bounded function in $H^2 \ominus BH^2$. Suppose that $g^2 - g$ vanishes on the zero set of B. Then $0 \le g(0) \le 1 - |B(0)|^2$.*

Proof. First, we prove that $g(0) \in \mathbb{R}$. Theorem 1.4 implies that

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 \log |e^{2\pi i g}| \, d\boldsymbol{m} \ge 0$$

and

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 \log |e^{-2\pi i g}| \, d\boldsymbol{m} \ge 0,$$

whence

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 \log |e^{2\pi i g}| \, d\boldsymbol{m} = 0.$$

Therefore $\mathrm{Re}(2\pi i g(0)) = 0$ by Lemma 2.1, i. e., $g(0) \in \mathbb{R}$.

Note that the function $1 - \overline{B(0)}B - g$ also satisfies the assumptions of Lemma 2.5. Thus, it suffices to prove that $g(0) \ge 0$. Suppose that $g(0) < 0$. By the Nevanlinna theorem (see [G], Chapter 4, Theorem 4.1), there exists an inner function I such that

$$I(a) = \begin{cases} 0, & \text{if } B(a) = 0, \;\; g(a) = 0; \\ (2\|g\|_\infty)^{-1}, & \text{if } B(a) = 0, \;\; g(a) = 1. \end{cases}$$

Consequently, for a sufficient large positive λ, there exists an inner function Ψ such that

$$\Psi(a) = \begin{cases} e^{\lambda g(0)}, & \text{if } B(a) = 0, \;\; g(a) = 0; \\ e^{\lambda(g(0)-2)}, & \text{if } B(a) = 0, \;\; g(a) = 1. \end{cases}$$

Applying Lemma 2.1, we get

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 \log |\Psi e^{\lambda(2g - g(0))}| \, d\boldsymbol{m}$$

$$= \lambda \,\mathrm{Re} \int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 (2g - g(0)) \, d\boldsymbol{m} \lambda \, g(0)(1 + |B(0)|^2) < 0.$$

This contradicts Theorem 1.4 because the function $\Psi e^{\lambda(2g - g(0))} - 1$ vanishes on the zero set of B. ■

Certainly, the next lemma is well known.

Lemma 2.6. *Let* $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$ *be a Blaschke product such that* $a_n B'(a_n) \neq 0$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} \frac{1}{|B'(a_n)|} < +\infty$. *Then*

$$1 - \overline{B(0)} B(z) = -\sum_{n=1}^{\infty} \frac{\overline{B(0)}}{\overline{a}_n \overline{B'(a_n)}} \cdot \frac{1}{1 - \overline{a}_n z}$$

for all $z \in \mathbb{D}$.

Proof. Put $B_N(z) = \prod_{n=1}^{N} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$ where N is a positive integer. Clearly,

$$1 - \overline{B_N(0)} B_N(z) = -\sum_{n=1}^{N} \frac{\overline{B_N(0)}}{\overline{a}_n \overline{B'_N(a_n)}} \cdot \frac{1}{1 - \overline{a}_n z}$$

for all $z \in \mathbb{D}$.

To justify the passage to the limit as $N \to +\infty$ it suffices to note that

$$\left| \frac{B_N(0)}{a_n B'_N(a_n)} \right| \leq \frac{1}{|a_n||B'(a_n)|}$$

for $1 \leq n \leq N$. ■

Let $N \in \mathbb{N} \cup \{\infty\}$. Put $\mathbb{N}_N \stackrel{\text{def}}{=} \{n \in \mathbb{N} : n \leq N\}$.

Theorem 2.7. *Let* $\{a_n\}_{n=1}^{N}$ *be a sequence in* $\mathbb{D} \setminus \{0\}$ *satisfying the Blaschke condition* $\sum_{n=1}^{N} (1 - |a_n|) < +\infty$, *where* $N \in \mathbb{N} \cup \{\infty\}$. *Suppose that* $a_k \neq a_l$ *for* $k \neq l$. *Let* B *denote the corresponding Blaschke product,* $B(z) = \prod_{n=1}^{N} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$. *The following statements are equivalent:*

(i) $p_B = 0$;

(ii) $\frac{a_n B'(a_n)}{B(0)} < 0$ *for all* $n \in \mathbb{N}_N$ *and* $\sum_{n=1}^{N} \frac{1}{|B'(a_n)|} < +\infty$;

(iii) *There exists a sequence* $\{\lambda_n\}_{n \in \mathbb{N}_N}$ *in* $[0, +\infty)$ *such that*

$$1 - \overline{B(0)} B(z) = \sum_{n=1}^{N} \frac{\lambda_n}{1 - \overline{a}_n z} \tag{2.3}$$

for all $z \in \mathbb{D}$;

(iv) *There exists a sequence* $\{\lambda_n\}_{n \in \mathbb{N}_N}$ *in* $[0, +\infty)$ *such that*

$$|1 - \overline{B(0)} B(z)|^2 = \sum_{n=1}^{N} \lambda_n \frac{1 - |a_n|^2}{|1 - \overline{a}_n z|^2} \tag{2.4}$$

for almost all $z \in \mathbb{T}$;

(v) *Every bounded harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $u(a_n) \geq 0$ for all $n \in \mathbb{N}_N$ satisfies the inequality*

$$\int\limits_{\mathbb{T}} u|1 - \overline{B(0)}B|^2 \, d\boldsymbol{m} \geq 0,$$

where $u(\zeta) \stackrel{\text{def}}{=} \lim\limits_{r \to 1-} u(r\zeta)$ for $\zeta \in \mathbb{T}$;

(vi) *For every bounded function $f \in H^2 \ominus BH^2$ the inequality*

$$|f(0)| \leq (1 - |B(0)|^2) \sup_{n \in \mathbb{N}_N} |f(a_n)|$$

holds.

Proof. We prove that (i)⇒(ii)⇒(iii)⇒(iv) ⇒(v)⇒(vi), (vi)⇒(ii) and (v)⇒(i).

Prove that (i) implies (ii). Put $g(z) = \dfrac{B(z)}{B'(a_n)(z - a_n)}$. By Lemma 2.5, we have $g(0) \geq 0$. Consequently, $\dfrac{B(0)}{a_n B'(a_n)} < 0$. Now applying Lemma 2.5 to the function

$$g \stackrel{\text{def}}{=} \sum_{k=1}^{n} \frac{B(z)}{B'(a_k)(z - a_k)},$$

we get

$$-\sum_{k=1}^{n} \frac{B(0)}{a_k B'(a_k)} \leq 1 - |B(0)|^2$$

for all $n \in \mathbb{N}_N$. Hence,

$$\sum_{n=1}^{N} \frac{1}{|B'(a_n)|} \leq \frac{1 - |B(0)|^2}{|B(0)|} < +\in fty.$$

To prove the implication (ii)⇒(iii) it suffices to make use of Lemma 2.6.

Prove that (iii) implies (iv). Substituting $z = 0$ in the formula (2.3), we get $\sum\limits_{n=1}^{N} \lambda_n = 1 - |B(0)|^2$. Now the formula (2.3) implies the equality

$$2\operatorname{Re}(1 - \overline{B(0)}B(z)) - (1 - |B(0)|^2) = \sum_{n \in \mathbb{N}_N} \lambda_n \Big(2 \operatorname{Re} \frac{1}{1 - \overline{a}_n z} - 1\Big),$$

i. e.,

$$1 - 2\operatorname{Re}(\overline{B(0)}B(z)) + |B(0)|^2 = \sum_{n \in \mathbb{N}_N} \lambda_n \operatorname{Re} \frac{1 - |a_n z|^2}{|1 - \overline{a}_n z|^2}$$

for all $z \in \mathbb{D}$. Hence,

$$|1-\overline{B(0)}B(z)|^2 = \sum_{n\in\mathbb{N}_N} \lambda_n \frac{1-|a_n|^2}{|1-\overline{a}_n z|^2}$$

for almost all $z \in \mathbb{T}$.

