OPERADIC DESCRIPTION OF STEENROD OPERATIONS

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ABSTRACT. In this paper, we introduce the Adem-Cartan operad related to level algebras and $\mathcal{E}_\infty$-structures. This operad enables an explicit proof of the Cartan-Adem relations for the Steenrod squares (at the prime $p = 2$). Moreover, it gives an operadic approach to secondary cohomological operations.

INTRODUCTION

The Steenrod Algebra $\mathcal{A}_p$ is one of the main computational tool of homotopy theory. Steenrod's operations were first introduced by N.E. Steenrod in 1947 [22] for $p = 2$ and for an odd prime in 1952 [23]. The relations between these cohomological operations were determined by J. Adem [2] and H. Cartan [6]. Cartan's proof relies on the computation of the singular cohomology of the Eilenberg-Mac Lane spaces at the prime p. Adem's proof is based on the computation of the homology of the symmetric group Σ_{p^2} acting on p^2 elements at the prime p. Later on, J. P. May [18] generalized Steenrod operations and gave a complete algebraic treatment of these operations. An operadic formulation of Steenrod operations seemed to be hidden behind this approach, as pointed out in [15].

Our main goal is to give a complete operadic definition of the Steenrod squares, for $p = 2$. This approach gives a totally explicit proof of the Adem-Cartan relations. As a by-product of these computations, we also obtain an operadic definition of secondary cohomological operations. These secondary operations introduced by J.F. Adams [1] are an efficient tool to deal with realisability problems of unstable modules over $\mathcal{A}_2$.

The framework of this paper is the theory of operads, which gives us a good context to deal with structures up to homotopy. For our purpose, we want to control the commutativity "up to homotopy" in order to deal with

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cup-i products (cochains analogues of the Steenrod squares) and to find relations between them. One of this operad is $\mathcal{E}_\infty$, a cofibrant replacement of the operad $\mathcal{C}$om whose algebras are commutative and associative algebras: using J. P. May's approach, one can prove that the cohomology of an $\mathcal{E}_\infty$-algebra is an unstable algebra over $\mathcal{B}_2$, the algebra of generalized Steenrod powers or extended Steenrod algebra (see [15]). Since the combinatorics of $\mathcal{E}_\infty$ is not simple (see [3] and [20]) and since the associativity of the product is not involved in Cartan-Adem relations ([2]), we relax the associativity condition. That is to say, the operad $\mathcal{C}$om is replaced by the operad $\mathcal{L}$ev whose algebras (called *level algebras*) are commutative algebras which are not necessarily associative but satisfy the following 4-terms relation:

$$(a * b) * (c * d) = (a * c) * (b * d).$$

Like in the classical case, we define an *unstable level algebra over* $\mathcal{B}_2$ as a module over $\mathcal{B}_2$ satisfying the instability condition as well as the Cartan formula. A specific cofibrant $\mathcal{L}\mathrm{ev}^{AC}$ is built, making the following diagram commutative

$$\begin{array}{ccccc} & & & & \mathcal{E}_\infty \\ & & & \nearrow & \downarrow \\ \mathcal{L}\mathrm{ev}^{AC} & \longrightarrow & \mathcal{L}\mathrm{ev} & \longrightarrow & \mathcal{C}\mathrm{om} \end{array}$$

This operad $\mathcal{L}\mathrm{ev}^{AC}$ has its importance in the following theorem:

Theorem 3.4: *Let A be a graded algebra over the operad $\mathcal{L}\mathrm{ev}^{AC}$ then A is an unstable level algebra over the extented Steenrod algebra.*

The proof relies on an explicit computation of generalized Cartan-Adem relations lying at the cochain level. They are in fact relations between composition of cup-i products (the cup-i products are in $\mathcal{L}\mathrm{ev}^{AC}(2)$ and control the commutativity) appearing in $\mathcal{L}\mathrm{ev}^{AC}(4)$ as boundaries of operadic generators G_n^m. Furthermore, these G_n^m's define the operadic secondary cohomological operations (see theorem 4.3.1) .

The paper is presented as follows. Section 1 contains the background needed. In Section 2, level algebras are defined, and the operad $\mathcal{L}\mathrm{ev}^{AC}$ is built in subsection 2.2 and described in theorem 2.2.6. We introduce also a specific cofibrant replacement $\mathcal{L}\mathrm{ev}_\infty$ of $\mathcal{L}$ev and explain the link with $\mathcal{E}_\infty$. Section 3 is devoted to the fondamental theorem 3.4 and its corollaries. Section 4 is concerned with secondary cohomological operations and section 5 is devoted to proofs of technical lemmas stated in the different sections.

1. Recollections

The ground field is $\mathbb{F}_2$. The symbol Σ_n denotes the symmetric group acting on n elements. Any p-cycle $\sigma \in \Sigma_n$ is written $(\sigma(1)\ldots\sigma(p))$.

1.1. **Operads.** ([9], [10], [15], [16]) In this article, a vector space means a differential $\mathbb{Z}$-graded vector space over $\mathbb{F}_2$, where the differential is of degree 1. A Σ_n*-module* is a $\mathbb{F}_2[\Sigma_n]$-differential graded module. A Σ*-module* $\mathcal{M} = \{\mathcal{M}(n)\}_{n>0}$ is a collection of Σ_n-modules. Any Σ_n-module M gives rise to a Σ-module $\mathcal{M}$ by setting $\mathcal{M}(q) = 0$ if $q \neq n$ and $\mathcal{M}(n) = M$.

An *operad* is a Σ-module $\{\mathcal{O}(n)\}_{n>0}$ such that $\mathcal{O}(1) = \mathbb{F}_2$, together with composition products:

$$\begin{array}{rcl} \mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \ldots \otimes \mathcal{O}(i_n) & \longrightarrow & \mathcal{O}(i_1 + \ldots + i_n) \\ o \otimes o_1 \otimes \ldots \otimes o_n & \mapsto & o(o_1, \ldots, o_n). \end{array}$$

These compositions are subject to associativity conditions, unitary conditions and equivariance conditions with respect to the action of the symmetric group. There is another definition of operads via $\circ_i$ operations

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1),$$

where $p \circ_i q$ is p composed with $n-1$ copies of the unit $1 \in \mathcal{O}(1)$ and with q at the i-th position.

The forgetful functor from the category of operads to the category of Σ-modules has a left adjoint : the free operad functor, denoted by $\mathcal{F}$ree.

An *algebra over an operad* $\mathcal{O}$ or a $\mathcal{O}$*-algebra* A is a vector space together with evaluation maps

$$\begin{array}{rcl} \mathcal{O}(n) \otimes A^{\otimes n} & \longrightarrow & A \\ o \otimes a_1 \otimes \ldots \otimes a_n & \mapsto & o(a_1, \ldots, a_n) \end{array}$$

These evaluation maps are subject to associativity conditions and equivariance conditions.

1.2. **Homotopy of operads.** ([4], [11]) The category of operads is a closed model category. Weak equivalences are quasi-isomorphisms and fibrations are epimorphisms. Cofibrations can be defined by the left lifting property with respect to the acyclic fibrations. For background material on closed model categories we refer to [8], [13] and [21].

1.2.1. **Operadic cellular attachment.** Any morphism of operads

$$\mathcal{P} \longrightarrow \mathcal{Q}$$

can be factorized by a cofibration ($\rightarrowtail$) followed by an acyclic fibration ($\xrightarrow{\sim}\!\!\!\twoheadrightarrow$). This factorization can be realized using the inductive process of attaching cells (the category of operads is cofibrantly generated [11]). An operad is *cofibrant* if the morphism from the initial object $\mathcal{F}\mathrm{ree}(0)$ to the operad is a cofibration. In order to produce a *cofibrant replacement* to an operad $\mathcal{O}$, one applies the inductive process of attaching cells to the canonical morphism $\mathcal{F}\mathrm{ree}(0) \longrightarrow \mathcal{O}$.

Let S_p^n be the free Σ_p-module generated by δt in degree n considered as a Σ-module. Let D_p^{n-1} be the free Σ_p-module generated by t in degree $n-1$ and dt in degree n, the differential sending t to dt. We have a canonical inclusion $i_n : S_p^n \longrightarrow D_p^{n-1}$ of Σ-modules (sending δt to dt). Let $f : S_p^n \longrightarrow \mathcal{O}$ be a morphism of Σ-modules. The cell D_p^{n-1} is attached to $\mathcal{O}$ along the morphism f via the following push-out:

$$\begin{array}{ccc} \mathcal{F}\mathrm{ree}(S_p^n) & \xrightarrow{i_n} & \mathcal{F}\mathrm{ree}(D_p^{n-1}) \\ \downarrow{\scriptstyle \mathcal{F}\mathrm{ree}(f)} & & \downarrow \\ \mathcal{O} & \xrightarrow{i} & \mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(S_p^{n-1}). \end{array}$$

The main point of this process is that $f(\delta t)$, which was a cycle in $\mathcal{O}$, becomes a boundary in $\mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(S_p^{n-1})$.

By iterating this process of cellular attachment, one gets a *quasi-free extension*: $\mathcal{O} \xrightarrow{i} \mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(V)$.

