HOW (NON-)UNIQUE IS THE CHOICE OF COFIBRATIONS?

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ABSTRACT. The question of the title is to be understood in the context of Quillen's model categories: having fixed the category of models and the subcategory of weak equivalences, how much ambiguity is there in finding suitable fibration and cofibration classes?

The bad news: the choice is probably never unique. Even for simplicial sets, proper subclasses of the monomorphisms can serve as cofibrations (with the usual weak equivalences) so as to satisfy the axioms.

The good news: in a wide class of model categories, keeping the weak equivalences fixed, any two small-generated cofibration classes will give model structures that have the same Quillen equivalence type. Morally, the Quillen equivalence class of a model category is determined by the underlying category and the subcategory of weak equivalences.

Under Quillen's conception, a *category of models for homotopy theory* is to come equipped with three distinguished classes of morphisms — cofibrations, weak equivalences and fibrations — of which the weak equivalences alone determine the associated homotopy category. The first example of distinct model categories with the same homotopy category is due to Quillen, and appears in Homotopical Algebra. (It is chain complexes, modelling the derived category in two different ways.) Another example was found separately by Bousfield–Kan [8] and Heller [10], on simplicial diagrams. In both of these cases, the underlying category of models for the alternative structures is the same, as are the weak equivalences, but two (co)fibration classes are possible, one properly contained in the other. Here's a specimen with three cofibration classes, still ordered linearly.

Example 0.1. On the category $SSet^{\Delta}$ (the *cosimplicial spaces* of Bousfield–Kan [8]), let the weak equivalences be maps that are Δ-objectwise weak equivalences in $SSet$. Let cof_1 be the class of maps with the left lifting property with respect to all maps that are Δ-objectwise acyclic fibrations in $SSet$. Let cof_2 be the class of monos that induce isomorphisms on the

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maximal augmentation of the underlying cosimplicial space. Let cof_3 be the class of all monomorphisms. Any of these can serve as the class of cofibrations on $SSet^{\Delta}$ with the above weak equivalences and one has $\mathsf{cof}_1 \subsetneqq \mathsf{cof}_2 \subsetneqq \mathsf{cof}_3$.

Example 1.3 is a homotopy theory "from nature" that demonstrates that there is no a priori bound on the cardinality of possible cofibration classes (even having fixed both the category of models and the weak equivalences, hence, the homotopy category); nor do these classes have to be linearly ordered by inclusion. The following cheap argument shows the same. Let $\mathcal{M}$ be a Quillen model category as in Example 0.1, with several possible classes of cofibrations cof_i, $i \in I$. Let X be a set (considered as a discrete diagram), and let a map in $\mathcal{M}^X$ be a weak equivalence iff it is so for every $x \in X$. For any function $X \xrightarrow{f} I$, one can now declare a map in $\mathcal{M}^X$ to be a cofibration iff it falls into $\mathsf{cof}_{f(x)}$ for $x \in X$, and have a Quillen model structure on $\mathcal{M}^X$.

It has been asserted by several authors (starting from Quillen) that the formal homotopy theories attached to Quillen-equivalent model categories should be thought of as equivalent. It is immediate that different cofibration classes notwithstanding, the alternative homotopy model structures listed are all Quillen equivalent. Regarding the cartesian product example $\mathcal{M}^X$ above, note that there is a maximal cofibration class out of the *set* of choices exhibited, even if they are no longer linearly ordered by inclusion.

The goal of Section 2 of this paper is to show that this behavior is, in some sense, generic. It is doubtful that results of this type — on the possible patterns of class-sized data — can be derived without some auxiliary handle on the set theory of the situation, and this is the reason for the qualification *in some sense*. The property assumed of the model category is stronger than being *cofibrantly generated* in the sense of Dwyer–Hirschhorn–Kan [9]; it is called *combinatorial* by Jeff Smith, and was stumbled upon independently in [5] while considering the homotopy theory of sheaves of algebraic structures.

The results of Section 1 point in the opposite direction. For example, it is shown that even for simplicial sets, there are infinitely many cofibration classes to go with ("topological" or "combinatorial", i.e. the usual) weak equivalences. The structure of these cofibration classes, ordered by inclusion, seems to be complicated.

Why should one bother to think of such questions? While one seldom wishes to abandon a particular category of models and class of weak equivalences, fibrancy and cofibrancy conditions are something of a necessary evil. For example, one of the basic results in model categories is that if X is a cofibrant and Y a fibrant object, then every map $X \to Y$ in the homotopy category is realized by an actual morphism from X to Y. Is it possible to "adjust" the notion of cofibration in such a way that certain preferred objects become cofibrant — obviously, at the price of losing fibrancy of others? This particular question is not answered here, but perhaps the paper contributes more motivation not to take "fibrant replacement" and "cofibrant replacement" as a built-in. In turn, the main result of Section 2, Quillen uniqueness, suggests that functors that take as input a homotopy theory to produce a spectrum — for example, Waldhausen K-theory type machines — should have as their domain a Quillen equivalence class of homotopy theories. In good cases, that amounts to just a category of models and subcategory of weak equivalences; the choice of cofibrations, while necessary for the machine to work, does not affect the homotopy type of the output.

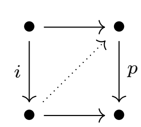
Acknowledgments. Cor. 2.6, or rather, that one may interpolate between different cofibration classes by a zig-zag of Quillen equivalences, was conjectured by Mike Hopkins. I am indebted to Jiří Rosický for many conversations and email-exchanges on set-theoretic category theory.

1. Many choices for cofibrations

The following theorem of J. Smith makes it sinfully easy to thin down cofibration classes. First, some notation.