Prove that (iv) implies (v). Let u be a bounded harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $u(a_n) \geq 0$ for all $n \in \mathbb{N}_N$. Then

$$\int_{\mathbb{T}} |1-\overline{B(0)}B|^2 u\, d\boldsymbol{m} = \sum_{n\in\mathbb{N}_N} \lambda_n \int_{\mathbb{T}} \frac{1-|a_n|^2}{|1-\overline{a}_n\zeta|^2} u(\zeta)\, d\boldsymbol{m}(\zeta)$$
$$= \sum_{n\in\mathbb{N}_N} \lambda_n u(a_n) \geq 0.$$

Prove that (v) implies (vi). We may suppose that $f(0) \geq 0$ and $\sup\limits_{n\in\mathbb{N}_N} |f(a_n)| = 1$. Note that $f(0) = \int\limits_{\mathbb{T}} |1-\overline{B(0)}B|^2 f\, d\boldsymbol{m}$ for all $f \in H^2 \ominus BH^2$ by Lemma 2.1. Consider the harmonic function $u \stackrel{\text{def}}{=} 1 - \operatorname{Re} f$. Clearly, $u(a_n) \geq 0$ for any $n \in \mathbb{N}_N$. Consequently, $\int\limits_{\mathbb{T}} u|1-\overline{B(0)}B|^2\, d\boldsymbol{m} \geq 0$, whence $f(0) \leq 1-|B(0)|^2$.

Now we prove that (vi) implies (ii). First, we consider the case where $N < +\infty$. By the Hahn-Banach theorem, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}_N}$ in $\mathbb{C}$ such that $\sum\limits_{n\in\mathbb{N}_N} |\lambda_n| \leq 1-|B(0)|^2$ and

$$f(0) = \sum_{n\in\mathbb{N}_N} \lambda_n f(a_n) \tag{2.5}$$

for all $f \in H^2 \ominus BH^2$. Applying the formula (2.5) for $f = \frac{B}{z-a_n}$, we get $\lambda_n = -\frac{B(0)}{a_n B'(a_n)}$. Applying the formula (2.5) for $f = 1-\overline{B(0)}B$, we get $1-|B(0)|^2 = \sum\limits_{n\in\mathbb{N}_N} \lambda_n$. Consequently, $\lambda_n \geq 0$ for all n, whence $\frac{a_n B'(a_n)}{B(0)} < 0$ for all n. Now consider the case, where $N = +\infty$. In this case, by the Hahn-Banach theorem, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}_N}$ in $\mathbb{C}$ such that $\sum\limits_{n=1}^{\infty} |\lambda_n| \leq 1-|B(0)|^2$ and the equality (2.5) is fulfilled for every bounded $f \in H^2 \ominus BH^2$ such that $\lim\limits_{n\to+\infty} f(a_n) = 0$. In particular, the formula (2.5) holds for any function f in the linear span of the family $\{\frac{B}{z-a_n}\}_{n=1}^{\infty}$. Note that this linear span is dense in the space $H^\infty \cap (H^2 \ominus BH^2)$ in the weak topology $\sigma(L^\infty, L^1)$. Indeed, the transformation $f \mapsto B\overline{f}\overline{z}$ is an antilinear homeomorphism of the space $H^\infty \cap (H^2 \ominus BH^2)$. Hence, the density of the linear span of $\{\frac{B}{z-a_n}\}_{n=1}^{\infty}$ follows from that of the linear span of $\{\frac{1}{1-\overline{a}_n z}\}_{n=1}^{\infty}$.

The latter property is well known. Thus, the formula (2.5) holds for every bounded function in $H^2 \ominus BH^2$; and repeating word for word the proof of the implication (vi)⇒(ii) for the case $N < +\infty$ we get that $\frac{a_n B'(a_n)}{B(0)} < 0$ for all n. Moreover,

$$\sum_{n=1}^{\infty} \frac{1}{|B'(a_n)|} = \sum_{n=1}^{\infty} \lambda_n \frac{|a_n|}{|B(0)|} \leq \frac{1}{|B(0)|} \sum_{n=1}^{\infty} \lambda_n = \frac{1 - |B(0)|^2}{|B(0)|} < \infty.$$

It remains to prove that (v) implies (i). By Theorem 1.4, it suffices to prove that (v) implies the inequality

$$\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta g| \, d\boldsymbol{m} \geq 0$$

for all $g \in H^\infty$. Let u_t be the bounded harmonic function $u : \mathbb{D} \to \mathbb{R}$ such that $u_t = \max(-t, \log|1 + \Theta g|)$ almost everywhere on $\mathbb{T}$, where $t > 0$. Clearly, $\log|1 + \Theta g| \leq u_t$ everywhere in $\mathbb{D}$. Hence, $u_t(a_n) \geq 0$ for all n. Thus, we have

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 u_t \, d\boldsymbol{m} \geq 0$$

for all $t > 0$, whence

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B|^2 \log|1 + \Theta g| \, d\boldsymbol{m} \geq 0. \ \blacksquare$$

Remark 1. Let the equivalent statements of Theorem 2.7 be fulfilled. Then (v) holds for the Poisson integral u of every summable function and (vi) holds for all $f \in H^1 \cap BH^1_-$. This follows from the proofs of the implications (iv)⇒(v) and (v)⇒(vi).

Remark 2. Let B denote the same as in the assumptions of Theorem 2.7. Then

$$\sup\{|f(0)| : f \in H^2 \ominus BH^2, |f(a_n)| \leq 1\} = \sum_{n \in \mathbb{N}_N} \left| \frac{B(0)}{a_n B'(a_n)} \right|.$$

Proof. First, we suppose that $\sum\limits_{n \in \mathbb{N}_N} \left| \frac{B(0)}{a_n B'(a_n)} \right| < +\infty$. Then

$$\sum_{n \in \mathbb{N}_N} \lambda_n f(a_n) = f(0) - B(0)\big(\overline{B} f\big)_+(0) \tag{2.6}$$

for all $f \in H^2$ such that the sequence $\{f(a_n)\}_{n\in\mathbb{N}_N}$ is bounded, where $\lambda_n = -\frac{B(0)}{a_n B'(a_n)}$. Indeed, (2.6) is evident if $\{n \in \mathbb{N}_N : f(a_n) \neq 0\}$ is finite. Thus, it suffices to consider the case where $N = +\infty$. We have

$$\sum_{j\in\mathbb{N}} \lambda_j B_n(a_j) f(a_j) = f(0)B_n(0) - B(0)\big(\overline{B}B_n f\big)_+(0),$$

where $B_n(z) \stackrel{\text{def}}{=} \prod\limits_{j>n} \frac{\overline{a}_j}{|a_j|}\frac{a_j - z}{1-\overline{a}_j z}$. Making $n \to +\infty$ we get (2.6). In particular,

$$\sum_{n\in\mathbb{N}_N} \lambda_n f(a_n) = f(0)$$

for all $f \in H^2 \ominus BH^2$ such that $\{f(a_n)\}_{n\in\mathbb{N}_N}$ is bounded. The case where $\sum\limits_{n\in\mathbb{N}_N} \left|\frac{B(0)}{a_n B'(a_n)}\right| = +\infty$ is evident. ■

Theorem 2.8. *Let $B \in \mathfrak{A}_+$. Then B is an interpolating Blaschke product. Moreover, $(1-|a|^2)|B'(a)| \geq \dfrac{4|B(0)|}{(1+|B(0)|)^2}$ for each zero a of B.*

Proof. Theorems 2.4 and 2.7 imply that

$$|1-\overline{B(0)}B(z)|^2 = \sum_{B(a)=0} \lambda_a \frac{1-|a|^2}{|1-\overline{a}z|^2}$$

almost everywhere on $\mathbb{T}$ with $\lambda_a = -\dfrac{B(0)}{aB'(a)} > 0$. Hence,

$$\lambda_a \frac{1-|a|^2}{|1-\overline{a}z|^2} \leq (1+|B(0)|)^2$$

for all $z \in \mathbb{T}$ and all zeros a of B. Substituting in the latter inequality $z = \frac{a}{|a|}$ we get

$$\lambda_a \frac{1+|a|}{1-|a|} \leq (1+|B(0)|)^2,$$

whence

$$(1-|a|^2)|B'(a)| \geq \frac{(1+|a|)^2}{|a|} \cdot \frac{|B(0)|}{(1+|B(0)|)^2} \leq \frac{4|B(0)|}{(1+|B(0)|)^2}$$

provided $B(a) = 0$. ■

Theorem 2.8 implies that the measure $\sum\limits_{B(a)=0} (1-|a|^2)\delta_a$ is a Carleson measure for every $B \in \mathfrak{A}_+$. The following theorem strengthens this assertion.

