# An example of a cubic Liénard system with linear damping having invariant algebraic curves of arbitrary degree


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**Abstract**

We present a cubic Liénard polynomial differential system with linear damping which has irreducible invariant algebraic curves of arbitrary high degree. Moreover, this differential system has a rational first integral.


## 1 Introduction

A differential system in $\mathbb{C}^2$ of the form

$$\dot{x} = P(x, y) \ , \qquad \dot{y} = Q(x, y) \ , \tag{1}$$

with $P, Q \in \mathbb{C}[x, y]$ (i.e. polynomials in the variables $x$ and $y$ with coefficients in $\mathbb{C}$) having maximum degree equal to $m$ is called a *polynomial differential system of degree $m$*.


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Let $f \in \mathbb{C}[x, y]$. An algebraic curve $f = f(x, y) = 0$ in $\mathbb{C}^2$ is a phase curve of system (1) if there exists a polynomial $K = K(x, y)$ satisfying

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf \ . \tag{2}$$

Such curves are called *invariant algebraic curves* of the polynomial differential system (1). The polynomial $K$ is called the *cofactor* of the invariant algebraic curve $f = 0$. Of course, the degree of the cofactor is at most $m-1$.

The existence of invariant algebraic curves for polynomial differential systems has been studied intensively due to the fact that such curves sometimes can be used to construct first integrals or integrating factors using the Darboux theory of integrability, for more details see [8, 17, 18, 4, 5, 12].

We shall study polynomial differential systems of the form

$$\dot{x} = y \ , \quad \dot{y} = -F_n(x)y - G_m(x) \ , \tag{3}$$

where $F_n(x)$ and $G_m(x)$ are polynomials of degree $n$ and $m$ in the variable $x$. We shall call such systems $(n, m)$*-Liénard systems.* The $(1, 3)$-Liénard systems are called the *cubic Liénard system with linear damping* and they have been studied recently by several authors. Thus, Dumortier and Rousseau [9] and Dumortier and Li [10] proved that these systems can have at most one limit cycle.

In 1995 Odani [15] started the study of the invariant algebraic curves for the $(n, m)$-Liénard systems. He shows that these systems have no invariant algebraic curves if $m \leq n$, and $g$ is not a constant multiple of $f$. In particular, this implies that the famous limit cycle of the van der Pol equation is not algebraic. One year later, Hayashi [11] studied the invariant algebraic curves of (3) when $m = n + 1$. But a good study of such curves is due to Żołądek [19]. His results apply to the remaining case $m > n$, where he shows that for any given $(n, m)$ there always exist systems with an invariant algebraic curve. He also shows that generic systems different from the linear one have no invariant algebraic curve.

An unknown problem for the $(n, m)$-Liénard systems is to know if they have irreducible invariant algebraic curves of arbitrary degree. Here, we shall prove that there are $(1, 3)$-Liénard systems having irreducible invariant algebraic curves of arbitrary degree, and rational first integrals. This problem for quadratic systems have been studied by Christopher and Llibre [6], by Moulin–Ollagnier [14] and Chavarriga and Grau [3], but in those cases they look for polynomial differential systems having irreducible invariant algebraic curves of arbitrary degree and without rational first integrals.

Our results are the following ones.

**Theorem 1.** *We consider the following cubic Liénard system with linear damping*

$$\dot{x} = y \ , \qquad \dot{y} = -2(2x+1)y + \frac{4n}{(n-1)^2}x - \frac{8}{3}x^2 - \frac{16}{9}x^3 \ . \tag{4}$$

*(a) This system has the following two invariant algebraic curves*

$$\begin{aligned} f = (1-n)^{1+n} &\left(-6x + 4(n-1)x^2 + 3(n-1t)y\right) \cdot \\ &\cdot \left(6nx + 4\left(n-1\right)x^2 + 3\left(n-1\right)y\right)^n \\ - &\left(9n - (n-1)^2(6x+4x^2+3y)\right)^{n+1} = 0 \ , \end{aligned}$$

*with cofactor* $K_f = -\frac{4}{3}(n+1)x$*, and*

$$g = 9n - (n-1)^2(6x+4x^2+3y) = 0 \ ,$$

*with cofactor* $K_g = -\frac{4}{3}x$.