Definition 1.1. Let $\mathcal{C}$ be a cocomplete category, I any class of morphisms of $\mathcal{C}$.

- Close the class of all pushouts of I under transfinite composition in $\mathcal{C}$. Add, by convention, all isomorphisms of $\mathcal{C}$. This defines the class $\mathsf{cell}(I)$ of *relative I-cellular maps.*
- The class $\mathsf{cof}(I)$ of I-cofibrations is defined as follows: $X \xrightarrow{c} Y \in \mathsf{cof}(I)$ iff c is a retract of an $X \xrightarrow{r} Z \in \mathsf{cell}(I)$ in the category of morphisms of $\mathcal{C}$.
- I-fibrations, or I-injectives, denoted $\mathsf{inj}(I)$, are the morphisms with the right lifting property w.r.t. I; that is, such that in any commutative square



with $i \in I$, $p \in \mathsf{inj}(I)$, a dotted lift making both triangles commute exists.

When the set I *permits the small object argument*, in the sense of Dwyer–Hirschhorn–Kan [9] (see Hirschhorn [11] or Hovey [12] for another write-up),

$\mathsf{cof}(I)$ is precisely the class of maps that have the left lifting property w.r.t. every member of $\mathsf{inj}(I)$.[1] Jeff Smith's theorem is:

Theorem 1.2. *Let $\mathcal{C}$ be a locally presentable category,* W *a subcategory, and I a set of morphisms of $\mathcal{C}$. Suppose they satisfy the criteria:*

- **c0)** W *is closed under retracts and has the 2-of-3 property (Quillen's axiom **M2**).*
- **c1)** $\mathsf{inj}(I) \subseteq \mathsf{W}$.
- **c2)** *The class* $\mathsf{cof}(I) \cap \mathsf{W}$ *is closed under transfinite composition and under pushout.*
- **c3)** *The inclusion of the full subcategory of* $\mathrm{Mor}(\mathcal{C})$ *with objects the* W *into* $\mathrm{Mor}(\mathcal{C})$ *(the category of morphisms of $\mathcal{C}$) satisfies Freyd's solution set condition at the objects I.*

Then setting weak equivalences $:= \mathsf{W}$*, cofibrations* $:= \mathsf{cof}(I)$ *and fibrations* $:= \mathsf{inj}(\mathsf{cof}(I) \cap \mathsf{W})$*, one obtains a cofibrantly generated Quillen model structure on $\mathcal{C}$.*

See [6] for background and the proof. In practice, W is an *accessible class* (cf. Adámek–Rosický [3]) of maps, whence the inclusion $\mathrm{Mor}(\mathsf{W}) \hookrightarrow \mathrm{Mor}(\mathcal{C})$ satisfies the solution set condition at every map. In that case, note that upon replacing the generating set I of cofibrations by any set $I' \subset \mathsf{cof}(I)$, conditions **c0-c2-c3** are still met. This will be exploited in the next example to produce mutations of the coarse equivariant model structure on simplicial sets with lots of distinct cofibration classes.

Example 1.3. Let G be a discrete group, $G - Set$ the category of sets with G-action. Fix a collection $\{G_i\}$ of subgroups of G, and let I be the set of maps $\emptyset \to G/G_i$ from the initial object to (say, left) cosets G/G_i. Then $\mathsf{cof}(I)$ can be seen to be the class of those monomorphisms $X \xrightarrow{f} Y$ in $G - Set$ where the stabilizers of points in the complement of the image of X in Y (if any such) is one of the G_i.

Consider now $SSet^G$. Define weak equivalences to be the same as those on the underlying map of simplicial sets, and define cofibrations to be those monomorphisms whose degree n part (corresponding to n-simplices) belongs to $\mathsf{cof}(I)$. If $\{G_i\}$ contains the identity (as one-element subgroup), then J. Smith's theorem applies. (See also Ex. 1.5.)

Creating Quillen model structures via right adjoints. The next result relies on a theorem of D. Kan. See Hirschhorn [11] or Hovey [12] for proofs.

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[1] With no mention to the contrary, we employ the terminology of the above references. "(Quillen) model category" will mean "closed model category". We write "acyclic (co)fibration" instead of "trivial".

Theorem 1.4. *Let $\mathcal{M}$ be a cofibrantly generated Quillen model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{N}$ be a finitely complete and cocomplete category, and $\mathcal{N} \overset{L}{\underset{R}{\leftrightarrows}} \mathcal{M}$ a pair of adjoint functors. Suppose the sets of maps $L(I)$, $L(J)$ each permit the small object argument and $R(f)$ is a weak equivalence in $\mathcal{M}$ for every $f \in \mathsf{cell}(F(J))$. Then there is a Quillen model structure on $\mathcal{N}$, with $f \in \operatorname{mor} \mathcal{N}$ being a weak equivalence (resp. fibration) iff $R(f)$ is one in $\mathcal{M}$. This structure is cofibrantly generated, with generating cofibrations $L(I)$ and generating acyclic cofibrations $L(J)$. Following M. Hopkins, say that this model structure is* created *by R.*

Example 1.5. Keeping the notations of Ex. 1.3, let $\mathcal{O}$ be the set of (say, left) cosets G/G_i, considered as objects of $G - Set$. The inclusion of the discrete set of objects $\mathcal{O} \hookrightarrow G - Set$ induces an adjunction $SSet^G \leftrightarrows SSet^{\mathcal{O}}$ whose right adjoint part takes the G_i-fixed points, degreewise. This right adjoint creates a model structure on $SSet^G$. The corresponding cofibration class is precisely the one described in Ex. 1.3. (Note, however, that the underlying class of weak equivalences was chosen to be the 'coarse' one there, regardless of the existence of 'fine equivariant weak equivalences'.)