Theorem 2.9. *Let $B \in \mathfrak{A}_+$. Then*

$$\sum_{B(a)=0} (1-|a|^2)u(a) \le \frac{(1+|B(0)|)^2}{|B(0)|}u(0)$$

for every function u positive and harmonic in the unit disk $\mathbb{D}$.

Proof. It suffices to note that

$$\sum_{B(a)=0} \frac{(1-|a|^2)^2}{|1-\overline{a}z|^2} \le \sum_{B(a)=0} \frac{1}{|a||B'(a)|} \cdot \frac{1-|a|^2}{|1-\overline{a}z|^2}$$

$$= \frac{|1-\overline{B(0)}B(z)|^2}{|B(0)|} \le \frac{(1+|B(0)|)^2}{|B(0)|}. \ \blacksquare$$

3. Construction of Blaschke products B of the class $\mathfrak{A}_+$

Theorem 3.1. *Let $\{r_n\}_{n=1}^N$ be a sequence in $(0,1)$ such that $\sum\limits_{n=1}^{N}(1-r_n) < +\infty$, where $N \in \mathbb{N}\cup\{\infty\}$. Then there exists a Blaschke product $B(z) = \prod\limits_{n=1}^{N} \frac{\overline{a}_n}{|a_n|}\frac{a_n - z}{1-\overline{a_n}z}$ of the class $\mathfrak{A}_+$ such that $|a_n| = r_n$ for all n.*

We need the following well-known topological assertion.

Lemma 3.2. *Let $f : \mathbb{T}^N \to \mathbb{T}^N$ be a continuous mapping, where $N \in \mathbb{N}$. Suppose that there exists a group epimorphism $\varphi : \mathbb{T}^N \to \mathbb{T}^N$ which is homotopic to f. Then $f(\mathbb{T}^N) = \mathbb{T}^N$.* $\blacksquare$

Proof of Theorem 3.1. Theorem 1.9 allows us to assume that $N \in \mathbb{N}$ and $0 < r_1 < r_2 < \cdots < r_N$. Let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N) \in \mathbb{T}^N$. Put

$$B_\zeta(z) \stackrel{\text{def}}{=} \prod_{n=1}^{N} \frac{z-a_n}{1-\overline{a}_n z},$$

where $a_n = \zeta_n r_n$. Clearly,

$$a_n B'_\zeta(a_n) = a_n(1-r_n^2)^{-1} \prod_{1\le j<n} \frac{a_n - a_j}{1-a_n\overline{a}_j} \prod_{n<j\le N} \frac{a_n - a_j}{1-a_n\overline{a}_j}$$

for $n = 1, 2, \ldots, N$. Define the mapping $f = (f_1, \ldots, f_N) : \mathbb{T}^N \to \mathbb{T}^N$ as follows

$$f_n(\zeta) \stackrel{\text{def}}{=} \operatorname{sgn}\left(a_n B'_\zeta(a_n)\right)$$

$$= \operatorname{sgn}\left(\zeta_n \prod_{1\le j<n} \frac{r_n\zeta_n - r_i\zeta_j}{1-r_n r_j \zeta_n\overline{\zeta}_j} \prod_{n<j\le N} \frac{r_n\zeta_n - r_i\zeta_j}{1-r_n r_j\zeta_n\overline{\zeta}_j}\right), \quad (n = 1, 2, \ldots, N),$$

where $\operatorname{sgn} w \overset{\text{def}}{=} w|w|^{-1}$. Consider the group homomorphism $\varphi = (\varphi_1, \dots, \varphi_N) : \mathbb{T}^N \to \mathbb{T}^N$ defined as follows

$$\varphi_n(\zeta) = \zeta_n^n \prod_{j=n+1}^{N} \zeta_j \qquad (n = 1, 2, \dots, N).$$

It is easy to see that φ is homotopic to f. Indeed, it suffices to put

$$f_n^t \overset{\text{def}}{=} \operatorname{sgn}\left(\zeta_n \prod_{1\le j<n} \frac{r_n\zeta_n - (1-t)r_i\zeta_j}{1-(1-t)r_n r_j \zeta_n \overline{\zeta}_j} \prod_{n<j\le N} \frac{(1-t)r_n\zeta_n - e^{t\pi i} r_i \zeta_j}{1-(1-t)r_n r_j\zeta_n\overline{\zeta}_j}\right),$$

where $t \in [0,1]$, $n = 1, 2, \dots, N$. For given $\xi \in \mathbb{T}^N$, we may solve the equation $\varphi(\zeta) = \xi$. Indeed, first we find ζ_N, after that we find ζ_{N-1} and so on. Thus, φ is an epimorphism. Hence, $f(\mathbb{T}^N) = \mathbb{T}^N$ by Lemma 3.2. In particular, $(1, 1, \dots, 1) \in f(\mathbb{T}^N)$. Thus, we have proved that there exists a sequence $\{a_n\}_{n=1}^N$ such that $|a_n| = r_n$ for $n = 1, 2, \dots, N$ and $a_n B'(a_n) > 0$ for $n = 1, 2, \dots, N$. Using the equality

$$\sum_{n=0}^{N} \frac{B(0)}{a_n B'(a_n)} = |B(0)|^2 - 1$$

we get $B(0) < 0$. Hence, $B \in \mathfrak{A}_+$. ∎

In a similar way we can prove a stronger version of Theorem 3.1.

Theorem 3.3. *Let $\{G_n\}_{n=1}^N$ be an increasing sequence of domains in $\mathbb{D}$, where $N \in \mathbb{N} \cup \{\infty\}$. Suppose that*
(a) $0 \in G_n$ for all n,
(b) $\partial G_n \cap \mathbb{T} = \varnothing$ for all n,
(c) ∂G_n is a Jordan curve for all n,
(d) $\sum_{n=1}^{N} \max_{z\in\partial G_n} (1-|z|) < +\infty$

Then there exists a Blaschke product $B(z) = \prod_{n=1}^{N} \frac{\overline{a}_n}{|a_n|} \frac{a_n - z}{1-\overline{a_n}z}$ of the class $\mathfrak{A}_+$ such that $a_n \in \partial G_n$ for all n. ∎

4. Properties of $\epsilon_p^0(\Theta)$

Let Θ be an inner function. Recall that

$$\epsilon_p^0(\Theta) \overset{\text{def}}{=} \inf\{\|1+\Theta f\|_p^p : f \in H^p, 1+\Theta(z)f(z) \neq 0 \ \ \forall z \in \mathbb{D}\}.$$

Clearly, $\epsilon_p(\Theta) \le \epsilon_p^0(\Theta) \le \varepsilon_p(\Theta) = 1 - |\Theta(0)|^2$. Hence, $\epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2$ for $p \ge 1$.

