A Stroock formula for a certain class of Lévy processes

M. Eddahbi[1]
Département de Math. & Info.
FSTG, Université Cadi Ayyad
B.P. 549, Marrakech, Maroc

J. L. Solé[2] and J. Vives[2]
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Spain

Abstract

We find a Stroock formula in the setting of the chaos expansion introduced by Nualart and Schoutens for a certain class of Lévy processes, using a Malliavin derivative based on the chaotic approach. As an application, we get the chaotic decomposition of the local time of a simple Lévy process.

1 Introduction

In general a Lévy process has not the chaotic representation property (say CRP for brevity), but Nualart and Schoutens (2000) have developed in [9] a kind of generalized CRP for a large class of Lévy processes. This work enabled to define a Malliavin derivative using this chaotic approach in a recent article of Léon *et al.* (2002) in [6]. The main goal of the present work is to get a Stroock formula in this setting. This formula gives the kernels of the chaotic decomposition of smooth random variables in the Malliavin sense, generated by the underlying Lévy process. Afterwards we apply it to find the decomposition of the local time $L(t,x)$ defined as the density of the occupation measure of a simple Lévy process. These processes have been studied in [6] as useful models in finance. They can also be used to approximate the sum of a Brownian motion plus a compound Poisson process.

More precisely a simple Lévy process is the sum of a Wiener component and m independent Poisson process which are also independent of the

[1]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: grant SB2000–0060.
[2]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: grant BFM2000–0009 and by the Catalan CIRIT, Grant 2001SGR00174.

Wiener process. For a complete survey on Lévy processes we refer to Bertoin [2] and Sato [11].

The paper is organized as follows: The second section is devoted to recall some definitions and results of the paper cited before with some new remarks and definitions that will be used in the sequel. In the third section we give the Stroock formula, and the last section deals with the chaotic decomposition of the local time of a simple Lévy process.

2 Basic elements of a Lévy chaotic calculus

2.1 Teugel's martingales associated to a Levy proces

Let $X = \{X_t : t \geq 0\}$ be a real valued Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Henceforth we always assume that we are using the càdlàg version. Let $\{\mathcal{F}_t : t \geq 0\}$ be the natural filtration of X completed with the null sets of $\mathcal{F}$.

We also assume that the Lévy measure ν of X satisfies the following condition: there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_{(-\varepsilon,\varepsilon)^c} e^{\delta\,|x|}\,\nu(dx) < +\infty \tag{2.1}$$

where $(-\varepsilon, \varepsilon)^c$ stands for the complement of the interval $(-\varepsilon, \varepsilon)$. This implies that X_t has moments of all orders and that the polynomials are dense in $L^2(\mathbb{R}, \mathbb{P} \circ X_1^{-1})$ (see [9]).

Define

$$X_t^{(1)} = X_t, \quad X_t^{(i)} = \sum_{0<s\leq t} \left(\Delta X_s\right)^i, \quad i \geq 2.$$

We have:

- The processes $X^{(i)} = \{X_t^{(i)} : t \geq 0\}$, $i = 1, 2, \dots$, are Lévy processes, that jump at the same points as X.
- $\mathbb{E}[X_t^{(i)}] = m_i t$, where $m_1 = \mathbb{E}[X_1]$ and $m_i = \int_{-\infty}^{\infty} x^i\,\nu(dx)$, $i \geq 2$.

Now define

$$Y_t^{(i)} = X_t^{(i)} - m_i\,t, \quad i = 1, 2, \dots$$

- The processes $Y^{(i)} = \{Y_t^{(i)} : t \geq 0\}$ are martingales. The predictable quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\langle Y^{(i)}, Y^{(j)} \rangle_t = m_{i+j}\,t \quad i, j \geq 2.$$

Finally, we introduce the so called Teugels's martingales,

$$H_t^{(i)} = \sum_{j=1}^{i} a_{ij} Y_t^{(j)} \quad i = 1, 2, \dots \tag{2.2}$$

where the constants a_{ij} are chosen in such a way that $a_{i1} = 1$ and the martingales $H^{(i)}$, $i = 1, 2, \dots$ are strongly orthogonal, that means, for $i \neq j$, the process $H^{(i)} H^{(j)}$ is a martingale. In particular, since $\langle H^{(i)}, H^{(j)} \rangle_t$ is a predictable process such that $H_t^{(i)} H_t^{(j)} - \langle H^{(i)}, H^{(j)} \rangle_t$ is a martingale, we have that

$$\langle H^{(i)}, H^{(j)} \rangle_t = 0, \quad i \neq j.$$

Moreover,

- The processes $\{H_t^{(i)} : t \geq 0\}$ are normal martingales with predictable quadratic variation process given by

$$\langle H^{(i)}, H^{(i)} \rangle_t = q_i \, t,$$

where

$$q_i = \sum_{j,j'=1,\dots,i} a_{ij} a_{ij'} \, m_{j+j'} + a_{i1}^2 \, \sigma^2.$$

2.2 Iterated integrals

In this article we work with iterated integrals (instead of multiple ones).