If we forget the differential on V then $\mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(V)$ is the coproduct of $\mathcal{O}$ by a free operad over a free graded Σ-module V. A *quasi-free operad* is an operad which is free over a free Σ-module if we forget the differential.

1.2.2. **Proposition.** ([11])

a) *Any cofibration is a retract of a quasi-free extension.*
b) *Any morphism of operads $f : \mathcal{P} \longrightarrow \mathcal{Q}$ can be factorized by a quasi-free extension followed by an acyclic fibration:*

$$\mathcal{P} \overset{i}{\rightarrowtail} \mathcal{P}\coprod_\tau \mathcal{F}\mathrm{ree}(V) \underset{p}{\overset{\sim}{\twoheadrightarrow}} \mathcal{Q}$$

c) *Any operad $\mathcal{O}$ has a cofibrant replacement which is a quasi-free operad. And any cofibrant operad is a retract of a quasi-free operad.*

d) *Let* $f : \mathcal{P} \longrightarrow \mathcal{Q}$ *be a morphism of operads and*

$$\mathcal{P} \overset{i}{\rightarrowtail} \mathcal{P}\coprod_\tau \mathcal{F}\mathrm{ree}(V) \underset{p}{\overset{\sim}{\twoheadrightarrow}} \mathcal{Q}$$

its factorization. If $\mathcal{P}$ *is cofibrant, the operad* $\mathcal{P}\coprod_\tau \mathcal{F}\mathrm{ree}(V)$ *is a cofibrant replacement of* $\mathcal{Q}$*. Moreover if* $f(k)$ *is an acyclic fibration of* Σ_k*-modules for* $k < n$ *then* V *may be chosen such that* $V(k) = 0$ *for* $k < n$*.*

The following proposition will be fundamental for our applications.

1.2.3. **Proposition.** *Let* V *be a free graded* Σ_p*-module together with*

$$d_V : V \longrightarrow \mathcal{O}(p) \oplus V$$

such that $d_V + d_{\mathcal{O}}$ *is of square zero. Then if* V *is bounded above the morphism* $\mathcal{O} \to \mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(V)$ *is a cofibration.*

Proof. The boundness assumption is needed in order to build a map

$$\mathcal{O} \rightarrowtail \mathcal{O}\coprod_\tau \mathcal{F}\mathrm{ree}(V)$$

by induction on the degree of V, using the cellular attachment process. In order to add cells the assumption $(d_V + d_{\mathcal{O}})^2 = 0$ is needed. □

1.3. **Homotopy invariance principle.** ([4], [5] ,[11], [19]) Let $\mathcal{O}$ be a cofibrant operad. The category of $\mathcal{O}$-algebras is also a closed model category where weak equivalences are quasi-isomorphisms and fibrations are epimorphisms.

Recall that the category of vector spaces is a closed model category, where weak equivalences are quasi-isomorphisms and fibrations are epimorphisms. In this category all objects are fibrant and cofibrant.

1.3.1. **Theorem.** (**Homotopy invariance principle.**) *Let* $\mathcal{O}$ *be a cofibrant operad and assume that the morphism of vector spaces*

$$f : X \longrightarrow Y$$

is a weak equivalence between vector spaces. Assume that X *is a* $\mathcal{O}$*-algebra. Then* Y *is also provided with a* $\mathcal{O}$*-algebra structure. For any cofibrant replacement* $\tilde{X}$ *of* X*, there exists a sequence of quasi-isomorphisms of* $\mathcal{O}$*-algebras:*

$$X \longleftarrow \tilde{X} \longrightarrow Y.$$

1.3.2. **Corollary.** *Let $\mathcal{O}$ be a cofibrant operad. Let (C, d) be a bounded cochain complex which is a $\mathcal{O}$-algebra and let H be its cohomology. Then H is a $\mathcal{O}$-algebra. Moreover there is a sequence of quasi-isomorphisms of $\mathcal{O}$-algebras:*

$$C \longleftarrow \tilde{C} \longrightarrow H$$

where $\tilde{C}$ is a cofibrant replacement of C.

Proof. Set $X = (C, d)$ and $Y = H$. There exists $f : X \to H$ which is a weak equivalence. Since bounded cochain complexes are cofibrant in the category of vector spaces, we can apply the previous theorem. □

2. A cofibrant replacement of the operad $\mathcal{L}$ev

2.1. **The Operad $\mathcal{L}$ev.** A *level algebra* A is a vector space together with a commutative product $*$ (non necessarily associative) satisfying the relation

$$(a * b) * (c * d) = (a * c) * (b * d), \ \forall a, b, c, d \in A. \tag{2.1}$$

2.1.1. **Definition.** Let $\mathbb{F}_2$ be the trivial representation of Σ_2 (generated by the operation μ) and $R_{\mathcal{L}\text{ev}}$ be the sub-Σ_4-module of $\mathcal{F}\text{ree}(\mathbb{F}_2)(4)$ generated by the elements $(\text{Id} + \sigma) \cdot \mu(\mu, \mu)$ for all $\sigma \in \Sigma_4$. Then the operad $\mathcal{L}$ev is the operad

$$\mathcal{L}\text{ev} = \mathcal{F}\text{ree}(\mathbb{F}_2) / < R_{\mathcal{L}\text{ev}} > .$$

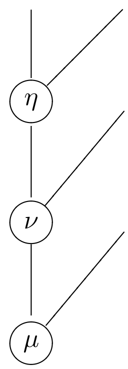
Algebras over this operad are level algebras.

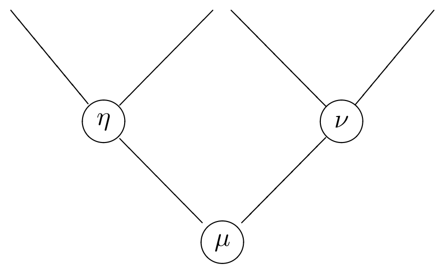
2.1.2. **Remark.** Since a commutative and associative algebra is trivially a level algebra, there is a morphism of operads

$$\mathcal{L}\text{ev} \longrightarrow \mathcal{C}\text{om},$$

where $\mathcal{C}$om denotes the operad defining commutative and associative algebras.

2.1.3. **Definition.** For any Σ_2-module M, the vector space $\mathcal{F}\text{ree}(M)(4)$ is a direct sum of two Σ_4-modules: the one indexed by trees of *shape 1* denoted by $\mathbb{T}_1$, that is the Σ_4-module generated by all the compositions $\mu \circ_1 \gamma$ for $\mu \in M$ and $\gamma \in \mathcal{F}\text{ree}(M)(3)$ and the one indexed by trees of *shape 2* denoted by $\mathbb{T}_2$, that is the Σ_4-module generated by all the compositions $\mu(\nu, \eta)$ for $\mu, \nu, \eta \in M$.





$\mathbb{T}_1$ $\mathbb{T}_2$

As an example, since there is only one generator $\mu \in \mathcal{F}\text{ree}(\mathbb{F}_2)(2)$, the dimension of $\mathbb{T}_1(\mathcal{F}\text{ree}(\mathbb{F}_2))$ is 12 and the dimension of $\mathbb{T}_2(\mathcal{F}\text{ree}(\mathbb{F}_2))$ is 3, a basis being for instance,

, ,

whereas the dimension of $\mathbb{T}_1(\mathcal{L}\text{ev})$ is 12 and the dimension of $\mathbb{T}_2(\mathcal{L}\text{ev})$ is 1.

2.2. **Construction of $\mathcal{L}\text{ev}^{AC}$.** This is one of the main step in order to state and prove the fondamental theorem 3.4. We build the *Adem-Cartan operad* $\mathcal{L}\text{ev}^{AC}$. The construction of this operad is explicit and is done by the process of attaching cells (see 1.2) to the standard bar resolution $\mathcal{E}$ of $\mathbb{F}_2[\Sigma_2]$.

2.2.1. **The standard bar resolution.** Let τ be the non trivial permutation of Σ_2. The standard bar resolution of Σ_2 over $\mathbb{F}_2$ is given by

$$\mathcal{E}^{-i} = \begin{cases} < e_i, \tau e_i >, & \text{if } i \geq 0 \\ 0, & \text{if } i < 0 \end{cases}$$

$$d(e_i) = e_{i-1} + \tau e_{i-1}, \text{with } e_{-1} = 0.$$

Denote again by $\mathcal{E}$ the free operad generated by the Σ-module $\mathcal{E}$. Since $\mathcal{L}\mathrm{ev}(2) = \mathbb{F}_2$, there is a fibration of operads:

$$p : \mathcal{E} \twoheadrightarrow \mathcal{L}\mathrm{ev}\,.$$

By definition of $\mathcal{L}\mathrm{ev}$, $p(n)$ is clearly a quasi-isomorphism for $n < 4$. Furthermore $p(4)$ is a quasi-isomorphism on pieces of shape 1. Hence we focus on pieces of shape 2.