*(b) The system has the rational first integral* $f(x,y)/g(x,y)^{n+1}$.

We note that the polynomial $f(x,y)$ defined in statement (a) of Theorem 1 has maximal degree $2n$ in the variable $x$ and maximal degree $n$ in the variable $y$.

From statement (b) it follows that all trajectories of system (4) (except the trajectories contained in $g(x,y)=0$) are contained in invariant algebraic curves of the form $f(x,y) + cg(x,y)^{n+1} = 0$ with $c \in \mathbb{C}$. We will show that except for finitely many values of $c$ all these algebraic curves are irreducible of degree $2(n+1)$. More precisely,

**Theorem 2.** *Let* $f(x,y)=0$ *and* $g(x,y)=0$ *be the two invariant algebraic curves of the polynomial differential system (4) defined in Theorem 1. Then except for finitely many values of* $c \in \mathbb{C}$ *all the invariant algebraic curves of the form* $f(x,y) + cg(x,y)^{n+1} = 0$ *are irreducible.*

We note that the $(1,3)$-Liénard systems are the unique $(n,m)$-Liénard systems for which is unknown if they have or not algebraic limit cycles, see Żołądek [19]. Our result showing that the $(1,3)$-Liénard systems can have invariant algebraic curves of arbitrary degree says that the answer to the above open problem can be difficult.

The paper is organized as follows. In Section 4 we prove Theorems 1 and 2. Finally, in Section 6 we study the phase portraits of the cubic Liénard system with linear damping (4).

## 2 Invariant algebraic curves

In this section we summarize some general facts concerning the invariant algebraic curves of polynomial differential systems. We consider a polynomial differential system (1) of degree $m$ with $P$ and $Q$ coprime.

The *multiple points* of an algebraic curve $f(x,y)=0$ are the points satisfying the following three equations:

$$f(x,y)=0\ , \quad \frac{\partial f}{\partial x}(x,y)=0\ , \quad \frac{\partial f}{\partial y}(x,y)=0\ .$$

The following two propositions are well–known, see for instance [5] and [6], respectively.

**Proposition 3.** *Let $f=0$ be an invariant algebraic curve with cofactor $K$ for the polynomial differential system (1).*

*(a) All multiple points of $f=0$ are singular points of system (1).*

*(b) All singular points of system (1) are contained in $\{K=0\}\cup\{f=0\}$.*

**Proposition 4.** *We suppose that $f\in\mathbb{C}[x,y]$ and let $f=f_1^{n_1}\cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{C}[x,y]$. Then, for a polynomial differential system (1), $f=0$ is an invariant algebraic curve with cofactor $K_f$ if and only if $f_i=0$ is an invariant algebraic curve for each $i=1,\ldots,r$ with cofactor $K_{f_i}$. Moreover $K_f=n_1K_{f_1}+\ldots+n_rK_{f_r}$.*

From Proposition 4 and the definition of invariant algebraic curve it follows easily the next result.

**Corollary 5.** *Let $f=0$ be an invariant algebraic curve of a polynomial differential system (1) with a cofactor $K$. If $g=0$ is another invariant algebraic curve of system (1), then $fg=0$ satisfies (2) with a cofactor $K+\frac{1}{g}\left(P\frac{\partial g}{\partial x}+Q\frac{\partial g}{\partial y}\right)$. If $g=y^n$, then the cofactor of $fg=0$ is $K+n\frac{Q}{y}$.*

## 3 General results about $(n,m)$–Liénard systems

In the next proposition we summarize the results on the $(n,m)$-Liénard systems that we shall need.

**Proposition 6.** *Let $f=0$ be an invariant algebraic curve of a $(n,m)$-Liénard system (3) with cofactor $K$.*

*(a) The polynomial $f$ is of the form* $ay^M + \sum_{i=0}^{M-1} f_i(x)y^i$*, with* $a \in \mathbb{C} \setminus \{0\}$.