Theorem 4.1. *Let Θ be an inner function and let $p \in (0, +\infty)$. Then the following statements are equivalent:*

(i) $\epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2$*;*

(ii) $\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 (|1 + \Theta f|^p - 1)\, d\boldsymbol{m} \geq 0$ *for every* $f \in H^p$ *such that* $1 + \Theta(z) f(z) \neq 0$ *for all* $z \in \mathbb{D}$.

Proof is similar to that of Theorem 1.1. ■

Corollary 4.2. *Let Θ be an inner function. Suppose that $\epsilon_q^0(\Theta) = 1 - |\Theta(0)|^2$ for some $q \in (0, +\infty)$. Then $\epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2$ for all $p \in (q, +\infty)$.*

Proof. See the proof of Corollary 1.2. ■

Put $p_\Theta^0 \overset{\text{def}}{=} \inf\{p > 0 : \epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2\}$. Clearly, $0 \leq p_\Theta^0 \leq p_\Theta \leq 1$.

Theorem 4.3. *Let Θ be an inner function. The following statements are equivalent:*

(i) $\epsilon_p^0(\Theta) = 1 - |\Theta(0)|^2$ *for all* $p > 0$, *i. e.,* $p_\Theta^0 = 0$*;*

(ii) $\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta f|\, d\boldsymbol{m} \geq 0$ *for all* $f \in \bigcup\limits_{p>0} H^p$ *such that* $1 + \Theta(z) f(z) \neq 0$ *for all* $z \in \mathbb{D}$.

Proof is similar to that of Theorem 1.4. ■

Theorem 4.4. *Let Θ and Ψ be inner functions. Then*

$$\epsilon_p^0(\Theta(\Psi)) \leq \frac{1 + |\Psi(0)|}{1 - |\Psi(0)|} \epsilon_p^0(\Theta eta).$$

In particular, $\epsilon_p^0(\Theta(\Psi)) \leq \epsilon_p^0(\Theta)$ if $\Psi(0) = 0$.

Proof. See the proof of Theorem 1.10. ■

Theorem 4.5. *Let Θ be an inner function. Put $\Theta_n(z) = \Theta(z^n)$ for a positive integer n. Then $\epsilon_p^0(\Theta_n) \leq \epsilon_p^0(\Theta) \leq \epsilon_{pn}^0(\Theta_n)$ for all $p \in (0, +\infty)$.*

Proof. See the proof of Theorem 1.11. ■

Corollary 4.6. *Let Θ be an inner function and let n be a positive integer. Then $p_\Theta^0 = 0$ if and only if $p_{\Theta_n}^0 = 0$.* ■

Lemma 4.7. *Let Θ and I be inner functions such that I is a divisor of Θ and $p_\Theta^0 = 0$. Suppose that $fI^{-1}\Theta + gI = 1$ for some $f, g \in H^\infty$. Then*

$$g(0)I(0) + \Theta(0) \int\limits_{\mathbb{T}} f\overline{I}\, d\boldsymbol{m} \in \mathbb{R}.$$

Proof. It is easy to see that $e^{2\pi i g I} - 1 \in \Theta H^\infty$. Consequently,

$$\int_{\mathbb{T}} gI|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} \in \mathbb{R},$$

see the beginning of the proof of Lemma 2.5. It remains to note that

$$\int_{\mathbb{T}} gI|1 - \overline{\Theta(0)}\Theta|^2 \, d\boldsymbol{m} = \int_{\mathbb{T}} gI(1 - \overline{\Theta(0)}\Theta)(1 - \Theta(0)\overline{\Theta}) \, d\boldsymbol{m}$$

$$= \int_{\mathbb{T}} gI(1 - \overline{\Theta(0)}\Theta + |\Theta(0)|^2) \, d\boldsymbol{m} - \Theta(0) \int_{\mathbb{T}} gI\overline{\Theta} \, d\boldsymbol{m}$$

$$= g(0)I(0) - \Theta(0) \int_{\mathbb{T}} (1 - fI^{-1}\Theta)\overline{\Theta} \, d\boldsymbol{m} = g(0)I(0) + \int_{\mathbb{T}} f\overline{I} \, d\boldsymbol{m} + |\Theta(0)|^2. \quad \blacksquare$$

Theorem 4.8. *Let Θ be an inner function such that $p_\Theta^0 = 0$ and $\Theta(0) \neq 0$. Then $\operatorname{Res}_a \frac{\Theta(0)}{z\Theta} \in \mathbb{R}$ for each zero a of Θ.*

Proof. Let $\Theta(a) = 0$. Denote by n be the multiplicity of the zero a. Put $I(z) \stackrel{\text{def}}{=} \frac{(z-a)^n}{(1-\overline{a}z)^n}$. Clearly, there exist functions $f, g \in H^\infty$ such that $fI^{-1}\Theta + gI = 1$. By Lemma 4.7, we have

$$g(0)I(0) + \Theta(0) \int_{\mathbb{T}} f\overline{I} \, d\boldsymbol{m} \in \mathbb{R}.$$

It remains to note that

$$g(0)I(0) + \Theta(0) \int_{\mathbb{T}} f\overline{I} \, d\boldsymbol{m} = g(0)I(0) + \frac{\Theta(0)}{2\pi i} \oint_{\mathbb{T}} \frac{f(z)\, dz}{zI(z)}$$

$$= g(0)I(0) + \operatorname{Res}_0 \frac{\Theta(0)f(z)}{zI(z)} + \operatorname{Res}_a \frac{\Theta(0)f(z)}{zI(z)}$$

$$= g(0)I(0) + \frac{\Theta(0)f(0)}{I(0)} + \operatorname{Res}_a \frac{\Theta(0)}{z\Theta(z)} - \operatorname{Res}_a \frac{I(z)g(z)}{\Theta(z)}$$

$$= 1 + \operatorname{Res}_a \frac{\Theta(0)}{z\Theta(z)}. \quad \blacksquare$$

5. Finite Blaschke products B with $p_B^0 = 0$

Denote by $\mathfrak{A}$ the set of all inner functions Θ such that $p_\Theta^0 = 0$ and $\Theta(0) \neq 0$. In Section 6 we shall see that $\mathfrak{A}$ contains singular inner functions. In this section we obtain a description of the finite Blaschke products of the class $\mathfrak{A}$.

Let $B(z) = \prod_{a\in\sigma}\left(\frac{a-z}{1-\overline{a}z}\right)^{k(a)}$, where k is a $\mathbb{Z}_+$-valued function on $\mathbb{D}$, $\sigma = \{a \in \mathbb{D} : k(a) > 0\}$. We always suppose that σ is a finite subset of $\mathbb{D}$ and $0 \notin \sigma$. Clearly, the function $1 - \overline{B(0)}B$ can be represented in the form

$$1 - \overline{B(0)}B(z) = \sum_{a\in\sigma}\sum_{l=1}^{k(a)} \frac{\lambda_a^{(l)} z^{l-1}}{(1-\overline{a}z)^l}. \tag{5.1}$$

Hence,

$$1 - B(0)\overline{B(z)} = \sum_{a\in\sigma}\sum_{l=1}^{k(a)} \frac{\overline{\lambda}_a^{(l)} \overline{z}^{l-1}}{(1-a\overline{z})^l}, \tag{5.2}$$

whence

$$1 - \frac{B(0)}{B(z)} = z\sum_{a\in\sigma}\sum_{l=1}^{k(a)} \frac{\overline{\lambda}_a^{(l)}}{(z-a)^l}. \tag{5.3}$$

Put $\lambda_a \overset{\text{def}}{=} \lambda_a^{(1)}$. The formula (5.3) implies that $\overline{\lambda}_a = -\operatorname{Res}_a \frac{B(0)}{zB(z)}$ for $a \in \sigma$. Suppose that $\lambda_a \in \mathbb{R}$ for all $a \in \sigma$. Then (5.1) and (5.2) imply the following equality

$$|1 - \overline{B(0)}B(z)|^2 = \sum_{a\in\sigma} \frac{\lambda_a(1-|a|^2)}{|1-\overline{a}z|^2} + \sum_{a\in\sigma_+}\sum_{l=2}^{k(a)} \frac{\lambda_a^{(l)} z^{l-1}}{(1-\ ovaz)^l}$$

$$+ \sum_{a\in\sigma_+}\sum_{l=2}^{k(a)} \frac{\overline{\lambda}_a^{(l)} \overline{z}^{l-1}}{(1-a\overline{z})^l} \tag{5.4}$$

for $z \in \mathbb{T}$, where $\sigma_+ \overset{\text{def}}{=} \{a \in \mathbb{D} : k(a) \geq 2\}$.