Let $\Sigma_n = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : 0 < t_1 < t_2 < \cdots < t_n\}$ be the positive simplex of $\mathbb{R}^n$.Given $f \in L^2(\mathbb{R}_+^n)$ we will denote by $J_n^{(i_1,\dots,i_n)}(f)$ the iterated integral of f with respect to $H^{(i_1)}, \dots, H^{(i_n)}$:

$$J_n^{(i_1,\dots,i_n)}(f) = \int_0^\infty \left(\int_0^{t_n-} \cdots \left(\int_0^{t_2-} f(t_1, \dots, t_n) dH^{(i_1)}(t_1) \right) \cdots dH^{(i_{n-1})}(t_{n-1}) \right) dH^{(i_n)}(t_n).$$

We will omit all the parenthesis to simplify the notation and remark that all these integrals are well defined since all the processes $H^{(i)}$, $i = 1, 2, \dots$ are martingales with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

Proposition 1 *Let f and g belong to $L^2(\mathbb{R}_+^n)$. Then*

$$\mathbb{E}\left[J_n^{(i_1,\dots,i_n)}(f) \, J_m^{(j_1,\dots,j_m)}(g) \right] = \begin{cases} q_{i_1} \cdots q_{i_n} \int_{\Sigma_n} f(t_1, \dots, t_n) \cdot \\ \qquad \cdot g(t_1, \dots, t_n) \, dt_1 \cdots dt_n, \\ \qquad \text{if } n = m \text{ and} \\ \qquad (i_1, \dots, i_n) = (j_1, \dots, j_n), \\ 0, \quad \text{otherwise.} \end{cases}$$

Now, let us recall one of the main results in [9] (see also [6]) which is the chaotic representation property of the square integrable random variables:

Proposition 2 *Let $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then F has a unique representation of the form*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) \tag{2.3}$$

where $f_{i_i,\ldots,i_j} \in L^2(\Sigma_j)$.

2.3 Derivative operators

This subsection is devoted to the study of the properties of the derivatives in the context of a calculus of variations.

Definition 3 *Let $f \in L^2(\Sigma_j)$. Set*

$$D_t^{(\ell)} J_n^{(i_1,\ldots,i_n)}(f) = \sum_{k=1}^{n} \mathbb{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\ldots,\widehat{i}_k,\ldots,i_n)} \left(f(\underbrace{\cdots}_{k-1}, t, \cdots) \mathbb{1}_{\Sigma_n^{(k)}(t)}(\cdot) \right),$$

where

$$\Sigma_n^{(k)}(t) = \{(t_1, \ldots, \widehat{t}_k, \ldots, t_n) \in \Sigma_{n-1} : \\ 0 < t_1 < \cdots < t_{k-1} < t \le t_{k+1} < \cdots < t_n\}$$

and $\widehat{i}$ means that the i–th index is omitted.

Observe that if $k \neq k'$ then $\Sigma_n^{(k)}(t) \cap \Sigma_n^{(k')}(t) = \emptyset$.

Now we define the spaces of the random variables that are differentiable in the ℓ–th direction. For this, we define the following subset of $L^2(\Omega)$:

$$\mathbb{D}_\ell^{1,2} = \Bigg\{ F \in L^2(\Omega),\ F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}) : \\ \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} \sum_{k=1}^{n} \mathbb{1}_{\{i_k=\ell\}} q_{i_1} \cdots \widehat{q}_{i_k} \cdots q_{i_n} \\ \times \int_0^\infty \|f_{i_1,\ldots,i_n}(\cdots, t, \cdots) \mathbb{1}_{\Sigma_n^{(k)}(t)}\|^2_{L^2([0,\infty)^{n-1})}\, dt < \infty \Bigg\}.$$

Definition 4 *Given $F \in \mathbb{D}_\ell^{1,2}$,*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n} J_n^{(i_1,\dots,i_n)}(f_{i_1,\dots,i_n}),$$

we define the derivative of F in the ℓ-th direction as the element of $L^2(\Omega\times\mathbb{R})$ given by

$$D_t^{(\ell)} F = \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n} \sum_{k=1}^{n} \mathbb{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\dots,\widehat{i}_k,\dots,i_n)} \left(f_{i_1,\dots,i_n}(\cdots,t,\cdots) \mathbb{1}_{\Sigma_n^{(k)}(t)}(\cdot) \right)$$

Observe that, as in the classical situation for Gaussian processes, $\mathbb{D}_\ell^{1,2}$ is dense in $L^2(\Omega)$, since the elements of $L^2(\Omega)$ with a finite chaotic expansion are in $\mathbb{D}_\ell^{1,2}$.

Set $\mathbb{D}^{1,2} = \cap_{\ell\in\mathbb{N}} \mathbb{D}_\ell^{1,2}$.

Define also for $r > 1$

$$\mathbb{D}_\ell^{r,2} = \left\{ F \in L^2(\Omega), D_\cdot^{(\ell_1,\dots,\ell_r)} F \in L^2([0,\infty)^r, \Omega]) \right\}$$

and $\mathbb{D}^\infty = \cap_{r\in\mathbb{N}} \cap_{\ell\in\mathbb{N}} \mathbb{D}_\ell^{r,2}$

2.4 Simple Lévy processes

A simple Lévy process is given by

$$X_t = \sigma W_t + \alpha_1 N_t^1 + \cdots + \alpha_m N_t^m, \quad t \geq 0 \tag{2.4}$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion, $\{N_t^j : t \geq 0\}$, $j = 1,\dots,m$, are independent Poisson processes (and independent of the Brownian motion) of parameter $\lambda_1,\dots,\lambda_m$, respectively, $\sigma > 0$ and $\alpha_1,\dots,\alpha_m$ are different non–null numbers. The Lévy measure of X is $\nu = \sum_{j=1}^m \lambda_j \delta_{\alpha_j}$ and satisfies the condition (2.1) of [9] for the validity of the chaotic representation property.