If $\mathcal{N}$ is a locally presentable category, it is complete and cocomplete and any set of maps in $\mathcal{N}$ permit the small object argument; in fact, any object X of $\mathcal{N}$ has a *rank*, that is, $\hom_{\mathcal{N}}(X, -)$ commutes with all κ-filtered colimits for some κ. (See Adámek–Rosický [3] or Borceux [7] vol.II. for an introduction to locally presentable categories.)

Corollary 1.6. *Let $\mathcal{N} \overset{L}{\underset{R}{\leftrightarrows}} \mathcal{M}$ be a Quillen pair between the model categories $\mathcal{N}$, $\mathcal{M}$. Assume $\mathcal{N}$ is a locally presentable category, $\mathcal{M}$ is cofibrantly generated, and R preserves and reflects weak equivalences (i.e. $R(f)$ is a weak equivalence in $\mathcal{M}$ iff f is one in $\mathcal{N}$). Then R creates, in the sense of 1.4, a Quillen model structure on $\mathcal{N}$ that has the same weak equivalences and a cofibration class equal to or smaller than the original one.*

Indeed, by the assumption $L \dashv R$ is a Quillen pair, L takes acyclic cofibrations to acyclic cofibrations; these are closed under the operations defining $\mathsf{cell}(-)$, and R is assumed to take weak equivalences into weak equivalences. Being a Quillen right adjoint, R preserves fibrations; hence the new class of fibrations in $\mathcal{N}$, the maps taken into $\mathcal{M}$-fibrations by R, contains the original. $\square$

Specialize further by assuming $\mathcal{N} = \mathcal{M}$. Note that in this case the requisite properties of the endo-adjunction are preserved by iterating it finitely many times: $\mathcal{M} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M} \underset{R}{\overset{L}{\leftrightarrows}} \dots \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$. Note also that if there exists a natural transformation $X \xrightarrow{\Theta_X} RX$ *or* a natural transformation $RX \xrightarrow{\Theta_X} X$ that is a weak equivalence at all objects X then (by the 2-of-3 axiom) R preserves and reflects weak equivalences. Such a pair R, Θ may be thought of as a *partial fibrantization functor*, at least if R does take some non-fibrant objects to fibrant ones.

In the paper where he defined his extension functor [13], D. Kan verified that $SSet \xrightarrow{\mathrm{Ex}} SSet$ preserves fibrations and the natural inclusion $X \to \mathrm{Ex}(X)$ is a weak equivalence. Any presheaf category is locally presentable and $SSet$ is cofibrantly generated, so the remarks following Cor. 1.6 apply.

We still need to verify that a properly decreasing chain of cofibration classes arises by iterating the construction. This is the hardest (or more cautiously, the most time-consuming) part of this paper, though this may be due to just the author's clumsiness. See Conj. 1.12 for a more elegant guess.

Proposition 1.7. *For any $n \in \mathbb{N}$, there exists a simplicial set X such that $\mathrm{Ex}^n(X)$ does not satisfy the Kan extension condition, but $\mathrm{Ex}^{n+1}(X)$ does.*[2]

Proof. Let Sd be the left adjoint of Ex; by adjunction, we seek X that has the right lifting property w.r.t. all the $n+1$-subdivided horn inclusions $\mathrm{Sd}^{n+1} \Lambda^i_k \hookrightarrow \mathrm{Sd}^{n+1} \Delta_k$, but fails it for some n-subdivided one. Let $\mathcal{D}$ be the diagram of two parallel arrows with a common section, thought of as the full subcategory of Δ^{op} with objects [0] and [1]. Write DirGr for $Set^{\mathcal{D}}$, thought of as the category of directed graphs (with an identity arrow for every vertex), and edge for the forgetful functor $SSet \to \mathrm{DirGr}$. Since edge has a right adjoint too, it suffices to find a $G \in \mathrm{DirGr}$ that has the right lifting property w.r.t. the edge restrictions of the $n+1$-subdivided horn inclusions $\mathsf{edge}(\mathrm{Sd}^{n+1} \Lambda^i_k) \hookrightarrow \mathsf{edge}(\mathrm{Sd}^{n+1} \Delta_k)$, but fails it for the edge graph of some n-subdivided one.

By Quillen's small object argument, any $H \in \mathrm{DirGr}$ can be mapped to one that has the right lifting property w.r.t. the edge graphs of the $n+1$-subdivided horn inclusions; namely, take the colimit $R_\infty(H)$ of the chain

$$H =: R_0(H) \to R_1(H) \to R_2(H) \to R_3(H) \to \dots$$

[2] By convention, set Ex^0 to be the identity.

where $R_{j+1}(H)$ arises from $R_j(H)$ by pushing on all filling conditions

$$\begin{array}{ccc} \mathsf{edge}(\mathrm{Sd}^{n+1}\Lambda^i_k) & \longrightarrow & R_j(H) \\ \big\downarrow & & \\ \mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_k) & & \end{array}$$

that exist at that stage. Set $G := R_\infty(\mathsf{edge}(\mathrm{Sd}^n \Lambda^0_2))$. We now only have to provide a specific lifting problem w.r.t. the edge graph of an n-subdivided horn inclusion that G *fails*. That will be

$$\begin{array}{ccc} \mathsf{edge}(\mathrm{Sd}^n \Lambda^0_2) & \xrightarrow{\text{canonical}} & R_\infty(\mathsf{edge}(\mathrm{Sd}^n \Lambda^0_2)) \\ \big\downarrow & \overset{(?)}{\nearrow} & \\ \mathsf{edge}(\mathrm{Sd}^n \Delta_2) & & \end{array} \tag{1.1}$$

To prove the lift impossible, we need

Lemma 1.8. *For vertices A, B of a connected graph, write $d(A,B)$ for the least integer p such that there exists a chain of p edges connecting A and B: $A - \bullet - \bullet \cdots \bullet - \bullet - B$. (If the graph happens to be oriented, ignore the orientation of the edges.)*

Write $\partial\Delta_k$ for the boundary of the standard k-simplex. Let A and B be vertices of $\mathsf{edge}(\mathrm{Sd}^{n+1}\partial\Delta_k)$, thought of as a subgraph of $\mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_k)$. If $d(A,B) \leqslant 2^n$ in $\mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_k)$, then the distance of A and B in $\mathsf{edge}(\mathrm{Sd}^{n+1}\partial\Delta_k)$ equals their distance in $\mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_k)$.

