Regularity and asymptotic behaviour of the local time for the d–dimensional fractional Brownian motion with N–parameters

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Abstract

We give the Wiener–Itô chaotic decomposition for the local time of the d–dimensional fractional Brownian motion with N–parameters. We study its smoothness in the Sobolev–Watanabe spaces and its asymptotic behaviour when the space variable tends to zero and the time variable tends to infinity.

AMS 1991 subject classifications. 60H05, 60H10

1 Introduction

Let $B^H = \{B_t^H : t \in \mathbb{R}_+\}$ be a standard fractional Brownian motion (fBm for brevity) with Hurst parameter $H \in (0,1)$. It is well known that this process is a centered Gaussian process which admits an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)dW_s,$$

[1]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant SB2000–0060.

[2]Supported by the Spanish *Ministerio de Educación, Cultura y Deporte*: Grant BFM2000–0009 and by the Catalan CIRIT: Grant 2001SGR00174.

where $W = \{W_t : t \in [0,T]\}$ is a Wiener process. The kernel $K_H(t,s)$, (see Alòs *et al.* [1]), for $s < t$ is given by, c_H being a constant and $\mu = H - \frac{1}{2}$,

$$K_H(t,s) = c_H(t-s)^{\mu} - \mu c_H \int_s^t (r-s)^{\mu-1} \left(1 - \left(\frac{s}{r}\right)^{-\mu}\right) dr, \tag{1}$$

The covariance function of B_t^H can be represented as

$$R_H(s,t) = \mathbb{E}\left(B_s^H B_t^H\right) = \int_0^{s\wedge t} K_H(t,r) K_H(s,r) dr,$$

and has the explicit form

$$R_H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

Moreover, for every $\beta \in (0,H)$ its sample–paths are a.s. Hölder continuous with exponent β.

We define the local time $L(t,x)$ of the fBm at time $t \in [0,T]$ and level $x \in \mathbb{R}$ as the density at the point x of the occupation measure

$$A \longmapsto \int_0^t \mathbb{1}_A(B_s^H) ds, \text{ where } A \in \mathcal{B}(\mathbb{R}).$$

Berman [4] and Geman and Horowitz [6] show that this local time exists, has a bi–continuous modification and it is Hölder continuous of order $\beta < \frac{1}{2H} - \frac{1}{2}$ in space and of order $\beta < 1 - H$ in time.

Another version of the local time of fBm, called weighted local time is defined in Coutin *et al.* [5] and Hu *et al.* [8] as the density of the occupation measure

$$A \longmapsto 2H \int_0^t \mathbb{1}_A(B_s^H) s^{2H-1} ds.$$

This local time satisfies a Tanaka–type formula and plays a role in financial applications in the context of fractional market models.

We consider the N–parameter, d–dimensional fractional Brownian field $B^{\overline{H}} = \{B_t^{\overline{H}} : t \in [0,T]^N\}$ ((N,d)–fBm for brevity), where $\overline{H} = (\overline{H}_1, \ldots, \overline{H}_d)$ and $\overline{H}_i = (H_{i,1}, \ldots, H_{i,N}) \in (0,1)^N$, for $i = 1, \ldots, d$. This is an extension of the fBm sheet introduced by Bardina *et al.* [3]. For $N = 2$ and $d = 1$, (N,d)–fBm has the same law as the fractional sheet defined by Ayache *et al.* [2] if the constants are chosen in a suitable form.

Similarly, for any $t \in [0,T]^N$ and $x \in \mathbb{R}^d$, the local time $L(t,x)$ of the (N,d)–fBm can be defined as the density of the occupation measure

$$A \longmapsto \int_{[0,t]} \mathbb{1}_A(B_s^{\overline{H}}) ds, \; A \in \mathcal{B}(\mathbb{R}^d).$$

Formally one can write

$$L(t,x) = \int_{[0,t]} \delta_x(B_s^{\overline{H}})ds$$

where δ_x denotes the Dirac function and $\delta_x(B_s^{\overline{H}})$ is therefore a distribution in the Watanabe sense [16].

The aim of this paper is to study the chaotic decomposition, the regularity in some Sobolev–Watanabe spaces and the asymptotic behaviour when $|x|$, the euclidean norm of x in $\mathbb{R}^d$, goes to 0 for the local time of the (N,d)–fBm.

In the $(1,1)$–fBm case we go further in the regularity properties of the local time that Coutin *et al.* [5] do for the weighted case and make an extension to (N,d)–fBm. We also generalize the results of Imkeller and Weisz [9] about the Wiener chaos expansion for the local time of the (N,d)–Brownian motion and its asymptotic behaviour near the origin.

In section 2 we give some preliminaries and basic definitions.

Section 3 deals with the chaotic decomposition of the local time $L(t,x)$ of $(1,1)$–fBm. We follow at the beginning the ideas of Coutin *et al.* [5], but finish in a way that enabled us to improve the result about the Sobolev–Watanabe regularity. In fact we prove that our local time belongs to the Sobolev–Watanabe space $\mathbb{D}^{\alpha,2}$, for all α less than $\frac{1}{2H} - \frac{1}{2}$. This shows that more regularity of the process B^H (H greater) implies less regularity for its local time.

In section 4 we consider the $(N,1)$–fBm, B^H with values in $\mathbb{R}$ and $H = (H_1,\ldots,H_N)$. We establish the chaotic decomposition of its local time $L(t,x)$ with respect to the multi–parameter Wiener process. We show that $L(t,x)$ is in $\mathbb{D}^{\alpha,2}$ for every $\alpha < \sum_{j=1}^N \frac{1}{2H_j} - \frac{1}{2}$.

Section 5 deals with the generalization (N,d)–fBm, $B^{\overline{H}}$. We also establish, when $\sum_{j=1}^N \frac{1}{\max_{1\le i\le d} H_{i,j}} > d$, the chaotic decomposition of $L(t,x)$ using the ideas of the previous sections. As before, we show that it belongs to the space $\mathbb{D}^{\alpha,2}$ for every $\alpha < \sum_{j=1}^N \frac{1}{2\max_{1\le i\le d} H_{i,j}} - \frac{d}{2}$ provided that x is different from the origin.

Section 6 is devoted to the study of the asymptotic behaviour of the local time of (N,d)–fBm with $H_{i,j} = H$, when $|x|$ goes to 0. Based on the study of the Wiener case, done by Imkeller and Weisz [9], and using the results of previous sections we show that the norm of $L(t,x)$ in $\mathbb{D}^{\alpha,2}$ explodes when $dH \ge 1$ and find the power of this explosion.

The proofs of some Lemmas found to be too long or too technical, were placed in an appendix at the end of the paper.

2 Preliminaries and notations

Let $I_n(f_n)$ denote the multiple Itô stochastic integral of a symmetric kernel $f_n \in L^2([0,T]^n)$ with respect to the Wiener process W. The Wiener chaos expansion of a square integrable Brownian random variable F is given by $F = \sum_{n=0}^{\infty} I_n(f_n)$. Let $\mathbf{L}$ be the Ornstein–Uhlenbeck operator

$$\mathbf{L}F = -\sum_{n=0}^{\infty} n I_n(f_n).$$

If $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$ we define the Sobolev–Watanabe spaces $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(\mathrm{Id} - \mathbf{L})^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where Id stands for the mapping identity.

We denote by D the derivative operator, defined on multiple integrals as

$$D_t(I_n(f_n)) = n I_{n-1}(f_n(\cdot, t)).$$

This operator is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(L^2([0,T]))$.

It is known that a random variable F belongs to $\mathbb{D}^{\alpha,2}$ if and only if its chaotic decomposition $\sum_{n=0}^{\infty} I_n(f_n)$ satisfies

$$\sum_{n=0}^{\infty} (1+n)^{\alpha} \|I_n(f_n)\|_2^2 < +\infty.$$

Set $\mathbb{D}^{\infty,2} = \cap_{\alpha \in \mathbb{R}} \mathbb{D}^{\alpha,2}$. The Stroock formula that gives the Wiener chaos decomposition of a functional $F \in \mathbb{D}^{\infty,2}$ is

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n \left(\mathbb{E} \left(D^n F \right) \right).$$

For a complete survey of this materials we refer the reader to the book by Nualart [13].

Let $p_\varepsilon(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. Consider also, for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\varepsilon^d(x) = \prod_{i=1}^{d} p_\varepsilon(x_i), \ x = (x_1, \ldots, x_d).$$

We denote by $\mathbf{H}_n$ the n–th Hermite polynomial, defined for $n \geq 1$, by

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right)\right), \ x \in \mathbb{R}$$

and $\mathbf{H}_0(x) = 1$.