Observe that

$$X_t^{(1)} = X_t = \sigma W_t + \sum_{j=1}^m \alpha_j N_t^j$$

and

$$X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i = \sum_{j=1}^m \alpha_j^i \ N_t^j, \quad i \geq 2,$$

because the fact that two Poisson processes adapted to the same filtration are independent is equivalent to they do not jump at the same time (see [2]). Since

$$Y_t^{(i)} = X_t^{(i)} - m_i\, t, \quad i \geq 1,$$

we have

$$Y_t^{(1)} = \sigma W_t + \sum_{j=1}^{m} \alpha_j \left(N_t^j - \lambda_j t \right)$$

and

$$Y_t^{(i)} = \sum_{j=1}^{m} \alpha_j^i \left(N_t^j - \lambda_j t \right), \ i \geq 2.$$

Then, the martingales $Y^{(i)} : i \geq 1$, are linear combinations of W_t, $N_t^1 - \lambda_1 t, \ldots, N_t^m - \lambda_m t$. Since the martingales $H^{(i)}$ are a linear combination of $Y^{(1)}, \ldots, Y^{(i)}$ it follows that $H^{(i)}$ are also a linear combination of W_t, $\widetilde{N}_t^1, \ldots, \widetilde{N}_t^m$, where $\widetilde{N}_t^j = N_t^j - \lambda_j t$. Therefore, we have a chaotic representation property in terms of the iterated integrals with respect W_t, $\widetilde{N}_t^1, \ldots, \widetilde{N}_t^m$. Recall also that by Proposition 1.10 in [6], $H^{(i)} = 0$, for $i \geq m+2$. Furthermore, since we are assuming that $\alpha_1, \ldots, \alpha_m$ are different, there is one and only one way to express $W_t, \widetilde{N}_t^1, \ldots, \widetilde{N}_t^m$ as a linear combination of $Y^{(1)}, \ldots, Y^{(m+1)}$ which is deduced by the uniqueness of the chaotic representation property in terms of the W_t, $\widetilde{N}_t^1, \ldots, \widetilde{N}_t^m$.

To unify the notations, we will write

$$G_0(t) = W_t, \quad \text{and} \quad G_j(t) = \widetilde{N}_t^j \ \ j = 1, \ldots, m.$$

Also, we will denote by $L_n^{(i_1,\ldots,i_n)}(f)$ the iterated integral of f with respect to $G_{i_1}, \ldots, G_{i_n}$:

$$L_n^{(i_1,\ldots,i_n)}(f) = \int_0^{\infty} \int_0^{t_n^-} \cdots \int_0^{t_2^-} f(t_1, \ldots, t_n) dG_{i_1}(t_1) \cdots dG_{i_{n-1}}(t_{n-1})\, dG_{i_n}(t_n).$$

Thus, we have a chaotic representation property in terms of the G_i's

Proposition 5 *Let $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then F has a representation of the form*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{0 \leq i_1,\ldots,i_n \leq m} L_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}),$$

where $f_{i_1,\ldots,i_j} \in L^2(\Sigma_j)$.

In this setting we recall a result of Léon *et al.* [6] which says that is possible to compute the derivatives in the directions W, $N^1, \ldots, N^m$, following the classical rules on each space. Recall that in the Poisson setting the derivative is a difference operator (see [7]).

We use the space $\Omega = \Omega_0 \times \Omega_1 \times \ldots \times \Omega_m$, where Ω_0 is the canonical space of the Wiener process and Ω_i, $i = 1, \ldots, m$ are respectively the canonical spaces of Poisson process N^i, $i = 1, \ldots, m$, that is the space of all possible paths of the Poisson's.

Remark 6 *If we iterate the derivative with respect to the Poisson's i and j $(i < j)$ we obtain*

$$\begin{aligned} D^i_{t_2} D^j_{t_1} F &= F\left(\omega_0, \ldots, \omega_i + \delta_{t_2}, \ldots, \omega_j + \delta_{t_1}, \ldots, \omega_m\right) \\ &\quad - F\left(\omega_0, \ldots, \omega_i, \ldots, \omega_j + \delta_{t_1}, \ldots, \omega_m\right) \\ &\quad - F\left(\omega_0, \ldots, \omega_i + \delta_{t_2}, \ldots, \omega_j, \ldots, \omega_m\right) \\ &\quad + F\left(\omega_0, \ldots, \omega_m\right). \end{aligned}$$

Observe that $D^i_{t_2} D^j_{t_1} F = D^j_{t_1} D^i_{t_2} F$, but this equality is not true in general if we only interchange the super-indexes or the sub-indexes.
If the iteration is done with respect to the same direction $i \geq 1$, we obtain that $D^{i,\ldots,i}_{t_1,\ldots t_s} F$, denoted by $D^{i,s}_{J_s} F$ where $J_s = \{t_1, \ldots, t_s\}$, is equal to

$$\sum_{B \subset J_s} (-1)^{s-(\#B)} F(\omega_0, \ldots, \omega_i + \sum_{t_j \in B} \delta_{t_j}, \ldots, \omega_m)$$

where we have used the convention $\sum_{t_j \in \emptyset} \delta_{t_j} = 0$.

Remark 7 *It is clear that $D^0_{t_1} D^i_{t_2} F = D^i_{t_2} D^0_{t_1} F$, for all $i \geq 1$. We want to compute $D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F$, $i_1, \ldots, i_n \in \{0, 1, \ldots, m\}$. We denote $I_s = \{j, i_j = s\}$ and $J_s = \{t_j, j \in I_s\}$, $s \in \{0, 1, \ldots, m\}$. Then $k_s = \#I_s$ will be the order of derivation with respect G_s and hence $\sum_{i=0}^m k_i = n$ and $\cup_{i=1}^m J_i = \{t_1, \ldots, t_n\}$. Therefore*

$$D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F = D^{0,k_0}_{J_0} D^{1,k_1}_{J_1} \cdots D^{m,k_m}_{J_m} F,$$

with the convention that $D^{i,0}_{J_i}$ is the identity, and

$$D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F = D^{0,k_0}_{J_0} \left[\sum_{B_i \subset J_i, 1 \leq i \leq m} (-1)^{\sum_{i=1}^m (k_i - (\#B_i))} F \right. \\ \left. \left(\omega_0, \omega_1 + \sum_{t_j \in B_1} \delta_{t_j}, \ldots, \omega_m + \sum_{t_j \in B_m} \delta_{t_j} \right) \right]$$