Proof. (a) Suppose there is a facet (i.e. top-dimensional face) $\Delta_{k-1} \hookrightarrow \Delta_k$ of our k-simplex such that $\mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_{k-1})$ contains both A and B. We claim there is a retraction $\mathrm{Sd}\,\Delta_k \xrightarrow{r} \mathrm{Sd}\,\Delta_{k-1}$ in *SSet*, a fortiori a retraction $\mathsf{edge}\,\mathrm{Sd}^n(r)$ in DirGr. This proves, in this case, that the distance of A and B cannot be greater in $\mathsf{edge}(\mathrm{Sd}^{n+1}\partial\Delta_k)$ than in $\mathsf{edge}(\mathrm{Sd}^{n+1}\Delta_k)$, since the distance of vertices cannot increase under graph homomorphisms.

Denote the vertices of Δ_k by $[0], [1], \ldots, [k]$, Δ_{k-1} being the facet opposite [k]. Vertices of $\mathrm{Sd}\,\Delta_k$ can be identified with non-empty subsets $\{i_0, i_1, \ldots, i_q\}$ of $\{0, 1, \ldots, k\}$ (listed in increasing order), and p-simplices of $\mathrm{Sd}\,\Delta_k$ can be identified with $p+1$-tuples of subsets that are non-decreasing under containment; for example, $\{1,3\} \subseteq \{0,1,2,3,5\} \subseteq \{0,1,2,3,5\} \subseteq \{0,1,2,3,5,8\}$ is a (degenerate) 3-simplex.

The requisite retraction $\mathrm{Sd}\,\Delta_k \to \mathrm{Sd}\,\Delta_{k-1}$ is now defined by sending $\{i_0, i_1, \ldots, i_q\}$ to itself if $i_q < k$, and to $\{0, 1, 2, 3, \ldots, (k-1)\}$ otherwise; this does extend to a map of tuples, hence to a simplicial map.

(b) If there is no facet of Δ_k as in case (a), then without loss of generality we may assume that A lies on the facet opposite the vertex [0], B lies on the facet opposite the vertex [1], and neither lies on the intersection of these facets, the (codimension 2) face $\mathcal{F}$ with vertices $[2], [3], \ldots, [k]$. Consider a distance-minimizing path in $\mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_k)$ between A and B. If this path contains a vertex V on the subdivided face $\mathcal{F}$, then the argument of (a) can be applied separately to the paths AV and VB to deduce that the distance-minimizing edge path between A and B can proceed on $\mathsf{edge}(\mathrm{Sd}^{n+1}\,\partial\Delta_k)$, as desired.

(c) The missing case is when the path avoids $\mathcal{F}$. We show that any such path must be at least of length 2^{n+1}, contradicting our assumption that $d(A, B) \leqslant 2^n$.

It is enough to verify this for the case of subdivisions of a triangle, $\mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_2)$. Indeed, there is a simplicial map $\Delta_k \to \Delta_2$ that sends [0] to [0], [1] to [1], and [i] to [2] for $i > 1$. Under the induced graph map $\mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_k) \to \mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_2)$, A, B and the "common-face avoiding path between them will be sent to an analogous configuration in $\mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_2)$, and the path length will decrease (at worst).

From now on, we will treat $\mathrm{Sd}^n\,\Delta_2$ as a planar figure, i.e. identify it with its geometric realization. By induction on n, we show that the (non-degenerate) triangles of $\mathrm{Sd}^n\,\Delta_2$ can be partitioned into 2^n disjoint classes, denoted H_i, $i = 0, 1, \ldots, 2^n - 1$, with the following properties:

- Vertices of $\mathrm{Sd}^n\,\Delta_2$ lying on the side opposite the vertex [0] belong to the region H_0.
- Vertices of $\mathrm{Sd}^n\,\Delta_2$ lying on the side opposite the vertex [1] belong to the region H_{2^n-1}.
- Any other vertex of $\mathrm{Sd}^n\,\Delta_2$, with the exception of [2], lies on the common boundary of regions H_i and H_{i+1} for some $0 \leqslant i < 2^n - 1$.

A few pictures should make this more palatable.

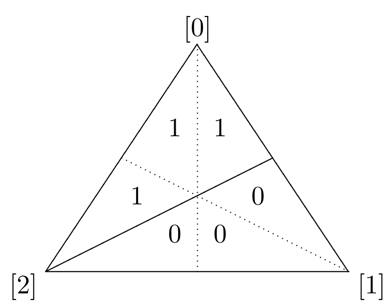

FIG. 1. The partitioning of $\mathrm{Sd}^1\,\Delta_2$.
Triangles belonging to H_i are marked i.

