INVARIANT ALGEBRAIC CURVES OF LARGE DEGREE FOR QUADRATIC SYSTEMS

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Abstract

In this paper we present for the first time examples of algebraic limit cycles and saddle loops of degree greater than 4 for planar quadratic systems. In particular, we give examples of algebraic limit cycles of degree 5 and 6, and algebraic saddle loops of degree 3 and 5 surrounding a strong focus. We also give an example of an invariant algebraic curve of degree 12 for which the quadratic system has no Darboux integrating factors or first integrals.

1 Introduction

We shall study polynomial differential systems in $\mathbb{R}^2$ defined by

$$\dot{x} = p(x, y), \qquad \dot{y} = q(x, y), \tag{1}$$

where p, q are coprime polynomials of degree 2, more explicitly

$$p(x, y) = \sum_{i,j=0}^{2} p_{i,j} x^i y^j, \qquad q(x, y) = \sum_{i,j=0}^{2} q_{i,j} x^i y^j. \tag{2}$$

We shall call such differential systems *quadratic systems*. The object of our study will be the limit cycles of such systems, namely, algebraic limit cycles given by zeroes of some irreducible polynomial φ, where

$$\varphi(x,y) = \sum_{i,j=0}^{n} \varphi_{i,j} x^i y^j.$$

If the zero set of the algebraic curve $\varphi(x,y) = 0$ has a non–trivial real branch then gives an invariant curve of system (1) if and only if there exists a polynomial $k = k(x,y)$ satisfying

$$p\frac{\partial \varphi}{\partial x} + q\frac{\partial \varphi}{\partial y} - k\varphi = 0. \tag{3}$$

The polynomial k is called a *cofactor* of the curve $\varphi = 0$. In case of quadratic systems the cofactor can be at most linear. A limit cycle which is also an irreducible invariant algebraic curve of degree n for system (1) is called an *algebraic limit cycle* of degree n. A similar definition also holds for algebraic saddle–loops.

Until now only five different families of algebraic limit cycles for quadratic systems have been found: one of degree 2 [9], and four of degree 4 (for details see [2]). It is known that there are no algebraic limit cycles of degree 3, see Evdokimenco [5, 6, 7], or see Theorem 11 of [4] for a short proof. Recently it has also been proved in [3], that there are no other algebraic limit cycles of degree 4.

The question of whether quadratic systems can have algebraic limit cycles of higher degree has remained open since the first examples of algebraic limit cycles of degree 4 appeared in 1966 by Yablonskii [10] and in 1973 by Filiptsov [8]. In this paper we shall give explicit examples of algebraic limit cycles of degree 5 and 6 in Theorems 2 and 3.

In Theorem 5 we classify all the saddle–loops of degree 3 for quadratic systems which have a focus in their interior, and in Theorem 6 we show the existence of a saddle–loop of degree 5. Interestingly, this saddle–loop does not appear to be part of a larger family of algebraic limit cycles.

In the final section we give an example of a quadratic system with an invariant algebraic curve of degree 12 which is not Darboux integrable (i.e. it has no Darboux integrating factors or first integrals; or alternatively, it has no Liouvillian first integral). Until recently, the highest degree examples known of such curves was at most 8.

2 Algebraic limit cycles of degrees 5 and 6

The main idea used in this section is to apply a change of variables to a known quadratic system with an algebraic limit cycle, which preserves the degree of the system, but increases the degree of the algebraic curve. For this purpose we use the birational transformation

$$(x, y) \to (x/y^2, 1/y), \tag{4}$$

after an appropriate translation. In fact, this transformation is an involution.

If the system is of the form

$$\begin{aligned} \dot{x} &= \alpha x + \beta y + 2ex^2 + bxy + cy^2, \\ \dot{y} &= \gamma x + \delta y + exy + fy^2, \end{aligned} \tag{5}$$

then it is easy to see that we can apply the transformation above and still remain in the class of quadratic systems. Furthermore, if we move any of the singular points of system (9) to the origin, then the new system has the form (5).