3 Smoothness of the local time of the $(1,1)$–fBm

Now, let $B^H = \{B_t^H : t \in [0,T]\}$ be a real valued fBm with Hurst parameter H. We will give the chaotic expansion for the local time of this process, and study its regularity in the sense of Sobolev–Watanabe.

Let $R_H(s,t)$ be the covariance function of the fBm B^H, and set

$$Q_H(z) = \begin{cases} \dfrac{R_H(1,z)}{z^H} & \text{if } z \in (0,1] \\ \\ 0 & \text{if } z = 0. \end{cases}$$

The following two lemmas, which will be needed in the sequel, study the properties of the function $Q_H(\cdot)$ and the behaviour of $\int_0^1 Q_H(z)^n \frac{dz}{z^H}$ when n goes to infinity. Their proofs are given in the appendix.

Lemma 1 *The function $Q_H(\cdot)$ is continuous with values in $[0,1]$, $Q_H(1) = 1$ and strictly increasing.*

Lemma 2 *There exists a constant $c(H)$, independent of n, such that*

$$\int_0^1 Q_H(z)^n \frac{dz}{z^H} \leq \frac{c(H)}{n^{\frac{1}{2H}}}. \tag{2}$$

In order to establish the chaotic expansion of the local time of the fBm, we recall the following lemma (Nualart and Vives [14]).

Lemma 3 *Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of square integrable random variables with the expansions*

$$F_\varepsilon = \sum_{n=0}^{\infty} I_n\left(f_n^\varepsilon\right),$$

where f_n^ε are symmetric and belong to $L^2([0,T]^n)$.
Assume that
i) f_n^ε converges in $L^2([0,T]^n)$, when ε goes to zero, to some symmetric function $f_n \in L^2([0,T]^n)$
ii) $\sum_{n=1}^{\infty} \sup_\varepsilon n! ||f_n^\varepsilon||_2^2$ is convergent.
Then the family F_ε converges in $L^2(\Omega)$ to $F = \sum_{n=0}^{\infty} I_n\left(f_n\right)$.

Proposition 4 *Let $L(t,x)$ be the local time of the $(1,1)$–fBm B^H. Then $L(t,x)$ has the following Wiener chaos expansion :*

$$L(t,x) = \sum_{n=0}^{\infty} \int_0^t \frac{p_{s^{2H}}(x)}{s^{nH}} \mathbf{H}_n\left(\frac{x}{s^H}\right) I_n\left(K_H(s,\cdot)^{\otimes n}\right) ds. \tag{3}$$

Proof: The proof follows the same ideas as in Proposition 2 of [5]. Applying the Stroock formula to $p_\varepsilon(B_s^H - x)$ we get the following chaos expansion

$$\int_0^t p_\varepsilon(B_s^H - x)ds =$$
$$= \sum_{n=0}^{\infty} \int_0^t \frac{p_{s^{2H}+\varepsilon}(x)}{(s^{2H}+\varepsilon)^{\frac{n}{2}}} \mathbf{H}_n\left(\frac{x}{\sqrt{s^{2H}+\varepsilon}}\right) I_n\left(K_H(s,\cdot)^{\otimes n}\right) ds. \tag{4}$$

Then we must prove that when ε tends to 0 the above series converges in $L^2(\Omega)$ to the expansion (3).

This decomposition can be found also by the method used in Proposition 1 of Imkeller *et al.* [10] using the fact that Hermite polynomials form an orthogonal basis in $L^2([0,T], p_1(x)dx)$ and $\mathbf{H}_n(W(h)) = \frac{1}{n!} I_n(h^{\otimes n})$. Note that our definition of the Hermite polynomial is different from the one used in [10].

Denote by f_n^ε and f_n the n–th kernels of (3) and (4), respectively. In order to prove the convergence of (4) it suffices to verify the satisfaction of conditions i) and ii) of Lemma 3.

We start by checking condition ii). Set

$$\beta_{n,\varepsilon}(s) = \frac{p_{s^{2H}+\varepsilon}(x)}{\left[s^{2H}+\varepsilon\right]^{\frac{n}{2}}} \mathbf{H}_n\left(\frac{x}{\sqrt{s^{2H}+\varepsilon}}\right).$$

Hence

$$\|f_n^\varepsilon\|_2^2 = \int_{[0,T]^n} \left(\int_0^t \beta_{n,\varepsilon}(s) \prod_{i=1}^n K_H(s,t_i) ds\right)^2 dt_1 \cdots dt_n.$$

Now, writing the square of the integral as the product of two integrals and using Fubini theorem, we obtain

$$\|f_n^\varepsilon\|_2^2 = 2\int_0^t du \int_0^u \beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v) R_H(u,v)^n dv.$$

Using the fact that

$$\left|\mathbf{H}_n(y)\exp\left(-\frac{y^2}{2}\right)\right| \leq \frac{2^{\frac{n}{2}+1}}{n!\sqrt{\pi}} \cdot \Gamma\left(\frac{n+1}{2}\right) = c_n,$$

we get

$$\left|\beta_{n,\varepsilon}(s)\right| \leq c_n s^{-(n+1)H}.$$

Therefore

$$n!\,\|f_n^\varepsilon\|_2^2 \le 2n!c_n^2 \int_0^t du \int_0^u R_H(u,v)^n (uv)^{-(n+1)H} dv,$$

and by Sterling formula $n!c_n^2$ behaves like $\frac{c}{\sqrt{n}}$ when n is large.

As $R_H(u,v) = R_H(1, \frac{v}{u})u^{2H}$, making the change of variable $\frac{v}{u} = z$, and using Lemma 2 it can be shown that the series $\sum_{n=1}^\infty n!\,\|f_n^\varepsilon\|_2^2$ is less than

$$\sum_{n=1}^\infty \frac{c}{\sqrt{n}} \frac{t^{2(1-H)}}{2(1-H)} \int_0^1 Q_H(z)^n \frac{dz}{z^H} \le \sum_{n=1}^\infty \frac{c(t,H)}{n^{\frac{1}{2H}+\frac{1}{2}}},$$

which is always convergent since $\frac{1}{2H} + \frac{1}{2} > 1$.

Now, let us check condition i). We have

$$\|f_n^\varepsilon - f_n\|_2^2 = \int_{[0,T]^n} \left(\int_0^t \left(\beta_{n,\varepsilon}(s) - \beta_{n,0}(s) \right) \prod_{i=1}^n K_H(s,t_i) ds \right)^2 dt_1 \cdots dt_n,$$

where $\beta_{n,0}(\cdot)$ is the function $\beta_{n,\varepsilon}(\cdot)$ for $\varepsilon = 0$.

Similar techniques allow us to write

$$\begin{aligned} \|f_n^\varepsilon - f_n\|_2^2 &= 2\int_0^t du \int_0^u \beta_{n,\varepsilon}(u)\beta_{n,\varepsilon}(v) R_H(u,v)^n dv \\ &\quad +2\int_0^t du \int_0^u \beta_{n,0}(u)\beta_{n,0}(v) R_H(u,v)^n dv \\ &\quad -4\int_0^t du \int_0^u \beta_{n,\varepsilon}(u)\beta_{n,0}(v) R_H(u,v)^n dv. \end{aligned}$$

We estimate the first and the last integrals and get the bound

$$\frac{t^{2(1-H)}}{2(1-H)} \int_0^1 Q_H(z)^n \frac{dz}{z^H},$$

which is finite because $Q_H(z) \le 1$ by Lemma 1 and $H \in (0,1)$. Hence, by the dominated convergence theorem $\|f_n^\varepsilon - f_n\|_2^2$ goes to zero as ε tends to zero. □

Theorem 5 *The local time $L(t,x)$ of $(1,1)$–fBm B^H belongs to the space $\mathbb{D}^{\alpha,2}$ for every $\alpha < \frac{1}{2H} - \frac{1}{2}$.*

Remark 6 *It is known that the paths of fBm are β–Hölder continuous for all $\beta < H$, and its local time is β–Hölder continuous for all $\beta < \frac{1}{2H} - \frac{1}{2}$ in space and for all $\beta < 1 - H$ in time. Hence when H becomes smaller, the trajectory of B^H has less regularity but its local time becomes more regular. In the case $H = \frac{1}{2}$, which corresponds to standard Brownian motion, we obtain the known regularity result : $\alpha < \frac{1}{2}$.*

Proof: In Proposition 4 we have proved the chaotic expansion of $L(t, x)$. Hence to prove the Theorem we must check that

$$\sum_{n=0}^{\infty}(1+n)^{\alpha}\mathbb{E}\left(\int_0^t \frac{p_{s^{2H}}(x)}{s^{nH}}\, \mathbf{H}_n\left(\frac{x}{s^H}\right) I_n\left(K_H(s,\cdot)^{\otimes n}\right) ds\right)^2$$

is convergent for all $\alpha < \frac{1}{2H} - \frac{1}{2}$.