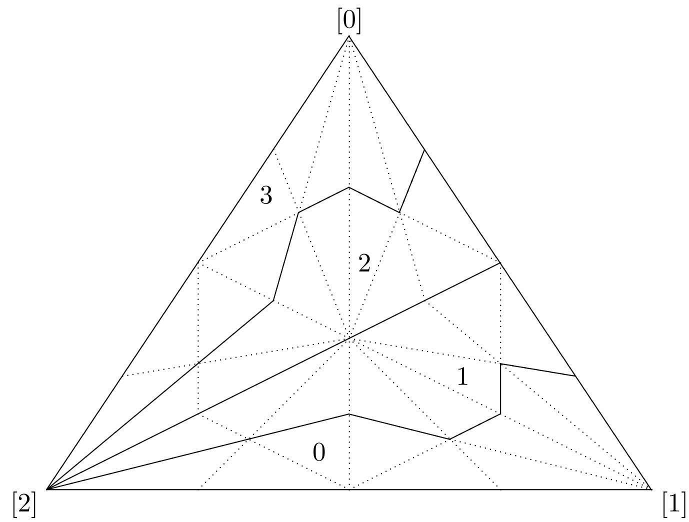

FIG. 2. The partitioning of $\mathrm{Sd}^2\,\Delta_2$.
Only one triangle in each region H_i is marked i.

Recalling that A lies on the side opposite the vertex [0] in $\mathrm{Sd}^{n+1}\,\Delta_2$, B lies on the side opposite the vertex [1], and the path between them is supposed to avoid the vertex [2], this proves the claim: such an edge path must indeed be at least of length 2^{n+1}, since it has to cross 2^{n+1} regions.

The inductive step in defining the partitions H_i is as follows. Suppose it has been done for $\mathrm{Sd}^n\,\Delta_2$. Consider a triangle τ of $\mathrm{Sd}^{n+1}\,\Delta_2$. It is a subdivision of a unique triangle T of $\mathrm{Sd}^n\,\Delta_2$. Let T belong to the partition H_i. τ will belong either to H_{2i} or to H_{2i+1} according to the rules

- If the edge τ and T have in common borders region H_{i-1} in $\mathrm{Sd}^n\,\Delta_2$, then τ will belong to H_{2i} in $\mathrm{Sd}^{n+1}\,\Delta_2$.
- If the edge τ and T have in common borders region H_{i+1} in $\mathrm{Sd}^n\,\Delta_2$, then τ will belong to H_{2i+1} in $\mathrm{Sd}^{n+1}\,\Delta_2$.
- If the edge τ and T have in common borders only region H_i in $\mathrm{Sd}^n\,\Delta_2$, then τ will belong to H_{2i} or H_{2i+1} in $\mathrm{Sd}^{n+1}\,\Delta_2$ according to whether the unique vertex τ and T have in common lies on the region H_{i-1} or H_{i+1} in $\mathrm{Sd}^n\,\Delta_2$. (Exactly one of these must hold by the induction hypothesis.)

Here, for convenience, the edge of $\mathrm{Sd}^n\,\Delta_k$ opposite the vertex [0] should be thought of as bordering a region labelled '-1', and the edge opposite the vertex [1] as bordering a region labelled '2^n'.

The induction hypotheses are verified by a finite amount of labor, enumerating the patterns of color assignments to the boundary of T. □

Now, to finish the proof of Prop. 1.7, $\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0)$ is precisely a zig-zag of length 2^{n+1}. Call its extreme vertices X and Y. (X and Y were those vertices of Δ_2 that bounded the edge missing in Λ_2^0.) Certainly $d(X,Y) > 2^n$ in $\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0)$. We will continue to denote the image of X and Y in $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ under the canonical $\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0) \to R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ by the same letters, and claim that $d(X,Y) > 2^n$ in $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ for all $j > 0$ too. Indeed, $R_{j+1}(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ is defined from $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ via finitely many pushouts of the type

$$\begin{array}{ccc} \mathsf{edge}(\mathrm{Sd}^{n+1}\,\Lambda_k^i) & \longrightarrow & R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0)) \\ \big\downarrow & & \\ \mathsf{edge}(\mathrm{Sd}^{n+1}\,\Delta_k) & & \end{array}$$

and from the Lemma, one sees that two vertices of distance $> 2^n$ in $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ cannot have images of distance $\leqslant 2^n$ in the pushout either: no paths shorter than 2^n are glued on, save perhaps between vertices that already are of that distance (or closer) in $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$. (Note that the distance of vertices cannot increase under graph homomorphism.)

If a lift (?) existed in the diagram (1.1), then a lift would have to exist into $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ for some finite j already, since $\mathsf{edge}(\mathrm{Sd}^n\,\Delta_2)$ is finite. But that cannot happen: for the vertices X, Y of $\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0)$ defined above, the distance between their images in $\mathsf{edge}(\mathrm{Sd}^n\,\Delta_2)$ is 2^n, and the distance between their images in $R_j(\mathsf{edge}(\mathrm{Sd}^n\,\Lambda_2^0))$ is greater than 2^n. □

Corollary 1.9. *There exists a countable, properly increasing chain of fibration classes in SSet each of which, together with the usual weak equivalences, models (in Quillen's sense) the homotopy theory of spaces. Namely, in the n^{th} such define f to be a fibration iff $\mathrm{Ex}^n(f)$ is a Kan fibration.*

One can show (see Prop. 1.11 below) that e.g. the standard simplices are fibrant in these model structures for $n > 0$. (Δ_k itself is a Kan complex only for $k = 0$.) On the other hand, it will no longer be true that every object is cofibrant.

Digression on nerves and subdivision. The edge graph counterexample was artificial, only to serve the purpose of Prop. 1.7. It may be of some combinatorial interest to understand "how gradually" other families of simplicial sets become fibrant. Let $N\mathcal{C}$ denote the nerve of the category $\mathcal{C}$.

