Chaotic expansion and smoothness of some functionals of the fractional Brownian motion

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Abstract

This paper deals with some additive functionals of the fractional Brownian motion that arise as limits in law of some occupation times of this process. In concrete, this functionals are obtained via the Cauchy principal value and the Hadamard finite part. We derive some regularity properties of theses functionals in Sobolev–Watanabe sense.

Key words: Fractional Brownian motion, Additive functionals, Local time, Chaotic expansion, Fractional derivative, Hilbert transform, Sobolev–Watanabe spaces.

1 Introduction

Fractional Brownian motion (fBm for brevity) is a generalization of ordinary Brownian motion that has been used successfully to model a variety of natural phenomena such as terrains, coast lines, rivers, financial data and clouds. It is a self–similar Gaussian process with stationary increments. The main difference between fBm and standard Brownian motion is that while the increments in Brownian motion are independent they are dependent in fBm. This process and related transformations are actually the main subject of a lot of research groups in the theory of stochastic analysis, financial

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mathematics, fractal analysis of computer traffic and telecommunications, etc...

Let $\{B_t^H : t \in [0,T]\}$ be a real valued fBm. For all $0 \leq t \leq T$, we define the random occupation measure $\mu_t(\cdot)$ by $\mu_t(A) = \int_0^t \mathbb{1}_A(B_s^H)ds$, where A is a Borel set of $\mathbb{R}$ and $\mathbb{1}_A(\cdot)$ is its characteristic function. It is well known by Berman (1973) and Geman and Horowitz (1980) that $\mu_t(\cdot)$ has a density with respect to the Lebesgue measure. We will denote it here by $\ell^H(t,x)$. The family $\{\ell^H(t,x) : t \geq 0, x \in \mathbb{R}\}$ is called the local time associated with the fBm B^H. Moreover, $\ell^H(t,x)$ has a version which is almost surely continuous as well as uniformly continuous on any compact set, and satisfies the so called occupation density formula

$$\int_0^t f(B_s^H)ds = \int_{\mathbb{R}} f(x)\ell^H(t,x)dx,$$

for any bounded Borel function f. Moreover, the two processes $\{\ell^H(\lambda t, \lambda^H x) : t \geq 0\}$ and $\{\lambda^{1-H}\ell^H(t,x) : t \geq 0\}$ have the same law for every $\lambda > 0$. This is a consequence of the self–similarity of B^H i.e. $\{B_{ct}^H, t \geq 0\}$ and $\{c^H B_t^H, t \geq 0\}$ have the same law.

One can also write

$$\ell^H(t,x) = \int_0^t \delta_x(B_s^H)ds.$$

For ease of notations we shall omit the superscript H for $\ell^H(t,x)$.

In what follows we shall consider certain important singular operators, certain inequalities and properties for those operators such as fractional derivative and Hilbert transform of a real function.

Let $0 < \delta < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a function that belongs to $\mathcal{C}^\delta(\mathbb{R}) \cap L^1(\mathbb{R})$ where $\mathcal{C}^\delta(\mathbb{R})$ is the space of locally δ–Hölder continuous functions on $\mathbb{R}$. For $\delta > \gamma > 0$ we define $D_\pm^\gamma f$ by

$$D_\pm^\gamma f(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^\infty \frac{f(x) - f(x \mp y)}{y^{1+\gamma}} dy.$$

The operators D_+^γ and D_-^γ are called right–handed and left–handed *Marchaud fractional derivatives* of order γ respectively.

We put $D^\gamma := D_+^\gamma - D_-^\gamma$.

It is known form Hardy and Littlewood (1928) that $D_\pm^\gamma f$ is $(\delta - \gamma)$–Hölder continuous when f is δ–Hölder continuous for any $0 < \gamma < \delta$.

Fractional derivatives and integrals usually known as *fractional calculus*, have many uses and they themselves have arisen from certain requirements in applications, such as fractional integro–differentiation which has now become a significant topic in mathematical analysis. Applied mathematicians and scientists found the fractional calculus useful in various fields namely quantitative biology, electrochemistry, transport theory, probability and potential theory to mention a few. We refer the reader for a complete survey on the fractional integrals and derivatives to the book by Samko *et al.* (1993) (and the references therein). This book was focused on the evaluation of fractional integrals and derivatives of concrete functions and to applications to diffusion problems.

For $\gamma = 0$ we define D^0 as

$$D^0 f(x) := \int_0^\infty \frac{f(x+y) - f(x-y)}{y} dy.$$

Therefore for any $0 \leq \gamma < 1$ we have

$$D^\gamma f(x) = \kappa(\gamma) \int_0^\infty \frac{f(x+y) - f(x-y)}{y^{1+\gamma}} dy,$$

where $\kappa(\gamma) = \frac{\gamma}{\Gamma(1-\gamma)}$ for $0 < \gamma < 1$ and $\kappa(0) = 1$.

Let the function f belongs to $L^2(\mathbb{R})$. We consider the Hilbert transform of the function f defined by

$$\mathcal{H} f(x) := \frac{1}{\pi} \left(v.p. \left(\frac{1}{\cdot} \right) * f \right) (x).$$

Then $D^0 = \pi \mathcal{H}$.

From the theory of singular integrals it is known that the operator D^0 maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover for any $f \in L^p(\mathbb{R})$, $p > 1$

$$\left\| D^0 f \right\|_{L^p(\mathbb{R})} \leq c_p \left\| f \right\|_{L^p(\mathbb{R})}, \tag{1.1}$$

where c_p depends only on p. However (1.1) fails in the case $p = 1$ in which f belongs to $L^1(\mathbb{R})$. It is also known that $D^0 f$ still exists for almost everywhere in this case, but $D^0 f$ does not necessary belong to $L^1(\mathbb{R})$. If f belongs to $L^p(\mathbb{R}) \cap \mathcal{C}^\delta(\mathbb{R})$ for $p > 1$ then $D^0 f$ exists for all x and preserves the class of Hölder continuous functions of order $\delta > 0$. In the particular case ($p = 2$) the operator $\pi^{-1} D^0 = \mathcal{H}$ is an isometry on $L^2(\mathbb{R})$ and $\mathcal{H}^{-1} = -\mathcal{H}$. For more deep properties of this operator we refer the reader to Titchmarsh (1948) Chap. V.

Integrals transformations including Fourier and Hilbert transforms play a significant role in signal processing. For a complete description of selected applications of Hilbert transforms which serves as a theoretical basis of the complex notations of signals we refer to the book by Hahn (1996).

The above mentioned operators $D_{\pm}^{\gamma}$ and $\mathcal{H}$ appeared also in the following limit results.