Lemma 5.1. *Let u be the Poisson integral of a measure $\mu \in M(\mathbb{T})$. Suppose that $\lambda_a \in \mathbb{R}$ for all $a \in \sigma$, and $d_a^n u = 0$ for all $a \in \sigma_+$ and $n = 1, 2, \dots, k(a) - 1$. Then*

$$\int_{\mathbb{T}} |1 - \overline{B(0)}B(z)|^2\, d\mu(z) = \sum_{a\in\sigma} \lambda_a u(a).$$

Proof. We may suppose that u is real. Clearly,

$$\int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z} = \frac{1}{2}u(0) + \frac{1}{2}u(z) + \frac{1}{2}\widetilde{u}(z)\, i \overset{\text{def}}{=} \frac{1}{2}u(0) + f(z).$$

Hence,

$$f^{(n)}(z) = n! \int\limits_{\mathbb{T}} \frac{\overline{\zeta}^n d\mu(\zeta)}{(1-\overline{\zeta}z)^{n+1}}.$$

Note that the equality $d_a^n u = 0$ implies that $f^{(n)}(a) = 0$ provided $n \geq 1$. Finally, apply (5.4). ■

Theorem 5.2. *The following statements are equivalent:*

(i) $p_B^0 = 0$;

(ii) $\operatorname{Res}_a \frac{B(0)}{zB(z)} \in \mathbb{R}$ *for all* $a \in \sigma$;

(iii) *There exists a family* $\{\lambda_a\}_{a\in\sigma}$ *of real numbers and a family* $\{\lambda_a^{(l)}\}_{a\in\sigma_+, 2\leq l\leq k(a)}$ *of complex numbers such that*

$$1 - \overline{B(0)}B(z) = \sum_{a\in\sigma}\sum_{l=1}^{k(a)} \frac{\lambda_a^{(l)} z^{l-1}}{(1-\overline{a}z)^l};$$

(iv) *There exists a family* $\{\lambda_a\}_{a\in\sigma}$ *of real numbers and a family* $\{\lambda_a^{(l)}\}_{a\in\sigma_+, 2\leq l\leq k(a)}$ *of complex numbers such that*

$$|1 - \overline{B(0)}B(z)|^2 = \sum_{a\in\sigma} \frac{\lambda_a(1-|a|^2)}{|1-\overline{a}z|^2} + \sum_{a\in\sigma_+}\sum_{l=2}^{k(a)} \frac{\lambda_a^{(l)} z^{l-1}}{(1- \ ovaz)^l}$$

$$+ \sum_{a\in\sigma_+}\sum_{l=2}^{k(a)} \frac{\overline{\lambda}_a^{(l)} \overline{z}^{l-1}}{(1-a\overline{z})^l} \tag{5.5}$$

everywhere on $\mathbb{T}$;

(v) *Every bounded harmonic function* $u : \mathbb{D} \to \mathbb{R}$ *such that* $d_a^n u = 0$ *for all* $a \in \sigma$ *and* $n = 0, 1, \ldots, k(a)-1$ *satisfies the equality*

$$\int\limits_{\mathbb{T}} u|1 - \overline{B(0)}B|^2 \, d\boldsymbol{m} = 0,$$

where $u(\zeta) \stackrel{\text{def}}{=} \lim_{r\to 1-} u(r\zeta)$ *for* $\zeta \in \mathbb{T}$;

(vi) *If* $f \in H^2 \ominus BH^2$, $f(a) \in \mathbb{R}$ *for all* $a \in \sigma$, *and* $f^{(n)}(a) = 0$ *for all* $a \in \sigma_+$ *and* $n = 1, 2, \ldots, k(a)-1$, *then* $f(0) \in \mathbb{R}$.

Proof. We prove that (i)⇒(ii)⇒(iii)⇒(iv) ⇒(v)⇒(vi), (vi)⇒(ii) and (ii)⇒(i).

The implication (i)⇒(ii) is a particular case of Theorem 4.8. The implications (ii)⇒(iii) and (iii)⇒(iv) were essentially proved at the beginning of this section. The implication (iv)⇒(v) is essentially Lemma 5.1. To prove the implication (v)⇒(vi) it suffices to note that $f(0)=\int\limits_{\mathbb{T}}|1-\overline{B(0)}B|^2 f\,d\boldsymbol{m}$ for all $f\in H^2\ominus BH^2$ by Lemma 2.1.

Let us prove that (vi)⇒(ii). Let $(\cdot,\cdot)_{\mathbb{R}}$ denote the inner product in the space $H^2\ominus BH^2$, i. e., $(f,g)_{\mathbb{R}}\stackrel{\text{def}}{=}\operatorname{Re}\int\limits_{\mathbb{R}} f\overline{g}\,d\boldsymbol{m}$. For $Z\subset H^2\ominus BH^2$, we put

$$Z^{\perp}\stackrel{\text{def}}{=}\{f\in H^2\ominus BH^2:(f,g)_{\mathbb{R}}=0\ \ \forall g\in\mathbb{R}\}.$$

Denote by X the space of all functions $f\in H^2\ominus BH^2$ such that $f(a)\in\mathbb{R}$ for all $a\in\sigma$ and $f^{(n)}(a)=0$ for all $a\in\sigma_+$ and $n=1,2,\dots,k(a)-1$. Put

$$Y\stackrel{\text{def}}{=}\{f\in H^2\ominus BH^2: f(0)\in\mathbb{R}\}.$$

The inclusion $X\subset Y$ implies that $Y^{\perp}\subset X^{\perp}$. Consequently, $i(1-\overline{B(0)}B)\in X^{\perp}$. It is easy to see that $X^{\perp}$ is the $\mathbb{R}$-linear span of the families $\left\{\dfrac{i}{1-\overline{a}z}\right\}_{a\in\lambda}$, $\left\{\dfrac{z^{n-1}}{(1-\overline{a}z)^n}\right\}_{a\in\lambda_+,2\le n\le k(a)}$ and $\left\{\dfrac{iz^{n-1}}{(1-\overline{a}z)^n}\right\}_{a\in\lambda_+,2\le n\le k(a)}$. Hence, there exists a decomposition

$$1-\overline{B(0)}B(z)=\sum_{a\in\sigma}\sum_{l=1}^{k(a)}\frac{\lambda_a^{(l)}z^{l-1}}{(1-\overline{a}z)^l}$$

with $\lambda_a^{(1)}\in\mathbb{R}$ for all $a\in\sigma$, i. e., $\operatorname{Res}_a\frac{B(0)}{zB(z)}\in\mathbb{R}$ for all $a\in\sigma$.

It remains to prove that (ii)⇒(i). Let $f\in H^p$. Suppose that $1+f(z)B(z)\ne 0$ for all $z\in\mathbb{D}$. There exists a positive singular measure μ on $\mathbb{T}$ such that

$$\log|1+B(z)f(z)|=\int\limits_{\mathbb{T}}\frac{1-|z|^2}{|1-\overline{\zeta}z|^2}\log|1+B(\zeta)f(\zeta)|\,d\boldsymbol{m}(\zeta)-\int\limits_{\mathbb{T}}\frac{1-|z|^2}{|1-\overline{\zeta}z|^2}\,d\mu(\zeta)$$

for all $z\in\mathbb{D}$. Hence, by Lemma 5.1, we have

$$\int\limits_{\mathbb{T}}|1-\overline{B(0)}B|^2\log|1+Bf|\,d\boldsymbol{m}=\int\limits_{\mathbb{T}}|1-\overline{B(0)}B|^2\,d\mu\ge 0.\ \blacksquare$$

In particular, we have the following result for finite Blaschke products with simple zeros.