As

$$\mathbb{E}\left[I_n\left(K_H(u,\cdot)^{\otimes n}\right) I_n\left(K_H(v,\cdot)^{\otimes n}\right)\right] = n!\, R_H(u,v)^n$$

and using the same ideas as in Proposition 4, it suffices to check the convergence of the series

$$\sum_{n=1}^{\infty}(1+n)^{\alpha}\frac{1}{\sqrt{n}}\int_0^t\int_0^t \frac{R_H(u,v)^n}{(uv)^{nH}}\cdot\frac{dudv}{(uv)^H} =$$

$$= \sum_{n=1}^{\infty}(1+n)^{\alpha}\frac{c(t,H)}{\sqrt{n}}\int_0^1 Q_H(z)^n\frac{dz}{z^H},$$

which is equivalent to

$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}-\alpha}}\int_0^1 Q_H(z)^n\frac{dz}{z^H}. \tag{5}$$

Now, by Lemma 2 the series (5) is bounded by $\sum_{n=1}^{\infty} n^{\alpha-\frac{1}{2H}-\frac{1}{2}}$, and hence it is convergent for $\frac{1}{2} - \alpha + \frac{1}{2H} > 1$, that is $\alpha < \frac{1}{2H} - \frac{1}{2}$. □

Remark 7 *For the weighted local time $\ell_t^H(x) = \int_0^t \delta_x(B_s^H)s^{2H-1}ds$, doing the same computations we get the series, $\sum_{n=1}^{\infty} n^{\alpha-\frac{1}{2}}\int_0^1 Q_H(z)^n z^{H-1}dz$ which leads exactly to the same regularity as $L(t, x)$.*

4 Smoothness of the local time of $(N,1)$–fBm

For a fixed point T in $\mathbb{R}_+$, let $\{W_t : t \in [0,T]^N\}$ be a real valued N–parameter Wiener process. Given a vector $H = (H_1, \dots, H_N)$ with all his components in $(0,1)$, we define the fractional Brownian field ($(N,1)$–fBm), B^H as

$$B_t^H = \int_{[0,t]} K_{H_1}(t_1, s_1) \cdots K_{H_N}(t_N, s_N) dW_s, \tag{6}$$

where K_{H_j}, $j = 1, \dots, N$ are the kernels introduced in (1).

One can show that B^H is a centered Gaussian process with the covariance function

$$R_H(s,t) = R_{H_1}(s_1, t_1) \cdots R_{H_N}(s_N, t_N)$$

for $s = (s_1, \dots, s_N)$ and $t = (t_1, \dots, t_N)$, where

$$R_{H_j}(s_j, t_j) = \frac{1}{2}(s_j^{2H_j} + t_j^{2H_j} - |t_j - s_j|^{2H_j}) \text{ for } j = 1, \dots, N.$$

We shall denote by $K_H(t,s)$ the function $K_{H_1}(t_1, s_1) \cdots K_{H_N}(t_N, s_N)$.

The $(N,1)$–fBm is an extension of the definition of the fractional Brownian sheet introduced by Bardina *et al.* [3]. Recently Ayache *et al.* [2] gave another definition to the fractional Brownian sheet with Hurst parameters $H = (H_1, H_2)$ as

$$\overline{B}_{(t_1,t_2)}^H := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{H_1}(t_1, u) f_{H_2}(t_2, v) dW_{(u,v)}, \ (t_1, t_2) \in \mathbb{R}_+^2,$$

where the kernels f_{H_i} are given by

$$f_{H_i}(s,u) = c_{H_i} \left((s-u)_+^{H_i - \frac{1}{2}} - (-u)_+^{H_i - \frac{1}{2}} \right), \text{ for } i = 1, 2,$$

and c_{H_i}, for $i = 1,2$ are some suitable constants. Here $\overline{B}^H$ is a centered Gaussian process with covariance function $R_{H_1}(s_1, t_1) R_{H_2}(s_2, t_2)$ if the constants are chosen in such a way that

$$\int_{-\infty}^{+\infty} f_{H_i}(1, u) du = 1, \text{ for } i = 1, 2.$$

It has the same law as the fractional anisotropic Wiener field introduced by Kamont [11].

In this section the multiple stochastic integrals I_n are interpreted with respect to the N–parameter Wiener process W, and $du = du_1 \dots du_N$.

Our aim is, as in the previous section, to find the chaotic decomposition of the local time and its Sobolev–Watanabe regularity.

Proposition 8 *Let $L(t,x)$ be the local time of the $(N,1)$–fBm B^H and $H=(H_1,\ldots,H_N)$. Then $L(t,x)$ has the following chaos expansion :*

$$L(t,x)=\sum_{n=0}^{\infty}\int_{[0,t]}\frac{p_{R_H(s,s)}(x)}{[R_H(s,s)]^{\frac{n}{2}}}\,\mathbf{H}_n\left(\frac{x}{\sqrt{R_H(s,s)}}\right)I_n\left(K_H(s,\cdot)^{\otimes n}\right)ds$$

for $x\in\mathbb{R}$, ($R_H(s,s)=\prod_{j=1}^{N}s_j^{H_j}$).

Proof: Following the proof of Proposition 4, we apply the Stroock formula (see Nualart and Vives [15]) to $p_\varepsilon(B_s^H-x)$ and obtain

$$p_\varepsilon(B_s^H-x)=\sum_{n=0}^{\infty}\frac{p_{R_H(s,s)+\varepsilon}(x)}{[R_H(s,s)+\varepsilon]^{\frac{n}{2}}}\,\mathbf{H}_n\left(\frac{x}{\sqrt{R_H(s,s)+\varepsilon}}\right)I_n\left(K_H(s,\cdot)^{\otimes n}\right).$$

Set, for $(t^1,\ldots,t^n)\in([0,T]^N)^n$

$$f_n^\varepsilon(t^1,\ldots,t^n,t):=$$

$$=\int_{[t^1\vee\ldots\vee t^n,t]}\frac{p_{R_H(s,s)+\varepsilon}(x)}{[R_H(s,s)+\varepsilon]^{\frac{n}{2}}}\,\mathbf{H}_n\left(\frac{x}{\sqrt{R_H(s,s)+\varepsilon}}\right)\prod_{i=1}^{n}K_H(s,t^i)ds.$$

It is easily seeing that

$$\begin{aligned}\|I_n(f_n^\varepsilon)\|_2^2 &= n!\,\|f_n^\varepsilon\|_2^2\\ &\le b_n\prod_{j=1}^{N}\int_0^{t_j}du_j\int_0^{u_j}R_{H_j}(u_j,v_j)^n(u_jv_j)^{-(n+1)H_j}dv_j,\end{aligned}$$

where b_n behaves like $\frac{c}{\sqrt{n}}$ when n is large. Given that $R_{H_j}(u_j,v_j)=R_{H_j}(1,\frac{v_j}{u_j})u_j^{2H_j}$, making the change of variables $u_j=u_j$, $v_j=z_ju_j$ for $j=1,\ldots,N$, and applying Lemma 2 we get

$$\begin{aligned}\sum_{n=1}^{\infty}n!\sup_\varepsilon\|f_n^\varepsilon\|_2^2 &\le \sum_{n=1}^{\infty}\frac{c(t,H)}{\sqrt{n}}\prod_{j=1}^{N}\int_0^1 Q_{H_j}(z)^n\frac{dz}{z^{H_j}}\\ &\le \sum_{n=1}^{\infty}\frac{c(t,H)}{n^{\frac{1}{2}+\sum_{j=1}^{N}\frac{1}{2H_j}}}<\infty.\end{aligned}$$

This shows that condition ii) of Lemma 3 is satisfied.