Proposition 1.10. *$N\mathcal{C}$ is a Kan complex iff $\mathcal{C}$ is a groupoid.*

A proof of this (no doubt classical) fact can be found in Lee [15]. Note that a category being a groupoid amounts to the solvability of two lifting conditions in the category of (small) categories: these ensure the possibility of left and right "division".

Proposition 1.11. $\mathrm{Ex}(N\mathcal{C})$ *is a Kan complex iff* $\mathcal{C}$ *has the right lifting property w.r.t. the following functors:*

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array} \quad\Longrightarrow\quad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

$$\bullet \longrightarrow \bullet \rightrightarrows \bullet \quad\Longrightarrow\quad \bullet \longrightarrow \bullet \rightrightarrows \bullet \longrightarrow \bullet$$

This is Latch–Thomason–Wilson [14], remark 5.8. In words, these are the criteria for a left calculus of fractions on $\mathcal{C}$. For purely aesthetic reasons, I venture

Conjecture 1.12. For any n, there exist finite categories $\mathcal{C}$ such that $\mathrm{Ex}^n(N\mathcal{C})$ is not a Kan complex, but $\mathrm{Ex}^{n+1}(N\mathcal{C})$ already is.

Now the method of this paper applies to functors other than Ex. Suppose one has a compatible system of subdivisions of the affine simplices (thought of just as simplicial complexes with ordered vertices, to be subdivided into ordered simplicial complexes again). That amounts to a functor $\Delta \xrightarrow{\mathrm{sd}} SSet$. If the subdivided simplices allow a systematic "collapse" to the originals (analogous to the last vertex map) then one gets a natural transformation from sd to the Yoneda embedding $\Delta \xrightarrow{y} SSet$. As observed by Kan, these data give rise to the following: a "singular" functor $SSet \xrightarrow{R} SSet$ defined by $X \mapsto \hom_{SSet}(\mathrm{sd}(-), X)$; the Kan extension of sd along y, this being a left adjoint to R; and a natural transformation Θ from the identity to R. There are infinitely many candidates for partial fibrantization functors (one does have to show that Θ_X is a weak equivalence, and R preserves, say, Kan fibrations) and — since any subdivision can be subdivided further, any two subdivisions have a common refinement, but for any subdivision, there is an incompatible one — one expects that the induced cofibration classes, all subclasses of the monomorphisms, form a complicated poset under inclusion. The hard part in all this seems to be explicitly comparing the strengths of different fibrantizations.

The exotic model structures on $SSet$ can be transported, by the usual methods, to groupoids, categories, simplicial universal algebras, sheaves thereof… They may give rise to the same category of cofibrations in the target category, nonetheless; for example, this happens with topological spaces, owing to the fact that the geometric realization of a subdivided simplex is homeomorphic to the original.

2. Quillen uniqueness

The statement is nearly longer than its proof:

Proposition 2.1. *Let $\mathcal{M}$ be a cofibrantly generated model category with weak equivalences* W*. Assume that every set of maps in $\mathcal{M}$ permits the small object argument, and that the subcategory* W *is closed under transfinite composition. Suppose that for each $\lambda \in \Lambda$, where Λ is a set, the class of maps* fib_λ *forms — together with* W *— a cofibrantly generated Quillen model structure on $\mathcal{M}$. Then so does* $\bigcap_{\lambda\in\Lambda} \mathsf{fib}_\lambda$*.*

Corollary 2.2. *Dually (keeping the assumptions on $\mathcal{M}$ and* W*) suppose* cof_λ *forms — together with* W *— a cofibrantly generated Quillen model structure on $\mathcal{M}$ for each $\lambda \in \Lambda$. There exists a least class* cof *containing all* cof_λ *such that* cof *is part of a model structure on $\mathcal{M}$ with the same class* W *of weak equivalences. This structure is cofibrantly generated as well.*

Proof. Consider the adjunction $\mathcal{M} \underset{\Delta}{\overset{\coprod}{\leftrightarrows}} \mathcal{M}^\Lambda$ where Δ is the diagonal functor. Its left adjoint sends a Λ-indexed family of objects to their coproduct. Put the product model structure on $\mathcal{M}^\Lambda$, i.e. let a morphism $\langle m_\lambda \mid \lambda \in \Lambda\rangle$ be a weak equivalence iff each $m_\lambda \in \mathsf{W}$, and a fibration iff m_λ is a fibration in $\mathcal{M}$ in the model structure corresponding to cof_λ. This gives a cofibrantly generated model structure on $\mathcal{M}^\Lambda$. Use Thm. 1.4 to conclude that Δ creates a Quillen structure on $\mathcal{M}$. Indeed, suppose $\mathsf{W} \cap \mathsf{cof}_\lambda = \mathsf{cof}(J_\lambda)$, J_λ a set, for $\lambda \in \Lambda$. The set J of generating acyclic cofibrations for $\mathcal{M}^\Lambda$ can be taken to be those Λ-tuples that contain an element of J_λ in the λ-coordinate, and are the initial map (the identity map on the initial object) at every other $\lambda' \in \Lambda$. $f \in \mathsf{cell}(\coprod(J)) \subset \operatorname{mor} \mathcal{M}$ is a transfinite composition of pushouts of elements of J_λ. But each such pushout is a weak equivalence (by the assumption that each J_λ is part of a model structure) and thus so is their composition (by the assumption that weak equivalences are closed under transfinite composition). $\Delta(f)$ will be a weak equivalence in $\mathcal{M}^\Lambda$ by definition. So Thm. 1.4 does apply. □

Remark 2.3. It does not seem to follow from Quillen's original axioms that weak equivalences are closed under transfinite composition, but I am aware of no counterexamples. If the model structure is simplicial (or if it is defined by the algebraic-logical means in [6]) then weak equivalences are closed under filtered colimits in the category of morphisms (a fortiori closed under transfinite composition).