Theorem 5.3. *Let $\{a_n\}_{n=1}^N$ be a finite sequence in $\mathbb{D}\setminus\{0\}$. Suppose that $a_k\ne a_l$ for $k\ne l$. Let B denote the corresponding Blaschke product, $B(z)=\prod\limits_{n=1}^{N}\dfrac{z-a_n}{1-\overline{a_n}z}$. The following statements are equivalent:*

(i) *$p_B^0=0$;*

(ii) $\dfrac{a_n B'(a_n)}{B(0)} \in \mathbb{R}$ *for all* $n \in \mathbb{N}_N$*;*

(iii) *There exists a sequence* $\{\lambda_n\}_{n\in\mathbb{N}_N}$ *in* $\mathbb{R}$ *such that*

$$1 - \overline{B(0)}B(z) = \sum_{n=1}^{N} \frac{\lambda_n}{1 - \overline{a}_n z};$$

(iv) *There exists a sequence* $\{\lambda_n\}_{n\in\mathbb{N}_N}$ *in* $\mathbb{R}$ *such that*

$$|1 - \overline{B(0)}B(z)|^2 = \sum_{n=1}^{N} \lambda_n \frac{1 - |a_n|^2}{|1 - \overline{a}_n z|^2}$$

for all $z \in \mathbb{T}$*;*

(v) *Every bounded harmonic function* $u : \mathbb{D} \to \mathbb{R}$ *such that* $u(a_n) = 0$ *for all* $n \in \mathbb{N}_N$ *satisfies the equality*

$$\int_{\mathbb{T}} u|1 - \overline{B(0)}B|^2 \, d\boldsymbol{m} = 0,$$

where $u(\zeta) \stackrel{\text{def}}{=} \lim\limits_{r\to 1-} u(r\zeta)$ *for* $\zeta \in \mathbb{T}$*;*

(vi) *For every function* $f \in H^2 \ominus BH^2$*, the inclusions* $f(a_n) \in \mathbb{R}$ *for all* $n \in \mathbb{N}_N$ *imply the inclusion* $f(0) \in \mathbb{R}$. ■

6. Inner functions with real spectrum

Let $\rho(\Theta)$ denote the spectrum of an inner function Θ, i. e.,

$$\rho(\Theta) \stackrel{\text{def}}{=} \{\zeta \in \overline{\mathbb{D}} : \liminf_{z\to\zeta} |\Theta(z)| = 0\}.$$

In this section we are going to prove that $p^0_\Theta = 0$ for all inner functions Θ such that $\rho(\Theta) \subset \mathbb{R}$. In particular, $p^0_S = 0$ for $S(z) = \exp\left(\dfrac{z+1}{z-1}\right)$.

We need some notation and a series of auxiliary assertions.

Denote by N_A the Smirnov class, i. e., the set of all functions $f : \mathbb{D} \to \mathbb{C}$ representable in the form $f(z) = \lim\limits_{n\to+\infty} f_n(z)$, where $f_n \in H^\infty$ and $|f_n| \le |f_{n+1}|$ everywhere in $\mathbb{D}$.

Let f be a function analytic in $\mathbb{D}$. For $\zeta \in \mathbb{T}$, we denote by $f(\zeta)$ the angular boundary values of Θ at the point ζ.

Let μ_+ denote the Cauchy integral of a measure $\mu \in M(\mathbb{T})$, i. e.,

$$\mu_+(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}$$

for $z \in \mathbb{D}$. It is easy to verify that $\operatorname{Re}\mu_+ = \dfrac{1}{2}\Big(\mu_+(0) + \dfrac{d\mu}{d\boldsymbol{m}}\Big)$ almost everywhere on $\mathbb{T}$ provided μ is a real measure. Here $\dfrac{d\mu}{d\boldsymbol{m}}$ denotes the density of μ, i. e., such a function $h \in L^1(\mathbb{T})$ that the measure $\mu - h \cdot \boldsymbol{m}$ is singular.

Lemma 6.1. *Let Θ be an inner function and let μ be a real measure on $\mathbb{T}$ such that $\Theta^{-1}\mu_+ \in N_A$. Then Θ has angular boundary values $|\mu|$-almost everywhere on $\mathbb{T}$ and* $\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \, d\mu = 0$.

Proof. Note that the function $\Theta^{-1}(\Theta - \Theta(0))(1 - \overline{\Theta(0)}\Theta)\mu_+$ vanishes at 0 and belongs to the weak H^1-Hardy space. Also we have

$$\operatorname{Re}(\Theta^{-1}(\Theta - \Theta(0))(1 - \overline{\Theta(0)}\Theta)\mu_+) \in L^1(\mathbb{T}).$$

Consequently (see [A1] or [A2]), there exists a real measure ν such that

$$\Theta^{-1}(\Theta - \Theta(0))(1 - \overline{\Theta(0)}\Theta)\mu_+ = \nu_+$$

in $\mathbb{D}$. By a theorem of Poltoratski (see [P]), the function Θ has angular boundary values $|\mu|$-almost everywhere on $\mathbb{T}$, $|\Theta| = 1$ μ-almost everywhere and $|1 - \overline{\Theta(0)}\Theta|^2\mu_s = \nu_s$, where μ_s and ν_s are the singular parts of the measures μ and ν respectively. Let μ_a and ν_a be the absolutely continuous parts of the measures μ and ν respectively. Clearly, $\nu_a = 2(|1 - \overline{\Theta(0)}\Theta|^2 \operatorname{Re}\mu_+)\boldsymbol{m} = |1 - \overline{\Theta(0)}\Theta|^2\mu_a$. It remains to note that $\nu(\mathbb{T}) = 0$ because $\nu_+(0) = 0$. ■

Corollary 6.2. *Let μ and Θ satisfy the assumptions of Lemma 6.1. Suppose that $\mu_s \le 0$. Then*

$$\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \operatorname{Re}\mu_+ \, d\boldsymbol{m} \ge 0\,.$$

Proof. It suffices to note that

$$2\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \operatorname{Re}\mu_+ \, d\boldsymbol{m} = -\int_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \, d\mu_s \ge 0. \ ■$$

Let $\tau > 0$. Put $S_\tau(z) \stackrel{\text{def}}{=} e^{\tau\frac{z+1}{z-1}}$.

Lemma 6.3. *Let $f \in N_A$. The following statements are equivalent:*

(i)

$$f(x) = o(e^{\frac{2\lambda}{x-1}}) \quad (x \to 1\,,\ x \in [0,1))$$

for all $\lambda < \tau$,

(ii)

$$fS_\tau^{-1} \in N_A.$$

Certainly, this lemma is well known. We omit an elementary proof of this lemma.