As a simple example, we show that the example of Yablonskii with an algebraic limit cycle of degree 4 can be obtained from the well–known example of an algebraic limit cycle of degree 2 due to Qin Yuan–Xun [9].

Proposition 1. *The system of Yablonskii,*

$$\begin{aligned} \dot{x} &= -4abcx + 3(a+b)cx^2 - (a+b)y + 4xy, \\ \dot{y} &= ab(a+b)x + (4ab - 3(a+b)^2/2 + 4abc^2)x^2 - \\ &\quad -4abcy + 8(a+b)cxy + 8y^2, \end{aligned} \tag{6}$$

with irreducible invariant algebraic curve

$$x^2(-a+x)(-b+x) + (cx^2+y)^2 = 0, \tag{7}$$

can be transformed to the system

$$\begin{aligned} \dot{x} &= -3\,(a+b)\,cx + 4abcx^2 - 4y + (a+b)\,xy, \\ \dot{y} &= \left(a\left(b + 4bc^2\right) - (3a^2+3b^2)/2\right)x + 2\,(a+b)\,cy + ab\,(a+b)\,x^2 + \\ &\quad +4abcxy + 2\,(a+b)\,y^2, \end{aligned}$$

with the invariant algebraic curve

$$-\frac{(a-b)^2}{4ab} + ab\left(x - \frac{a+b}{2ab}\right)^2 + (y+c)^2 = 0, \tag{8}$$

by the transformation $(x, y) \to (1/x, y/x^2)$.

The algebraic curves (7) and (8) both give limit cycles when $abc \neq 0$, $a \neq b$, $ab > 0$, *and* $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$, *and the transformation maps the one limit cycle onto the other.* ∎

We now apply the transformation to the system

$$\dot{x} = 2(1 + 2x - 2ax^2 + 6xy), \qquad \dot{y} = 8 - 3a - 14ax - 2axy - 8y^2, \tag{9}$$

which has the algebraic curve

$$\frac{1}{4} + x - x^2 + ax^3 + xy + x^2y^2 = 0, \tag{10}$$

which defines an algebraic limit cycle of degree 4 for $0 < a < 1/4$. This is the system discovered by Chavarriga et al. [2, 3].

By first shifting one of the singular points of system (9) to the origin and then applying transformation (4) we get the following result.

Theorem 2. *System*

$$\begin{aligned} \dot{x} &= 28x - \tfrac{12}{\alpha+4}y^2 - 2(\alpha^2 - 16)(12 + \alpha)x^2 + 6(3\alpha - 4)xy, \\ \dot{y} &= (32 - 2\alpha^2)x + 8y - (\alpha + 12)(\alpha^2 - 16)xy + (10\alpha - 24)y^2, \end{aligned} \tag{11}$$

has an irreducible algebraic invariant curve of degree 5 *given by*

$$\begin{aligned} &x^2 + (16 - \alpha^2)x^3 + (\alpha - 2)x^2y + \frac{1}{(4+\alpha)^2}y^4 - \frac{6}{(4+\alpha)^2}y^5 - \frac{2}{4+\alpha}xy^2 + \\ &\frac{(\alpha-4)(12+\alpha)}{4}x^2y^2 + \frac{(12+\alpha)}{4+\alpha}xy^4 + \frac{8-\alpha}{4+\alpha}xy^3 = 0. \end{aligned} \tag{12}$$

For $\alpha \in (3\sqrt{7}/2, 4)$ *the curve (12) contains an algebraic limit cycle of degree* 5.

Proof: Let $a = 16 - \alpha^2$. When we make the change of coordinates

$$(x, y) = \left(\frac{u}{v^2} - \frac{1}{\alpha+4}, \frac{1}{v} + \frac{\alpha - 2}{2} \right), \tag{13}$$

multiply by v, and replace (u, v) again with (x, y), the system (9) becomes (11). The curve (12) is obtained from (10) by means of the same change of coordinates and multiplication by v^6. The irreducibility of (12) follows from the irreducibility of (10).

Since the curve (10) contains an algebraic limit cycle for $a \in (0, 1/4)$, one may easily check, that the above oval does not intersect the singular line of the transformation (13), so the theorem follows. ∎

By changing coordinates to a different singular point and applying a linear transformation which preserves the form (5) we obtain the following.