2.2.2. **Notation.** –The element $\sigma \cdot e_k(e_i, e_j)$ in $\mathbb{T}_2^{-(i+j+k)}(\mathcal{E})$ is written $[(\sigma(1)\sigma(2))_i(\sigma(3)\sigma(4))_j]_k$. For $\{a, b, c, d\} = \{1, 2, 3, 4\}$ the picture is

$$[(ab)_i(cd)_j]_k = \qquad \in \mathcal{E}(4)^{-(i+j+k)}$$

By notation, $[\tau(ab)_i(cd)_j]_k$ (resp. $[(ab)_i\tau(cd)_j]_k$, resp. $\tau[(ab)_i(cd)_j]_k$) means $[(ba)_i(cd)_j]_k$ (resp. $[(ab)_i(dc)_j]_k$, resp. $[(cd)_j(ab)_i]_k$). Since

$$\mathrm{d}e_k(e_i, e_j) = (\mathrm{d}e_k)(e_i, e_j) + e_k(\mathrm{d}e_i, e_j) + e_k(e_i, \mathrm{d}e_j)$$

one has the differentiation rule

$$\begin{aligned} d([(ab)_i(cd)_j]_k) =& [(\mathrm{Id} + \tau)(ab)_{i-1}(cd)_j]_k \\ &+ [(ab)_i(\mathrm{Id} + \tau)(cd)_{j-1}]_k \\ &+ (\mathrm{Id} + \tau)[(ab)_i(cd)_j]_{k-1}, \end{aligned} \tag{2.2}$$

with the convention that if one parameter i, j or k is zero the corresponding -1 term is zero.

We use May's convention: for any integers i and j the symbol (i, j) denotes $\frac{(i+j)!}{i!j!} \in \mathbb{F}_2$, if $i \geq 0$ and $j \geq 0$ and $(i, j) = 0$ otherwise. If the 2-adic expansion of i and j is $i = \sum a_k 2^k$ and $j = \sum b_k 2^k$ then $(i, j) = 0 \in \mathbb{F}_2$ if and only if there exists k such that $a_k = b_k = 1$.

2.2.3. **Definition.** We define some elements $u_n^m \in \mathcal{E}(4)^{-n}$ which will be responsible for Adem and Cartan relations. More explicitly, $u_n^m \in$

$\mathbb{T}_2(\mathcal{E})^{-(i+j+0)}$ is of the form $u_n^m(1,2,3,4) = \sum_{k,\sigma} \lambda_{k,\sigma}^m \sigma \cdot e_0(e_k, e_{n-k})$ also written $u_n^m(1,2,3,4) = \sum_{k,\sigma} \lambda_{k,\sigma}^m (\sigma(1)\sigma(2))_k(\sigma(3)\sigma(4))_{n-k}$. By definition

$$u_0^0(1,2,3,4) = (13)_0(42)_0,$$
$$u_n^0(1,2,3,4) = 0,\ n > 0$$

and for any m such that $2^k \leq m \leq 2^{k+1}-1$,

$$\begin{aligned} u_n^m(1,2,3,4) = \tau^m[\sum_{i=0}^{2^{k+1}-1} &\sum_{0\leq 2^{k+1}\delta - i\leq n} \\ &(n-m+i, m-1)(i,m)(13)_{2^{k+1}\delta-i}(42)_{n+i-2^{k+1}\delta} \\ &+ (n-m+i,m-1)(i-1,m)(13)_{2^{k+1}\delta-i}(24)_{n+i-2^{k+1}\delta}] \end{aligned} \tag{2.3}$$

The symbol $[u_n^m]_x$ means $\sum_{k,\sigma} \lambda_{k,\sigma}^m \sigma \cdot e_x(e_k, e_{n-k})$.

2.2.4. **Proposition.** *The u_n^m's satisfy the following properties:*

$$\tau u_0^0(1,2,3,4) = u_0^0(4,3,2,1) \tag{2.4}$$
$$u_n^m = 0,\ \text{for } n < m \tag{2.5}$$
$$u_m^m = \tau^m[(13)_m(42)_0 + (13)_0(24)_m] \tag{2.6}$$
$$\begin{aligned} \mathrm{d}u_{n+1}^{m+1}(1,2,3,4) =& u_n^{m+1}(1,2,3,4) + u_n^{m+1}(3,4,1,2) \\ &+ \tau u_n^m(1,2,3,4) + u_n^m(2,1,4,3),\ \forall m,n \geq 0. \end{aligned} \tag{2.7}$$

The proof of this proposition is postponed to last section.

2.2.5. **Remark.** By the definition of u_n^m we have

$$\begin{aligned} \tau u_n^1(1,2,3,4) &= \sum_k (13)_{2k}(42)_{n-2k} + \sum_l (13)_{2l+1}(24)_{n-2l-1} \\ &= (234)\cdot e_0(\psi(e_n)) \end{aligned}$$

where ψ is the coproduct in the standard bar resolution $\mathcal{E}$ of $\mathbb{F}_2$ (see [18], [3]).

2.2.6. **Theorem.** *There exists a cofibrant operad $\mathcal{L}\mathrm{ev}^{AC}$ satisfying the following properties:*

a) $\mathcal{L}\mathrm{ev}^{AC}(2) = \mathcal{E}$;
b) *there is a fibration $f : \mathcal{L}\mathrm{ev}^{AC} \to \mathcal{L}\mathrm{ev}$ such that $f(n)$ is a quasi-isomorphism for $n < 4$ and induces an isomorphism* $H^0(\mathcal{L}\mathrm{ev}^{AC}(n)) \simeq \mathcal{L}\mathrm{ev}(n)$;

c) *there are elements* $G_n^m \in \mathcal{L}\mathrm{ev}^{AC}(4)$ *of degree* $-n$*, for* $m \geq 1$ *and* $n \geq m$ *such that*

$$\begin{aligned} \mathrm{d}G_m^m = \phi^m(((\mathrm{Id} + (12)(34))G_{m-1}^{m-1} + (\mathrm{Id} + (13)(24))G_{m-1}^{m-2} \\ + \sum_s [u_{p+s}^s(1,2,3,4) + u_{p+s}^{s+1}(4,3,2,1)]_{m-1-p-s} \\ + \sum_s [u_{m-1-p+s}^s(1,3,2,4) + u_{m-1-p+s}^{s+1}(4,2,3,1)]_{p-s}), \end{aligned} \tag{2.8}$$

and for $n \geq m$

$$\begin{aligned} \mathrm{d}G_{n+1}^m = \phi^m(((\mathrm{Id} + (12)(34))G_n^m + (\mathrm{Id} + (13)(24))G_n^{m-2} \\ + \sum_s [u_{p+s}^s(1,2,3,4) + u_{p+s}^{s+1}(4,3,2,1)]_{n-p-s} \\ + \sum_s [u_{n-p+s}^s(1,3,2,4) + u_{n-p+s}^{s+1}(4,2,3,1)]_{p-s}) \end{aligned} \tag{2.9}$$

where p *is the integer part of* $\frac{m-1}{2}$*, and* ϕ *is the operation exchanging symbols* 2 *and* 3 *in the formula.*

Proof. We build a sequence of cofibrant differential graded operads $\mathcal{L}\mathrm{ev}_m^{AC}$ satisfying a) and b) by attaching cells G_n^m to $\mathcal{L}\mathrm{ev}_{m-1}^{AC}$ satisfying c). The initial operad $\mathcal{L}\mathrm{ev}_0^{AC}$ is $\mathcal{E}$. As a consequence $\mathcal{L}\mathrm{ev}^{AC} = \lim_m \mathcal{L}\mathrm{ev}_m^{AC}$ will satisfy a), b) and c). All the operads involved are cofibrant, thanks to proposition 1.2.3. In order to perform the process of attaching cells we need to prove that $\mathrm{d}^2 = 0$ (cf proposition 1.2.3). The main ingredient is the relation (2.7), except for the term $\mathrm{d}u_0^0$. But this term appears only if $p = 0$, i.e. if $m = 0$ or $m = 1$. Let $n \geq 1$,

$$\begin{aligned} (\mathrm{d})^2(G_{n+1}^1) =& \mathrm{d}((\mathrm{Id} + (13)(24))G_n^1 \\ &+ \sum_s [u_s^s(1,3,2,4)]_{n-s} + [u_n^1(4,3,2,1)]_0) \\ =& \sum_s \underbrace{[u_s^s(1,3,2,4)]_{n-1-s}}_{A_1} + [u_{n-1}^0(1,2,3,4)]_0 \\ &+ [u_{n-1}^1(4,3,2,1)]_0 + \sum_s \underbrace{[u_s^s(3,1,4,2)]_{n-1-s}}_{A_2} \\ &+ [u_{n-1}^0(3,4,1,2)]_0 + [u_{n-1}^1(2,1,4,3)]_0 \\ &+ \mathrm{d}(\sum_s [u_s^s(1,3,2,4)]_{n-s} + [u_n^1(4,3,2,1)]_0) \end{aligned}$$

If $n = 1$, we have

$$\begin{aligned}(\mathrm{d})^2(G_2^1) =& u_0^0(1,3,2,4) + u_0^0(3,1,4,2) + u_0^0(1,2,3,4) + u_0^0(3,4,1,2)\\ &+\mathrm{d}[u_0^0(1,3,2,4)]_1 + \mathrm{d}[u_1^1(1,3,2,4)]_0 + \mathrm{d}[u_1^1(4,3,2,1)]_0\end{aligned}$$

And we conclude, using relation (2.4) by computing

$$\begin{aligned}\mathrm{d}[u_1^1(4,3,2,1)]_0 =& u_0^0(1,2,3,4) + u_0^0(3,4,1,2)\\ \mathrm{d}[u_0^0(1,3,2,4)]_1 + \mathrm{d}[u_1^1(1,3,2,4)]_0 =& u_0^0(1,3,2,4) + u_0^0(4,2,3,1)\\ &+u_0^0(4,2,3,1) + u_0^0(3,1,4,2).\end{aligned}$$

If $n > 1$,

$$\begin{aligned}\mathrm{d}[u_s^s(1,3,2,4)]_{n-s} =& [u_s^s(1,3,2,4)]_{n-s-1} + \tau[u_s^s(1,3,2,4)]_{n-s-1}\\ &+\tau[u_{s-1}^{s-1}(1,3,2,4)]_{n-s} + [u_{s-1}^{s-1}(3,1,4,2)]_{n-s}\end{aligned}$$

which eliminates $A_1 + A_2$ and $\mathrm{d}[u_n^1(4,3,2,1)]_0$ eliminates the other terms by relation (2.7), since $u_{n-1}^0 = 0$.