Let $\ell(t,x)$ be the local time of the fBm B^H at level x and time t. If f is a function belonging to $L^1(\mathbb{R})$ such that $\overline{f} := \int_{\mathbb{R}} f(x)dx \neq 0$ then the sequence of processes

$$\frac{1}{n^{1-H}} \int_0^{nt} f(B_s^H)ds,\ t \geq 0$$

converges in law as n go to infinity to the process $\overline{f}\ell(t,0)$, $t \geq 0$.

If f is a function such that $\mathcal{H}f$ belongs to $L^1(\mathbb{R})$ and $\overline{\mathcal{H}f} \neq 0$ then the sequence of processes

$$\frac{1}{n^{1-H}} \int_0^{nt} f(B_s^H)ds,\ t \geq 0$$

converges in law as n go to infinity to the process $\overline{\mathcal{H}f}\mathcal{H}\ell(t,\cdot)(0)$, $t \geq 0$.

Let $0 < \gamma < \delta < \frac{1}{2H} - \frac{1}{2}$ and f belongs to $\mathcal{C}^{\delta}(\mathbb{R}) \cap L^1(\mathbb{R})$, then the sequence of processes

$$\frac{1}{n^{1-H(1+\gamma)}} \int_0^{nt} D_{-}^{\gamma} f(B_s^H)ds,\ \ t \geq 0,$$

converges in law as $n \to +\infty$ to the process $\overline{f}D_{+}^{\gamma}\ell(t,\cdot)(0)$, $t \geq 0$.

More discussions on limit theorems of this kind can be found in Yamada (1986, 1996) for Brownian motion, Fitzsimmons and Getoor (1992) for stable Lévy processes and Shieh (1996) for fractional Brownian motion.

Theses transformations for the local time of the Brownian motion have also been considered by a number of authors (Ezawa *et al.* (1975), Yor (1982), Biane and Yor (1987) and Bertoin (1989, 1990)) for various motivations and different points of view.

The above description represents the motivation of the study of the chaotic expansion and regularity properties of the local time and related transformations.

In this paper we go one step further by presenting another application of the fractional derivatives and integrals. Concretely the operators $D_{\pm}^{\gamma}$ and D^0 are power full to give simple representations of some additive functionals of stochastic processes (see e.g. Yamada (1984), for the Brownian motion).

The main goal of this paper is to establish a unified result on the expansion in Wiener chaos of the fractional derivative and the Hilbert transform of the local time of the fBm (Theorem 3.1). Regularity in Sobolev–Watanabe spaces are given for theses transformations. Our main concrete application is the study of the chaotic expansion and the regularity in Sobolev–Watanabe spaces of some additives functionals which arise as limits of suitable normalized occupation times of the fBm. (Theorem 4.1, Theorem 4.2 and Corollary 4.1). An advantage using Sobolev–Watanabe spaces is the ability to use Wiener chaos expansion and related properties and techniques to further study the regularity of some functionals of the fBm or roughly speaking of the underlying Wiener process thank to the representation (2.3) and to obtain approximation results of the local time in some functional spaces (Proposition 5.2 and Proposition 5.3).

We briefly explain the structure of the rest of the paper. The second section provides the background material needed and the statement of the preliminary result. The third section contains the main result and the forth section deals with main applications. Finally in the last section we state and prove some technical intermediate results and then explain how the main result can be deduced.

2 Preliminary results

Let us first recall some facts on the Wiener chaos expansion.

Let $W = \{W_t : t \in [0,T]\}$ be a standard Brownian motion defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, $(\mathcal{F}_t)_{0\le t\le T}$ being the completed standard Brownian filtration. Let $I_n(f_n)$ denote the multiple Itô stochastic integral of a symmetric kernel $f_n \in L^2([0,T]^n)$ with respect to the Wiener process W.

In the theory of stochastic analysis on Wiener space it is well known that any square integrable random variable F can be written as $F = \sum_{n=0}^{\infty} I_n(f_n)$. This form is called the chaotic expansion of F. The Ornstein–Uhlenbeck operator $\mathbf{L}$ is defined by

$$\mathbf{L}F = -\sum_{n=0}^{\infty} nI_n(f_n).$$

If $p \in (1,\infty)$ and $\alpha \in \mathbb{R}$ we define the Sobolev–Watanabe spaces (see, Watanabe (1984)) $\mathbb{D}^{\alpha,p}$, as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(\mathrm{Id} - \mathbf{L})^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where Id stands for the mapping identity.

It is known that a random variable F belongs to $\mathbb{D}^{\alpha,2}$ if and only if its chaotic expansion $\sum_{n=0}^{\infty} I_n(f_n)$ satisfies

$$\sum_{n=0}^{\infty}(1+n)^{\alpha} n! \left\| f_n \right\|_2^2 < +\infty,$$

where $\|f_n\|_2 = \|f_n\|_{L^2([0,T]^n)}$.

For a complete survey on this topic we refer the reader to the book by Nualart (1995).

Let $p_\sigma(x)$ be the centered Gaussian kernel with variance $\sigma > 0$. We denote by $\mathbf{H}_n$ the n–th Hermite polynomial defined for $n \geq 1$ by

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right) \quad \text{and} \quad \mathbf{H}_0(x) = 1.$$

Let $B^H = \{B_t^H : t \in [0,T]\}$ be a real valued standard fractional Brownian motion (1–fBm for brevity) with Hurst parameter $H \in (0,1)$. It is well known that B^H is a centered Gaussian process and admits the following integral representation

$$B_t^H = \int_0^t K_H(t,s) dW_s.$$

The kernel $K_H(t,s)$ for $s < t$ is given by,

$$K_H(t,s) = c_H (t-s)^{\mu} - \mu c_H \int_s^t (r-s)^{\mu-1}\left(1 - \left(\frac{s}{r}\right)^{-\mu}\right) dr, \tag{2.2}$$

c_H being a constant and $\mu = H - \frac{1}{2}$.

The covariance function $R_H(s,t) = \mathbb{E}\left(B_s^H B_t^H\right)$ of B^H has the explicit form

$$\begin{aligned} R_H(s,t) &= \int_0^{s\wedge t} K_H(t,r) K_H(s,r) dr \\ &= \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \end{aligned}$$

Moreover, the sample paths of B^H are a.s. $(H-\varepsilon)$–Hölder continuous for any $\varepsilon > 0$.

Berman (1973) and Geman and Horowitz (1980) show that the local time $\ell(t,x)$ of B^H exists and has Hölder continuous modification of order $\frac{1}{2H} - \frac{1}{2} - \varepsilon$ in space and of order $1 - H - \varepsilon$ in time for any $\varepsilon > 0$.

Let $\{W_t : t \in [0,T]^N\}$ be a real valued N–parameter Wiener process. Given a vector $H = (H_1, \dots, H_N)$ with all his components in $(0,1)$ and a fixed point T in $\mathbb{R}_+$, we define the N–parameter fractional Brownian motion (N–fBm for brevity), denoted by $\{B_t^H : t \in [0,T]^N\}$, as

$$B_t^H = \int_{[0,t]} \prod_{j=1}^{N} K_{H_j}(t_j, s_j) dW_s, \tag{2.3}$$

for $s = (s_1, \dots, s_N)$ and $t = (t_1, \dots, t_N)$ and where K_{H_j}, $j = 1, \dots, N$ are the kernels given in (2.2).