Remark 8 *If F is of the form $F = f(W_t, \widetilde{N}_t^1, \ldots, \widetilde{N}_t^m)$, where f is a smooth function, then the derivative $D_s^i F$, $i \geq 1$ and $s \leq t$, is the difference $f(W_t, \widetilde{N}_t^1, \ldots, \widetilde{N}_t^i + 1, \ldots, \widetilde{N}_t^m) - f(W_t, \widetilde{N}_t^1, \ldots, \widetilde{N}_t^m)$. So in general*

$$
\begin{aligned}
D_{t_1,\ldots,t_n}^{i_1,\ldots,i_n} F &= D_{J_0}^{0,k_0} D_{J_1}^{1,k_1} \cdots D_{J_m}^{m,k_m} F \\
&= D_{J_0}^{0,k_0} \Bigg[\sum_{0 \leq j_i \leq k_i, 1 \leq i \leq m} (-1)^{n-k_0-\sum_{i=1}^m j_i} \\
&\qquad \prod_{i=1}^m \binom{k_i}{j_i} f\left(W_t, \widetilde{N}_t^1 + j_1, \ldots, \widetilde{N}_t^m + j_m\right) \Bigg] \\
&= \sum_{0 \leq j_i \leq k_i, 1 \leq i \leq m} (-1)^{n-k_0-\sum_{i=1}^m j_i} \\
&\qquad \prod_{i=1}^m \binom{k_i}{j_i} \frac{\partial^{k_0} f}{\partial x_1^{k_0}} \left(W_t, \widetilde{N}_t^1 + j_1, \ldots, \widetilde{N}_t^m + j_m\right) 1\!\!1_{\{J_0 \subset [0,t]^{k_0}\}}.
\end{aligned}
$$

Remark 9 *For only one Poisson and without the Brownian part, different authors as Léon* et al. *[5] and Privault [10], have used the iterated derivatives to find the chaotic decomposition of functionals of the d first jump times of the process. In the present article we consider different Poisson processes and a Brownian part. Moreover we are not restricted to a specified finite number of jump times.*

3 Stroock formula for Lévy processes

The aim of this section is to derive a Stroock formula for Lévy processes in $\mathbb{D}^\infty$. First we deal with the case that F is an element of a specific chaos and finally we will extend it to F in $\mathbb{D}^\infty$.

We start with the following lemma.

Lemma 10 *Let F be $J_n^{i_1,\ldots,i_n}(f)$, then*

$$
D_{t_1,\ldots,t_n}^{j_1,\ldots,j_n} F = \sum_{\sigma \in \mathcal{P}\{1,\ldots,n\}} \delta_{j_{\sigma(1)},i_1} \cdots \delta_{j_{\sigma(n)},i_n} f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}) 1\!\!1_{\{t_{\sigma(1)} < \ldots < t_{\sigma(n)}\}}, \tag{3.5}
$$

where $\mathcal{P}\{1,\ldots,n\}$ is the set of permutations of $\{1,\ldots,n\}$.

Remark 11 *In the latter sum only one summand is different from zero.*

Proof of Lemma 10. We use induction on n. For $n=2$, we have

$$\begin{aligned} &D_{t_1}^{j_1}\left(\int_0^1\int_0^{r_2-} f(r_1,r_2)dH^{(i_1)}(r_1)dH^{(i_2)}(r_2)\right)\\ &\quad=\delta_{j_1,i_1}\int_{t_1}^1 f(t_1,r_2)dH^{(i_2)}(r_2)+\delta_{j_1,i_2}\int_0^{t_1} f(r_1,t_1)dH^{(i_1)}(r_1)\\ &\quad=\delta_{j_1,i_1}\int_0^1 f(t_1,r_2)1\!\!1_{[t_1,1]}(r_2)dH^{(i_2)}(r_2)\\ &\qquad\quad+\delta_{j_1,i_2}\int_0^1 f(r_1,t_1)1\!\!1_{[0,t_1]}(r_1)dH^{(i_1)}(r_1)\end{aligned}$$

And applying now the operator $D_{t_2}^{j_2}$ we get

$$\begin{aligned} &D_{t_2}^{j_2}D_{t_1}^{j_1}\left(J_2^{i_1,i_2}(f)\right)\\ &\qquad=\delta_{j_2,i_2}\delta_{j_1,i_1}f(t_1,t_2)1\!\!1_{[t_1,1]}(t_2)+\delta_{j_2,i_1}\delta_{j_1,i_2}f(t_2,t_1)1\!\!1_{[0,t_1]}(t_2).\end{aligned}$$

Hence, if $t_1<t_2$, we have

$$D_{t_2}^{j_2}D_{t_1}^{j_1}\left(\int_0^1\int_0^{r_2-} f(r_1,r_2)dH^{(i_1)}(r_1)dH^{(i_2)}(r_2)\right)=\delta_{j_1,i_1}\delta_{j_2,i_2}f(t_1,t_2).$$

Therefore the formula (3.5) is satisfied for $n=2$.
Besides

$$\begin{aligned} &D_{t_1,\ldots,t_n}^{j_1,\ldots,j_n}\left(J_n^{i_1,\ldots,i_n}(f)\right)\\ &=D_{t_{j_n}}^{j_n}\left(\cdots\left(D_{t_{j_1}}^{j_1}\left(J_n^{i_1,\ldots,i_n}(f)\right)\right)\right)\\ &=D_{t_{j_n}}^{j_n}\cdots D_{t_{j_2}}^{j_2}\left[\sum_{k=1}^n\delta_{i_kj_1}J_{n-1}^{i_1,\ldots,\widehat{i}_k,\ldots,i_n}\left(f(\ldots,t_1,\ldots)1\!\!1_{\sum_n^{(k)}(t_1)}(\cdot)\right)\right].\end{aligned}$$