*(b) The cofactor $K$ is independent of $y$; i.e.* $K = K(x)$.

*(c) The degree of $K$ is less or equal to* $\max\{n, m-1\}$.

*Proof*: Let $M$ be the degree of $f$ as a polynomial with respect to the variable $y$; i.e. $f(x,y) = \sum_{i=0}^{M} f_i(x)y^i$ with $f_M(x) \neq 0$. Assume that $K(x,y) = \sum_{i=0}^{N} K_i(x)y^i$. Then, from (2), we get that the highest order terms in $y$ of

$$y\frac{\partial f}{\partial x} \ , \quad -(F_n(x)y + G_m(x))\frac{\partial f}{\partial y} \quad \text{and} \quad Kf \ ,$$

are

$$f_M'(x)y^{M+1} \ , \quad -MF_n(x)f_M(x)y^M \quad \text{and} \quad K_N(x)f_M(x)y^{M+N} \ ,$$

respectively. Therefore, either $N = 1$ and $f_M'(x) = K_1(x)f_M(x)$, or $N = 0$, $f_M(x) = a \in \mathbb{C} \setminus \{0\}$ and $f_{M-1}'(x) - aMF_n(x) = aK_0(x)$. Clearly, only the second possibility can occur. Hence, statements (a) and (b) follow.
For the proof of the statement (c) let $\hat{f}(x,y)$, $\hat{F}_n(x)$, $hG_n(x)$, $\hat{K}$ denote homogenous terms of highest order of $f$, $F_n$, $G_n$ and $K$ respectively. Now the highest order terms of

$$y\frac{\partial f}{\partial x} \ , \quad -(F_n(x)y + G_m(x))\frac{\partial f}{\partial y} \quad \text{and} \quad Kf \ ,$$

are

$$\hat{f}_x'y \ , \quad -(\hat{F}_n(x)y + \hat{G}_m(x))\hat{f}_y' \quad \text{and} \quad \hat{K}(x)\hat{f} \ ,$$

In case $m > n+1$ this gives $-\hat{G}_m(x)\hat{f}_y' = \hat{K}(x)\hat{f}$, and clearly it only can take place iff $\hat{f} = \hat{f}(x)$ and $\deg \hat{K}(x) < m$. The remaining cases are treated similarly. ∎

# 4 Proof of the main results

For proving Theorem 1 we shall need the following result of the Darbouxian theory of integrability, for more details see for instance [8, 5, 18].

**Lemma 7.** *Suppose that the polynomial differential system (1) admits $p$ invariant algebraic curves $f_i = 0$ with cofactors $K_i$ for $i = 1, \cdots, p$. Then there exist $\lambda_i \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i = 0 \tag{5}$$

*if and only if the (multi–valued) function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p}, \tag{6}$$

*is a first integral of system (1).*

*Proof of Theorem 1*: A tedious but easy computation allows to check that $f = 0$ and $g = 0$ are invariant algebraic curves of system (4) with cofactors $K_f$ and $K_g$, respectively. So, statement (a) follows.

From statement (a) and Lemma 7 it follows (b) because $K_f$–$(n+1)K_g$=0. ∎

Before proving Theorem 2 we need some preliminary notions and results.

We shall need the following result well–known result, see for instance [5].

**Proposition 8.** *We suppose that $f \in \mathbb{C}[x, y]$ and let $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization in irreducible factors over $\mathbb{C}[x, y]$. Then, for a polynomial differential system (1), $f = 0$ is an invariant algebraic curve with cofactor $K_f$ if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor $K_{f_i}$. Moreover $K_f = n_1 K_{f_1} + \ldots + n_r K_{f_r}$.*

Let $X = (P, Q)$ be the vector field associated to the polynomial differential system (1). The *$n$–th extactic curve of $X$*, $\mathbf{E}_n(X) = 0$, is defined by the equation

$$\det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix} = 0\,, \tag{7}$$

where $v_1, v_2, \ldots, v_l$ is a basis of the $\mathbb{C}$–vector space of polynomials in $\mathbb{C}[x, y]$ of degree at most $n$ (hence $l = (n+1)(n+2)/2$), $X^0(v_i) = v_i$ and $X^j(v_i) = X^{j-1}(X(v_i))$.