One can show that $\{B_t^H : t \in [0,T]^N\}$ is a centered Gaussian process with the covariance function

$$R_H(s,t) = \prod_{j=1}^{N} R_{H_j}(s_j, t_j) = R_H(t,s).$$

To simplify the notations, we will assume $R_H(s) = R_H(s,s) = \prod_{j=1}^N s_j^{2H_j}$.

For a given victor $H = (H_1, \dots, H_N)$ we shall denote by $K_H(t,s)$ the function $\prod_{j=1}^N K_{H_j}(t_j, s_j)$, $H^* = \max\{H_1, \dots, H_N\}$, $\overline{H} = \frac{1}{N}\sum_{j=1}^N H_j$ and $\frac{1}{\widehat{H}} = \frac{1}{N}\sum_{j=1}^N \frac{1}{H_j}$.

The local time of B^H is defined, for any $t \in [0,T]^N$ and for every $x \in \mathbb{R}$, as

$$\ell(t,x) = \int_{[0,t]} \delta_x(B_s^H) ds$$

where δ_x denotes the Dirac function on $\mathbb{R}$. The function $\ell(t,x)$ has Hölder continuous modification of order $\frac{N}{2\widehat{H}} - \frac{1}{2} - \varepsilon$ in space and of order $N(1-\overline{H})-\varepsilon$ in time for any $\varepsilon > 0$ (see e.g. Xiao (1997)).

In the sequel the multiple stochastic integrals I_n are interpreted with respect to the N–parameter Wiener process W and $du = du_1 \dots du_N$.

Let us now recall a recent result due to Eddahbi *et al.* (2002) concerning the chaotic decomposition of the local time of the fBm at any level $x \in \mathbb{R}$ and its Sobolev–Watanabe regularity.

Theorem 2.1 *Let $\ell(t,x)$ be the local time of the N–fBm B^H. Then $\ell(t,x)$ has the following chaos expansion :*

$$\ell(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot; t, x))$$

where, for $(t^1, \dots, t^n) \in ([0,T]^N)^n$

$$f_n(t^1, \dots, t^n; t, x) = \int_{[t^1 \vee \cdots \vee t^n, t]} \frac{p_{R_H(s)}(x)}{[R_H(s)]^{\frac{n}{2}}} \mathbf{H}_n \left(\frac{x}{[R_H(s)]^{\frac{1}{2}}} \right) \prod_{i=1}^{n} K_H(s, t^i) ds.$$

Consequently $\ell(t,x)$ *belongs to* $\mathbb{D}^{\alpha,2}$ *for every* $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2}$.

Proof: See Eddahbi *et al.* (2002). □

3 Main result

We establish the chaotic expansion for the fractional derivative of $\ell(t,x)$ and for its Hilbert transform. This extends the results by Eddahbi *et al.* (2000) to the fBm setting.

For any $0 < \gamma < \frac{1}{2H^*} - \frac{1}{2}$ the fractional derivative $D^{\gamma}_{\pm}$ of $\ell(t,x)$ is given by

$$D^{\gamma}_{\pm} \ell(t, \cdot)(x) = \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{\infty} \frac{\ell(t,x) - \ell(t, x \mp y)}{y^{1+\gamma}} dy.$$

To simplify the notations we simply use $D^{\gamma}_{\pm}\ell(t,x)$ instead of $D^{\gamma}_{\pm}\ell(t,\cdot)(x)$.

The main result of this section generalizes those of Eddahbi *et al.* (2000) to a real valued N–fBm and presents an unified approach to the study of the fractional derivative and the Hilbert transform of local times.

Theorem 3.1 *Let* $0 < \gamma < \frac{1}{2H^*} - \frac{1}{2}$ *and* $\mathbf{D} \in \{D^{\gamma}_{+},\ D^{\gamma}_{-},\ D^{\gamma},\ D^{0}\}$. *Then for every* $t \in [0,T]^N$ *and* $x \in \mathbb{R}$, $\mathbf{D}\ell(t,x)$ *has the following chaos expansion*

$$\mathbf{D}\ell(t,x) = \sum_{n=0}^{\infty} I_n \left(\mathbf{D} f_n \left(\cdot ; t, x \right) \right),$$

where

$$f_n(t^1, \dots, t^n; t, x) = \int_{[t^1 \vee \cdots \vee t^n, t]} \frac{p_{R_H(s)}(x)}{[R_H(s)]^{\frac{n}{2}}} \mathbf{H}_n \left(\frac{x}{[R_H(s)]^{\frac{1}{2}}} \right) \prod_{i=1}^{n} K_H(s, t^i) ds.$$

Moreover $\mathbf{D}\ell(t,x)$ *belongs to* $\mathbb{D}^{\alpha,2}$, *for all* $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \gamma$.

Remark 1 *As a consequence of the Rademacher–Menchov lemma (see e.g. Stout (1984)) if* $\{J_n : n \geq 1\}$ *is an orthogonal sequence of random variables such that* $\sum_{n=1}^{\infty} (\log n)^2 \mathbb{E}[J_n]^2$ *is finite, then the series* $\sum_{n=1}^{\infty} J_n$ *converges almost surely. Then the expansions in both Theorems 2.1 and 3.1 converge a.s. to* $\ell(t,x)$ *and* $\mathbf{D}\ell(t,x)$ *respectively.*

Remark 2 *In the spatial case where $H_j = H$ for all $j = 1, \dots, N$, $\mathbf{D}\ell(t,x)$ belongs to $\mathbb{D}^{\alpha,2}$, for all $\alpha < \frac{N}{2H} - \frac{1}{2} - \gamma$. Hence when H is close to zero the local time and its fractional derivative are more regular as functions on ω (the random component). Theorem 3.1 recovers the results of Eddahbi* et al. *(2000) in the particular case when $N = 1$ and $H = \frac{1}{2}$ (Brownian motion case).*

Remark 3 *In the case where $N = 1$ the regularity of $D^\gamma \ell(t,x)$ for $\gamma \geq 0$ in $\mathbb{D}^{\alpha,2}$ depends on the bound $\frac{1}{2H} - \frac{1}{2} - \gamma$ which is exactly the bound of the Hölder regularity of the $D^\gamma \ell(t,x)$ as a function in the space variable. This observation is not clear in the case of the Brownian motion ($H = \frac{1}{2}$) where the bound for both space and time variables of $\ell(t,x)$ and $D^\gamma \ell(t,x)$ are $\frac{1}{2}$ and $\frac{1}{2} - \gamma$ respectively (see e.g. Eddahbi* et al. *(2000)).*

4 Applications: some examples

This section deals with tree additive functionals of the 1–fBm (which means that we take $N = 1$). We deduce the chaotic decomposition and regularity properties for this functionals of the fBm.