The same holds for $(\mathrm{d})^2(G_n^2)$ since the definition is the same as $\mathrm{d}(G_n^1)$ except that 2 and 3 are exchanged. We have just to check that $(\mathrm{d})^2(G_2^2) = 0$.

$$\begin{aligned}(\mathrm{d})^2(G_2^2) =& \mathrm{d}((\mathrm{Id} + (12)(34))G_1^1)\\ &+ \underbrace{\mathrm{d}([u_0^0(1,2,3,4)_1 + [u_1^1(1,2,3,4)]_1 + [u_1^1(4,2,3,1)]_0))}_{X}\\ =& u_0^0(1,3,2,4) + u_0^0(1,2,3,4) + u_0^0(2,4,1,3)\\ &+ u_0^0(2,1,4,3) + X\\ =& 0\end{aligned}$$

For $m > 2$ the computation of $(\mathrm{d})^2(G_m^n)$ is different if $n = m$ or $n > m$, since we have set in the definition $G_{m-1}^m := G_{m-1}^{m-1}$. We compute only the second one (for m even), and leave the first one to the reader. By relation (2.9),

$$\begin{aligned}(\mathrm{d})^2 G_{n+1}^m = \mathrm{d}(&(\mathrm{Id} + (12)(34))G_n^m + (\mathrm{Id} + (13)(24))G_n^{m-2}\\ &\sum_s [u_{p+s}^s(1,2,3,4) + u_{p+s}^{s+1}(4,3,2,1)]_{n-p-s}\\ &\sum_s [u_{n-p+s}^s(1,3,2,4) + u_{n-p+s}^{s+1}(4,2,3,1)]_{p-s})\end{aligned}$$

On the one hand, the coefficient of G_{n-1}^m is $(\mathrm{Id} + (12)(34))^2 = 0$, the one of G_{n-1}^{m-2} is $2(\mathrm{Id} + (12)(34) + (13)(24) + (14)(23)) = 0$ and the one of G_{n-1}^{m-4} is $(\mathrm{Id} + (13)(24))^2 = 0$. On the other hand, we consider the coefficient of the terms $u_x^y(1,2,3,4), u_x^y(4,3,2,1)$, $u_x^y(2,1,4,3)$ and $u_x^y(3,4,1,2)$; let A be the coefficient given by $\mathrm{d}G_n^m$, B the one given by $\mathrm{d}G_n^{m-2}$, and C the one given by the others terms. That is

$$A = \sum_s \underbrace{[u^s_{p+s}(1,2,3,4)]_{n-1-p-s}}_{\alpha_1^s} + \underbrace{[u^{s+1}_{p+s}(4,3,2,1)]_{n-1-p-s}}_{\beta_1^s}$$
$$+ \sum_s \underbrace{[u^s_{p+s}(2,1,4,3)]_{n-1-p-s}}_{\alpha_2^s=(12)(34)\alpha_1^s} + \underbrace{[u^{s+1}_{p+s}(3,4,1,2)]_{n-1-p-s}}_{\beta_2^s=(12)(34)\beta_1^s}$$

$$B = \sum_s \underbrace{[u^s_{p-1+s}(1,2,3,4)]_{n-p-s}}_{\gamma_1^s} + \underbrace{[u^{s+1}_{p-1+s}(4,3,2,1)]_{n-p-s}}_{\delta_1^s}$$
$$+ \sum_s \underbrace{[u^s_{p-1+s}(3,4,1,2)]_{n-p-s}}_{\gamma_2^s=(13)(24)\gamma_1^s} + \underbrace{[u^{s+1}_{p-1+s}(2,1,4,3)]_{n-p-s}}_{\delta_2^s=(13)(24)\delta_1^s}$$

Using relation (2.7) and the rules of differentiation (2.2)

$$\begin{aligned}\mathrm{d}[u^s_{p+s}(1,2,3,4)]_{n-p-s} = & \\ & \underbrace{[u^s_{p+s}(1,2,3,4)]_{n-p-1-s}}_{\alpha_1^s} + [\tau u^s_{p+s}(1,2,3,4)]_{n-1-p-s} \\ & + \underbrace{[u^s_{p+s-1}(1,2,3,4)]_{n-p-s}}_{\gamma_1^s} + \underbrace{[u^s_{p+s-1}(3,4,1,2)]_{n-p-s}}_{\gamma_2^s} \\ & + [\tau u^{s-1}_{p+s-1}(1,2,3,4)]_{n-p-s} + \underbrace{[u^{s-1}_{p+s-1}(2,1,4,3)]_{n-p-s}}_{\alpha_2^{s-1}}\end{aligned}$$

As a consequence

$$C_1 = \sum_s \mathrm{d}[u^s_{p+s}(1,2,3,4)]_{n-p-s} = \sum_s \alpha_1^s + \alpha_2^s + \gamma_1^s + \gamma_2^s$$

Computing the other part of C, we get

$$C_2 = \sum_s \mathrm{d}[u^{s+1}_{p+s}(4,3,2,1)]_{n-p-s} = \sum_s \beta_1^s + \beta_2^s + \delta_1^s + \delta_2^s$$

Since $C = C_1 + C_2$ we get $A+B+C=0$. The other terms in $(1,3,2,4)$, $(4,2,3,1)$, $(2,4,1,3)$ and $(3,1,4,2)$ are computed in the same way. Hence $(\mathrm{d})^2 = 0$.

Let us prove that the operads $\mathcal{L}\mathrm{ev}_m^{AC}$ satisfy b). Proposition 1.2.2 implies that $\mathcal{L}\mathrm{ev}_m^{AC}(n) \to \mathcal{L}\mathrm{ev}(n)$ is a quasi-isomorphism for $n < 4$, since $\mathcal{E}$ satisfies this property (see 2.2.1). The same holds for $\mathcal{L}\mathrm{ev}^{AC}$. Consider the map $\mathcal{L}\mathrm{ev}_1^{AC}(4) \to \mathcal{L}\mathrm{ev}(4)$: the cells G_1^1 are attached in order to "kill" the extra terms in $\mathcal{E}(4)^0$; let $f_1 = [(12)_0(34)_0]_0$, $f_2 = [(13)_0(24)_0]_0$ and $f_3 = [(14)_0(32)_0]_0$ be in $\mathcal{E}(4)^0$. The generators of shape 2 in $\mathcal{E}(4)^0$ are $\sigma \cdot f_1, \sigma \cdot f_2$ and $\sigma \cdot f_3$, where σ is in the dihedral group acting on the vertices of the trees of shape 2. Any element $\sigma \cdot f_i$ is in the same cohomology class as f_i according to the differential in $\mathcal{E}(4)$. Furthermore, $\mathrm{d}G_1^1 = [(12)_0(43)_0]_0 + [(13)_0(42)_0]_0$ implies that f_1 and f_2 are in the same cohomology class, and so is f_3 (compute $(34) \cdot \mathrm{d}G_1^1$). As a consequence $H^0(\mathcal{L}\mathrm{ev}_1^{AC}(4)) \to \mathcal{L}\mathrm{ev}(4)$ is an isomorphism. Moreover, the relations defining $\mathcal{L}\mathrm{ev}$ are generated by a Σ_4-module, thus $H^0(\mathcal{L}\mathrm{ev}_1^{AC}(n)) \to \mathcal{L}\mathrm{ev}(n)$ is an isomorphism, for all n. If $m > 1$, we do not introduce cells in degree -1, hence $H^0(\mathcal{L}\mathrm{ev}_m^{AC}(n)) \to \mathcal{L}\mathrm{ev}(n)$ is an isomorphism for all n. The same holds for $\mathcal{L}\mathrm{ev}^{AC}$. □

2.3. **Definition.** A *Adem-Cartan algebra* is an algebra over $\mathcal{L}\mathrm{ev}^{AC}$.