Let us now set

$$f_n(t^1,\dots,t^n,t) :=$$

$$= \int_{[t^1\vee\ldots\vee t^n,t]} \frac{p_{R_H(s,s)}(x)}{[R_H(s,s)]^{\frac{n}{2}}}\ \mathbf{H}_n\left(\frac{x}{\sqrt{R_H(s,s)}}\right)\prod_{i=1}^{n} K_H(s,t^i)ds.$$

In order to see that $\|f_n^\varepsilon - f_n\|_2^2$ goes to zero as ε tends to zero, we apply the same ideas as in Proposition 4 and use the dominated convergence theorem. Hence condition i) is proved. □

Theorem 9 *The local time $L(t,x)$ of the $(N,1)$–fBm B^H belongs to the space $\mathbb{D}^{\alpha,2}$ for every $\alpha < \sum_{j=1}^{N}\frac{1}{2H_j}-\frac{1}{2}$.*

Proof: Reasoning as above, it suffices to check that the following quantity is finite

$$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}-\alpha}}\int_{[0,t]}\int_{[0,t]}\frac{R_H(u,v)^n}{[R_H(u,u)R_H(v,v)]^{\frac{n}{2}}}\cdot\frac{dudv}{[R_H(u,u)R_H(v,v)]^{\frac{1}{2}}} =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}-\alpha}}\prod_{j=1}^{N}\int_0^{t_j}\int_0^{t_j}\frac{R_{H_j}(u_j,v_j)^n}{u_j^{nH_j}v_j^{nH_j}}\cdot\frac{du_jdv_j}{u_j^{H_j}v_j^{H_j}}$$

$$= \sum_{n=1}^{\infty}\frac{c(H,t,N)}{n^{\frac{1}{2}-\alpha}}\prod_{j=1}^{N}\int_0^{1} Q_{H_j}(z)^n\frac{dz}{z^{H_j}}.$$

Now, by Lemma 2,

$$\prod_{j=1}^{N}\int_0^1 Q_{H_j}(z)^n\frac{dz}{z^{H_j}} \le \frac{c(H)}{n^{\sum_{j=1}^{N}\frac{1}{2H_j}}}.$$

We conclude that the local time belongs to the space $\mathbb{D}^{\alpha,2}$ with α less than $\sum_{j=1}^{N}\frac{1}{2H_j}-\frac{1}{2}$. □

Remark 10 *If the vector H has all the components equal to H^* the bound is $\frac{N}{2H^*}-\frac{1}{2}$, and if $N=1$ (the 1–parameter case) we recover the bound of the previous section. Moreover, the bound for the regularity increases with N and decreases with respect to the Hurst parameter H^*. Consequently, the local time of an $(N,1)$–fBm is more regular in x and in ω when H is close to zero.*

5 Smoothness of the local time of (N,d)–fBm

This section is devoted to study the local time of a d–dimensional fractional Brownian motion with N–parameters for $d \geq 2$. We obtain its chaotic decomposition and its Sobolev–Watanabe regularity.

Consider the $N \times d$ matrix $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ where for $i = 1, \dots, d$ and $j = 1, \dots, N$, $\overline{H}_i = (H_{i,1}, \cdots, H_{i,N})$ and $H_{i,j} \in (0,1)^N$ is a column vector. We will say that $B^{\overline{H}} = (B_t^{\overline{H}_1}, \dots, B_t^{\overline{H}_d})_{t\in[0,T]^N}$ is an (N,d)–fBm if its components are independent and for every $i = 1, \dots, d$, $B^{\overline{H}_i}$ is a $(N,1)$–fBm with Hurst parameter $\overline{H}_i$. This implies that

$$R_{\overline{H}_i}(s,t) = \mathbb{E}(B_s^{\overline{H}_i} B_t^{\overline{H}_i}) = \prod_{j=1}^{N} R_{H_{i,j}}(s_j, t_j).$$

Recall that

$$B_t^{\overline{H}_i} = \int_{[0,t]} K_{\overline{H}_i}(t,s) dW_s^i,$$

where $K_{\overline{H}_i}(t,s) = \bigotimes_{j=1}^N K_{H_{i,j}}(t_j, s_j)$, $s,t \in [0,T]^N$ and W^i, $i = 1, \dots, d$ are independent N–parameter Wiener processes.
The local time of $B^{\overline{H}}$ is defined, for any $t \in [0,T]^N$ and for every $x \in \mathbb{R}^d \setminus \{0\}$, as

$$L(t,x) = \int_{[0,t]} \delta_x(B_s^{\overline{H}}) ds$$

where δ_x denotes the Dirac function on $\mathbb{R}^d$.

This situation, when $N = 1$, has been considered by Hu and Øksendal in [7] for the fBm, using a white noise approach to give a chaotic expansion for the local time and to study the asymptotic behaviour of its second order moment in time T. Our approach mainly follows the ideas of Imkeller and Weisz in [9] for the Wiener case and it uses the representation (6), which allows us to use the Wiener–Itô multiple stochastic integrals and to make computations much easier than in [7]. We will prove here more precise and general results, for the d–dimensional and N–parameter case, that recover the results of Imkeller and Weisz [9] when $H_{i,j} = H = \frac{1}{2}$.

We recall first the next lemma that is proved in the Wiener case ($H_{i,j} = \frac{1}{2}$) in Propositions 3 and 6 of Imkeller *et al.* [10] and Imkeller and Weisz [9] and which plays a crucial role in this section.

In what follows we set for $t \in \mathbb{R}_+^N$ and $\overline{H}_i = (H_{i,1}, \dots, H_{i,N}) \in (0,1)^N$, $\underline{t}^{\overline{H}_i} := \prod_{j=1}^N t_j^{H_{i,j}}$ and $H_j^* = \max\{H_{i,j} : i = 1, \dots, d\}$

Lemma 11 *For $u \in [0,1]^N$, $\varepsilon, \beta > 0$, $n_i \in \mathbb{N}$ and $x_i \in \mathbb{R}$, we set*

$$\begin{aligned} \mathbf{S}(u, x_i, \beta, \varepsilon, n_i) &= \mathbf{H}_{n_i}\left(\frac{x_i}{\sqrt{\underline{u}^{2\overline{H}_i} + \varepsilon}}\right) \exp\left(\frac{-\beta x_i^2}{\underline{u}^{2\overline{H}_i} + \varepsilon}\right) \\ \mathbf{T}(u, x_i, \beta, \varepsilon, n_i) &= \frac{1}{\sqrt{\underline{u}^{2\overline{H}_i} + \varepsilon}} \exp\left(-\left(\frac{1}{2} - \beta\right) \frac{x_i^2}{\underline{u}^{2\overline{H}_i} + \varepsilon}\right) \end{aligned}$$

1) For any $\frac{1}{4} \leq \beta < \frac{1}{2}$, there exists a universal constant c such that

$$\begin{aligned} &\sup_{u,v \in [0,1]^N} \sup_{\varepsilon > 0, x \in \mathbb{R}^d} \sum_{n_1 + \cdots + n_d = m} \prod_{i=1}^{d} |\mathbf{S}(u, x_i, \beta, \varepsilon, n_i) \mathbf{S}(v, x_i, \beta, \varepsilon, n_i)| \\ &\leq \sum_{n_1 + \cdots + n_d = m} \prod_{i=1}^{d} \frac{c}{(n_i \vee 1)^{\frac{8\beta - 1}{6}}} \leq \frac{c}{(m \vee 1)^{1 - d(1 - \frac{8\beta - 1}{6})}} \end{aligned}$$

2) If $\mathbf{K}$ is a compact set in $\mathbb{R}^d$ not containing the origin, then the function

$$(u, x) \in \mathbb{R}_+^N \times \mathbf{K} \longmapsto \prod_{i=1}^{d} \mathbf{T}(u, x_i, \beta, 0, n_i)$$

is bounded.

Now we find the chaotic expansion of $L(t, x)$.

Proposition 12 *Let $I_{n_i}^i$ denotes the multiple integral with respect to the Wiener process W^i. If $B^{\overline{H}}$ is an (N, d)–fBm, then its local time $L(t, x)$ admits the chaotic decomposition*

$$L(t, x) = \sum_{n_1, \ldots, n_d \geq 0} \int_{[0,t]} \prod_{i=1}^{d} \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i}} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{s}^{\overline{H}_i}}\right) I_{n_i}^i \left(K_{\overline{H}_i}(s, \cdot)^{\otimes n_i}\right) ds,$$

for all t in $[0, T]^N$ and all x in $\mathbb{R}^d \setminus \{0\}$ provided that $\sum_{j=1}^{N} \frac{1}{H_j^*} > d$.