For the local time $\ell(t,x)$ we can then write

$$\mathcal{H}\ell(t,\cdot)(x) = \frac{1}{\pi} \int_0^\infty \frac{\ell(t,x+y) - \ell(t,x-y)}{y} dy.$$

Therefore

$$\begin{aligned} D^0 \ell(t,\cdot)(x) &= \int_0^\infty \frac{\ell(t,x+y) - \ell(t,x-y)}{y} dy \\ &= \pi \mathcal{H}\ell(t,\cdot)(x). \end{aligned}$$

Let $A_H(t,x)$ be the continuous additive functional of the fBm B_t^H which corresponds to Cauchy's principal value $v.p.(\frac{1}{x})$. More precisely $A_H(t,x)$ is defined as

$$\begin{aligned} A_H(t,x) &= \int_0^t v.p. \left(\frac{1}{B_s^H - x} \right) ds \\ &= \pi \mathcal{H}\ell(t,\cdot)(x) \\ &= D^0 \ell(t,\cdot)(x). \end{aligned}$$

Next, we will define the additive functional of B^H which corresponds to the Hadamard finite part $p.f.(x_+^{-1-\gamma})$ for $0 < \gamma < \frac{1}{2H} - \frac{1}{2}$, by

$$A_H^\gamma(t,x) = \int_0^t p.f.\left(\frac{1}{(B_s^H - x)_+^{1+\gamma}}\right) ds.$$

Moreover $A_H^\gamma(t,x)$ can be represented in term of the fractional derivative and the Hilbert transform as

$$A_H^\gamma(t,x) = D_+^\gamma \ell(t,x)\cos\pi\gamma + \mathcal{H}(D_+^\gamma \ell(t,\cdot))(x)\sin\pi\gamma. \tag{4.4}$$

The processes $\ell(t,x)$, $A_H(t,x)$ and $A_H^\gamma(t,x)$ are additive functionals of the fBm B^H associated respectively with the Schwartz distributions δ_x, $v.p.(\frac{1}{x})$ and $p.f.(x_+^{-1-\gamma})$. Theses additive functionals appeared in some limit theorems as discussed by Shieh (1996) for the fBm and in Yamada (1996) for the Brownian motion ($H = \frac{1}{2}$). We recall the main limit theorems in which the above functionals appeared as limit processes.

Proposition 4.1 *Let f be a function in $L^1(\mathbb{R})$. Assume that $\overline{f} := \int_{\mathbb{R}} f(x)dx \neq 0$. Then the sequence*

$$\frac{1}{n^{1-H}} \int_0^{nt} f(B_s^H)ds, \quad t \geq 0$$

converges in law as n go to infinity to the processes $\overline{f}\ell(t,0)$, $t \geq 0$. Let f be a function such that $D^0 f$ belongs to $L^1(\mathbb{R})$. Assume that $\overline{D^0 f} \neq 0$. Then the sequence

$$\frac{1}{n^{1-H}} \int_0^{nt} f(B_s^H)ds, \quad t \geq 0$$

converges in law as n go to infinity to the process $\pi^{-2}\overline{D^0 f} A_H(t,0)$, $t \geq 0$.

Proposition 4.2 *Let $0 < \gamma < \beta < \frac{1}{2H} - \frac{1}{2}$. Assume that $f \in \mathcal{C}^\beta(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the sequence*

$$\frac{1}{n^{1-H(1+\gamma)}} \int_0^{nt} D_-^\gamma f(B_s^H)ds, \quad t \geq 0,$$

converges in law as $n \to +\infty$ to the process $\overline{f} D_+^\gamma \ell(t,0)$, $t \geq 0$, and the sequence

$$\frac{1}{n^{1-H(1+\gamma)}} \int_0^{nt} D_+^\gamma f(B_s^H)ds, \quad t \geq 0,$$

converges in law as $n \to +\infty$ to the process $\overline{f} A_H^\gamma(t,0)$, $t \geq 0$.

Now, we derive the chaos expansion and regularity in $\mathbb{D}^{\alpha,2}$ of the additive functionals $A_H^{\gamma}(t,x)$ and $A_H(t,x)$.

Theorem 4.1 *For any $0 < \gamma < \frac{1}{2H} - \frac{1}{2}$ the additive functional $A_H^{\gamma}(t,x)$ has the following chaotic decomposition*

$$A_H^{\gamma}(t,x) = \sum_{n=0}^{\infty} I_n \left(D_+^{\gamma} f_n(\cdot; t, x) \cos \pi\gamma + \mathcal{H}(D_+^{\gamma} f_n(\cdot; t, \cdot))(x) \sin \pi\gamma \right),$$

where

$$f_n(t_1, \ldots, t_n; t, x) = \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{p_{s^{2H}}(x)}{s^{nH}} \, \mathbf{H}_n \left(\frac{x}{s^H} \right) \prod_{i=1}^{n} K_H(s, t_i) ds.$$

Moreover $A_H^{\gamma}(t,x)$ belongs to $\mathbb{D}^{\alpha,2}$, for every $\alpha < \frac{1}{2H} - \frac{1}{2} - \gamma$.

Theorem 4.2 *The additive functional $A_H(t,x)$ has the following chaotic decomposition*

$$A_H(t,x) = \sum_{n=0}^{\infty} I_n \left(D^0 f_n(\cdot; t, x) \right),$$

where the kernels f_n are as in Theorem 4.1. Moreover $A_H(t,x)$ belongs to $\mathbb{D}^{\alpha,2}$, for every $\alpha < \frac{1}{2H} - \frac{1}{2}$.

Proof: The proof is a consequence of Theorem 3.1 and the formula (4.4).

At the end of this section we treat the continuous additive functional which corresponds to the function $(y-x)_+^{\beta-1}$, $x \in \mathbb{R}$ where, $0 < \beta < 1$, is defined by

$$A_{-\beta}^{\gamma}(t,x) = \int_0^t \left(B_s^H - x \right)_+^{\beta-1} ds,$$

and hence it belongs to the class of continuous additive functionals of B_t^H.

By the occupation density formula, we can write

$$\begin{aligned} A_{-\beta}^{\gamma}(t,x) &= \int_{\mathbb{R}} (y-x)_+^{\beta-1} \ell(t,y) dy \\ &= \left((\cdot)_+^{\beta-1} * \ell(t,\cdot) \right)(x). \end{aligned}$$

Let g be a continuous function with compact support. If $\mathrm{I}^{\beta} g$ denotes the β–th integral of the function g defined by

$$\mathrm{I}^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \left((\cdot)_+^{\beta-1} * g \right)(x),$$

then

$$A^{\gamma}_{-\beta}(t,x) = \Gamma(\beta)\mathrm{I}^{\beta}\ell(t,x)\cdot$$

Since, I^{β} is a linear continuous operator from the Banach space $\mathcal{C}(\mathbf{I})$ of continuous functions on the compact set $\mathbf{I}$ to $\mathcal{C}(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$, it is then a closed linear operator.