The induction hypothesis yields

$$\begin{aligned} &D_{t_1,\ldots,t_n}^{j_1,\ldots,j_n}\left(J_n^{i_1,\ldots,i_n}(f)\right)\\ &=\sum_{k=1}^n\delta_{i_kj_1}\sum_{\tau\in\mathcal{P}\{1,\ldots,\widehat{k},\ldots,n\}}^{\infty}\delta_{j_{\tau(2)},i_1}\cdots\delta_{j_{\tau(k-1)},i_{k-1}}\delta_{j_{\tau(k+1)},i_{k+1}}\cdots\delta_{j_{\tau(n)},i_n}\\ &\times f(t_{\tau(2)},\ldots t_{\tau(k-1)},t_1,t_{\tau(k+1)}\ldots,t_{\tau(n)})1\!\!1_{\{t_{\tau(2)}<\ldots<t_{\tau(k-1)}<t_1<t_{\tau(k+1)}\ldots<t_{\tau(n)}\}}\\ &=\sum_{\sigma\in\mathcal{P}\{1,\ldots,n\}}\delta_{j_{\sigma(1)},i_1}\cdots\delta_{j_{\sigma(n)},i_n}f(t_{\sigma(1)},\ldots,t_{\sigma(n)})1\!\!1_{\{t_{\sigma(1)}<\ldots<t_{\sigma(n)}\}}.\end{aligned}$$

Remark 12 *If $t_1 < \ldots < t_n$ we get the following equality*

$$D^{j_1,\ldots,j_n}_{t_1,\ldots,t_n}\left(J^{i_1,\ldots,i_n}_n(f)\right) = \delta_{j_1,i_1}\cdots\delta_{j_n,i_n} f(t_1,\ldots,t_n).$$

Now we apply the Lemma 10 to obtain the Stroock formula.

Proposition 13 *Let F be in $\mathbb{D}^\infty$ with the chaotic expansion*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{0\le i_1,\ldots,i_n\le m} J^{i_1,\ldots,i_n}_n(f_{i_1,\ldots,i_n}).$$

Then

$$f_{i_1,\ldots,i_n}(t_1,\ldots,t_n) = \mathbb{E}[D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n}F]$$

where $t_1 < \ldots < t_n$.

Proof: As the iterated derivatives of order n of elements belonging to chaos of order less than n are zero and applying the above lemma, we get

$$D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n}F = f_{i_1,\ldots,i_n}(t_1,\ldots,t_n) + M_n, \text{ with } t_1 < \cdots < t_n,$$

where M_n is a sum of variables in chaos of order greater or equal than n. Taking the expectation we obtain the desired formula.

4 Chaotic expansion of the local time of a simple Lévy process

The aim of this section is to apply the Stroock formula to find the chaotic decomposition of the local time of a simple Lévy process.

We denote by H_n the n–th Hermite polynomial defined for $n \geq 1$ by

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right),$$

and $H_0(x) = 1$.

We can define the local time $L(t,x)$ of a simple Lévy process X as the density of the occupation measure

$$m_t(A) = \int_0^t \mathbb{1}_A(X_s)ds, \ A \in \mathcal{B}(\mathbb{R}).$$

It is well known that $m_t(\cdot)$ has a density $L(t,x)$ which is a non–decreasing function of t, and the measure $L(dt,x)$ is concentrated on the level set

$\{s : X_s = x\}$. Moreover, Barlow [1] show that $L(t,x)$ has an almost surely jointly continuous version. We can write

$$L(t,x) = \int_0^t \delta_x(X_s)ds.$$

To apply the Stroock formula we consider

$$L_\varepsilon(t,x) = \int_0^t p_\varepsilon(X_s - x)ds,$$

where p_ε is the centered Gaussian kernel with variance $\varepsilon > 0$. The classical idea of approximating the Dirac distribution δ_x by p_ε has been used to calculate the chaotic decomposition of the local time in the case of the Brownian motion by Nualart and Vives [8] and for the fractional Brownian motion by Coutin *et al.* [3] and Eddahbi *et al.* [4]

Before stating precise results of this section, we prove some technical lemmas

Lemma 14 *Let* $F = p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}_s^i - x)$. *Then*

$$\begin{aligned} D^{j_1,\ldots,j_n}_{t_1,\ldots,t_n} F &= D^{0,k_0}_{J_0} D^{1,k_1}_{J_1} \cdots D^{m,k_m}_{J_m} F \\ &= \sum_{i=1}^m \sum_{\ell_i=0}^{k_i} \prod_{j=1}^m \binom{k_j}{\ell_j} (-1)^{n-k_0-\sum_{j=1}^m \ell_j} \sigma^{k_0} \\ &\quad \times p_\varepsilon^{(k_0)} \left(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}_s^i + \alpha_i \ell_i - x \right) \prod_{i=1}^n \mathbb{1}_{[0,s]}(t_i), \end{aligned}$$

where $k_i = \#\{j, i_j = i\}$, $i = 0, 1, \ldots, m$.

Proof: We only have to apply Remark 8 in section 2.

Now we shall compute the expectation of this expression.