Observe that the definition of extactic curve is independent of the chosen basis of the $\mathbb{C}$–vector space of polynomials of degree at most $n$. The extactic curves for polynomials vector fields were introduced by Lagutinskii in [13].

Essentially, they are the curves of inflection and higher order inflection points for the orbits of the vector field, see [16].

Let $H = F/G$ be a rational first integral of a polynomial differential system (1); i.e. $F$ and $G$ are polynomials in the variables $x$ and $y$. We say that $H$ has *degree* $n$ if $n$ is the maximum of the degrees of $F$ and $G$.

The following two propositions are due to Lagutinskii [13]. They have been rediscovered recently see Pereira [16], and Christopher, Llibre and Pereira [7]. Since these papers at this moment are not easy to find and their proofs are short we present them here.

**Proposition 9.** *Let $X$ be a vector field on $\mathbb{C}^2$. Then $\mathbf{E}_n(X) \equiv 0$ and $\mathbf{E}_{n-1}(X) \not\equiv 0$ if and only if $X$ admits a rational first integral of degree $n$.*

*Proof*: Let $p \in \mathbb{C}^2$ be a non–singular point of $X$. We may assume that $p$ is the origin of $\mathbb{C}^2$. Suppose that the solution passing through it is parametrized, locally, by $(x, y(x))$. Since $\mathbf{E}_n(X)$ vanishes identically, the composition of our local solution with the $n$–Veronese map,

$$(x, y) \to (x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n, x^{n-1}, \ldots, y^{n-1}, \ldots, x^2, xy, y^2, x, y, 1)$$

is contained in a hyperplane, therefore $(x, y(x))$ must be contained in an algebraic curve of degree less than or equal to $n$. The fact that $\mathbf{E}_{n-1}(X) \neq 0$ implies that the generic solution is of degree at least $n$. This completes the "only if" part of the statement.

If $X$ admits a first integral of degree $n$ then every invariant curve is of degree at most $n$ and hence every point is a $n$–inflection point, i.e., $\mathbf{E}_n(X) = 0$. Since not every invariant curve has degree $n-1$, $\mathbf{E}_{n-1}(X) \neq 0$. ∎

**Proposition 10.** *Every invariant algebraic curve of degree $n$ by the vector field $X$ is a factor of $\mathbf{E}_n(X)$.*

*Proof*: Let $f$ be an invariant algebraic curve of degree $n$. As it was observed the choice of the basis of the $\mathbb{C}$–vector space plays no role in the definitions of extactic curve, therefore we can take $v_1 = f$. Since

$$\begin{array}{rclcl} X(f) & = & L_f f \ , & & \\ X^2(f) & = & X(L_f f) & = & \left(L_f^2 + X(L_f)\right) f \ , \\ X^k(f) & = & X(X^{k-1}(f)) & = & (\,\text{polynomial}\,)\, f \ , \end{array}$$

$f$ is a factor of $\mathbf{E}_n(X)$. ∎

Let $H = F/G$ be a rational first integral of a polynomial differential system (1). According to Poincaré [17] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable* value of $H$ if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$. Here, if $c = \infty$ then $f + cg$ denotes $g$.

We say that the degree of the rational first integral $H$ is *minimal* between all the degrees of the rational first integrals of system (7) if any other rational first integral of (7) has degree $\geq n$. Then the rational first integrals of degree $n$ are called *minimal*.

The next result is stated by Poincaré in [17] without a proof. A proof of it appears in [2], since this proof is also short we give it.

**Proposition 11.** *Assume that a polynomial differential system (1) has a minimal rational first integral $H$. Then $H$ has finitely many remarkable values.*

*Proof*: Let $H = f/g$ be a minimal rational first integral of system (1) of degree $n$. We must prove that there are finitely many values of $c \in \mathbb{C}$ such that $f - cg$ is reducible in $\mathbb{C}[x, y]$.