Hence, Theorem 4.1 leads to the following result

Corollary 4.1 *The additive functional $A^{\gamma}_{-\beta}(t,x)$, $0<\beta<1$, has the following chaos expansion*

$$A^{\gamma}_{-\beta}(t,x) = \Gamma(\beta)\sum_{n=0}^{\infty} I_n\left(\mathrm{I}^{\beta} f_n\left(\cdot;t,\cdot\right)(x)\right).$$

Moreover, the functional $A^{\gamma}_{-\beta}(t,x)$ belongs to $\mathbb{D}^{\alpha,2}$, for all $\alpha < \frac{1}{2H} - \frac{1}{2} + \beta$.

Proof: Comparing the fractional integral I^{β} with the Marchaud fractional derivative, $\mathrm{I}^{\beta}\ell(t,x)$ is formally obtained from $D^{\beta}_{+}\ell(t,x)$ if we replace β with $-\beta$. Hence,

$$\mathrm{I}^{\beta}\ell(t,x) = D^{-\beta}_{+}\ell(t,x),$$

and the Corollary is proved.

5 Proof of the main result

In order to prove the Theorem 3.1, we need the following technical results.

Let $R_H(s,t)$ be the covariance function of 1–fBm. Set

$$Q_H(z) := \begin{cases} \dfrac{R_H(1,z)}{z^H} & \text{if } z \in (0,1] \\ 0 & \text{if } z = 0. \end{cases}$$

The following lemma studies the properties of the function $Q_H(\cdot)$ and the behaviour of

$$\int_0^1 Q_H(z)^n \frac{dz}{z^{H(1+\gamma)}}$$

when n goes to infinity and $0 \le \gamma < \frac{1}{2H^*} - \frac{1}{2}$.

Lemma 5.1 *The function $Q_H(\cdot)$ is continuous with values in $[0,1]$, $Q_H(1) = 1$, strictly increasing and there exists a constant $c(\gamma, H)$ independent of n such that*

$$\int_0^1 Q_H(z)^n \frac{dz}{z^{H(1+\gamma)}} \leq \frac{c(\gamma, H)}{n^{\frac{1}{2H}}}.$$

Proof: The Proof this Lemma can be found in Eddahbi *et al.* (2002).

Proposition 5.1 *Let $0 < \gamma < \frac{1}{2H^*} - \frac{1}{2}$, $\mathbf{D} \in \{D_+^\gamma,\ D_-^\gamma,\ D^\gamma,\ D^0\}$ and $f_n(\cdot;t,x)$ be as in Theorem 3.1. Then the series*

$$\sum_{n\geq 0} I_n\left(\mathbf{D} f_n(\cdot;t,x)\right),$$

is convergent in $L^2(\Omega)$. Moreover, the sum $\sum_{n\geq 0} I_n\left(\mathbf{D} f_n(\cdot;t,x)\right)$ belongs to $\mathbb{D}^{\alpha,2}$ for every $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \gamma$.

Proof: Let us give the proof for D_+^γ (the other cases can be proved similarly and by linearity).

We consider the series

$$\sum_{n\geq 0} I_n\left(D_+^\gamma f_n(\cdot;t,x)\right).$$

Let us first recall an important formula known from Szegö (1939), that is

$$\mathbf{H}_n(y)\exp\left(-\frac{y^2}{2}\right) = \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n}{2}+1}}{n!\sqrt{\pi}} G_{n,0}^g(y)$$

where

$$G_{n,\gamma}^g(y) := \int_0^\infty r^{n+\gamma}\exp\left(-r^2\right) g\left(yr\sqrt{2}\right) dr,\ y\in\mathbb{R} \text{ and } \gamma\geq 0$$

and

$$g(y) := \begin{cases} \cos(y) & \text{if } n\in 2\mathbb{N} \\ \sin(y) & \text{if } n\notin 2\mathbb{N}. \end{cases}$$

Thanks to the expression of the kernels f_n we have

$$D_+^\gamma f_n(t^1,\ldots,t^n;t,x)$$

$$= \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi n!}\int_{[t^1\vee\cdots\vee t^n,t]} \frac{\prod_{i=1}^n K_H(s,t^i)}{[R_H(s)]^{\frac{n+1}{2}}} D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right) ds.$$

Then

$$
\begin{aligned}
&\left\| D_+^\gamma f_n(\cdot;t,x)\right\|_2^2 \\
=\ & \frac{2^{n+1}}{(\pi n!)^2}\int_{[0,t]}\cdots\int_{[0,t]}\int_{[t^1\vee\cdots\vee t^n,t]}\int_{[t^1\vee\cdots\vee t^n,t]} \\
&\times\left[\frac{\prod_{i=1}^n K_H(u,t^i)}{[R_H(u)]^{\frac{n+1}{2}}} D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(u)]^{\frac12}}\right)\right. \\
&\left.\times\frac{\prod_{i=1}^n K_H(v,t^i)}{[R_H(v)]^{\frac{n+1}{2}}} D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(v)]^{\frac12}}\right)\right] dudv\, dt^1\cdots dt^n \\
=\ & \frac{2^{n+1}}{(\pi n!)^2}\int_{[0,t]}\int_{[0,t]}\frac{R_H(u,v)^n}{[R_H(u)R_H(v)]^{\frac{n+1}{2}}} \\
&\times D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(u)]^{\frac12}}\right) D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(v)]^{\frac12}}\right) dudv,
\end{aligned}
$$

where we have used in the last equality the Fubini theorem and the definition of $R_H(u,v)$.