Lemma 15 *Set* $g_n(y) = H_n(y)\exp(-y^2/2)$. *Then*

$$\mathbb{E}\left[p_\varepsilon^{(n)}(\sigma W_s - a)\right] = \frac{(-1)^n}{(s\sigma^2+\varepsilon)^{\frac{n+1}{2}}} \frac{\sqrt{n!}}{\sqrt{2\pi}} g_n\left(\frac{a}{\sqrt{s\sigma^2+\varepsilon}}\right).$$

Proof: As $p_\varepsilon^{(n)}(x) = \frac{1}{\sigma^{n+1}} p^{(n)}_{\frac{\varepsilon}{\sigma^2}}\left(\frac{x}{\sigma}\right)$, we have

$$\mathbb{E}\left[p_\varepsilon^{(n)}(\sigma W_s - a)\right] = \frac{1}{\sigma^{n+1}} \mathbb{E}\left[p^{(n)}_{\frac{\varepsilon}{\sigma^2}}\left(W_s - \frac{a}{\sigma}\right)\right].$$

Interchanging the derivative operator and the expectation and using the fact that

$$p_{\varepsilon}^{(n)}(x) = (-1)^n \sqrt{n!} \frac{p_{\varepsilon}(x)}{\varepsilon^{\frac{n}{2}}} H_n\left(\frac{x}{\sqrt{\varepsilon}}\right)$$

and

$$\mathbb{E}\left[p_{\varepsilon}(W_s - a)\right] = p_{s+\varepsilon}(a).$$

we get the formula of the lemma.

Lemma 16

$$\left|\mathbb{E}\left[p_{\varepsilon}^{(n)}(\sigma W_s - a)\right]\right| \leq \frac{1}{2\pi}\Gamma\left(\frac{n+1}{2}\right)\left(\frac{s\sigma^2}{2}\right)^{-\frac{n+1}{2}}.$$

Proof: We apply Lemma 15 and the standard inequality,

$$|g_n(y)| \leq \frac{2^{\frac{n}{2}}}{\sqrt{\pi n!}}\Gamma\left(\frac{n+1}{2}\right).$$

Proposition 17 *The chaos expansion of* $L_{\varepsilon}(t,x) = \int_0^t p_{\varepsilon}(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}_s^i - x)ds$ *is*

$$L_{\varepsilon}(t,x) = \sum_{n=0}^{\infty} \sum_{0\leq i_1,\dots i_n \leq m} L_n^{i_1,\dots,i_n}\left[\sum_{i=1}^{m}\sum_{\ell_i=0}^{k_i}\prod_{j=1}^{m}\binom{k_j}{\ell_j}\sigma^{k_0}\right.$$
$$\times(-1)^{n-k_0-\sum_{j=1}^m \ell_j}\frac{\sqrt{k_0!}}{\sqrt{2\pi}}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\int_{t_1\vee\cdots\vee t_n}^{t}\frac{e^{-\lambda s}}{(s\sigma^2+\varepsilon)^{\frac{k_0+1}{2}}}$$
$$\left.\times\prod_{j=1}^{m}\frac{(\lambda_j s)^{r_j}}{r_j!}g_n\left(\frac{x-\sum_{j=1}^m \alpha_j(r_j - \lambda_j s + \ell_j)}{\sqrt{s\sigma^2+\varepsilon}}\right)ds\right],$$

where $\lambda = \sum_{i=1}^m \lambda_i$.

Remark 18 *Note that the kernels depend only on* $k_0, k_1, \dots, k_m$. *Hence it is the same for all sets of indexes* $i_1, \dots, i_n$ *that derives to equals* $k_0, k_1, \dots, k_m$. *Observe also that although the kernel is the same, the iterated integral* $L_n^{i_1,\dots,i_n}$ *can depend on the order of the indexes as we see in the next example.*

$$\int_0^1\int_{u_2}^1 dW_{u_1}dN_{u_2} = \int_0^1 (W_1 - W_{u_2})\,dN_{u_2} = W_1N_1 - \sum_{i=1}^{N_1} W_{T_i},$$

where $T_i, i \in \mathbb{N}$ are the jump times of the Poisson. But

$$\int_0^1 \int_{u_2}^1 dN_{u_1} dW_{u_2} = \int_0^1 (N_1 - N_{u_2})\, dW_{u_2}$$

$$= W_1 N_1 - \left[N_1 \left(W_1 - W_{T_{N_1}}\right) + (N_1 - 1)\left(W_{T_{N_1}} - W_{T_{N_1 - 1}}\right) + \cdots + W_{T_2} - W_{T_1}\right]$$

$$= -\sum_{i=1}^{N_1} W_{T_i}$$

Proof: We apply the Stroock formula to $p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}_s^i - x)$ and use Lemma 14 for the expression of the iterated derivative and Lemma 15 for the expectation with respect to the Wiener part. Besides we calculate the expectation with respect to the Poisson part. □

In order to establish the chaotic expansion of the local time of the above simple Lévy process we state the following lemma, that generalizes in our setting a Lemma of [8].

Lemma 19 *Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of square integrable random variables with the expansions*

$$F_\varepsilon = \mathbb{E}[F_\varepsilon] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}^\varepsilon),$$

where $f_{i_1,\ldots,i_n}^\varepsilon$ belong to $L^2([0,\infty)^n)$. Assume that

(i) $f_{i_1,\ldots,i_n}^\varepsilon$ converges in $L^2([0,\infty)^n)$, when ε goes to zero, to some symmetric function $f_{i_1,\ldots,i_n} \in L^2([0,\infty)^n)$.

(ii) $\sum_{n=1}^\infty \sum_{i_1,\ldots,i_n} q_{i_1} \cdots q_{i_n} \sup_\varepsilon \|f_{i_1,\ldots,i_n}^\varepsilon(\cdot)\|_{L^2([0,\infty)^n)}^2$ is convergent.