2.4. **A cofibrant replacement of the operad Lev.**

2.4.1. **Proposition.** *There exists a cofibrant replacement of $\mathcal{L}\mathrm{ev}$, denoted by $\mathcal{L}\mathrm{ev}_\infty$ such that*

$$\mathcal{L}\mathrm{ev}^{AC} \rightarrowtail \mathcal{L}\mathrm{ev}_\infty = \mathcal{L}\mathrm{ev}^{AC} \textstyle\coprod_\tau \mathcal{F}\mathrm{ree}(V) \overset{\sim}{\twoheadrightarrow} \mathcal{L}\mathrm{ev}$$

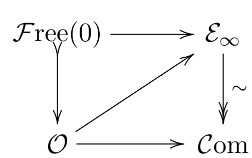
is a cofibration followed by an acyclic fibration. Moreover,

$$\begin{cases} \mathcal{L}\mathrm{ev}_\infty(n) = \mathcal{L}\mathrm{ev}^{AC}(n), & \text{for } n < 4 \\ \mathcal{L}\mathrm{ev}_\infty(4) = \mathcal{L}\mathrm{ev}^{AC}(4) \oplus W, & \end{cases}$$

where W is a free graded Σ_4-module.

Proof. Use proposition 1.2.2 on the map $\mathcal{L}\mathrm{ev}^{AC} \to \mathcal{L}\mathrm{ev}$. □

2.4.2. **$\mathcal{L}\mathrm{ev}_\infty$-algebras and commutative algebras up to homotopy.** According to remark 2.1.2, there is a morphism $\mathcal{L}\mathrm{ev} \longrightarrow \mathcal{C}\mathrm{om}$. By definition, a *commutative algebra up to homotopy* is an algebra over a cofibrant replacement $\mathcal{E}_\infty$ of the operad $\mathcal{C}\mathrm{om}$. The structure of closed model category on operad implies that for any cofibrant operad $\mathcal{O} \to \mathcal{C}\mathrm{om}$ there exists a morphism $\mathcal{O} \to \mathcal{E}_\infty$ such that the following diagram commutes:



Hence, there exist maps $\mathcal{L}\mathrm{ev}^{AC} \to \mathcal{E}_\infty$ and $\mathcal{L}\mathrm{ev}_\infty \to \mathcal{E}_\infty$ such that the previous diagram commutes.
Note that $\mathcal{E}_\infty$ does not need to be cofibrant for the existence of such a diagram. However, we need $\mathcal{E}_\infty(2) = \mathcal{L}\mathrm{ev}_\infty(2) = \mathcal{L}\mathrm{ev}^{AC}(2) = \mathcal{E}(2)$, in order to define cup-i products and to detect them (see the remark following corollary 3.5).

3. Adem-Cartan algebras and the extended Steenrod algebra

The aim of this section is to prove that Adem-Cartan algebras is a good framework for studying algebras provided with an action of the extended Steenrod algebra ([18], [17] also called the algebra of generalized Steenrod powers).

3.1. The extended Steenrod algebra $\mathcal{B}_2$.

3.1.1. **Generalized Steenrod powers.** The *extended Steenrod algebra*, denoted by $\mathcal{B}_2$, is a graded associative algebra over $\mathbb{F}_2$ generated by the generalized Steenrod squares Sq^i of degree $i \in \mathbb{Z}$. These generators satisfy the Adem relation, if $t < 2s$:

$$\mathrm{Sq}^t\,\mathrm{Sq}^s = \sum_i \binom{s-i-1}{t-2i} \mathrm{Sq}^{s+t-i}\,\mathrm{Sq}^i .$$

Note that negative Steenrod squares are allowed and that $\mathrm{Sq}^0 = \mathrm{Id}$ is not assumed. If $\mathcal{A}_2$ denotes the classical Steenrod algebra, then

$$\mathcal{A}_2 \cong \frac{\mathcal{B}_2}{< \mathrm{Sq}^0 + \mathrm{Id} >}.$$

3.1.2. **Definition.** As in the classical case, an *unstable module over* $\mathcal{B}_2$, is a graded $\mathcal{B}_2$-module together with the instability condition.

$$\mathrm{Sq}^n(x) = 0 \text{ if } |x| < n$$

An unstable level algebra over $\mathcal{B}_2$ is a graded level algebra $(A, *)$ which is an unstable module over $\mathcal{B}_2$, such that

$$\begin{cases} \mathrm{Sq}^{|x|}(x) = x * x, \\ \mathrm{Sq}^s(x*y) = \sum \mathrm{Sq}^t(x) * \mathrm{Sq}^{s-t}(y) \quad \text{(Cartan relation)} \end{cases}$$

3.1.3. **Remark.** The category of unstable algebras over $\mathcal{A}_2$ is a full subcategory of the category of unstable level algebras over $\mathcal{B}_2$.

3.2. **Graded algebras over a differential graded operad.** This section provides technical results concerning graded algebras (that means algebras whose differential is zero) over a differential graded operad, in order to prove the fondamental theorem 3.4 and its corollaries.

3.2.1. **Lemma.** *Let A be a graded algebra over a differential operad $\mathcal{O}$. Assume that $o \in \mathcal{O}(n)$ is a boundary. Then $o(a_1, \dots, a_n) = 0$, for all $a_1, \dots, a_n$ in A.*

Proof. There exists $\omega \in \mathcal{O}(n)$ such that $d\omega = 0$. The Leibniz rule implies that

$$d_A(\omega(a_1, \dots, a_n)) = o(a_1, \dots, a_n) + \sum_i \omega(a_1, \dots, d_A a_i, \dots, a_n)$$

and the result follows because $d_A = 0$. □

A direct consequence of this lemma is the following proposition:

3.2.2. **Proposition.** *Let A be a graded Adem-Cartan algebra, then it is a level algebra.*

Proof. An Adem-Cartan algebra A is provided with a product of degree 0 given by the element $e_0 \in (\mathcal{L}\mathrm{ev}^{AC}(2))^0$. Since $e_0 + \tau e_0$ is the boundary of $e_1 \in (\mathcal{L}\mathrm{ev}^{AC}(2))^{-1}$, we deduce that the product is commutative, by lemma 3.2.1. Again, the relation $dG_1^1 = [(12)_0(43)_0]_0 + [(13)_0(42)_0]_0 = ((34) + (234))e_0(e_0, e_0)$ in $(\mathcal{L}\mathrm{ev}^{AC}(4))^0$ implies the 4-terms relation (2.1) defining level algebras. □

3.3. **Cup-i products.** Let A be an algebra over an operad $\mathcal{O}$ such that $\mathcal{O}(2) = \mathcal{E}(2)$. For instance $\mathcal{O} = \mathcal{E}, \mathcal{L}\mathrm{ev}^{AC}, \mathcal{L}\mathrm{ev}_\infty, \mathcal{E}_\infty$. The evaluation map

$$\mathcal{O}(2) \otimes A^{\otimes 2} \longrightarrow A,$$

defines cup-i products $a \cup_i b = e_i(a, b)$. If $d_A = 0$, they are graded commutative products of degree $-i$. Steenrod squares action on A are defined by $\mathrm{Sq}^r(a) = a \cup_{|a|-r} a$. Following P. May [18], define $D_n(a) = e_n(a, a)$. In that terminology, Adem relations read

$$\sum_k (k, v - 2k) D_{w-v+2k} D_{v-k}(a) = \sum_l (l, w - 2l) D_{v-w+2l} D_{w-l}(a), \quad (3.1)$$

and Cartan relation read

$$D_n(x * y) = \sum_{k=0}^{n} D_k(x) * D_{n-k}(y). \quad (3.2)$$

3.4. **Theorem.** *Let A be a graded Adem-Cartan algebra then A is an unstable level algebra over the extented Steenrod algebra.*

The following lemma is useful for the proof of the main theorem; since it is technical, the proof of this lemma is postponed to the last section.

3.4.1. **Lemma.** *Let $\mathcal{O}$ be an operad such that $\mathcal{O}(2) = \mathcal{E}(2)$ (e.g. $\mathcal{O} = \mathcal{E}, \mathcal{L}\mathrm{ev}^{AC}, \mathcal{L}\mathrm{ev}_\infty, \mathcal{E}_\infty$). For every graded $\mathcal{O}$-algebra A and for every $a \in A$,*

$$[u_n^m]_x(a,a,a,a) = (n-2m, 2m-1) D_x D_{\frac{n}{2}}(a),$$

where $[u_n^m]_x$ denotes the image of $[u_n^m]_x \in \mathcal{E}$ by $\mathcal{E} \to \mathcal{O}$.