After some simple computations we can write for $\lambda > 0$

$$
\begin{aligned}
G_{n,0}^g\left(\frac{x}{\lambda}\right) - G_{n,0}^g\left(\frac{x-y}{\lambda}\right) &= G_{n,0}^g\left(\frac{x}{\lambda}\right)\left(1-\cos\left(\frac{yr\sqrt{2}}{\lambda}\right)\right) \\
&\quad + G_{n,0}^{g'}\left(\frac{x}{\lambda}\right)\sin\left(\frac{yr\sqrt{2}}{\lambda}\right).
\end{aligned}
$$

Let us set

$$
c_1(\gamma) := \frac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty \frac{1-\cos(y)}{y^{1+\gamma}} dy
$$

and

$$
c_2(\gamma) := \frac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty \frac{\sin(y)}{y^{1+\gamma}} dy.
$$

Then

$$
\frac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty \left(1-\cos\left(\frac{r\sqrt{2}}{\lambda}y\right)\right)\frac{dy}{y^{1+\gamma}} = \frac{r^\gamma 2^{\frac{\gamma}{2}}}{\lambda^\gamma} c_1(\gamma),
$$

and

$$
\frac{\gamma}{\Gamma(1-\gamma)}\int_0^\infty \sin\left(\frac{r\sqrt{2}}{\lambda}y\right)\frac{dy}{y^{1+\gamma}} = \frac{r^\gamma 2^{\frac{\gamma}{2}}}{\lambda^\gamma} c_2(\gamma).
$$

Hence

$$D_+^\gamma G_{n,0}^g\left(\frac{x}{\lambda}\right)=\frac{2^{\frac{\gamma}{2}}}{\lambda^\gamma}\left[c_1(\gamma)G_{n,\gamma}^g\left(\frac{x}{\lambda}\right)+c_2(\gamma)G_{n,\gamma}^{g'}\left(\frac{x}{\lambda}\right)\right].$$

Now using the fact that

$$\sup_{\lambda>0}\sup_{x\in\mathbb{R}}\left|G_{n,\gamma}^g\left(\frac{x}{\lambda}\right)\right|\leq\frac{1}{2}\Gamma\left(\frac{n+\gamma+1}{2}\right),$$

and

$$\sup_{\lambda>0}\sup_{x\in\mathbb{R}}\left|G_{n,\gamma}^{g'}\left(\frac{x}{\lambda}\right)\right|\leq\frac{1}{2}\Gamma\left(\frac{n+\gamma+1}{2}\right),$$

we deduce that

$$\left|D_+^\gamma G_{n,0}^g\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right)\right|\leq\frac{c(\gamma)2^{\frac{\gamma}{2}}}{[R_H(s)]^{\frac{\gamma}{2}}}\Gamma\left(\frac{n+\gamma+1}{2}\right),$$

where $c(\gamma)=\max\{|c_1(\gamma)|,|c_2(\gamma)|\}$.

Therefore

$$\begin{aligned} n!\left\|D_+^\gamma f_n(\cdot;t,x)\right\|_2^2 &\leq c(\gamma)c(n)\int_{[0,t]}\int_{[0,t]}\frac{R_H(u,v)^n\,dudv}{[R_H(u)R_H(v)]^{\frac{n+1+\gamma}{2}}}\\ &= c(\gamma,H,t)c(n)\prod_{j=1}^N\int_0^1 Q_{H_j}(z)^n\frac{dz}{z^{H_j(1+\gamma)}}. \end{aligned}$$

But Lemma 5.1 yields

$$n!\left\|D_+^\gamma f_n(\cdot;t,x)\right\|_2^2\leq c(\gamma,H,t)\frac{c(n)}{n^{\frac{N}{2\widehat{H}}}},$$

$c(\gamma,H,t)$ being a constant which may changes from line to line.

By Stirling formula we have,

$$c(n)\sim\frac{1}{n^{\frac{1}{2}-\gamma}}.$$

Thus

$$\sum_{n\geq1}n!\left\|D_+^\gamma f_n(\cdot;t,x)\right\|_2^2\leq\sum_{n\geq1}\frac{c(\gamma,H,t)}{n^{\frac{1}{2}-\gamma+\frac{N}{2\widehat{H}}}}. \tag{5.5}$$

The right hand side of the above inequality converges for any $\gamma<\frac{N}{2\widehat{H}}-\frac{1}{2}$, but this condition is satisfied since $\gamma<\frac{1}{2H^*}-\frac{1}{2}\leq\frac{N}{2\widehat{H}}-\frac{1}{2}$.

Now, from (5.5) we deduce that the

$$\sum_{n=0}^{\infty} I_n \left(D_+^{\gamma} f_n \left(\cdot ; t, x \right) \right)$$

converges in $\mathbb{D}^{\alpha,2}$, for all $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \gamma$ and the limit is uniform on $\mathbb{D}^{\alpha,2}$ in the space variable. The proof of the proposition is complete. □

Let $\mathcal{E}$ be a Banach space endowed with the norm $\|\cdot\|_{\mathcal{E}}$. We shall denote by $\mathcal{C}(\mathbb{R};\mathcal{E})$, the space of continuous functions on $\mathbb{R}$ endowed with the sup norm $\|\cdot\|_{\infty,\mathcal{E}}$ and by $\mathcal{C}^{\delta}(\mathbb{R};\mathcal{E})$, the space of δ–Hölder continuous $\mathcal{E}$–valued functions endowed with the norm $\|f\|_{\infty,\delta,\mathcal{E}} := \|f\|_{\infty,\mathcal{E}} + \|f\|_{\delta,\mathcal{E}}$, where $\|f\|_{\infty,\mathcal{E}} = \sup_x \|f(x)\|_{\mathcal{E}}$ and $\|f\|_{\delta,\mathcal{E}} = \sup_{x \neq y} \frac{\|f(x)-f(y)\|_{\mathcal{E}}}{|x-y|^{\delta}}$.

Lemma 5.2 *Let γ and δ be real numbers such that $0 < \gamma < \delta$. Let $\mathbf{D} \in \{D_+^{\gamma},\ D_-^{\gamma},\ D^{\gamma}\}$. Then $\mathbf{D}$ is a bounded linear operator from $\mathcal{C}^{\delta}(\mathbb{R};\mathcal{E})$ to $\mathcal{C}(\mathbb{R};\mathcal{E})$. Consequently $\mathbf{D}$ is continuous, hence is a closed operator.*

Proof: Let A be a positive constant and f be a function in the space $\mathcal{C}^{\delta}(\mathbb{R};\mathcal{E})$. Assume that, $\mathbf{D} = D_-^{\gamma}$. We have

$$\begin{aligned}
\left\| D_-^{\gamma} f(x) \right\|_{\mathcal{E}} &= \frac{\gamma}{\Gamma(1-\gamma)} \left\| \int_0^{\infty} \frac{f(x) - f(x+y)}{y^{1+\gamma}} dy \right\|_{\mathcal{E}} \\
&\leq \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{A} \frac{\|f(x) - f(x+y)\|_{\mathcal{E}}}{y^{1+\gamma}} dy \\
&\quad + \frac{\gamma}{\Gamma(1-\gamma)} \int_A^{+\infty} \frac{\|f(x) - f(x+y)\|_{\mathcal{E}}}{y^{1+\gamma}} dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\| D_-^{\gamma} f \right\|_{\infty,\mathcal{E}} &\leq c(\gamma) \|f\|_{\delta,\mathcal{E}} \int_0^{A} y^{\delta-\gamma-1} dy + c(\gamma, A) \|f\|_{\infty,\mathcal{E}} \\
&\leq c(\delta,\gamma,A) \ \|f\|_{\infty,\delta,\mathcal{E}} \quad \text{since} \quad \gamma < \delta,
\end{aligned}$$

and the proof is done. □

In the case where $\gamma > 0$ we take $\mathcal{E} = \mathbb{D}^{\alpha,2}$, the Sobolev–Watanabe space.