Then the family F_ε converges in $L^2(\Omega)$ to

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}).$$

Proposition 20 *For each $x \in \mathbb{R}$, and $t \in [0,T]$, the random variables*

$$\int_0^t p_\varepsilon\left(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}_s^i - x\right) ds$$

converge to $L(t,x)$ in $L^2(\Omega)$ as ε tends to zero. Furthermore the local time $L(t,x)$ has the following chaotic decomposition

$$\begin{aligned} L(t,x) &= \sum_{n=0}^{\infty} \sum_{0\le i_1,\dots i_n\le m} L_n^{i_1,\dots,i_n} \Bigg[\sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \prod_{j=1}^{m} \binom{k_j}{\ell_j} \sigma^{k_0} \\ &\quad \times (-1)^{n-k_0-\sum_{j=1}^m \ell_j} \frac{\sqrt{k_0!}}{\sqrt{2\pi}} \sum_{r_1=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} \int_{t_1\vee\cdots\vee t_n}^{t} \frac{e^{-\lambda s}}{(s\sigma^2)^{\frac{k_0+1}{2}}} \\ &\quad \times \prod_{j=1}^{m} \frac{(\lambda_j s)^{r_j}}{r_j!} g_n \left(\frac{x-\sum_{j=1}^m \alpha_j (r_j - \lambda_j s + \ell_j)}{\sigma\sqrt{s}} \right) ds \Bigg] \end{aligned} \tag{4.6}$$

where $\lambda = \sum_{i=1}^m \lambda_i$.

Proof of Proposition 20: We must check the two hypotheses of Lemma 19. We start with (ii). By Lemma 14 we have

$$\begin{aligned} \left| \mathbb{E}\left[D_{t_1,\dots,t_n}^{i_1,\dots,i_n} L_\varepsilon(t,x) \right] \right| &\le \sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \prod_{j=1}^{m} \binom{k_j}{\ell_j} \\ &\times \sum_{r_1=0}^{\infty} \cdots \sum_{r_m=0}^{\infty} \int_{t_1\vee\cdots\vee t_n}^{t} e^{-\lambda s} \prod_{j=1}^{m} \frac{(\lambda_j s)^{r_j}}{r_j!} \sigma^{k_0} \\ &\times \mathbb{E}_W \, p_\varepsilon^{(k_0)} \left(\sigma W_s + \sum_{i=1}^{m} \alpha_i (r_i - s + \ell_i) - x \right) \prod_{i=1}^{n} \mathbb{1}_{[0,s]}(t_i) ds, \end{aligned} \tag{4.7}$$

and by Lemma 16, and since

$$e^{-\lambda s} \prod_{j=1}^{m} \sum_{r_j=0}^{\infty} \frac{(\lambda_j s)^{r_j}}{r_j!} = 1,$$

the right hand side of (4.7) is bounded by

$$\begin{aligned} &\sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \frac{\sigma^{k_0}}{2\pi} \prod_{j=1}^{m} \binom{k_j}{\ell_j} \int_{t_1\vee\cdots\vee t_n}^{t} \frac{\Gamma(\frac{k_0+1}{2})}{(\frac{s\sigma^2}{2})^{\frac{k_0+1}{2}}} ds \\ &\qquad = \frac{2^{n-\frac{k_0}{2}+\frac{1}{2}}}{2\pi\sigma} \Gamma\left(\frac{k_0+1}{2}\right) \int_{t_1\vee\cdots\vee t_n}^{t} \frac{ds}{s^{\frac{k_0+1}{2}}}, \end{aligned}$$

where we have used the identity

$$\sum_{i=1}^{m}\sum_{\ell_i=0}^{k_i}\prod_{j=1}^{m}\binom{k_j}{\ell_j}=2^{n-k_0}.$$

Therefore

$$\left|\mathbb{E}\left[D^{i_1,\dots,i_n}_{t_1,\dots,t_n}L_\varepsilon(t,x)\right]\right|^2\le\frac{2^{2n-k_0+1}}{(2\pi\sigma)^2}\Gamma\left(\frac{k_0+1}{2}\right)^2\left(\int_{t_1\vee\dots\vee t_n}^{t}\frac{ds}{s^{\frac{k_0+1}{2}}}\right)^2.$$

Then

$$\begin{aligned}
&\left\|\mathbb{E}\left[D^{i_1,\dots,i_n}_{t_1,\dots,t_n}L_\varepsilon(t,x)\right]\right\|^2_{L^2([0,T]^n)}\\
\le\ & \frac{2^{2n-k_0+1}}{(2\pi\sigma)^2}\Gamma\left(\frac{k_0+1}{2}\right)^2\int_0^t\frac{y^{n-1}dy}{(n-1)!}\left(\int_y^t s^{-\frac{k_0+1}{2}}ds\right)^2\\
=\ & \frac{2^{2n-k_0+1}}{(2\pi\sigma)^2}\Gamma\left(\frac{k_0+1}{2}\right)^2\int_0^t\int_0^t\int_0^t\frac{y^{n-1}\mathbb{1}_{[0,u\wedge v]}(y)dy}{(n-1)!}(uv)^{-\frac{k_0+1}{2}}dudv\\
=\ & \frac{2^{2n-k_0}}{n!(\pi\sigma)^2}\Gamma\left(\frac{k_0+1}{2}\right)^2\int_0^t v^{-\frac{k_0+1}{2}}dv\int_0^v u^{n-\frac{k_0+1}{2}}du\\
=\ & \frac{2^{2n-k_0}}{n!(\pi\sigma)^2}\Gamma\left(\frac{k_0+1}{2}\right)^2\frac{2}{2n-k_0+1}\frac{t^{n-k_0+1}}{n-k_0+1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{0\le i_1,\dots i_n\le m}\left\|\mathbb{E}\left[D^{i_1,\dots,i_n}_{t_1,\dots,t_n}L_\varepsilon(t,x)\right]\right\|^2_{L^2([0,T]^n)}\\
&\le\sum_{n=0}^{\infty}\sum_{k_0=0}^{n}\frac{t2^{k_0+1}}{(\pi\sigma)^2k_0!}\Gamma\left(\frac{k_0+1}{2}\right)^2 \qquad (4.8)\\
&\qquad\sum_{k_1+\ldots+k_m=n-k_0}\frac{(4t)^{n-k_0}}{k_1!\ldots k_m!}\frac{1}{2n-k_0+1}\frac{1}{n-k_0+1}\\
&\le\sum_{n=0}^{\infty}\sum_{k_0=0}^{n}\frac{2^{k_0+1}}{m(\pi\sigma)^2k_0!}\Gamma\left(\frac{k_0+1}{2}\right)^2\frac{(m4t)^{n-k_0+1}}{(n-k_0+1)!}\frac{1}{2n-k_0+1}\\
&\le\sum_{k_0=0}^{\infty}\frac{2^{k_0+1}}{m(\pi\sigma)^2k_0!}\Gamma\left(\frac{k_0+1}{2}\right)^2\frac{e^{m4t}-1}{k_0+1}. \qquad (4.9)
\end{aligned}$$