Remark 2.4. Chain complexes in Grothendieck abelian categories, simplicial sets, simplicial sheaves, simplicial algebras, symmetric spectra, their sheaves, diagram categories and localizations are all examples of 2.1, since they are locally presentable categories. There do exist categories where every set of maps permits the small object argument besides locally presentable ones; see [1] II. 14 for a broad class of examples.

Remark 2.5. In the Quillen model structure thus created on $\mathcal{M}$, cofibrations are of course $\mathsf{cof}(\bigcup_{\lambda\in\Lambda} I_\lambda)$ where I_λ is a set generating cof_λ. Still, it doesn't seem possible to deduce directly from Thm. 1.2 that one can pass to unions of generating (acyclic) cofibrations.

Corollary 2.6. (independence from (co)fibrations)
Let $\mathcal{M}$, W be as in Prop. 2.1. The cofibrantly generated homotopy model structures the pair may belong to are all in the same Quillen equivalence class of homotopy theories.

Proof. Suppose that both cof_1 and cof_2 are possible cofibration classes. Let cof_3 be their least upper bound, and denote by $\mathcal{M}_i$ $(i = 1, 2, 3)$ $\mathcal{M}$ as a model category with cof_i serving as cofibrations. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are Quillen-equivalent by a "zig-zag of identities" $\mathcal{M}_1 \leftrightarrows \mathcal{M}_3 \rightleftarrows \mathcal{M}_2$. (The identity on $\mathcal{M}$ with codomain $\mathcal{M}_3$ is to be thought of as the left adjoint.) □

3. A set or a class?

One could obviously ask many questions now about the po'set' of possible cofibration classes, under hypotheses of various strengths.

Question 3.1. For any Grothendieck topos $\mathcal{E}$, let $\mathsf{cof}_{\mathcal{E}^{\Delta^{op}}}$ be the collection whose elements are classes of morphisms in $\mathcal{E}^{\Delta^{op}}$ that, when taken to be cofibrations, form a cofibrantly generated Quillen model category on simplicial objects in $\mathcal{E}$ together with the usual (locally defined) weak equivalences. $\mathsf{cof}_{\mathcal{E}^{\Delta^{op}}}$ is non-empty, since the class of all monomorphisms belongs to it. Does $\mathsf{cof}_{\mathcal{E}^{\Delta^{op}}}$ have a proper class, or merely a set of elements?

The reason for singling out these test cases is that they are important in applications (they include simplicial sheaves and simplicial diagrams by definition) and — as far as categorical properties are concerned — they may be paradigmatic.

Remark 3.2. We have been informal (or, rather, we have neglected to say) in what axiomatic set theory we interpret statements about collections being *classes* as opposed to *sets*. It seems that Gödel-Bernays set theory is a powerful enough framework for these purposes, and Morse-Kelly set theory certainly is. Alternatively, use a (single) Grothendieck universe above the usual V.

Remark 3.3. Order the elements of $\mathsf{cof}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$ by inclusion. Prop. 2.1 implies that any set of elements in this collection has a least upper bound. If $\mathsf{cof}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$ is merely a set, then necessarily (if it is a proper class, then perhaps...) it has a maximal element. It would be interesting to know whether the class of monos is maximal.

One knows from [6] that any choice of a small site of definition for $\mathcal{E}$, i.e. small category $\mathcal{C}$ with a Grothendieck topology J such that $\mathcal{E}$ is equivalent to $\mathrm{Sh}(\mathcal{C}, J)$, gives an element of $\mathsf{cof}_{\mathcal{E}^{\Delta^{\mathrm{op}}}}$: take the sheafification of the class of Bousfield-Kan cofibrations in $\mathrm{Pre}(\mathcal{C})^{\Delta^{\mathrm{op}}}$. This map from sites for $\mathcal{E}$ to cofibration classes in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ is neither injective nor surjective. For example, none of the cofibration classes found for *SSet* in Cor. 1.9 (with $n > 0$) can be constructed this way, and in fact, the recipe yields the class of monos in *SSet* for any choice of site of definition for the topos *Set*.

The clues pertaining to Question 3.1 are meager, and not directly linked to simplicial Quillen model categories. For any category $\mathcal{C}$, define $\mathsf{fact}_{\mathcal{C}}$ as the class of small-generated weak factorization systems $\langle \mathsf{C}, \mathsf{F} \rangle$ ("cofibration-acyclic fibration" pairs) ordered by inclusion on C, dropping the requirement that they are cofibrations for some *prescribed* class of weak equivalences.

Remark 3.4. J. Rosický observed that any weak factorization system is part of *a* Quillen model category: namely, take the weak equivalences to be the class of *all* maps. Quillen's axioms (save the first, the existence of enough limits and colimits) are satisfied tautologously. If the category of models has an initial or a terminal object, then the homotopy category corresponding to this structure is a trivial groupoid (with exactly one morphism from any object to any other). See [2] Ex. 3.7.

For a special (but fairly important) class of toposes, a bit can be said about $\mathsf{fact}_{\mathcal{C}}$. Call a category *atomic* if it possesses a set of objects $\mathcal{A}$ (the "atoms") such that any monomorphism $X \xrightarrow{m} Y$ is isomorphic to a canonical inclusion $\coprod_{i \in I} A_i \to \coprod_{i \in J} A_i$ with $A_i \in \mathcal{A}$, $I \subseteq J$, from a coproduct of atoms (repetitions permitted) into the coproduct of a larger (or identical) collection of atoms. (This expression for m is not assumed to be unique.)