Lemma 6.4. *Let $f \in N_A$. Suppose that $1 + S(z)f(z) \neq 0$ for every $z \in \mathbb{D}$. Then there exists a function $g \in N_A$ such that $e^{Sg} = 1 + Sf$.*

Proof. Let us take a function $G \in N_A$ such that $e^G = 1 + Sf$ and $\lim\limits_{x\to 1-} G(x) = 0$. Clearly, $|G(x)| \le 2S(x)|f(x)|$ if x is close enough to 1. Put $g \stackrel{\text{def}}{=} GS^{-1}$. Note that $g \in N_A$ by Lemma 6.3. ■

Theorem 6.5. *Let Θ be an inner function such that $\rho(\Theta) \subset [-1, 1]$. Then $p^0_\Theta = 0$.*

Proof. It is easy to see that $\Theta(\overline{z}) = \overline{\Theta(z)}$. Suppose that

$$\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta f|\, d\boldsymbol{m} < 0$$

for some function $f \in N_A$ such that $1 + \Theta(z)f(z) \neq 0$ for all $z \in \mathbb{D}$. Then

$$\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta(z)|^2 \log|1 + \Theta(z)\overline{f}(\overline{z})|\, d\boldsymbol{m}(z)$$

$$= \int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta(z)|^2 \log|1 + \Theta(z)f(z)|\, d\boldsymbol{m}(z) < 0.$$

Hence,

$$\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta g|\, d\boldsymbol{m} < 0, \tag{6.1}$$

where $g(z) = f(z) + \overline{f(\overline{z})} + f(z)\overline{f(\overline{z})}$. Clearly, $1 + \Theta(x)g(x) \in (0, +\infty)$ for all $x \in (-1, 1)$. Take a holomorphic function $G : \mathbb{D} \to \mathbb{C}$ such that $G(x) \in \mathbb{R}$ for all $x \in (-1, 1)$ and $e^G = 1 + \Theta g$. Let $\Theta = BS$, where B is a Blaschke product and S is a singular inner function. Clearly, $GB^{-1} \in N_A$. Moreover, Lemma 6.3 implies that $GS^{-1} \in N_A$. Hence, $G\Theta^{-1} \in N_A$. Note that

$$G(z) = \int\limits_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z}\, d\mu(\zeta) = (2\mu - G(0)\boldsymbol{m})_+,$$

where $\mu \in M(\mathbb{T})$ is a real measure with $\mu_s \le 0$. Moreover, $\operatorname{Re} G = \log|1 + \Theta g|$ almost everywhere on $\mathbb{T}$. Hence, by Corollary 6.2, we have

$$\int\limits_{\mathbb{T}} |1 - \overline{\Theta(0)}\Theta|^2 \log|1 + \Theta g|\, d\boldsymbol{m} \ge 0,$$

which contradicts (6.1). ■

7. Blaschke products of degree two

Let $a, b \in \mathbb{D}$. Put $B(z) \stackrel{\text{def}}{=} \dfrac{(z-a)(z-b)}{(1-\overline{a}z)(1-\overline{b}z)}$. In this section we compute p_B^0. Clearly, $p_B^0 = 0$ if $ab = 0$. Moreover, Theorem 6.5 implies that $p_B^0 = 0$ if $a = b$. We will suppose that $a \neq b$ and $ab \neq 0$. Let $\zeta \in \mathbb{T}$. Denote by f_ζ the function in $H^2 \ominus BH^2$ such that $f_\zeta(a) = 1$ and $f_\zeta(b) = \zeta$. Clearly, $f_1 = 1 - \overline{B(0)}B$ and

$$f_\zeta = \frac{B}{B'(a)(z-a)} + \frac{\zeta B}{B'(b)(z-b)}$$

for every $\zeta \in \mathbb{T}$.

The proof of the following lemma is straightforward.

Lemma 7.1. *The following statements are equivalent:*

(i) $\|f_\zeta\|_2 < \|f_1\|_2$;

(ii) $\operatorname{Re}(\zeta(1-\overline{a}b)) > \operatorname{Re}(1-\overline{a}b)$. ■

Lemma 7.2. *Suppose that the equivalent statements of Lemma 7.1 are fulfilled. Then $f_\zeta(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$.*

Proof. Clearly, $f_\zeta(z) = 0$ if and only if $z = \frac{\zeta B'(a)a + B'(b)b}{\zeta B'(a) + B'(b)}$. We need to prove that $|z| > 1$, i.e., $|\zeta B'(a)a + B'(b)b| > |\zeta B'(a) + B'(b)|$. Hence, the required assertion is equivalent to each of the following inequalities

$$|B'(a)|^2|a|^2 + |B'(b)|^2|b|^2 + 2\operatorname{Re}\zeta B'(a)\overline{B'(b)}a\overline{b} >$$
$$> |B'(a)|^2 + |B'(b)|^2 + 2\operatorname{Re}\zeta B'(a)\overline{B'(b)},$$

$$2\operatorname{Re}\zeta B'(a)\overline{B'(b)}(a\overline{b} - 1) > (1-|a|^2)|B'(a)|^2 + (1-|b|^2)|B'(b)|^2,$$

$$2\operatorname{Re}\frac{\zeta|a-b|^2}{(1-|a|^2)(1-|b|^2)(1-a\overline{b})} >$$
$$> \frac{|a-b|^2}{(1-|a|^2)|1-\overline{a}b|^2} + \frac{|a-b|^2}{(1-|b|^2)|1-\overline{a}b|^2}.$$

Thus, we need to prove that $2\operatorname{Re}(\zeta(1-\overline{a}b)) > 2 - |a|^2 - |b|^2$. Note that $2\operatorname{Re}(1-\overline{a}b) > 2 - |a|^2 - |b|^2$. It remains to apply Lemma 7.1. ■

Theorem 7.3. *Let g be a function holomorphic in $\mathbb{D}$. Suppose that $g(a) = 0$, $g(b) = it$ with $t \in \mathbb{R}$ and $\|e^g\|_2^2 \le 1 - |B(0)|^2$. Then $|t| < \pi$.*

Proof. Suppose that $|t| \ge \pi$. Put $f = e^{\frac{2\pi g}{|t|}}$ and $p = \frac{|t|}{\pi}$. Then $f(a) = f(b) = 1$ and $\|f\|_p^p = \|e^g\|_2^2 \le 1 - |B(0)|^2$. Hence, by Corollary 1.3, we have

$f = (1 - \overline{B(0)}B)^{\frac{2}{p}}$, i. e., $g = \log(1 - \overline{B(0)}B)$. Hence, $g(b) = g(a) = 0$, and we get a contradiction. ■

Corollary 7.4. *Let $\zeta = e^{it}$ with $-\pi < t < \pi$. Suppose that $\|f_\zeta\|_2^2 < 1-|B(0)|^2$. Let g be the branch of $\log f_\zeta$ such that $g(a) = 0$. Then $g(b) = it$.* ■

Theorem 7.5. *Let $a, b \in \mathbb{D}$ and $p > 0$. Suppose that $1 - a\bar{b} = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$. Then the following statements are equivalent*

(i) $p < \frac{2|\theta|}{\pi}$,

(ii) *there exists a function $F \in H^2 \ominus BH^2$ such that $F(z) \neq 0$ for all $z \in \mathbb{D}$, $(F^{\frac{2}{p}} - 1)B^{-1} \in H^2$ and $\|F\|_2^2 < 1 - |B(0)|^2$,*

(iii) *there exists a function $f \in H^p$ such that $f(a) = f(b) = 1$, $f(z) \neq 0$ for any $z \in \mathbb{D}$ and $\|f\|_p^p < 1 - |B(0)|^2$.*

Proof. First, we prove that (i) implies (ii). Clearly, we may assume that $\theta > 0$. Put $F = f_\zeta$ for $\zeta = e^{ip\pi}$. Note that $\cos(p\pi - \theta) < \cos\theta$. Hence, $\|F\|_2^2 < 1 - |B(0)|^2$, and $(F^{\frac{2}{p}} - 1)B^{-1} \in H^2$ by Corollary 7.4. The implication (ii)⇒(iii) is trivial. It remains to prove that (iii) implies (i). Let g be the branch of $\log f$ such that $g(a) = 0$. Then $g(b) = 2\pi ni$ for $n \in \mathbb{Z}$. By Lemma 7.1, the inequality $\|f\|_p^p < 1 - |B(0)|^2$ implies that $\cos(pn\pi - \theta) > \cos\theta$. Moreover, $|pn\pi| < \pi$ by Corollary 7.4. Thus, $|pn\pi - \theta| < |\theta|$, whence $n \neq 0$ and $p < \frac{2|\theta|}{|n|\pi} \leq \frac{2|\theta|}{\pi}$. ■