3.4.2. **Proof of 3.4.** We have already proved in proposition 3.2.2 that a graded Adem-Cartan algebra is a level algebra. Assume first that it is a module over $\mathcal{B}_2$ (Adem relation (3.1)). Hence the instability condition reads

$$\mathrm{Sq}^n(x) = x \cup_{|x|-n} x = 0, \text{ if } |x| - n < 0.$$

The next equality is also immediate

$$\mathrm{Sq}^{|x|}(x) = x \cup_0 x = e_0(x,x) = x * x.$$

The Cartan relation is given by $\mathrm{d}G_n^1$; according to lemma 3.2.1, $\mathrm{d}G_{n+1}^1(x,y,x,y) = 0$.

$$\begin{aligned}\mathrm{d}G_{n+1}^1(x,y,x,y) =& (\mathrm{Id} + (13)(24)) G_n^1(x,y,x,y) \\ &+ \sum_s [u_s^s(1,3,2,4)]_{n-s}(x,y,x,y) \\ &+ [u_n^1(4,3,2,1)]_0(x,y,x,y).\end{aligned}$$

By relation (2.6) and because $\cup_i$ is commutative for all i, one has for $s > 0$

$$[u_s^s(1,3,2,4)]_{n-s}(x,y,x,y) = 2(x \cup_s y) \cup_{n-s} (x \cup_0 y) = 0$$

and $[u_0^0(1,3,2,4)]_n(x,y,x,y) = (x \cup_0 y) \cup_n (x \cup_0 y)$. Hence

$$\mathrm{d}G_{n+1}^1(x,y,x,y) = (x \cup_0 y) \cup_n (x \cup_0 y) + \sum_{k=0}^{n} (x \cup_k x) \cup_0 (y \cup_{n-k} y)$$

which gives the Cartan relation (3.2)

$$D_n(x * y) = \sum_{k=0}^{n} D_k(x) * D_{n-k}(y).$$

The proof of Adem relation (3.1) relies on lemma 3.4.1, and on the fact that $\mathrm{d}G^m_{n+1}(a,a,a,a)=0$. Combined with the relation (2.9) of theorem 2.2.6 we get

$$\sum_s [u^s_{p+s}(a,a,a,a)+u^{s+1}_{p+s}(a,a,a,a)]_{n-p-s} = \sum_t [u^t_{n-p+t}(a,a,a,a)+u^{t+1}_{n-p+t}(a,a,a,a)]_{p-t}$$

$$\Rightarrow \sum_s [(p-s,2s-1)+(p-s-2,2s+1)]D_{n-p-s}D_{\frac{p+s}{2}}(a) = \sum_t [(n-p-t,2t-1)+(n-p-t-2,2t+1)]D_{p-t}D_{\frac{n-p+t}{2}}(a).$$

But $(x,y-2)+(x-2,y)=(x,y)$, hence

$$\sum_s (p-s,2s+1)D_{n-p-s}D_{\frac{p+s}{2}}(a) = \sum_t (n-p-t,2t+1)D_{p-t}D_{\frac{n-p+t}{2}}(a).$$

Since the first term is zero as soon as $p-s$ is odd, we can set $s=p-2l$, and also $t=n-p-2k$. As a consequence,

$$\sum_l (2l,2p-4l+1)D_{n-2p+2l}D_{p-l}(a) = \sum_k (2k,2n-2p-4k+1)D_{2p-n+2k}D_{n-p-k}(a)$$

Using the 2-adic expansion, one gets $(2l,2p-4l+1)=(l,p-2l)$; by setting $w:=p$ and $v:=n-p$, one gets

$$\sum_l (l,w-2l)D_{v-w+2l}D_{w-l}(a) = \sum_k (k,v-2k)D_{w-v+2k}D_{v-k}(a)$$

which is the relation (3.1). □

3.5. **Corollary.**
a) Any graded $\mathcal{L}\mathrm{ev}_\infty$-algebra is an unstable level algebra over $\mathcal{B}_2$ and any graded $\mathcal{E}_\infty$-algebra is an unstable algebra over $\mathcal{B}_2$.
b) The cohomology of an Adem-Cartan algebra or a Lev$_\infty$-algebra is an unstable level algebra over $\mathcal{B}_2$ and the cohomology of any $\mathcal{E}_\infty$-algebra is an unstable algebra over $\mathcal{B}_2$.

Proof. As we have morphisms of operads $\mathcal{L}\mathrm{ev}^{AC}\to\mathcal{L}\mathrm{ev}_\infty$ and $\mathcal{L}\mathrm{ev}^{AC}\longrightarrow\mathcal{E}_\infty$ (section 2.4.2), it suffices to prove the results for Lev^{AC}.

Part a) is our main theorem. Concerning part b), in order to prove the Adem-Cartan relations we compute the boundaries of $G_n^1(a,b,a,b)$ and $G_n^m(a,a,a,a)$ for cocycles a and b that represents classes $[a]$ and $[b]$ in the cohomology. These boundaries give the Adem-Cartan relations between $e_o(a,b)$, $e_i(a,a)$, $e_j(b,b)$ which represent $[a]*[b]$, $\mathrm{Sq}^{|a|-i}(a)$ and $\mathrm{Sq}^{|b|-j}(b)$ respectively.

There is a second proof that uses the homotopy invariance principle. Let (C,d) be a $\mathcal{L}\mathrm{ev}^{AC}$-algebra, then by corollary 1.3.2, its cohomology H is a graded $\mathcal{L}\mathrm{ev}^{AC}$-algebra, and we can apply part a). Notice that both proves give the same unstable structure over $\mathcal{B}_2$. This follows from the sequence of quasi-isomorphisms of algebras between the cochain complex and its cohomology: $C \longleftarrow \tilde{C} \longrightarrow H$. □

Remark. According to 2.4.2, this result holds for any operad $\mathcal{O}$ together with an acyclic fibration $\mathcal{O} \to \mathcal{C}\mathrm{om}$ satisfying $\mathcal{O}(2) = \mathcal{E}(2)$. In particular it holds for the algebraic Barratt-Eccles operad $\mathcal{BE}$ (studied in [3]), which is not cofibrant. It would be nice to describe explicitly the map $\mathcal{L}\mathrm{ev}^{AC} \longrightarrow \mathcal{BE}$, by determining the images of the G_n^m's in $\mathcal{BE}$.

4. Operadic secondary cohomological operations.

4.1. **The classical approach.** Let $C^*(X;\mathbb{F}_2)$ denotes the singular cochains complex of topological space X and $H^*(X,\mathbb{F}_2)$ its cohomology. We recall that $C^*(X;\mathbb{F}_2)$ is endowed with a structure of $\mathcal{E}_\infty$-algebra [12].

Adams defined in an axiomatic way stable secondary cohomological operations [1]. His approach is topological, and uses the theory of so-called "universal examples". These operations correspond to Adem relations

$$R_{Ad} = \sum_i \mathrm{Sq}^{m_i}\,\mathrm{Sq}^{n_i},$$

and are denoted by Φ. Let recall Adams axioms:

Axiom 1. For any $u \in H^n(X;\mathbb{F}_2)$, $\Phi(u)$ is defined if and only if $\mathrm{Sq}^{n_i}(u) = 0$ for all n_i.

Axiom 2. If $\Phi(u)$ is defined then

$$\Phi(u) \in H^{m_i+n_i+n-1}(X;\mathbb{F}_2)/\oplus_i Im(\mathrm{Sq}^{m_i}).$$

Axiom 3. The operation Φ is natural.

Axiom 4. Let (X,A) be a pair of topological spaces, we have the long exact sequence

$$\ldots \xrightarrow{\delta^*} H^n(X,A;\mathbb{F}_2) \xrightarrow{j^*} H^n(X;\mathbb{F}_2) \xrightarrow{i^*} H^n(A;\mathbb{F}_2) \xrightarrow{\delta^*} \ldots$$

let $v \in H^n(X,A;\mathbb{F}_2)$ be a class such that ϕ is defined on $j^*(v) \in H^n(X;\mathbb{F}_2)$. Let $w_i \in H^*(A;\mathbb{F}_2)$ such that $\delta^*(w_i) = \mathrm{Sq}^{n_i}(v)$. Then, we have

$$i^*\Phi(j^*(v)) = \sum_i \mathrm{Sq}^{m_i}(w_i) \in H^*(A;\mathbb{F}_2)/\oplus_i Im(\mathrm{Sq}^{m_i}).$$

Axiom 5. The operation Φ commutes with suspension.

Later on, Kristensen proved in [14] that these operations can be defined at the cochain level, using the existence of a coboundary which creates the stable secondary cohomological operation defined by Adams ([14], chapter 6). More precisely, for an Adem relation R_{Ad} and a class $x \in \cap_i \mathrm{Sq}^i$, Kristensen defines cochain operations θ such that the differential of $\theta(c)$ (c is a representant of x) gives a cocycle representing an Adem relation $R_{Ad}(x)$. If one chooses b_i such that $db_i = \mathrm{Sq}^i(c)$, then one gets a cocycle, and a cohomology class

$$Qu^r(c) = [\theta(c) + \sum_i (e_{n-m_i+n_i}(1, e_{n-n_i})(b_i, c, c) + e_{n-m_i+n_i-1}(b_i, b_i))].$$

Then,

4.1.1. **Theorem.** (Kristensen, theorem 6.1 of [14]) *Any operation $x \mapsto Qu^r(c)$ satisfies axiom 1-5 of Adams.*

The idea underlying this section is the following one: if one can extends θ to a map $\tilde{\theta} : C^*(X;\mathbb{F}_2)^{\otimes 4} \to C^*(X;\mathbb{F}_2)$, where θ is $\tilde{\theta}$ composed with the diagonal, there is an operadic interpretation if regarding $\tilde{\theta}$ as an element of the endomorphism operad of the singular cochains functor End_{C^*}, where

$$End_{C^*}(n) = Hom(C^*(-)^{\otimes n}, C^*(-))$$

is the vector space of n-multilinear natural transformations of the functor $C^*(-)$ (see [12]). The following sequence of morphisms of operads

$$\mathcal{L}ev^{AC} \to \mathcal{E}_\infty \to End_{C^*}$$

suggests to find a pre-image of θ in $\mathcal{L}ev^{AC}$. The key point comes from theorem 3.4: there is a one to one correspondance between operadic cells $G_p^m \in \mathcal{L}ev^{AC}(4)$ and Adem relations.