In the case where $\gamma = 0$, we take $\mathcal{E} = L^2(\Omega)$ but we consider the norms in the space $L^2(\mathbb{R}, L^2(\Omega))$.

Lemma 5.3 *The operator D^0 linear and bounded from $L^2(\mathbb{R}, L^2(\Omega))$ to $L^2(\mathbb{R}, L^2(\Omega))$. Moreover the operator $\pi^{-1}D^0$ is an isometry on $L^2(\mathbb{R}, L^2(\Omega))$.*

Proof: Let us first prove the second point of the lemma. Assume that F is an element of $L^2(\mathbb{R}, L^2(\Omega))$. Since $\pi^{-1}D^0 = \mathcal{H}$, which is an isometry on $L^2(\mathbb{R})$, hence a closed linear operator we have

$$\begin{aligned}
\left\|D^0(F)(\cdot)\right\|^2_{L^2(\mathbb{R},L^2(\Omega))} &= \pi^2 \int_{\mathbb{R}} \mathbb{E}\left[\mathcal{H}(F)(x)\right]^2 dx \\
&= \pi^2 \mathbb{E} \int_{\mathbb{R}} \left[\mathcal{H}(F)(x)\right]^2 dx \\
&= \pi^2 \mathbb{E} \int_{\mathbb{R}} \left[F(x)\right]^2 dx \\
&= \pi^2 \left\|F\right\|^2_{L^2(\mathbb{R},L^2(\Omega))}.
\end{aligned}$$

Now, we establish two propositions which give approximation result of $\ell(t,x)$ in the spaces $\mathcal{C}^\delta(\mathbb{R}, \mathbb{D}^{\alpha,2})$ and $L^2(\mathbb{R}, L^2(\Omega))$.

Proposition 5.2 *Set $\ell_m(t,x) = \sum_{n=0}^m I_n(f_n(\cdot;t,x))$. Then $\ell_m(t,x)$ converges to $\ell(t,x)$ in $\mathcal{C}^\delta(\mathbb{R}, \mathbb{D}^{\alpha,2})$ for all $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \delta$ and $\delta < \frac{1}{2H^*} - \frac{1}{2}$. Moreover, for all fixed $m \in \mathbb{N}$, the mapping $x \mapsto \ell_m(t,x)$ is δ–Hölder continuous function for any $0 < \delta < \frac{1}{2H^*} - \frac{1}{2}$.*

Proof: We know from Theorem 2.1 that a.s.

$$\ell(t,x) = \sum_{n=0}^{\infty} I_n\left(f_n(\cdot;t,x)\right),$$

where

$$\begin{aligned}
&f_n(t^1,\dots,t^n;t,x) \\
&= \frac{(-1)^{[\frac{n}{2}]}2^{\frac{n+1}{2}}}{\pi n!} \int_{[t^1\vee\cdots\vee t^n,t]} \frac{\prod_{i=1}^n K_H(s,t^i)}{[R_H(s)]^{\frac{n+1}{2}}} G^g_{n,0}\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right) ds.
\end{aligned}$$

Let $0 < \delta < \frac{1}{2H^*} - \frac{1}{2}$ and $\alpha > 0$ such that $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \delta$, then we have

$$\|\ell_m(t,\cdot) - \ell(t,\cdot)\|^2_{\mathcal{C}^\delta(\mathbb{R},\mathbb{D}^{\alpha,2})} = \left\|\sum_{n=m+1}^{\infty} I_n(f_n(\cdot;t,\cdot))\right\|^2_{\mathcal{C}^\delta(\mathbb{R},\mathbb{D}^{\alpha,2})}$$

$$\leq \quad 2 \sup_{x\in\mathbb{R}} \left\| \sum_{n=m+1}^{\infty} I_n(f_n(\cdot;t,x)) \right\|^2_{\mathbb{D}^{\alpha,2}}$$

$$+2 \sup_{x\neq y} \frac{\left\|\sum_{n=m+1}^{\infty} \{I_n(f_n(\cdot;t,x)) - I_n(f_n(\cdot;t,y))\}\right\|^2_{\mathbb{D}^{\alpha,2}}}{|x-y|^{2\delta}}.$$

Set

$$h_n(s,x,y) := G^g_{n,0}\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right) - G^g_{n,0}\left(\frac{y}{[R_H(s)]^{\frac{1}{2}}}\right).$$

Using similar computations as in the proof of the Proposition 5.1 we can write

$$\mathbb{E}\left|I_n(f_n(\cdot;t,x)) - I_n(f_n(\cdot;t,y))\right|^2$$

$$= \quad \frac{2^{n+1}}{(\pi n!)^2}\int_{[0,t]}\int_{[0,t]} \frac{R_H(u,v)^n}{[R_H(u)R_H(v)]^{\frac{n+1}{2}}} h_n(u,x,y)h_n(v,x,y)dudv$$

$$\leq \quad c(\delta,H,t)\frac{2^{n+1+\delta}}{(\pi n!)^2}\Gamma\left(\frac{n+\delta+1}{2}\right)^2 \frac{|x-y|^{2\delta}}{n^{\frac{N}{2\widehat{H}}}},$$

where we have used in the last inequality the fact that

$$\sup_{u\in[0,T]^N} |h_n(u,x,y)| \leq 2^{\frac{n+1-\delta}{2}}\Gamma\left(\frac{n+\delta+1}{2}\right)|x-y|^{\delta}$$

and Lemma 5.1.
Therefore

$$\sup_{x\neq y}\frac{\sum_{n=1}^{\infty} n^{\alpha}\mathbb{E}\left|I_n(f_n(\cdot;t,x)) - I_n(f_n(\cdot;t,y))\right|^2}{|x-y|^{2\gamma}} \leq \sum_{n=1}^{\infty}\frac{c(\delta,H,t)}{n^{\frac{N}{2\widehat{H}}+\frac{1}{2}-\delta-\alpha}},$$

since by Stirling formula

$$\frac{2^{n+1+\delta}}{(\pi n!)^2}\Gamma\left(\frac{n+\delta+1}{2}\right)^2 \sim \frac{1}{n^{\frac{1}{2}-\delta}}.$$

Consequently the series $\sum_{n=0}^{\infty} I_n(f_n(\cdot;t,x))$ converges in the space $\mathcal{C}^{\delta}(\mathbb{R},\mathbb{D}^{\alpha,2})$ for any $\alpha < \frac{N}{2\widehat{H}} - \frac{1}{2} - \delta$ and the proof is complete. □

In order to establish the result of the Proposition 5.2 in the space $L^2(\mathbb{R},L^2(\Omega))$ we first state an important lemma.