Theorem 3. *System*

$$\dot{x} = 28(\beta-30)\beta x + y + 168\beta^2 x^2 + 3xy,$$
$$\dot{y} = 16\beta(\beta-30)\left(14(\beta-30)\beta x + 5y + 84\beta^2 x^2\right) + 24(17\beta-6)\beta xy + 6y^2, \quad (14)$$

has an irreducible algebraic invariant curve of degree 6 *given by*

$$-7y^3 + 3(\beta-30)^2\beta y^2 + 18(\beta-30)(-2+\beta)\beta xy^2 + 27(\beta-2)^2\beta x^2y^2 +$$
$$24(\beta-30)^3\beta^2 xy + 144(\beta-30)(\beta-2)^2\beta^2 x^3 y + 48(\beta-30)^4\beta^3 x^2 +$$
$$576(\beta-30)^2(-2+\beta)^2\beta^3 x^4 - 432(\beta-2)^2\beta^2(3+2\beta)x^4 y -$$
$$3456(\beta-30)(-2+\beta)^2\beta^3(3+2\beta)x^5 + 3456(\beta-2)^2\beta^3(12+\beta)(3+2\beta)x^6 +$$
$$24(\beta-30)^2\beta^2(9\beta-4)x^2y + 64(\beta-30)^3\beta^3(9\beta-4)x^3 = 0. \quad (15)$$

For $\beta \in (3/2, 2)$ *the curve (15) contains an algebraic limit cycle of degree* 6.

Proof: Let $a = (4-\beta^2)/7$. When we make the change of coordinates

$$(x, y) = \left(\frac{v + 4u\beta\left(-30 + 3u\left(-2+\beta\right)+\beta\right)}{12u^2\beta(\beta^2-4)}, \frac{30-\beta-u\left(8+3\beta\right)}{14u}\right), \quad (16)$$

multiply by $-21\beta u/2$, and replace (u, v) again with (x, y), system (9) becomes (14). The curve (15) is obtained from (10) by means of the same change of coordinates and multiplication by $2016\beta^2\left(\beta^2-4\right)^2 u^6$. The irreducibility of (12) is now obvious.

Since the curve (10) contains an algebraic limit cycle for $a \in (0, 1/4)$, the theorem follows in a way similar to the last part of Theorem 2. ∎

3 Invariant algebraic separatrix saddle–loops of degrees 3 and 5 surrounding a focus

When the system is integrable it is easy to find examples of a period annulus whose boundary is given by a saddle–loop. We are only interested in the case where the saddle–loop has a focus in its interior.

In this section, we classify all invariant algebraic saddle–loops of degree 3 surrounding a focus, and give an example of degree 5. We shall use the following classification:

Theorem 4. ([1], Theorem 8) *A quadratic system (1) which has an irreducible cubic algebraic invariant curve* $f(x, y) = 0$ *is affine–equivalent, scaling the variable* t *if necessary, to one of the following systems:*

(i) $f(x,y) = \sum_{i+j=0}^{2} m_{ij}x^i y^j + xy(\alpha x + \beta y)$, *with* $\alpha\beta \neq 0$, *and*
$\dot{x} = \partial f/\partial y$, $\dot{y} = -\partial f/\partial x$.

(ii) $f(x,y) = \sum_{i+j=0}^{2} m_{ij}x^i y^j + x(\alpha x^2 + \beta xy + \gamma y^2)$, *with* $\beta^2 < 4\alpha\gamma$, *and*
$\dot{x} = \partial f/\partial y$, $\dot{y} = -\partial f/\partial x$.

(iii) $f(x,y) = m_{00} + m_{10}x + m_{02}y^2 + x^3$ *and*
$\dot{x} = 2y(-b_{20}m_{02} + b_{02}x)/3, \dot{y} =$
$= (3b_{02}m_{00} + b_{20}m_{02}m_{10} + 2b_{02}m_{10}x)/(3m_{02}) + b_{20}x^2 + b_{02}y^2$.