Proof: From the results of the previous sections we can write the following decomposition for $p_\varepsilon^d(B_s^{\overline{H}} - x) = \prod_{i=1}^d p_\varepsilon(B_s^{\overline{H}_i} - x_i)$,

$$p_\varepsilon^d(B_s^{\overline{H}} - x) = \prod_{i=1}^{d} \sum_{n_i \geq 0} \beta_{n_i, \varepsilon}(s) I_{n_i}^i(K_{\overline{H}_i}(s, \cdot)^{\otimes n_i}), \text{ for any } s \in [0, T]^N,$$

where the $\beta_{n_i,\varepsilon}(s)$ are given by

$$\beta_{n_i,\varepsilon}(s) = \frac{p_{\underline{s}^{2\overline{H}_i}+\varepsilon}(x_i)}{\left[\underline{s}^{2\overline{H}_i}+\varepsilon\right]^{\frac{n_i}{2}}} \mathbf{H}_{n_i}\left(\frac{x_i}{\sqrt{\underline{s}^{2\overline{H}_i}+\varepsilon}}\right).$$

Then the approximation $L_\varepsilon(t,x)$ of the local time can be written as

$$\begin{aligned} L_\varepsilon(t,x) &= \int_{[0,t]} p_\varepsilon^d(B_s^{\overline{H}}-x)ds \\ &= \sum_{n_1,\dots n_d\geq 0} \int_{[0,t]} \prod_{i=1}^d \beta_{n_i,\varepsilon}(s) I_{n_i}^i(K_{\overline{H}_i}(s,\cdot)^{\otimes n_i})ds \end{aligned}$$

Next, we will show the convergence of $L_\varepsilon(t,x)$ to $L(t,x)$ in $L^2(\Omega)$. As usual we must check conditions i) and ii) of Lemma 3 and we only concentrate on condition ii).
Using the independence of $B^{\overline{H}_i}$ we have

$$\begin{aligned} \|L_\varepsilon(t,x)\|_2^2 &= \sum_{m\geq 0}\sum_{n_1+\cdots+n_d=m}\int_{[0,t]}du\int_{[0,t]}dv \times \\ &\quad \times\prod_{i=1}^d \beta_{n_i,\varepsilon}(u)\beta_{n_i,\varepsilon}(v)R_{\overline{H}_i}(u,v)^{n_i}. \end{aligned}$$

Now, for $\frac{1}{4}\leq\beta<\frac{1}{2}$ we can write

$$\begin{aligned} &\beta_{n_i,\varepsilon}(u)\beta_{n_i,\varepsilon}(v) = \\ &= \frac{\mathbf{S}(u,x_i,\beta,\varepsilon,n_i)\mathbf{S}(v,x_i,\beta,\varepsilon,n_i)\mathbf{T}(u,x_i,\beta,\varepsilon,n_i)\mathbf{T}(v,x_i,\beta,\varepsilon,n_i)}{2\pi\left[\left(\underline{u}^{2\overline{H}_i}+\varepsilon\right)\left(\underline{v}^{2\overline{H}_i}+\varepsilon\right)\right]^{\frac{n_i}{2}}}, \end{aligned} \tag{7}$$

and by Lemma 11, (7) is bounded by

$$\frac{1}{\underline{u}^{n_i\overline{H}_i}\underline{v}^{n_i\overline{H}_i}}\cdot\frac{c}{(n_i\vee 1)^{\frac{8\beta-1}{6}}}.$$

Then

$$\|L_\varepsilon(t,x)\|_2^2 \leq \sum_{m\geq 0}\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^d\frac{c}{(n_i\vee 1)^{\frac{8\beta-1}{6}}}\cdot\mathbf{R}(t,d,N),$$

where

$$\begin{aligned}
\mathbf{R}(t,d,N) &= \int_{[0,t]} du \int_{[0,t]} dv \prod_{i=1}^{d}\prod_{j=1}^{N} \frac{R_{H_{i,j}}(u_j,v_j)^{n_i}}{(u_jv_j)^{H_{i,j}n_i}} \\
&= 2^N \prod_{j=1}^{N} \int_0^{t_j} du_j \int_0^{u_j} dv_j \prod_{i=1}^{d} \frac{R_{H_{i,j}}(u_j,v_j)^{n_i}}{(u_jv_j)^{H_{i,j}n_i}} \\
&= 2^N \prod_{j=1}^{N} \int_0^{t_j} u_j du_j \int_0^1 \prod_{i=1}^{d} Q_{H_{i,j}}(z)^{n_i} dz \\
&= \underline{t}^2 \prod_{j=1}^{N} \int_0^1 \prod_{i=1}^{d} Q_{H_{i,j}}(z)^{n_i} dz.
\end{aligned}$$

Therefore $\|L_\varepsilon(t,x)\|_2^2$ is less than

$$\sum_{m\geq 0}\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d}\frac{c(\overline{H},d,t)}{(n_i\vee 1)^{\frac{8\beta-1}{6}}}\prod_{j=1}^{N}\int_0^1\prod_{i=1}^{d}Q_{H_{i,j}}(z)^{n_i}dz. \tag{8}$$

Recall that $H_j^* = \max\{H_{i,j} : i = 1,\dots,d\}$ and set $m = \sum_{i=1}^d n_i$. Using the same techniques as in the proof of the Lemma 2, we can show that for δ close to 0

$$\begin{aligned}
\int_{1-\delta}^{1} \prod_{i=1}^{d} Q_{H_{i,j}}(z)^{n_i} dz &\leq c_1(\overline{H}) \int_0^1 \exp\left(-\sum_{i=1}^{d} n_i t^{2H_{i,j}}\right) dt \\
&\leq c_1(\overline{H}) \int_0^1 \exp\left(-m t^{2H_j^*}\right) dt \\
&\leq c_2(\overline{H}) m^{-\frac{1}{2H_j^*}},
\end{aligned}$$

and by Lemma 1,

$$\int_0^{1-\delta} \prod_{i=1}^{d} Q_{H_{i,j}}(z)^{n_i} dz \leq c(\delta)a^m, \text{ for some } 0<a<1.$$

But one can choose a constant c in such a way that

$$c(\delta)a^m \leq c\cdot m^{-\frac{1}{2H_j^*}} \text{ for all } m.$$

Hence,

$$\prod_{j=1}^{N}\int_0^1 \prod_{i=1}^{d} Q_{H_{i,j}}(z)^{n_i}dz \le c(\overline{H},d)\prod_{j=1}^{N} m^{-\frac{1}{2H_j^*}} \tag{9}$$

Thus, by Lemma 11 (8), aside from the constants is bounded by

$$\sum_{m\ge1}\prod_{j=1}^{N} m^{-\frac{1}{2H_j^*}} \sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d}\frac{1}{(n_i\vee 1)^{\frac{8\beta-1}{6}}} \le$$

$$\le \sum_{m\ge1}\frac{c}{m^{1-d(1-\frac{8\beta-1}{6})+\sum_{j=1}^{N}\frac{1}{2H_j^*}}}.$$

Therefore series (8) converges for $0<\sum_{j=1}^{N}\frac{1}{2H_j^*}-d(1-\frac{8\beta-1}{6})$. But for β close to $\frac{1}{2}$ we obtain $d<\sum_{j=1}^{N}\frac{1}{H_j^*}$. □

Theorem 13 *If $B^{\overline{H}}$ is an (N,d)–fBm, then its local time $L(t,x)$ belongs to the space $\mathbb{D}^{\alpha,2}$ for every $\alpha<\sum_{j=1}^{N}\frac{1}{2H_j^*}-\frac{d}{2}$ for all t in $[0,T]^N$, and all x in $\mathbb{R}^d\setminus\{0\}$.*

Proof: Following previous ideas we only need to show the convergence of

$$\sum_{m\ge1}(1+m)^{\alpha}\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d}\frac{c(H,d)}{(n_i\vee1)^{\frac{8\beta-1}{6}}}\prod_{j=1}^{N}\int_0^1\prod_{i=1}^{d}Q_{H_{i,j}}(z)^{n_i}dz \tag{10}$$

which is bounded by

$$\sum_{m\ge1}(1+m)^{\alpha}\prod_{j=1}^{N} m^{-\frac{1}{2H_j^*}}\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d}\frac{1}{(n_i\vee1)^{\frac{8\beta-1}{6}}} \quad \le$$

$$\le\; c(\overline{H},d)\sum_{m\ge1}(1+m)^{\alpha}m^{d(1-\frac{8\beta-1}{6})-1}\prod_{j=1}^{N}m^{-\frac{1}{2H_j^*}}.$$

Hence series (10) converges for $\alpha<\sum_{j=1}^{N}\frac{1}{2H_j^*}-d(1-\frac{8\beta-1}{6})$ which for, β close to $\frac{1}{2}$, gives the desired condition. □

Remark 14 *If $H_{i,j}=\frac{1}{2}$ for $i=1,\ldots,d$, $j=1,\ldots,N$, then $\sum_{j=1}^{N}\frac{1}{2H_j^*}-\frac{d}{2}=N-\frac{d}{2}$, and we obtain the same condition as Imkeller and Weisz [9] for the N–parameter Wiener process in $\mathbb{R}^d$.*

6 Asymptotic behaviour of the local time of the (N, d)–fBm

We turn now our interest to the study of the asymptotic behaviour of the local time as $|x| \to 0$. We will show that, as in the case of N–parameter $\mathbb{R}^d$–valued Wiener process, it has a singularity when $dH > 1$.
An important question is how to renormalize the local time that means, to find a deterministic function $f(t,x)$ such that $f(t,x)||L(t,x)||_{\alpha,2}$ converges to a non–zero limit as $|x| \to 0$ and / or $\underline{t} = t_1 \cdots t_N \to \infty$.