Let $u$ be a factor of a reducible polynomial of the form $f - cg$. Then the degree of $u$ is smaller than $n$. By Proposition 8, $u = 0$ is an invariant algebraic curve of system (1). Since the degree of $H$ is minimal between all the degrees of the rational first integrals, by Proposition 9 the extactic curves $\mathbf{E}_k(X) \not\equiv 0$ for $k = 1, \ldots, n-1$. From the definition of extactic curve $\mathbf{E}_k(X)$ is a polynomial. This implies that the number of factors of all the reducible polynomials $f - cg$ is finite. Consequently, the number of remarkable values of $H$ is finite. ∎

**Proposition 12.** *Let $f(x, y) = 0$ and $g(x, y) = 0$ be the two invariant algebraic curves of the polynomial differential system (4) defined in Theorem 1. Then the polynomial $f(x, y) - g(x, y)^{n+1}$ is irreducible.*

Proposition 12 will be proved in the next section.

*Proof of Theorem 2*: Let $H = f/g^{n+1}$ be the rational first integral of the polynomial differential system (4) given in statement (b) of Theorem 1. Clearly, $f$ and $g$ are coprime because $g$ is irreducible and

$$f\left(x, \frac{3n}{(n-1)^2} - 2x - \frac{4}{3}x^2\right)$$

is not zero. So the degree of $H$ is $2(n+1)$. By Proposition 12, the algebraic curve $f = 0$ is irreducible. Therefore, the degree $2(n + 1)$ of $H$ is minimal. Hence, by Proposition 11, the theorem follows. ∎

## 5 Proof of Proposition 12

The Liénard system of the considered kind can be transformed into a quadratic system with an invariant straigth line. For the simplicity of our proof we shall use the following change of coordinates:

$$x = \frac{3}{v}, \quad y = \frac{-6(2+v) - 3n(u - 4(2+v) + 2n(2+v))}{(n-1)^2\, v^2}, \tag{8}$$

under which the system(4) (after multiplication by $-\frac{y}{2}$) transforms to

$$\dot{x} = -2x - \frac{n}{(n-1)^2}x^2 - 2xy + 2y^2, \quad \dot{y} = -y\left(2 + \frac{n}{2(n-1)^2}x + y\right). \tag{9}$$

In order to be able to look for the algebraic phase curves of (9) we shall need an analogue of Proposition 10 about the possible form of cofactor

**Proposition 13.** *There is a correspondence between the algebraic curves of the system (4) and the system (9). To every algebraic phase curve $\gamma$ of (4) with cofactor $k_1x + k_0$ there corresponds an algebraic phase curve $\delta$ of (9) with cofactor*

$$-2l - \frac{nN}{2(n-1)^2}x - (k+N)y, \tag{10}$$

*depending on 2 parameters $k_0 = 2k$, $k_1 = \frac{4}{3}(l - N)$, and $N$ is a constant depending on the degree of $\delta$ (see proof).*

*Proof*: Proposition 10 it follows, that the form $k_1x+k_0$ is a general form of a cofactor for (4). If $\gamma$ is an algebraic phase curve of (9) given by $\varphi(x,y) = 0$, the transformation (8) transforms it to a phase curve with linear cofactor $-\frac{yk_0}{2} - \frac{3k_1}{2}$, but the function $\Phi(u,v) = \varphi(x(u,v), y(u,v))$ defining it is no longer a polynomial, but a rational function, whose denominator is $y^N$. It becomes a polynomial after multiplication by $y^N$, and now from Corollary 5 the Proposition follows. ∎