Lemma 5.4 *For $u \in [0,T]^N$, $\beta > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we set*

$$\mathbf{S}_n(u,x,\beta) = \mathbf{H}_n\left(\frac{x}{[R_H(u)]^{\frac{1}{2}}}\right)\exp\left(\frac{-\beta x^2}{R_H(u)}\right),$$

$$\mathbf{T}_n(u,x,\beta) = \frac{1}{[R_H(u)]^{\frac{1}{2}}}\exp\left(-\left(\frac{1}{2}-\beta\right)\frac{x^2}{R_H(u)}\right).$$

a) For any $\frac{1}{4} \le \beta < \frac{1}{2}$, there exists a universal constant c such that

$$\sup_{u,v\in[0,T]^N}\sup_{x\in\mathbb{R}}|\mathbf{S}_n(u,x,\beta)\mathbf{S}_n(v,x,\beta)| \le \frac{c}{(n\vee 1)^{\frac{8\beta-1}{6}}}.$$

b)

$$\int_{\mathbb{R}}|\mathbf{T}_n(u,x,\beta)\mathbf{T}_n(v,x,\beta)|\,dx \le \frac{c(\beta)}{[R_H(u)\wedge R_H(v)]^{\frac{1}{2}}}.$$

Proof: The proof of this lemma is similar to that given in Imkeller *et al.* (1995), we omit it here.

Proposition 5.3 *Set $\ell_m(t,x) = \sum_{n=0}^m I_n(f_n(\cdot;t,x))$. Then*

$$\|\ell_m(t,\cdot) - \ell(t,\cdot)\|_{L^2(\mathbb{R},L^2(\Omega))}$$

converges to zero as m tends to infinity.

Proof: Let us fix the time $t \in [0,T]^N$ and recall the chaotic expansion of the local time

$$\ell(t,x) = \sum_{n=0}^{\infty}\int_{[0,t]}\frac{p_{R_H(s)}(x)}{[R_H(s)]^{\frac{n}{2}}}\mathbf{H}_n\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right)I_n\left(K_H(s,\cdot)^{\otimes n}\right)ds.$$

We have

$$\begin{aligned}
&\|\ell_m(t,\cdot) - \ell(t,\cdot)\|^2_{L^2(\mathbb{R},L^2(\Omega))} \\
&= \int_{\mathbb{R}}\sum_{n=m+1}^{\infty}\left\{\int_{[0,t]}\int_{[0,t]}\frac{R_H(u,v)^n}{[R_H(u)R_H(v)]^{\frac{n}{2}}}\right. \\
&\qquad \times \mathbf{H}_n\left(\frac{x}{[R_H(u)]^{\frac{1}{2}}}\right)p_{R_H(u)}(x) \\
&\qquad \left.\times\, \mathbf{H}_n\left(\frac{x}{[R_H(v)]^{\frac{1}{2}}}\right)p_{R_H(v)}(x)dudv\right\}dx.
\end{aligned}$$

Now, for $\frac{1}{4} \leq \beta < \frac{1}{2}$ we can write

$$\begin{aligned}
&\mathbf{H}_n\left(\frac{x}{[R_H(u)]^{\frac{1}{2}}}\right)p_{R_H(u)}(x)\mathbf{H}_n\left(\frac{x}{[R_H(v)]^{\frac{1}{2}}}\right)p_{R_H(v)}(x)\\
&= \frac{\mathbf{S}_n(u,x,\beta)\mathbf{S}_n(v,x,\beta)\mathbf{T}_n(u,x,\beta)\mathbf{T}_n(v,x,\beta)}{2\pi}\\
&= : J_n(u,v,x,\beta)
\end{aligned}$$

and by Lemma 5.4,

$$\int_{\mathbb{R}} |J_n(u,v,x,\beta)|\, dx \leq \frac{c(\beta)}{[R_H(u)\wedge R_H(v)]^{\frac{1}{2}}}\frac{1}{(n\vee 1)^{\frac{8\beta-1}{6}}}.$$

Then

$$\begin{aligned}
&\|\ell(t,\cdot)\|^2_{L^2(\mathbb{R},L^2(\Omega))}\\
&\leq \sum_{n\geq 0}\frac{c(\beta)}{(n\vee 1)^{\frac{8\beta-1}{6}}}\int_{[0,t]}\int_{[0,t]}\frac{R_H(u,v)^n}{[R_H(u)R_H(v)]^{\frac{n}{2}}}\\
&\qquad\qquad \times\frac{dudv}{[R_H(u)\wedge R_H(v)]^{\frac{1}{2}}}\\
&\leq \sum_{n\geq 0}\frac{c(H,t)}{(n\vee 1)^{\frac{8\beta-1}{6}}}\prod_{j=1}^{N}\int_0^1 Q_{H_j}(z)^n\frac{dz}{z^{H_j}}\\
&\leq c_1(H,N,t)+c_2(H,N,t)\sum_{n\geq 1}\frac{1}{n^{\frac{N}{2\widehat{H}}+\frac{8\beta-1}{6}}}.
\end{aligned}$$

But the series $\sum_{n\geq 1} n^{-\frac{N}{2\widehat{H}}-\frac{8\beta-1}{6}}$ is convergent for any β satisfying $\frac{N}{2\widehat{H}} + \frac{8\beta-1}{6} > 1$. Letting β tends to $\frac{1}{2}$ we get the convergence of the series

$$\sum_{n=0}^{\infty}\int_{[0,t]}\frac{p_{R_H(s)}(x)}{[R_H(s)]^{\frac{n}{2}}}\mathbf{H}_n\left(\frac{x}{[R_H(s)]^{\frac{1}{2}}}\right)I_n\left(K_H(s,\cdot)^{\otimes n}\right)ds$$

in the space $L^2(\mathbb{R},L^2(\Omega))$, hence $\|\ell_m(t,\cdot)-\ell(t,\cdot)\|_{L^2(\mathbb{R},L^2(\Omega))}$ converges to zero as m tends to infinity. □

Proof of Theorem 3.1:
Case where $\gamma > 0$. It is a consequence of Proposition 5.1, Lemma 5.2 and

Proposition 5.2, and the fact that

$$\mathbf{D}I_n\left(f_n\left(\cdot;t,\cdot\right)\right)(x) = I_n\left(\mathbf{D}f_n\left(\cdot;t,x\right)\right)$$

for $\mathbf{D} \in \{D_+^\gamma, D_-^\gamma, D^\gamma\}$ and

$$n^\alpha n! \left\|\mathbf{D}f_n\left(\cdot;t,x\right)\right\|_2^2 \le \frac{c(\gamma, H, t)}{n^{\frac{N}{2H}+\frac{1}{2}-\gamma-\alpha}}$$

for n large enough.

Case where $\gamma = 0$. Combining Proposition 5.1, Lemma 5.3 and Proposition 5.3 we finish the proof. □

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