Proposition 3.5. *Let $\mathcal{C}$ be an atomic category, C a subclass of the monomorphisms of $\mathcal{C}$ containing all isomorphisms, closed under pushout, transfinite composition and retracts. Then $\mathsf{C} = \mathsf{cof}(G)$ for some set G of morphisms. Moreover, there exists only a set of such 'cofibration classes' C.*

Proof. Consider the monos of $\mathcal{C}$ that arise as $\coprod_{i \in I} A_i \to \coprod_{i \in J} A_i$ with the property that any given atom occurs as A_i, with i running over I, at most once, and any particular atom occurs as the value A_i, i running over J, at most twice. Obviously, there is only a set M of such monos. Given C as above, let G be $\mathsf{C} \cap M$; then $\mathsf{cof}(G) \subseteq \mathsf{C}$. The claim is $\mathsf{cof}(G) = \mathsf{C}$. Indeed, given $m \in \mathsf{C}$, write it (up to isomorphism) in the form $\coprod_{i \in I} A_i \to \coprod_{i \in J} A_i$; we will find a retract (in the category of morphisms of $\mathcal{C}$) m' of m such that $m' \in \mathcal{M}$.

More precisely, for each atom $A \in \mathcal{A}$, let I_A be its 'multiplicity' in I, i.e. $I_A = \{i \in I \mid A_i = A\}$, and analogously for J_A. $\coprod_{I_A} A \to \coprod_{J_A} A$ retracts to the coproduct inclusion $A \to A \coprod A$ if I_A is non-empty and J_A properly contains I_A; $\coprod_{I_A} A \to \coprod_{J_A} A$ retracts to the identity $A \to A$ if $I_A = J_A$ is non-empty; $0 \to \coprod_{J_A} A$ retracts to $0 \to A$, where 0 is the initial object (the empty coproduct) if J_A is non-empty. $m \to m'$ is to be the coproduct of these retractions. Since C is closed under retracts of maps, $m' \in \mathsf{C} \cap M = G$.

Via a transfinite composite of coproducts of identity maps on atoms with m', one can build a $\coprod_{i \in I} A_i \to \coprod_{i \in K} A_i$, with $J \subseteq K$, that belongs to $\mathsf{cof}(G)$. (Note that any identity morphism is in $\mathsf{cof}(G)$, and $\mathsf{cof}(G)$ is closed under – even infinite – coproducts of maps, since those can be written as (transfinite) composites of pushouts.) Via another retract, $m \cong \coprod_{i \in I} A_i \to \coprod_{i \in J} A_i \in \mathsf{cof}(G)$ then. □

The main examples of atomic categories are the *atomic toposes* of Barr and Diaconescu [4]. In an atomic topos, the decomposition into atoms is essentially unique, and the subobject lattice of any object X is a complete atomic Boolean algebra (on the atoms in the decomposition of X). Probably the best-known example of such a topos is $G - Set$, where G is a discrete group. Here the atoms are the transitive G-sets; the decomposition into atoms is the decomposition into orbits. More generally, a presheaf topos Pre($\mathcal{C}$) is atomic iff $\mathcal{C}$ is a groupoid. If G is a topological group, continuous $G - Set$ is an atomic topos as well. One has a complete characterization for atomic topoi in terms of sites.

Unfortunately, Prop. 3.5 implies nothing for 3.1, since $\mathcal{E}^{\Delta^{\mathrm{op}}}$ is never an atomic category — though perhaps it supports the intuition that the answer is 'set' when one restricts cofibration classes to be contained within the monomorphisms. As a source of the opposite intuition, however, one has

Proposition 3.6. *Suppose a locally presentable category $\mathcal{C}$ possesses a weak factorization system $\langle \mathsf{C}, \mathsf{F} \rangle$ such that C is not generated by any* set *of morphisms. Then $\mathsf{fact}_{\mathcal{C}}$ has the size of a proper class, and in fact contains a properly increasing sequence of the order type of all ordinals.*

Proof. Select an arbitrary subset I of C. By the transfinite small object argument, it generates a weak factorization system on $\mathcal{C}$ whose left class $\mathsf{cof}(I)$ — namely, the closure of I under pushouts, transfinite compositions and retracts — must be contained in C; moreover, it must be a proper subclass of C, since C itself is not generated by any set. This allows one to properly extend $\mathsf{cof}(I)$ as $\mathsf{cof}(I \cup i)$, where i is any morphism from $\mathsf{C} - \mathsf{cof}(I)$. Do this at successor ordinals; at limit ordinals, take the supremum of the generating sets already constructed. □

Note that all the cofibration classes thus constructed are contained in C. (C is not an element of $\mathsf{fact}_{\mathcal{C}}$ itself, by our convention, since it is not small-generated.)

The following examples where Prop. 3.6 applies can be found in [2], as Prop. 3.4 and Cor. 3.5.

Theorem 3.7. • *Let Poset be the category of partially ordered sets and order-preserving maps. There is a weak factorization system on Poset whose left ("cofibration") class is the class of regular monomorphisms, i.e. inclusions of subposets. This factorization system is not generated by a set of cofibrations.*
• *There is a weak factorization system on Cat, the category of small categories and functors, whose left class consists of full functors. This factorization system is not generated by a set.*

So *Poset* does contain a proper class of small-generated cofibration classes within the monomorphisms. Note that neither *Poset* nor *Cat* is a topos, though they are locally finitely presentable categories.

Note that in any category $\mathcal{C}$, $\langle \text{all morphisms}, \text{isomorphisms} \rangle$ form a weak (in fact, orthogonal) factorization system that's small-generated if $\mathcal{C}$ is locally presentable. It is obviously maximal in $\mathsf{fact}_{\mathcal{C}}$.

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