(iv) $f(x,y) = y^2 + x^3$ *and*
$\dot{x} = 2(b_{01}x - y + b_{11}x^2 + b_{02}xy)/3, \quad \dot{y} = b_{01}y + x^2 + b_{11}xy + b_{02}y^2$.

(v) $f(x,y) = x^2 \pm y^2 + x^3$ *and*
$\dot{x} = 2(b_{01}x + (B_{02} \mp B_{20})y + b_{01}x^2 + 3B_{02}xy)$,
$\dot{y} = 2(B_{20} \mp B_{02})x + 2b_{01}y + 3B_{20}x^2 + 3b_{01}xy + 9B_{02}y^2$.

(vi) $f(x,y) = y + x^3$ *and*
$\dot{x} = (a + bx + cx^2 + dxy)/3, \quad \dot{y} = by - ax^2 + cxy + dy^2$.

(vii) $f(x,y) = 1 + xy + x^3$ *and*
$\dot{x} = 3a_{11} + a_{10}x + a_{20}x^2 + a_{11}xy, \quad \dot{y} = 3a_{20} - 9a_{11}x - a_{10}y - 3a_{10}x^2 + 2a_{20}xy + 2a_{11}y^2$.

(viii) $f(x,y) = 1 + x + x^2y$ *and*
$\dot{x} = -a_{11}/2 - (b_{01}/2 + a_{11}/4)x + (a_{11} - 4b_{11} - 2b_{01})x^2/8 + a_{11}xy$,
$\dot{y} = (4b_{11} - 2b_{01} + a_{11})/8 + b_{01}y + b_{11}xy - 2a_{11}y^2$.

(ix) $f(x,y) = m_{10}x + m_{01}y + x^2y$, *with* $m_{01} \neq 0$, *and*
$\dot{x} = \frac{1}{2}(2a_{20}m_{01} + b_{02}m_{10}) + a_{20}x^2$,
$\dot{y} = -(a_{20}m_{10} + \frac{1}{2m_{01}}b_{02}m_{10}^2) - (2a_{20} + \frac{1}{2m_{01}}b_{02}m_{10})xy + b_{02}y^2$.

(x) $f(x,y) = m_{00} + m_{10}x + m_{01}y + x^2y$, *with* $m_{00} \neq 0$, *and*
$\dot{x} = a_{20}m_{01} + \frac{1}{2m_{00}}b_{01}m_{01}m_{10} - \frac{1}{2}b_{01}x + a_{20}x^2$,
$\dot{y} = -a_{20}m_{10} - \frac{1}{2m_{00}}b_{01}m_{10}^2 + b_{01}y - (2a_{20} + \frac{b_{01}}{2m_{00}}m_{10})xy + \frac{1}{m_{00}}b_{01}m_{01}y^2$.

(xi) $f(x,y) = \mp x^2 + y^2 + x^2y$ *and*
$\dot{x} = 2ax + 2y + x^2 \mp axy, \quad \dot{y} = 2(1 \mp y)(\pm x + ay)$.

(xii) $f(x,y) = m_{10}x + m_{01}y + m_{02}y^2 + x^2y$ *and*
$\dot{x} = \partial f/\partial y$, $\dot{y} = -\partial f/\partial x$.

(xiii) $f(x,y) = -x^2/(2a) + 2axy + y^2 + x^2y$, *with* $a \neq 0$, *and*
$\dot{x} = (2a_{20} + 4a^3a_{20} + b_{11})x - 2a^2b_{11}y + 2a^2a_{20}x^2 - (2a_{20} + b_{11})axy$,
$\dot{y} = -ab_{11}x + (2a_{20} + 4a^3a_{20} + b_{11} + 4a^3b_{11})y + 2a^2b_{11}xy - 2(2a_{20} + b_{11})ay^2$.

(xiv) $f(x,y) = y^2 + x(x^2 + axy + by^2)$, *with* $b \neq 0$, $a^2 - 4b \neq 0$, *and*
$\dot{x} = 6x - (a + 4k)y + 2(3b - a^2 - ak)x^2 - b(a + 4k)xy$,
$\dot{y} = 9y + 6kx^2 + 2(3b - a^2 + 2ak)xy + b(2k - a)y^2$.