Let us recall the expression of the $\mathbb{D}^{\alpha,2}$–norm of the local time $L(T,x)$. For the sake of simplicity we take $T = \widetilde{1} := (1,\ldots,1)$.

We have

$$\|L(\widetilde{1},x)\|^2_{\alpha,2} = \sum_{m\geq 0}(1+m)^\alpha A_m(x) \tag{11}$$

where

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^d \left(\prod_{j=1}^N \frac{R_{H_{i,j}}(u_j,v_j)}{(u_jv_j)^{H_{i,j}}} \right)^{n_i} \times \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{u}^{\overline{H}_i}}\right) \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{v}^{\overline{H}_i}}\right) p_{\underline{u}^{2\overline{H}_i}}(x_i) p_{\underline{v}^{2\overline{H}_i}}(x_i).$$

and

$$A_0(x) = \left(\int_{[0,1]^N} ds \prod_{i=1}^d \frac{1}{\left(2\pi \prod_{j=1}^N s_j^{2H_{i,j}}\right)^{\frac{1}{2}}} \exp\left(\frac{-x_i^2}{2\prod_{j=1}^N s_j^{2H_{i,j}}} \right) \right)^2.$$

In the sequel we denote by $\Gamma(x,\gamma)$ and $\Gamma(x)$, respectively, the complementary Gamma and Gamma functions, that for every $x > 0$, $\gamma \geq 0$

$$\Gamma(x,\gamma) = \int_\gamma^\infty e^{-t} t^{x-1} dt \quad \text{and} \quad \Gamma(x) = \Gamma(x,0).$$

In all this section we confine our attention to the situation where $H_{i,j} = H$ for all $(i,j) \in \{1,\ldots,d\} \times \{1,\ldots,N\}$, and use the notation B^H for $B^{\overline{H}}$.

Hence $A_0(x)$ becomes

$$A_0(x) = \frac{1}{(2\pi)^d} \left(\int_{[0,1]^N} \frac{1}{\underline{s}^{dH}} \exp\left(\frac{-|x|^2}{2\underline{s}^{2H}} \right) ds \right)^2.$$

Our main result in this section is the following

Theorem 15 *Let B_t^H be (N,d)–fBm. Set $\lambda := d - \frac{1}{H}$. Then we have the following renormalization of the local time $L(t,x)$ for any $\alpha < \frac{N}{2H} - \frac{d}{2}$:*

1) If $\lambda > 0$, then

$$\lim_{|x|\to 0} \left\| L(\widetilde{1},x) \right\|_{\alpha,2} \left(\frac{2^{\frac{\lambda}{2}} \left(\frac{1}{2H}\right)^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}}(N-1)!} \left(\log \frac{1}{|x|^2}\right)^{N-1} \Gamma\left(\frac{\lambda}{2}\right) \right)^{-1} = 1.$$

2) If $\lambda = 0$, then

$$\lim_{|x|\to 0} \left\| L(\widetilde{1},x) \right\|_{\alpha,2} \left(\frac{\left(\frac{1}{2H}\right)^N}{(2\pi)^{\frac{d}{2}} N!} \left(\log \frac{1}{|x|^2}\right)^{N} \right)^{-1} = 1.$$

3) If $\lambda < 0$, then

$$\lim_{|x|\to 0} \left\| L(\widetilde{1},x) \right\|_{\alpha,2} \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1-Hd}\right)^{N} \right)^{-1} = 1.$$

Remark 16 *The previous results show that at the origin, for $\lambda \geq 0$ the local time explodes and for $\lambda < 0$ it has a finite limit. Observe that if $H = \frac{1}{2}$, that is the situation discussed by Imkeller and Weisz [9] we have always $dH \geq 1$ for $d \geq 2$ and the local time explodes at the origin.*

To prove this theorem we will study the behaviour of the chaos of order zero in the chaotic expansion of the local time and we will show that the convergence, as $|x| \to 0$ of the chaos $k \geq 1$ is governed by the chaos zero. We need some technical results which will be proved in the appendix.

Lemma 17 *Let $\lambda = d - \frac{1}{H}$ and consider*

$$B_0(\gamma) = \left(\int_{[0,1]^N} \exp\left(\frac{-\gamma}{\underline{s}^{2H}}\right) \frac{ds}{\underline{s}^{dH}} \right)^2 \quad \text{for } \gamma \neq 0.$$

Then we have the following asymptotic behaviour:

1) If $\lambda > 0$ and $N > 1$ then

$$\begin{aligned} B_0(\gamma) &= \frac{1}{((N-1)!)^2} \left(\frac{1}{2H}\right)^{2N} \gamma^{-\lambda} \left(\log \frac{1}{\gamma}\right)^{2N-2} \Gamma\left(\frac{\lambda}{2}, \gamma\right)^2 \\ &\quad + \gamma^{-\lambda} \mathcal{O}\left(\left(\log \frac{1}{\gamma}\right)^{2N-3} \right) \end{aligned}$$

2) If $\lambda = 0$ then

$$B_0(\gamma) = \frac{1}{(N!)^2}\left(\frac{1}{2H}\right)^{2N} e^{-2\gamma}\left(\log\frac{1}{\gamma}\right)^{2N} + \mathcal{O}\left(\left(\log\frac{1}{\gamma}\right)^{2N-1}\right)$$

3) If $\lambda < 0$, then

$$B_0(\gamma) = \left(\frac{1}{1-Hd}\right)^{2N} + o(\gamma).$$

The following result aims to estimate the chaos $A_m(x)$ for all $m \geq 1$ of (11).

Lemma 18 *Let $\lambda = d - \frac{1}{H}$ and put*

$$B_m(\gamma) = \int_{[0,1]^N}\int_{[0,1]^N} \frac{R_H(u,v)^m}{(\underline{u}\cdot\underline{v})^{H(m+d)}} \exp\left(\frac{-\gamma}{\underline{u}^{2H}}\right)\exp\left(\frac{-\gamma}{\underline{v}^{2H}}\right) dudv.$$

Then we have, $c(H,d,N)$ being a positive constant depending only on H, d and N,

1) If $\lambda > 0$ and $N > 1$, then

$$B_m(\gamma) \ \leq \ c(H,d,N)\gamma^{-\lambda}\left(\log\frac{1}{\gamma}\right)^{N-1}\Gamma(\lambda,\gamma)\, m^{-\frac{N}{2H}}.$$

2) If $\lambda = 0$, then

$$B_m(\gamma) \ \leq \ c(H,d,N)e^{-\gamma}\left(\log\frac{1}{\gamma}\right)^{N} m^{-\frac{N}{2H}}.$$

3) If $\lambda < 0$, then

$$B_m(\gamma) \ \leq \ c(H,d,N) m^{-\frac{N}{2H}}.$$

Proof of the Theorem 15: Consider first the situation $d > \frac{1}{H}$. The chaos of order 0 is given by

$$A_0(x) = \frac{1}{(2\pi)^d} B_0\left(\frac{1}{2}|x|^2\right)$$

and for $m \geq 1$

$$A_m(x) \leq c\cdot(m)^{d(1-\frac{8\beta-1}{6})-1} B_m\left(\left(\frac{1}{2}-\beta\right)|x|^2\right),$$

where we have used in the last inequality Lemma 11 and Lemma 18 with β close to $\frac{1}{2}$. By Lemma 18, 1) the right hand side above is less than

$$c(H,d,N)m^{d(1-\frac{8\beta-1}{6})-1}m^{-\frac{N}{2H}}\gamma^{-\lambda}\left(\log\frac{1}{\gamma}\right)^{N-1}\Gamma(\lambda,\gamma).$$

It obvious that the series

$$\sum_{m\geq 1}(1+m)^{\alpha}m^{d(1-\frac{8\beta-1}{6})-1}m^{-\frac{N}{2H}}$$

is convergent for every $\alpha < \frac{N}{2H}-\frac{d}{2}$. The conclusion follows.

The cases $Hd=1$ and $Hd<1$ can be treated similarly. □

We can also deduce the behaviour of the local time $L(t,x)$ as $\underline{t}\to\infty$ and $|x|\to 0$ in the Sobolev–Watanabe spaces $\mathbb{D}^{\alpha,2}$. For $\lambda = d-\frac{1}{H}$ we also have to distinguish three cases: $\lambda>0$, $\lambda=0$ and $\lambda<0$.