Now Theorem 1 is equivalent to the following

**Theorem 14.** *The quadratic polynomial differential system (9) has the following two irreducible invariant algebraic curves:*

$$f(x,y) = (x + 2(n-1)y)(nx - 2(n-1)y)^n + n^n(x+y^2)^{1+n},$$

*of degree $2(n+1)$ with respect to $y$ and degree $n+1$ with respect to $x$, its cofactor is*

$$K_f = -2(n+1) - \frac{n(n+1)}{(n-1)^2}x - 2(n+1)y \ ;$$

*and*

$$g(x,y) = x + y^2 = 0,$$

*with cofactor*

$$K_g = -\frac{n}{(n-1)^2}x - 2\,(1+y)\,.$$

*Proof*: It is easy to check that $f = 0$ and $g = 0$ are invariant algebraic curves of system (9) with cofactor $K_f$ and $K_g$, respectively. Clearly, $g$ is irreducible in $\mathbb{C}[x,y]$. So, it only remains to prove the irreducibility of $f$.

In order to do that we shall use the Proposition 4. If the polynomial $f$ could have been factorized, this would mean that there had to exist an irreducible polynomial $h \in \mathbb{C}[x,y]$ of degree smaller than the degree of $f$, such that

(1) $h$ divides $f$

(2) $h = 0$ is an invariant algebraic curve

(of course, in view of Proposition 4, (2) implies (1)). Using the property (1) we shall prove that cofactor $h$ must satisfy:

(i) $h$ must be of the form $h(x,y) = x^r + l.o.t.$, $r < 2(n+1)$, where *l.o.t.* means terms of lower degree in both variables $x$, $y$. Moreover, the parameter $k$ must be equal to 0.

(ii) $h(x,0) = x^m$, and $l = N/2$.

So, the cofactor of $h$ must be equal to $-N - \frac{nN}{2(n-1)^2}x - Ny$. Finally, we shall show (Lemma 15) that every polynomial satisfying (2) must be of a form $g(x,y) = x + y^2$ raised to some power. But as $g$ does not divide $f$ ($f(-y^2,y)$ does not vanish), this would mean contradiction with (1), and hence with the assumption of reducibility of $f$.

Substituting (9) and (**??**) into (1), and multiplying by $2(n-1)^2$ we obtain, that the algebraic phase curves of (9) are given by equation $\Xi(\varphi) \equiv 0$, where

$$\begin{aligned}\Xi(\varphi(x,y)) =& 4l(n-1)^2 + nNx + 2(n-1)^2(k+N)y\varphi - \\ & - 2\left[nx^2 - 2(n-1)^2y^2 + 2(n-1)^2x(y+1)\right]\frac{\partial\varphi}{\partial x} - \\ & - y\left[4(n-1)^2 + nx + 2(n-1)^2y\right]\frac{\partial\varphi}{\partial y}\,.\end{aligned}$$

We assume that $h$ divides $f$.

*Proof of (i)* It is easy to see that $f = y^{2(n+1)} + l.o.t.$. So (up to multiplication by a constant) $h$ must be of the form $g = y^r + l.o.t.$, $r < 2(n+1)$. Now we observe, that

$$\begin{aligned}\Xi(y^A x^B) = &- 4(A+B-l)(n-1)^2 x^B y^A - n(A+2B-N)x^{B+1}y^A - \\ &- 2(n-1)^2(A+2B-k-N)x^B y^{1+A} + \\ &+ 4B(n-1)^2 x^{B-1} y^{2+A} \ .\end{aligned} \tag{11}$$

Therefore, it contains terms of degree at most $A+B+1$. Moreover, $\Xi(y^s) = 4(l-s)(n-1)^2 y^s + n(N-s)xy^s + 2(n-1)^2(N+k-s)y^{s+1}$, so we get that $r = N$, and $k = 0$.

*Proof of (ii)* Since $f(x,0) = 2n^n x^{n+1}$, we see that $h(x,0) = \text{const}x^m$, $m \leq n$. Writing $h(x,y) = \sum_{s=0}^r h_s(x)y^s$ and substituting it into $\Xi(h) = 0$, we obtain

$$(4l(n-1)^2 + nNx)h_0(x) - x(2(n-1)^2 + nx)h_0'(x) = 0 \ .$$

That is, $h_0(x) = Cx^l(2(n-1)^2 + nx)^{\frac{N}{2}-l}$. Therefore we get $l = N/2$ and $h_0 = Cx^N/2$. As an immediate corollary we see, that $N$ must be an even number.