In order to simplify the notation we shall use slightly different symbols for the parameters than the ones used in Theorem 4.

Theorem 5. *A quadratic system (1) which has an irreducible cubic algebraic invariant curve which contains a saddle–loop is affine–equivalent, scaling the variable t if necessary, to one of the following systems.*

(a) The system

$$\begin{aligned} \dot{x} &= 2(bx + bx^2 + (A + B)y + 3Bxy), \\ \dot{y} &= 2(A + B)x + 3Ax^2 + 2by + 3bxy + 9By^2, \end{aligned} \tag{17}$$

with invariant curve $x^2 + x^3 - y^2 = 0$, *for*

(a.1) $(A + B)^2 - b^2 > 0$,

(a.2) $b^2 + 12B(2B - A) < 0$, *or* $b^2 + 12B(2B - A) \geq 0$ *and* $b^2 - 6B(A + B) - b\sqrt{b^2 + 12B(2B - A)} > 0$.

This is system (v) of Theorem 4 with b_{01}, B_{02} and B_{20} replaced by b, A and B, respectively.

(b) The system

$$\begin{aligned} \dot{x} &= (2A + 4a^3A + B)x - 2a^2By + 2a^2Ax^2 - (2A + B)axy\ , \\ \dot{y} &= -aBx + (2A + 4a^3A + B + 4a^3B)y + 2a^2Bxy - 2(2A + B)ay^2 \end{aligned} \tag{18}$$

with invariant curve $-x^2/(2a) + 2axy + y^2 + x^2y = 0$, *for*

(b.1) $a < -1/2^{1/3}$,

(b.2) $4A^2 + 8a^3A^2 + 4AB + 8a^3AB + B^2 > 0$,

(b.3) $A(2A + B) < 0$ *or* $A(2A + B) \geq 0$ *and* $\sqrt{-\frac{2}{a}} < \frac{-4A^2 - 8a^3A^2 - 4AB - 8a^3AB - B^2}{4a^2A(2A+B)} + 2a$.

This is system (xiii) of Theorem 4 with a_{20} and b_{11} replaced by A and B, respectively.

Proof: Systems (i), (ii) and (xii) are Hamiltonian, and system (iii) has a rational first integral. The curves appearing in systems (vi), (vii), (viii), (ix) and (x) are linear in y. The curves in systems (iv) and (xiv) have a cusp. Finally one may check, that curve of system (xi) does not contain loops. This leaves us with the two cases (v) and (xiii) to consider.

Consider first system (v): in order that our curve contains a loop we must choose a "$-$" sign in the formula from Theorem 4. The field has four critical points $p_0 = (0,0)$, $p_1 = (-2(A+B)/(3A), 2b/(9A))$, p_3 and p_4.

The point p_0 is located at the multiple point of the curve, and condition (a.1) implies that it is a saddle. The points p_2 and p_3 also lie on the invariant curve, and the condition (a.2) guarantees that they do not lie on the loop of the curve.

For system (xiii) the field has four critical points (taking into account their multiplicities), two of which are located on the invariant algebraic curve, namely

$$p_0 = (0,0), \quad p_1 = \left(\frac{-2a(A+B)}{2A+B} - \frac{2A+B}{4a^2A}, \frac{1}{2a} + \frac{4a^2A(A+B)}{(2A+B)^2} \right).$$

The other two singular points remain outside the curve, or coincide with one of the first two.

Condition (b.1) means that the invariant algebraic curve contains a loop: Indeed, the equation $\frac{-x^2}{2a} + 2axy + y^2 + x^2y = 0$, for $a < -1/2^{1/3}$, has two connected components, one with a loop with multiple point at p_0, and the second one nonsingular.