Corollary 7.6. *Suppose that $1-a\bar{b} = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$. Then $p_B^0 = \frac{2|\theta|}{\pi}$.* ■

Corollary 7.7. *Suppose that $1-a\bar{b} = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$. Then $p_B \geq \frac{2|\theta|}{\pi}$.* ■

Corollary 7.8. *For every $p \in [0, 1)$ there exist points a and b in the unit disk $\mathbb{D}$ such that $p_B = p$.*

Proof. It suffices to note that $p_B = 0$ if $\frac{a}{b} < 0$, and to apply Corollary 7.7. ■

8. Concluding Remarks and Open Problems

1. Blaschke products of degree two with a multiple zero. Let $B(z) \stackrel{\text{def}}{=} \dfrac{(z-r)^2}{(1-rz)^2}$ with $r \in (0, 1)$. Note that $p_B^0 = 0$ by Theorem 5.2 and $p_B > 0$ for every $r \in (0, 1)$ by Theorem 2.4. Put $\beta(r) = p_B$. One can prove that

$$1 - \beta(r) \sim \frac{3}{2}\sqrt{1-r} \quad \text{as} \quad r \to 1-$$

and

$$\beta(r) \sim r^2 \Big(\log \frac{1}{r} \Big)^{-1} \quad \text{as} \quad r \to 0+ .$$

The proofs of these asymptotic formulae are based on rather long computations, and we omit them. Moreover, one can prove that $\beta(r) \le 1 - \sqrt{1-r^2}$ for all $r \in [0,1)$.

2. Blaschke products with simple zeros lying on a circle $r\mathbb{T}$. Let B be a finite Blaschke product $B(z) = \prod_{n=1}^{N} \frac{z-a_n}{1-\overline{a}_n z}$ such that $|a_n| = r$ for all n and $a_k \ne a_l$ for $k \ne l$. One can prove that if $p_B^0 = 0$, then $a_1^N = a_2^N = \dots = a_N^N$. The converse is also true because if $a_1^N = a_2^N = \dots = a_N^N$, then $B(z) = \frac{z^N - a_1^N}{1 - \overline{a}_1^N z} \in \mathfrak{A}_+$ by Corollary 4.6. Hence, $p_B^0 \le p_B = 0$.

Let B be a finite Blaschke product with simple zeros $\{a_n\}_{n=1}^N$. Then the function $1 - \overline{B(0)}B$ can represented as follows

$$1 - \overline{B(0)}B(z) = \sum_{n=1}^{N} \frac{\lambda_n}{1-\overline{a}_n z}.$$

Suppose that $\lambda_1 = \lambda_2 = \dots = \lambda_N$. In other words, we have $a_1 B'(a_1) = a_2 B'(a_2) = \dots = a_N B'(a_N)$. Then $|a_1| = |a_2| = \dots = |a_N|$.

Also, it is interesting to note that the equalities $B'(a_1) = B'(a_2) = \dots = B'(a_N)$ imply $N = 1$.

These observations generate the following questions.

Question 1. Does there exist an infinite Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1-\overline{a}_n z}$ with simple zeros such that $a_1 B'(a_1) = a_2 B'(a_2) = \dots = a_n B'(a_n) = \dots$?

Question 2. Does there exist an infinite Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1-\overline{a}_n z}$ with simple zeros such that $B'(a_1) = B'(a_2) = \dots = B'(a_n) = \dots$?

3. Blaschke products of the class $\mathfrak{A}_+$. Let $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1-\overline{a}_n z}$ be an infinite Blaschke product with simple zeros. Then by Theorem 2.7, we have $B \in \mathfrak{A}_+$ if and only if $\frac{a_n B'(a_n)}{B(0)} < 0$ for all n and $\sum_{n=1}^{\infty} \frac{1}{|B'(a_n)|} < +\infty$.

Question 3. Does there exist an infinite Blaschke product $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$ with simple zeros such that $\frac{a_n B'(a_n)}{B(0)} < 0$ for all n and $\sum_{n=1}^{\infty} \frac{1}{|B'(a_n)|} = +\infty$?

Theorem 2.8 allows us to reformulate this question as follows.

Does the condition $\frac{a_n B'(a_n)}{B(0)} < 0$ for all n imply that B is an interpolating Blaschke product?

Question 4. Let $f \in H^2 \ominus BH^2$, where $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$ be an infinite Blaschke product. Put $\lambda_n \stackrel{\text{def}}{=} (f, \frac{B}{B'(a_n)(z-a_n)})$ where $(\cdot, \cdot)$ denote the inner product in $H^2 \ominus BH^2$, i.e., $(f, g) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f\overline{g}\, d\boldsymbol{m}$. Is it true that f belongs to the closure of the cone generated by the family $\left\{ \frac{\lambda_n}{1 - \overline{a}_n z} \right\}_{n=1}^{\infty}$?

If the answer to this question is in positive, then the answer to Question 3 is in negative.

Question 5. Let $f \in H^2 \ominus BH^2$, where $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$ be an infinite Blaschke product. Put $\lambda_n \stackrel{\text{def}}{=} (f, \frac{B}{B'(a_n)(z-a_n)})$. Is it true that f belongs to the closure of the $\mathbb{R}$-linear span of the family $\left\{ \frac{\lambda_n}{1 - \overline{a}_n z} \right\}_{n=1}^{\infty}$?

4. Inner functions of the class $\mathfrak{A}$. The author does not know any description of the inner functions of the class $\mathfrak{A}$.

Conjecture. Let $B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \cdot \frac{a_n - z}{1 - \overline{a}_n z}$ be an infinite Blaschke product with $B(0) \neq 0$. Then $B \in \mathfrak{A}$ if and only if $\operatorname{Res}_{a_n} \frac{B(0)}{zB(z)} \in \mathbb{R}$ for all n.

We can prove this conjecture only in some special cases. By Theorem 6.5, this conjecture is true in the case where the zeros of B lie on a diameter of the disk $\mathbb{D}$. Theorem 5.2 states that this conjecture is true in the case where B is a finite Blaschke product. In fact, one can prove that the conjecture is true in the case where B is a finite product of interpolating Blaschke products.

5. Inner functions Θ with $p_\Theta = 1$. By Theorem 1.7, $p_B < 1$ for every finite Blaschke product B.

Question 5. Does there exist an inner function Θ such that $p_\Theta = 1$?

Put $h(\delta) \stackrel{\text{def}}{=} \sup\{p_\Theta : \Theta \text{ is an inner function}, |\Theta(0)| \leq \delta\}$.

Question 6. Is it true that $h(\delta) < 1$ for all $\delta \in (0,1)$?

Question 7. Is it true that $\lim\limits_{\delta \to 0+} h(\delta) = 0$?

References

[G] J. B. Garnett, *Bounded Analytic Functions*, Academic Press (1981).

[A1] A. B. Aleksandrov, *Essays on non locally convex Hardy classes*, Lecture Notes in Math. **864** (1981), 1-89, Springer-Verlag, Berlin–Heidelberg-New York.

[A2] A. B. Aleksandrov, *A-integrability of the boundary values of harmonic functions* (Russian), Mat. Zametki **30** (1981) no. 1, 59–72; translation in Math. Notes **30** (1982), 515–523.

[D] K. M. Dyakonov, *Moduli and arguments of analytic functions from subspaces in H^p that are invariant under the backward shift operator* (Russian), Sibirsk. Mat. Zh. **31** (1990), no. 6, 64–79; translation in Siberian Math. J. **31** (1990), no. 6, 926–939

[P] A. G. Poltoratski, *Boundary behavior of pseudocontinuable functions* (Russian), Algebra i Analiz **5** (1993), no. 2, 189–210; translation in St. Petersburg Math. J. **5** (1994), no. 2, 389–406.

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