The result is

Theorem 19 *Let $\{L(t,x): t\in\mathbb{R}_+^N, x\in\mathbb{R}^d\}$ be the local time of the (N,d)–fBm B^H. Then the following limits hold for any $\alpha<\frac{N}{2H}-\frac{d}{2}$:*

1) For $\lambda>0$ and $N>1$

$$\lim_{\underline{t}\to+\infty,|x|\to 0}\|L(t,x)\|_{\alpha,2}\left(\frac{2^{\frac{\lambda}{2}}\left(\frac{1}{2H}\right)^N|x|^{-\lambda}}{(2\pi)^{\frac{d}{2}}(N-1)!}\left(\log\frac{\underline{t}^{2H}}{|x|^2}\right)^{N-1}\Gamma\left(\frac{\lambda}{2}\right)\right)^{-1}=1.$$

2) For $\lambda=0$,

$$\lim_{\underline{t}\to+\infty,|x|\to 0}\|L(t,x)\|_{\alpha,2}\left(\frac{\left(\frac{1}{2H}\right)^N}{(2\pi)^{\frac{d}{2}}N!}\left(\log\frac{\underline{t}^{2H}}{|x|^2}\right)^{N}\right)^{-1}=1.$$

3) For $\lambda<0$,

$$\lim_{\underline{t}\to+\infty,|x|\to 0}\|L(t,x)\|_{\alpha,2}\left(\frac{\underline{t}^{2(1-dH)}}{(2\pi)^{\frac{d}{2}}}\left(\frac{1}{1-dH}\right)^{N}\right)^{-1}=1.$$

Proof: From the scaling property of the (N,d)–fBm of the Hurst parameter H one can show that the two processes

$$\left\{L(t,x): t\in\mathbb{R}_+^N,\ x\in\mathbb{R}^d\right\}$$

and

$$\left\{ \prod_{j=1}^{N} t_j^{1-\sum_{i=1}^{d} H_{i,j}} L\left(\widetilde{1}, \underline{t}^{-\overline{H}} x\right) : t \in \mathbb{R}_+^N,\ x \in \mathbb{R}^d \right\}$$

have the same law.

Hence when $H_{i,j} = H$ we have

$$\|L(t,x)\|_{\alpha,2}^2 = \underline{t}^{2(1-dH)} \left\| L\left(\widetilde{1}, \underline{t}^{-H} x\right) \right\|_{\alpha,2}^2 .$$

Then the conclusion follows easily from Theorem 15. □

7 Appendix

Proof of Lemma 1: Applying Schwartz inequality, we have

$$R_H(u,v) \le \sqrt{R_H(u,u)}\sqrt{R_H(v,v)}.$$

In particular $R_H(1,z) \le z^H$, so $Q_H(\cdot)$ has values in $[0,1]$.
In order to see that $Q_H(\cdot)$ is strictly increasing, we note first that for any $z \in (0,1)$,

$$\begin{aligned} &\frac{2}{H} Q_H'(z) = \\ &= -z^{-H-1} + z^{H-1} + z^{-H-1}(1-z)^{2H} + 2z^{-H}(1-z)^{2H-1}, \end{aligned} \tag{12}$$

and it suffices to prove that this expression is strictly positive.
Multiplying (12) by z^{H+1}, we see that it is equivalent to prove the following inequality

$$(1-z)^{2H-1}(1+z) + z^{2H} > 1. \tag{13}$$

Now, if $H < \frac{1}{2}$, we can write

$$(1-z)^{2H} \frac{1+z}{1-z} + z^{2H} > 1 + 2z > 1,$$

since $z^{2H} > z$ for any $z \in [0,1]$.
If $H > \frac{1}{2}$, as $2H-1 \in (0,1)$ then

$$(1-z)^{2H-1}(1+z) + z z^{2H-1} > (1-z)(1+z) + z^2 = 1.$$

For $H = \frac{1}{2}$ the inequality is immediate. So, for all H, the inequality (13) is satisfied and hence $Q_H(\cdot)$ is strictly increasing.

Proof of Lemma 2: The proof of this lemma is inspired in [12]. For any arbitrary $\delta \in (0,1)$, we can write

$$\begin{aligned} \int_0^1 Q_H(z)^n \frac{dz}{z^H} &\leq \int_0^{1-\delta} Q_H(z)^n \frac{dz}{z^H} + \int_{1-\delta}^1 Q_H(z)^n \frac{dz}{z^H} \\ &=: J_1 + J_2. \end{aligned}$$

By Lemma 1, $Q_H(\cdot)$ is a continuous increasing function on $[0,1]$. Then we have immediately

$$J_1 \leq c(H,\delta) Q_H(1-\delta)^n,$$

where $c(H,\delta)$ is a constant depending only on H and δ. Below we will use the same notation $c(H,\delta)$ for different constants.

Now we consider the term J_2 which is bounded by

$$(1-\delta)^{-H} \int_{1-\delta}^1 Q_H(z)^n dz = (1-\delta)^{-H} \int_{1-\delta}^1 \exp\left(n \log Q_H(z)\right) dz.$$

Using Taylor's formula, we have $-\log z \geq 1 - z$, for all $z \in (0,1]$. Therefore $-\log Q_H(z) \geq 1 - Q_H(z)$ for all $z \in [1-\delta, 1]$, since $Q_H(\cdot)$ has values in $(0,1]$ on $[1-\delta,1]$.
So

$$J_2 \leq (1-\delta)^{-H} \int_{1-\delta}^1 \exp\left(-n\left(1 - Q_H(z)\right)\right) dz.$$

Moreover, applying again Taylor's formula we obtain for all $H \in (0,1)$ and $z \in [1-\delta, 1]$,

$$z^H \leq 1 - H(1-z) \text{ and } z^{-H} \geq 1 + H(1-z).$$

Also,

$$z^{-H} = 1 + H(1-z) + \frac{1}{2} H\left(1+H\right) \xi^{-H-2} \left(1-z\right)^2 \text{ for some } \xi \in [z,1]$$

and hence

$$z^{-H} \leq 1 + H(1-z) + \frac{1}{2} H\left(1+H\right)\left(1-\delta\right)^{-H-2}\left(1-z\right)^2.$$

Now, using all the previous inequalities we have

$$Q_H(z) \leq 1 - \frac{1}{2}(1-z)^{2H} + \frac{1}{4} H\left(1+H\right)\left(1-\delta\right)^{-H-2}\left(1-z\right)^2.$$

Since, for every $z \in [1-\delta, 1]$

$$(1-z)^2 = (1-z)^{2H} (1-z)^{2(1-H)} \leq (1-z)^{2H} \delta^{2(1-H)},$$

we deduce that

$$1 - Q_H(z) \geq \frac{1}{2}(1-z)^{2H} (1 - o(\delta)),$$

where $o(\delta)$ is given by

$$o(\delta) := \frac{1}{2} H (1+H) \frac{\delta^{2(1-H)}}{(1-\delta)^{H+2}},$$

which goes to zero when δ tends to 0. Hence

$$J_2 \leq (1-\delta)^{-H} \int_{1-\delta}^{1} \exp\left(-\frac{n}{2}(1-z)^{2H} (1 - o(\delta))\right) dz.$$

Making the change of variable $t^{2H} = \frac{1}{2}(1-z)^{2H} (1 - o(\delta))$ we get

$$J_2 \leq c(H,\delta) \int_0^{c(H,\delta)} \exp\left(-nt^{2H}\right) dt,$$

and using now the change $u = nt^{2H}$, we obtain

$$J_2 \leq c(H,\delta) n^{-\frac{1}{2H}} \Gamma\left(\frac{1}{2H}\right).$$

To finish the proof it suffices to remark that for n large enough, say for all $n \geq n_0$, $(Q_H(1-\delta))^n \leq n^{-\frac{1}{2H}}$, but for all $1 \leq n \leq n_0$ we can find a constant c_{n_0} such that $Q_H(1-\delta)^n \leq c_{n_0} n^{-\frac{1}{2H}}$, and (2) is established.