Condition (b.2) means that p_0 is a saddle, and condition (b.3) guarantees that p_1 is located either on the component of the curve not containing the loop, or on the component with the loop, but not on the loop itself. ∎

Theorem 6. *Let* $-\sqrt{10} < f < \sqrt{10}$ *and* $r = \sqrt{10 - f^2}$*, then the system*

$$\begin{aligned}
&\dot{x} = -1 + x + xy,\\
&\dot{y} = (f+1)(6-3f-r)/24 + (f+1)(1+15f-4f^2+(3+f)r)x/48-\\
&(3(4+f)-r)y/12 - (f+1)^2(13-4f+r)x^2/48+\\
&(f+1)(5-2f-r)xy/24 + 2y^2,
\end{aligned}$$

has an irreducible algebraic invariant phase curve of degree 5 *given by*

$$\begin{aligned}\varphi(x,y) = {} & 64(-29350+22709f+792f^2-3503f^3+604f^4+ \\ & r(6047+f(-4234+(1271-168f)f)))+ \\ & 160(6043-3229f-3743f^2+4055f^3-1328f^4+146f^5+ \\ & (1+f)r(-1382+f(1525+f(-524+57f))))x+ \\ & 40(35393+2547f-35526f^2+8910f^3+7461f^4-3681f^5+448f^6 \\ & -(1+f)r(7429+f(-6622+f(2072+f(-314+27f)))))x^2+ \\ & 20(1+f)^2(-7777+26686f-31976f^2+16642f^3-3919f^4+344f^5+ \\ & (-1+f)r(-4661+f(3609+f(-887+67f))))x^3- \\ & 10(1+f)^3(13-4f+r)^3(-5+2f+r)^2x^4+ \\ & (1+f)^4(13-4f+r)^4(-5+2f+r)x^5+ \\ & 1920(1726-2111f+808f^2-99f^3+(-173-5(-14+f)f)r)y+ \\ & 960(6043-4094f-804f^2+950f^3-151f^4+ \\ & 2r(-691+f(503+f(-157+21f))))xy- \\ & 480(1+f)(13-4f+r)^2(-5+2f+r)(-14+5f+3r)x^2y+ \\ & 240(1+f)^2(13-4f+r)^3(-5+2f+r)x^3y+ \\ & 5760(13-4f+r)^2(-14+5f+3r)xy^2 = 0\end{aligned}$$

with cofactor $5y$. *For* $f \in \left(-11/\sqrt{13}, -3\right)$ *the invariant algebraic curve* $\varphi = 0$ *forms a saddle–loop surrounding the focus at the origin.*

Proof: From the expression for φ, we see that it is quadratic in y. Tedious calculation shows that the discriminant of this quadratic is a polynomial Δ of degree 6 in x. After the substitution $X = x - 2/(1+f)$, $\Delta = (f+1)X^3Q_3(X)$, where Q_3 is a polynomial of degree 3, and

$$\begin{aligned}Q_3(0) = {} & 80(f-1)^2(2985980+\sqrt{10-f^2}(-1639183+f(3021297+ \\ & f(-2077726+f(678778+f(-106179+6373f)))))+f(-6791799+ \\ & f(7165501+f(-4113758+f(1303614+f(-213787+14169f)))))).\end{aligned}$$

It can be shown that this expression is nonzero for $f^2 \neq 1$, and hence the discriminant cannot be a square, and so φ is irreducible for $f \in \left(-11/\sqrt{13}, -3\right)$.

The system has four singular points, p_0 located at the origin, and three others belonging to the curve $\varphi = 0$. For $f = -3$ the origin undergoes a bifurcation, and is a focus for $f \in (-53/17, -3)$. The analysis of the phase portrait show us, that for $f \in \left(-11/\sqrt{13}, -3\right)$ the curve $\varphi = 0$ contains a saddle–loop surrounding p_0. For $f = -11/\sqrt{13}$ two singular points lying on the curve, the saddle and the node coincide and a saddle–node bifurcation takes place. For smaller f there appears a singular point (saddle) on the part of the loop surrounding the origin, and the double point of our curve is a node. ∎

4 Open questions

Natural question is, if that kind of a transformations can be applied to other known examples of quadratic polynomial systems with algebraic limit cycles? Of course, the authors have tried it. As far as we know, the only other case admitting such transformation is the example of Yablonskii, but the transformation changes the degree of a cycle from 2 to 4, producing the example of Yablonskii from the known example of an invariant ellipse.

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