where we have used the facts that the bound of each term of the series depends only on $k_0, k_1, \ldots, k_m$,

$$\sum_{k_1+\ldots+k_m=r} \frac{1}{k_1!\ldots k_m!} = \frac{m^r}{r!}$$

and the interchange of summations in (4.9) and $2n - k_0 + 1 \geq k_0 + 1$.
Now by Stirling formula

$$\frac{2^{k_0+1}}{\pi^2(k_0+1)!}\Gamma\left(\frac{k_0+1}{2}\right)^2 \sim ck_0^{-3/2} \quad \text{for } k_0 \quad \text{large.}$$

Hence, the general term of the right hand side of (4.9) behaves as $ck_0^{-3/2}$ and the corresponding series is convergent.

Note that in our setting $q_0 = \cdots = q_m = 1$, because we do not work with the H_i's but use directly the Wiener and Poisson processes.

Now it remains to check condition (i) of the Lemma 19. We have

$$\begin{aligned}\left\|f^{\varepsilon}_{i_1,\ldots,i_n} - f_{i_1,\ldots,i_n}\right\|^2_{L^2([0,T]^n)} &= \left\|f^{\varepsilon}_{i_1,\ldots,i_n}\right\|^2_{L^2([0,T]^n)} + \left\|f_{i_1,\ldots,i_n}\right\|^2_{L^2([0,T]^n)} \\ &\quad -2\left\langle f^{\varepsilon}_{i_1,\ldots,i_n}, f_{i_1,\ldots,i_n}\right\rangle_{L^2([0,T]^n)}.\end{aligned}$$

It is clear that $f^{\varepsilon}_{i_1,\ldots,i_n}$ converges to $f_{i_1,\ldots,i_n}$ pointwise and using the dominated convergence we see that condition (i) holds.

Finally we will show, following standard arguments, that the limit of $\int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}^i_s - x)ds$, denoted by Λ^x_t is the local time $L(t,x)$. The above estimates are uniform in $x \in \mathbb{R}$. Therefore, we can conclude that the convergence of $\int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}^i_s - x)ds$ to Λ^x_t holds in $L^2(\Omega\times\mathbb{R}, \mathbb{P}\otimes\mu)$, for any finite measure μ. As a consequence, for any continuous function g in $\mathbb{R}$ with compact support we have that

$$\int_{\mathbb{R}}\left(\int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}^i_s - x)ds\right) g(x)dx$$

converges in $L^2(\Omega)$ to $\int_{\mathbb{R}} \Lambda^x_t g(x)dx$. But, this expression also converges to

$$\int_0^t g\left(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}^i_s\right) ds.$$

Hence,

$$\int_{\mathbb{R}} \Lambda^x_t g(x)dx = \int_0^t g\left(\sigma W_s + \sum_{i=1}^m \alpha_i \widetilde{N}^i_s\right) ds,$$

which implies that $\Lambda^x_t = L(t,x)$. □

References

[1] Barlow, M.T. (1988) Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Prob.* **16**, 1389–1427.

[2] Bertoin, J. (1996) *Lévy Processes.* Cambridge University Press, Cambridge.

[3] Coutin, L. Nualart, D. and Tudor, C.A. (2000) The Tanaka formula for the fractional Brownian motion. *Stoc. Proc. Appl.* **94** (2) 301–315.

[4] Eddahbi, M., Lacayo, R., Solé, J.L., Tudor, C.A. and Vives, J. (2002) *Regularity and asymptotic behaviour of the local time for the d–dimensional fractional Brownian motion with N–parameters* Preprint Núm. 14/2002, Universitat Autònoma de Barcelona.

[5] León, J.A., Solé, J.L. and Vives, J. (1998) The chaos decomposition of functionals of the jump times for the Poisson process over a compact time interval. *Stochastic models* (Guanajuato, 1998), pp. 269–282, *Aportaciones Mat. Investig.*, 14, *Soc. Mat. Mexicana, México.*

[6] León, J.A. Solé, J.L., Utzet, F. and Vives, J. (2002) On Lévy processes, Malliavin calculus and market models with jumps. *Finance Stoch.* **6** (2), 197–225.

[7] Nualart, D. and Vives, J. (1990) Anticipative calculus for the Poisson Process based on the Fock space, In *Séminaire de Probabilités* XXIV, *Lecture Notes on Mathematics*, **1426**, pp. 154–165, Springer, Berlin.

[8] Nualart, D. and Vives, J. (1992) Chaos expansions and local times. *Publicacions Matemàtiques*, **36**, 827–836.

[9] Nualart, D. and Schoutens, W. (2000) Chaotic and predictable representation for Lévy processes. *Stoc. Proc. Appl.* **90** (1), 109–122.

[10] Privault, N. (2002) Distribution-valued iterated gradient and chaotic decompositions of Poisson jump times functionals. *Publ. Mat.* **46** (1), 27–48.

[11] Sato, K.I. (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.