4.2. **Construction Definition.** Let A be a differential graded Adem-Cartan algebra, and let $x \in H^n(A)$. Let

$$\mathrm{d}G_{p+1}^m(x,x,x,x) = \sum_i Sq^{m_i} Sq^{n_i}(x) = 0$$

be an Adem relation with $x \in \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i})$. The integer m controls the sequence $\{m_i, n_i\}$ and $p = 3n - m_i - n_i$. Choose $c \in A^n$ a representant of $x \in H^n(A)$. Since $\mathrm{Sq}^{n_i}(x) = 0$, there exists $b_i \in C^{n+n_i-1}(A)$ such that $db_i = e_{n-n_i}(c,c)$. The element

$$b = \sum_i e_{n-m_i+n_i}(1, e_{n-n_i})(b_i, c, c) + e_{n-m_i+n_i-1}(b_i, b_i)$$

satisfies $d(G^m_{p+1}(c,c,c,c) + b) = 0$. Hence, $G^m_{p+1}(c,c,c,c) + b$ is a cocycle. We would like to associate to x the class of $G^m_{p+1}(c,c,c,c) + b$ in $H^{m_i+n_i+n-1}(A)/\oplus_i Im(\mathrm{Sq}^{m_i})$. In order to do so, let us prove that it is independant on the choice of the b_i 's.
Let $b'_i \in C^{n+n_i-1}(A)$ such that $\mathrm{d}b'_i = \mathrm{d}b_i = e_{n-n_i}(c,c)$, and

$$b' = \sum_i e_{n-m_i+n_i}(1, e_{n-n_i})(b'_i, c, c) + e_{n-m_i+n_i-1}(b'_i, b'_i)$$

Then $\mathrm{d}(b + b') = 0$ and the following relation implies the result:

$$b + b' = \sum_i \mathrm{Sq}^{m_i}(b_i + b'_i) + \mathrm{d}e_{n-m_i+n_i}(b_i + b'_i, b_i).$$

Moreover the class $[G^m_{p+1}(c,c,c,c)+b] \in H^{m_i+n_i+n-1}(A)$ does not depend on the choice of a representant c of x. Using the homotopy invariance principle, there is a zig-zag of acyclic fibrations of Adem-Cartan algebras

$$H^*(A) \leftarrow \widetilde{A} \rightarrow A.$$

Let c and c' be two cocycles that represent x, and pick lifts of $G^m_{p+1}(c,c,c,c)+b$ and $G^m_{p+1}(c',c',c',c')+b'$ in $\widetilde{A}$, obtained by lifting c, c', b, b'. These two lifts have the same image in $H^*(A)$ which is $G^m_{p+1}(x,x,x,x)$; using the acyclicity of the fibration, there exists a cochain $z \in \widetilde{A}$ such that $\mathrm{d}z$ is the sum of these two lifts. If z' denotes the image of z in A, then

$$dz' = G^m_{p+1}(c,c,c,c) + b + G^m_{p+1}(c',c',c',c') + b'.$$

This defines a map

$$\psi^{m,p} : \bigcap_i \mathrm{Ker}(\mathrm{Sq}^{n_i}) \subset H^n(A) \longrightarrow H^{n+m_i+n_i-1}(A)/\bigoplus_i Im(\mathrm{Sq}^{m_i}),$$

by setting $\psi^{m,p}(x) = [G^m_{p+1}(c,c,c,c) + b]$.

4.2.1. **Proposition.** *If $A = C^*(X, \mathbb{F}_2)$ the maps $\psi^{m,p}$ coincide with the stable secondary cohomological operations of Adams.*

Proof. The proof relies on theorem 4.1.1 with $\theta(c) = G^m_{p+1}(c,c,c,c)$. □

4.3. **Small models.** For any topological space X, the singular cochains $C^*(X;\mathbb{F}_2)$ is a $\mathcal{E}_\infty$-algebra [12]. Under mild conditions on X, this structure on the singular cochains determines a good algebraic model for its 2-adic homotopy type [17]. This result remains for $H^*(X,\mathbb{F}_2)$ [7]: using the homotopy invariance principle the cohomology of X is endowed with a $\mathcal{E}_\infty$-structure, denoted by $\mathcal{H}_X$ and named the *small model* of X. By definition $\mathcal{H}_X$ is isomorphic to $H^*(X,\mathbb{F}_2)$ as a graded vector space, indeed as an unstable algebra over $\mathcal{A}_2$ ([7], propositon 4.3).

Each element $e \in \mathcal{E}_\infty(n)$ gives rise to a cohomological operation:

$$e : \mathcal{H}_X^{\otimes n} \to \mathcal{H}_X.$$

As the homotopy invariance principle is not natural, this operations are not natural. Assume that the operad $\mathcal{E}_\infty$ is quasi-free i.e. $\mathcal{E}_\infty \cong \mathcal{F}\mathrm{ree}(V)$. Since any operation e can be decomposed as a sum of composition products of cells $v \in V$, the first step in order to understand operadic cohomological operations is to understand the action of cells. Recall that any cell v produces a relation between these cohomological operations, by computing $\mathrm{d}v$ (see lemma 3.2.1).

The morphism $\mathcal{L}\mathrm{ev}^{AC} \longrightarrow \mathcal{E}_\infty$ enables to consider $\mathcal{H}_X$ as an Adem-Cartan algebra. Besides, $\mathcal{L}\mathrm{ev}^{AC}$ is quasi-free on a Σ-module V^{AC} generated by the 2-cells e_i and the 4-cells G_n^m. Hence there are cohomological operations

$$\begin{array}{rc} e_i : & \mathcal{H}_X^{k_1} \otimes \mathcal{H}_X^{k_2} \to \mathcal{H}_X^{k_1+k_2-i} \\ G_n^m : & \mathcal{H}_X^{k_1} \otimes \mathcal{H}_X^{k_2} \otimes \mathcal{H}_X^{k_3} \otimes \mathcal{H}_X^{k_4} \to \mathcal{H}_X^{k_1+k_2+k_3+k_4-n} \end{array}$$

The 2-cells e_i give rise to the primary cohomological operations, the Steenrod squares $\mathrm{Sq}^i : \mathcal{H}_X^k \otimes \to \mathcal{H}_X^{k+i}$ defined by $\mathrm{Sq}^i(x) = e_{i-k}(x,x)$. The 4-cells G_{p+1}^m give rise to operations $\psi^{m,p} : \mathcal{H}_X^n \otimes \to \mathcal{H}_X^{4n-p-1}$ defined by $\psi^{m,p}(x) = G_{p+1}^m(x,x,x,x)$. We prove in the following theorem that these operations coincide with the stable secondary cohomological operations.

4.3.1. **Theorem.** *The stable secondary cohomological operations ϕ of Adams extend to maps $H^n(X,\mathbb{F}_2) \to H^{n+m_i+n_i-1}(X,\mathbb{F}_2)$ via the action of the G_{p+1}^m's.*

Proof. This relies on the following sequence of acyclic fibrations of Adem-Cartan algebras given by the homotopy invariance principle:

$$C^*(X;\mathbb{F}_2) \longleftarrow \widetilde{C^*(X,\mathbb{F}_2)} \longrightarrow \mathcal{H}_X^*.$$

We use the notations of the preceding proposition. As the left morphism is an epimorphism we can lift c, and the b_i's in $\widetilde{C^*(X,\mathbb{F}_2)}$ in $\tilde{c}$ and $\tilde{b}_i$.

The images of $\tilde{c}$ in $\mathcal{H}_X$ is x and of $\tilde{b}_i$ is zero. The result follows from the computation of the images of $G^m_{p+1}(\tilde{c},\tilde{c},\tilde{c},\tilde{c})+\tilde{b}$ which are $G^m_{p+1}(x,x,x,x)$ in $\mathcal{H}_X$ and $G^m_{p+1}(c,c,c,c)+b$ in $C^*(X;\mathbb{F}_2)$. $\square$

5. Proof of technical lemmas

In this section proposition 2.2.4 and lemma 3.4.1.