Proof of Lemma 17: Let us set

$$I_N(\gamma) = \int_{[0,1]^N} \exp\left(\frac{-\gamma}{\underline{s}^{2H}}\right) \frac{ds}{\underline{s}^{dH}}.$$

Using the change of variables $u_1 = s_1 \cdots s_N$, $u_2 = s_2 \cdots s_N$, $\ldots$, $u_N = s_N$, with the Jacobi determinant $\frac{1}{u_2 \cdots u_N}$, we obtain

$$\begin{aligned} I_N(\gamma) &= \int_{\{0 \leq u_1 \leq \cdots \leq u_N \leq 1\}} u_1^{-dH} \exp\left(\frac{-\gamma}{u_1^{2H}}\right) \frac{du_n \cdots du_1}{u_2 \cdots u_N} \\ &= \frac{1}{(N-1)!} \int_0^1 \left(\log \frac{1}{r}\right)^{N-1} \frac{1}{r^{dH}} \exp\left(\frac{-\gamma}{r^{2H}}\right) dr. \end{aligned}$$

Making the change of variable $\gamma r^{-2H} = t$ and with the notation $\lambda = d - \frac{1}{H}$, we get

$$I_N(\gamma) = \frac{1}{(N-1)!}\left(\frac{1}{2H}\right)^N \gamma^{-\frac{\lambda}{2}} A_{N-1}\left(\gamma, \frac{\lambda}{2}\right), \tag{14}$$

where we denoted

$$A_{N-1}(\gamma, \alpha) = \int_\gamma^\infty t^{\alpha-1} e^{-t} \left(\log \frac{t}{\gamma}\right)^{N-1} dt.$$

Since the convergence of integral (14) depends on the relation between the exponent $\frac{\lambda}{2} - 1$ and -1 we must discuss different situations. Note first that

$$\left(\log \frac{t}{\gamma}\right)^{N-1} = \sum_{k=0}^{N-1} C_{N-1}^k \left(\log \frac{1}{\gamma}\right)^{N-1-k} (\log t)^k. \tag{15}$$

Let us consider the following cases.

- $\lambda > 0$ ($\frac{\lambda}{2} - 1 > -1$). Observe that the function

  $$t \longmapsto t^{\frac{\lambda}{2}-1} e^{-t} (\log t)^k$$

  is always integrable on $[0, \infty)$ for any $k \in \mathbb{N}$ and, therefore, by (15)

  $$\begin{aligned} I_N(\gamma) &= \frac{1}{(N-1)!}\left(\frac{1}{2H}\right)^N \gamma^{-\frac{\lambda}{2}} \left(\log \frac{1}{\gamma}\right)^{N-1} \Gamma\left(\frac{\lambda}{2}, \gamma\right) \\ &\quad + \mathcal{O}\left(\left(\log \frac{1}{\gamma}\right)^{N-2}\right) \end{aligned}$$

  Then

  $$\begin{aligned} B_0(\gamma) &= \frac{1}{((N-1)!)^2}\left(\frac{1}{2H}\right)^{2N} \gamma^{-\lambda} \left(\log \frac{1}{\gamma}\right)^{2N-2} \Gamma\left(\frac{\lambda}{2}, \gamma\right)^2 \\ &\quad + \mathcal{O}\left(\left(\log \frac{1}{\gamma}\right)^{2N-3}\right). \end{aligned}$$

- $\lambda = 0$. We need to estimate the integral

  $$A_{N-1}(\gamma, 0) = \int_\gamma^\infty t^{-1} e^{-t} \left(\log \frac{t}{\gamma}\right)^{N-1} dt$$

Integration by parts gives

$$\begin{aligned} A_{N-1}(\gamma,0) &= \frac{1}{N}\int_\gamma^\infty e^{-t}\left(\log\frac{t}{\gamma}\right)^N dt \\ &= \frac{e^{-\gamma}}{N}\left(\log\frac{1}{\gamma}\right)^N + \mathcal{O}\left(\left(\log\frac{1}{\gamma}\right)^{N-1}\right) \end{aligned}$$

and thus

$$I_N(\gamma) = \frac{1}{N!}\left(\frac{1}{2H}\right)^N e^{-\gamma}\left(\log\frac{1}{\gamma}\right)^N + \mathcal{O}\left(\left(\log\frac{1}{\gamma}\right)^{N-1}\right)$$

and

$$B_0(\gamma) = \frac{1}{(N!)^2}\left(\frac{1}{2H}\right)^{2N} e^{-2\gamma}\left(\log\frac{1}{\gamma}\right)^{2N} + \mathcal{O}\left(\left(\log\frac{1}{\gamma}\right)^{2N-1}\right).$$

- $\lambda < 0$. One can see that the mapping $\gamma \mapsto B_0(\gamma)$ is continuous in 0, hence
$$B_0(\gamma) = \left(\frac{1}{\lambda H}\right)^{2N} + o(\gamma).$$

This proves Lemma 17. □

Proof of Lemma 18: We can write, since $R_H(u_k, v_k) = R_H(1, \frac{v_k}{u_k})u_k^{2H}$,

$$\begin{aligned} B_m(\gamma) &= 2^N \int_{[0,1]^N} du \int_0^{u_N} dv_N \cdots \int_0^{u_1} dv_1 \prod_{k=1}^N \frac{R_H(u_k,v_k)^m}{(u_k v_k)^{H(m+d)}} \times \\ &\qquad \times \exp\left(\frac{-\gamma}{\underline{u}^{2H}}\right)\exp\left(\frac{-\gamma}{\underline{v}^{2H}}\right) \\ &= 2^N \int_{[0,1]^N} du \int_0^{u_N} dv_N \cdots \int_0^{u_1} dv_1 \prod_{j=1}^N \frac{R_H(1,\frac{v_j}{u_j})^m u_j^{2Hm}}{(u_j v_j)^{H(m+d)}} \times \\ &\qquad \times \exp\left(\frac{-\gamma}{\underline{u}^{2H}}\right)\exp\left(\frac{-\gamma}{\underline{v}^{2H}}\right). \end{aligned}$$

Putting $\frac{v_j}{u_j} = z_j$ and computing iteratively the previous integral, we find

$$B_m(\gamma) = 2^N \int_{[0,1]^N}\left(\frac{R_H(\widetilde{1},z)^m}{\underline{z}^{H(m+d)}}\int_{[0,1]^N} \underline{u}^{1-2Hd}\exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}\right)du\right)dz$$

where $\kappa(r) = 1 + \frac{1}{r^{2H}}$.

We compute first

$$J_N(\gamma, \underline{z}) = \int_{[0,1]^N} \underline{u}^{1-2Hd} \exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}}\right) du$$

We can estimate the integral $J_N(\gamma, \underline{z})$ as in Lemma 17. Hence, by replacing γ with $\kappa(\underline{z})\gamma$, we get

$$J_N(\gamma, \underline{z}) = c(N, H) \int_0^1 s^{1-2Hd} \exp\left(-\frac{\gamma\kappa(\underline{z})}{s^{2H}}\right) \left(\log \frac{1}{s}\right)^{N-1} ds,$$

where $c(N, H)$ is a constant depending only on N and H (in the sequel the form of this constant is not important).

By the change of variables $\frac{\gamma\kappa(\underline{z})}{s^{2H}} = t$, with $s = (\frac{\gamma\kappa(\underline{z})}{t})^{\frac{1}{2H}}$ and $dt = -2H\gamma\kappa(\underline{z})s^{-2H-1}ds$, we obtain

$$J_N(\gamma, \underline{z}) = c(N, H)(\gamma\kappa(\underline{z}))^{-\lambda} \int_{\gamma\kappa(\underline{z})}^{\infty} e^{-t} t^{\lambda-1} \left(\log \frac{t}{\gamma\kappa(\underline{z})}\right)^{N-1} dt$$

Using the change of variables $s = \frac{t}{\kappa(\underline{z})}$ in the integral from above, we get,

$$J_N(\gamma, \underline{z}) = c(N, H)\gamma^{-\lambda} \int_{\gamma}^{\infty} e^{-s\kappa(\underline{z})} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} ds.$$

Therefore

$$B_m(\gamma) = 2^N \int_{[0,1]^N} J_N(\gamma, \underline{z}) \frac{R_H(\widetilde{1}, z)^m}{\underline{z}^{H(m+d)}} dz$$

$$= c(N, H)\gamma^{-\lambda} \int_{\gamma}^{\infty} \left(\int_{[0,1]^N} \frac{R_H(\widetilde{1}, z)^m}{\underline{z}^{Hm}} \cdot \frac{e^{-s(\kappa(\underline{z})-1)}}{\underline{z}^{Hd}} dz\right) e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma}\right)^{N-1} ds.$$

¿From Lemma 11, 2) we deduce that $\sup_{z\in[0,1]} z^{-dH} e^{-\gamma(\kappa(z)-1)}$ is finite. Hence

$$B_m(\gamma) \leq c(N, H)\gamma^{-\lambda} A_{N-1}(\gamma, \lambda) \left(\int_0^1 Q_H(z)^m dz\right)^N.$$

Since the estimation of the term $A_{N-1}(\gamma, \lambda/2)$ was done in the proof of Lemma 17. Now, by (9) we obtain the desired inequalities.

Observe that the convergence of the term of the zero chaos as $\gamma \to 0$ is faster than the convergence of other terms. □

Acknowledgment. M. Eddahbi is grateful to the *Centre de Recerca Matemàtica (IEC) and the Department of Mathematics of the UAB, Bellaterra, Barcelona*, Spain for the kind support and hospitality he received while this work was carried out. The authors would like to thank Juanjo Donaire for fruitful discussions.

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