We denote by $h^m(x,y)$ the homogeneous terms in $h$ of degree $m$.

**Lemma 15.** *For $m = N/2, N/2+1, \ldots, N$ it holds that*

$$h^m(x,y) = \binom{N/2+1}{N-m} y^{2m-N} x^{N-m} \ ,$$

*i.e.* $h(x,y) = (x+y^2)^{N/2}$.

*Proof*: We use induction with respect to $m$ going downwards. From (i) it follows, that the assertion is true for $m = N$. Now we assume that $h^m(x,y) = \binom{N/2+1}{N-m} y^{2(m+1)-N} x^{N-m-1}$. As $k = 0$ (i), and $l = N/2$ (ii), $\Xi(y^{2(m+1)-N} x^{N-m-1})$ is equal to

$$\begin{aligned}&4(n-1)^2(N/2-m-1)x^{N-m-1}y^{2(m+1)-N} + \\ &\qquad + 4(n-1)^2(N-m-1)x^{N-m-2}y^{2(m+2)-N} \ ,\end{aligned}$$

and $\Xi(y^{2m-N}x^{N-m})$ is equal to

$$\begin{aligned}&- 4(n-1)^2(m-N)x^{N-m-1}y^{2+2m-N} - \\ &\qquad - 4(n-1)^2(m-N/2)x^{N-m}y^{2m-N} \ ,\end{aligned}$$

we see that that $h^m$ must be of the form $\binom{N/2+1}{N-m} x^{N-m}(y^{2m-N} + l.o.t.)$, and from (11) it follows immediately that it cannot contain any other monomial except

$$\binom{N/2+1}{N-m} x^{N-m} y^{2m-N} .$$

So, the lemma is proved. ∎

Going back through the changes of variables we can write system (9) as the cubic Liénard system with linear damping (4) having the two irreducible invariant algebraic curves giving in Theorem 1. In short, statement (a) of Theorem 1 is a corollary of Theorem 14. Hence, Theorem 1 is proved.

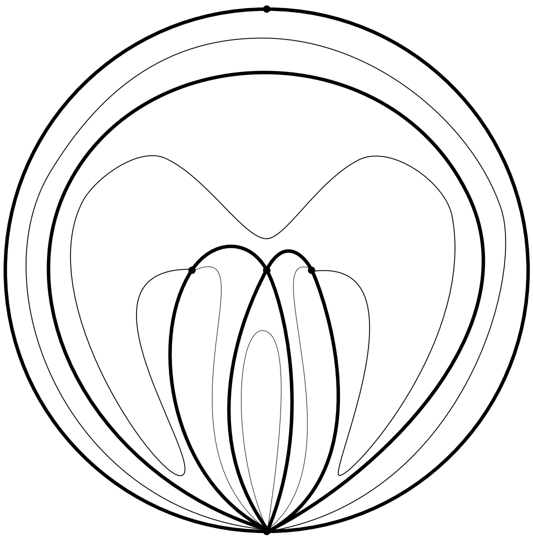

Figure 1: Phase portrait of system (4).

# 6 Real phase portraits of system (4)

Suppose that the cubic Liénard system with linear damping (4) is real; i.e. it is defined in $\mathbb{R}^2$. Then it has 3 real singular points: $p_0 = (0,0)$ (a saddle), $p_+ = \left(\frac{3}{2n-2}, 0\right)$ (source) and $p_- = \left(\frac{-3n}{2n-2}, 0\right)$ (a sink). There is also a singular point at the end of the $y$–axis on the line at infinity. The latter is quite complicated, it can be studied by means of a series of (two) blow ups, see for instance [1]. But the existence of the first integral, which is not continuous at that point, provides a phase portrait at the neighbourhood of that point. In this way, we can draw the whole phase portrait of system (4), see Figure 1. We note that the phase portrait does not depend on the degree $n$ of the first integral.

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