5.1. **Lemma.** *For any $i,j\leq 2^p-1$ one has*

$$\begin{aligned}(i,j)=&0, \text{ if } i+j\geq 2^p, \text{ and}\\ (i,j)=&(2^p-i-j-1,j).\end{aligned}$$

Proof. Let $\sum_{l=0}^{p-1}a_l2^l$ and $\sum_{l=0}^{p-1}b_l2^l$ be the 2-adic expansion of i and j respectively. Recall that $(i,j)=1$ if and only if the 2-adic expansion of $i+j$ is $\sum(a_l+b_l)2^l$. If $i+j\geq 2^p$ this is not the case, thus $(i,j)=0$.
If $(i,j)=1$ then the 2-adic expansion of $2^p-1-i-j$ is $\sum_{l=0}^{p-1}(1-a_l-b_l)2^l$, thus the 2-adic expansion of $(2^p-1-i-j)+j$ has for coefficients $(1-a_l-b_l)+(b_l)$. Consequently $(2^p-1-i-j,j)=1$. The converse is true by symmetry.
Note that the first assertion is a consequence of the second one, because if $(i+j)\geq 2^p$ then $(2^p-1-i-j)<0$, and $(\alpha,\beta)=0$ if $\alpha<0$ or $\beta<0$. $\square$

5.2. **Lemma.** *Let*

$$\begin{aligned}\alpha_i&=(n-m+i,m-1)(i,m)(13)_{2^{k+1}\delta-i}(42)_{n+i-2^{k+1}\delta}\\ \beta_i&=(n-m+i,m-1)(i-1,m)(13)_{2^{k+1}\delta-i}(24)_{n+i-2^{k+1}\delta}\end{aligned}$$

be some summands of u^m_n. Let $0\leq j\leq 2^{k+1}-1$, such that $j\equiv -n-i\ [2^{k+1}]$. There exists δ' such that $n+i-2^{k+1}\delta=2^{k+1}\delta'-j$. Then

$$\begin{aligned}\alpha_i=&(j,m-1)(n+j-m-1,m)(13)_{n+j-2^{k+1}\delta'}(42)_{2^{k+1}\delta'-j}\\ \beta_i=&(j,m-1)(n+j-m,m)(13)_{n+j-2^{k+1}\delta'}(24)_{2^{k+1}\delta'-j}\end{aligned}$$

Proof. One has

$$\alpha_i=(2^{k+1}(\delta+\delta')-j-m,m-1)(2^{k+1}(\delta+\delta')-j-n,m)\\ (13)_{n+j-2^{k+1}\delta'}(42)_{2^{k+1}\delta'-j}.$$

Since $(\delta+\delta')\neq 0$, the 2-adic expansion of $2^{k+1}(\delta+\delta')$ is $\sum_{p\geq k+1}a_p2^p$. Let a_r be the first non-zero coefficient. Then by lemma 5.1

$$\begin{aligned}(2^{k+1}(\delta+\delta')-j-m,m-1)&=(2^r-j-m,m-1)\\ &=(j,m-1)\end{aligned}$$

The same argument implies $(2^{k+1}(\delta+\delta')-j-n,m)=(j+n-m-1,m)$, and it holds also for β_i. □

5.3. **Proof of Lemma 3.4.1.** By the definition of u_n^m, one has

$$\begin{aligned}u_n^m(a,a,a,a) = &\sum_{i=0}^{2^{k+1}-1}\sum_{0\le 2^{k+1}\delta-i\le n} \\ &\quad (n-m+i,m-1)(i,m)D_{2^{k+1}\delta-i}(a)*D_{n+i-2^{k+1}\delta}(a)\\ &\quad + (n-m+i,m-1)(i-1,m)D_{2^{k+1}\delta-i}(a)*D_{n+i-2^{k+1}\delta}(a)\\ = &\sum_{i=0}^{m}\sum_{0\le 2^{k+1}\delta-i\le n}(n-m+i,m-1)(i,m-1)D_{2^{k+1}\delta-i}(a)*D_{n+i-2^{k+1}\delta}(a).\end{aligned}$$

Let j such that $n+i\equiv -j\ [2^{k+1}]$ be the one defined in lemma 5.2. If $i\neq j$, since the product $*$ is commutative and by virtue of lemma 5.2 one has

$$\begin{aligned}&(n-m+i,m-1)(i,m-1)D_{2^{k+1}\delta-i}(a)*D_{2^{k+1}\delta'-j}(a)+\\ &\qquad (n-m+j,m-1)(j,m-1)D_{2^{k+1}\delta'-j}(a)*D_{2^{k+1}\delta-i}(a)=0\end{aligned}$$

Then

$$\begin{aligned}u_n^m(a,a,a,a) = &\sum_{i=0}^{2^{k+1}-1}\sum_{\{(\delta,\delta')\ \mid\ 2^{k+1}(\delta+\delta')-2i=n\}}\\ &\qquad (n-m+i,m-1)(i,m-1)\underbrace{D_{2^{k+1}\delta-i}(a)*D_{2^{k+1}\delta'-i}(a)}_{X(\delta,\delta')}\end{aligned}$$

If $\delta\neq\delta'$ then $X(\delta,\delta')+X(\delta',\delta)=0$. Hence if there exists (r,δ) such that $2^{k+2}\delta-2r=n$, we have

$$u_n^m(a,a,a,a)=(n-m+r,m-1)(r,m-1)D_0D_{\frac{n}{2}}(a).$$

But $(n-m+r,m-1)=(2^{k+2}\delta-r-m,m-1)=(r,m-1)$ by lemma 5.1. As a consequence, $(n-m+r,m-1)(r,m-1)=(n-m+r,m-1)=(2n-2m+2r,2m-1)=(n-2m,2m-1)$. Furthermore, if $(n-2m,2m-1)=1$, then n is even and we can pick $0\le r\le 2^{k+1}-1$ such that $r\equiv -\frac{n}{2}\ [2^{k+1}]$. The same holds for $[u_n^m]_x$. □

5.4. **Proof of proposition 2.2.4.** We have to prove the relations (2.4), (2.5), (2.6), (2.7). The relation (2.4) is immediate by the definition of u_0^0.

Relation (2.5) and (2.6)– Assume $n\le m$. By lemma 5.1, $(i,m)=0$ if $i+m\ge 2^{k+1}$. The condition $2^{k+1}\delta-i\le n\le m\le 2^{k+1}-1$ implies δ equals 0 or 1. If $\delta=0$, then $i=0$ and $(n-m+i,m-1)=1$ only if $n=m$. That means that $\tau^m(13)_0(42)_m$ is a summand in u_m^m. If $\delta = 1$, then $i+m \geq 2^{k+1}$ and $(i,m) = 0$. If $n < m$, then $i+m-1 \geq 2^{k+1}$ and $(i,m-1) = 0$ which proves relation (2.5). If $n = m$, then for $i = 2^{k+1}-n$, the term $(2^{k+1}-m,m-1)(2^{k+1}-1-m,m)(24)_m(13)_0$ is a summand in u_m^m; by virtue of lemma 5.1, $(2^{k+1}-m,m-1) = (0,m-1)$ and $(2^{k+1}-m-1,m) = (0,m)$. This proves relation (2.6).

Relation (2.7)– It is easy to check that

$$\mathrm{d}u_1^1(1,2,3,4) = \tau u_0^0(1,2,3,4) + u_0^0(2,1,4,3).$$

Let assume that $m+1$ is even (the computation is similar when it is odd). For the convenience of the reader, $(ab)(cd)_{\delta,i}$ means $(ab)_{2^{k+1}\delta-i}(cd)_{r+i-2^{k+1}\delta}$ for an appropriate r. There are two cases to consider: if $2^k \leq m \leq 2^{k+1}-2$ or if $m = 2^{k+1}-1$. Since computation are long but not difficult, we'll present only the first case.

$$\begin{aligned}
\mathrm{d}u_{n+1}^{m+1} = &\sum_i \sum_\delta \underbrace{(n-m+i,m)(i,m+1)}_{a_{i+1}}(\mathrm{Id}+\tau)(13)(42)_{\delta,i+1} \\
&+ \underbrace{(n-m+i,m)(i,m+1)}_{b_i}(13)(\mathrm{Id}+\tau)(42)_{\delta,i} \\
&+ (n-m+i,m)(i-1,m+1)(\mathrm{Id}+\tau)(13)(24)_{\delta,i+1} \\
&+ \underbrace{(n-m+i,m)(i-1,m+1)}_{c_i}(13)(\mathrm{Id}+\tau)(42)_{\delta,i}
\end{aligned}$$

$$\begin{aligned}
&u_n^{m+1}(1,2,3,4) + u_n^{m+1}(3,4,1,2) = \sum_i \sum_\delta \\
&\underbrace{(n-m+i-1,m)(i,m+1)}_{d_i}(13)(42)_{\delta,i} \\
&+ (n-m+i-1,m)(i,m+1)(31)(24)_{\delta,i} \\
&+ (n-m+i-1,m)(i-1,m+1)(13)(24)_{\delta,i} \\
&+ (n-m+i-1,m)(i-1,m+1)(31)(42)_{\delta,i}
\end{aligned}$$

And using lemma 5.2

$$\tau u_n^m(1,2,3,4) + u_n^m(2,1,4,3) = \sum_i \sum_\delta$$
$$\underbrace{(n-m+i,m-1)(i,m)}_{f_i}(13)(42)_{\delta,i}$$
$$+ (n-m+i,m-1)(i-1,m)(13)(24)_{\delta,i}$$
$$+ (n-m+i-1,m)(i,m-1)(31)(24)_{\delta,i}$$
$$+ (n-m+i,m)(i,m-1)(13)(24)_{\delta,i}$$

For example, let us compute the coefficients A of $(13)(42)_{\delta,i}$, i.e. $a_i + b_i + c_i + d_i + f_i$

$$A = (n-m+i-1,m)(i-1,m+1) + (n-m+i,m)(i,m+1)$$
$$+(n-m+i,m)(i-1,m+1) + (n-m+i-1,m)(i,m+1)$$
$$+(n-m+i,m-1)(i,m).$$

We have the following

$$a_i + c_i =(n-m+i,m-1)(i-1,m+1)$$
$$b_i + d_i =(n-m+i,m-1)(i,m+1)$$

Then

$$a_i + b_i + c_i + d_i = (n-m+i,m-1)(i,m) = f_i.$$

All the others computation go the same. □

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