

# NON REALITY AND NON CONNECTIVITY OF COMPLEX POLYNOMIALS

ARNAUD BODIN

ABSTRACT. With the same method we provide negative answers to the following questions: It is possible to find real equations for complex polynomials in two variables up to topological equivalence? Can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials?

## 1. INTRODUCTION

Two polynomials  $f, g \in \mathbb{C}[x, y]$  are *topologically equivalent*, and we will denote  $f \approx g$ , if there exists homeomorphisms  $\Phi$  and  $\Psi$  such that we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\Phi} & \mathbb{C}^2 \\ f \downarrow & & \downarrow g \\ \mathbb{C} & \xrightarrow{\Psi} & \mathbb{C}. \end{array}$$

They are *algebraically equivalent*, and we will denote  $f \sim g$ , if we have  $\Phi \in \text{Aut } \mathbb{C}^2$  and  $\Psi = \text{id}$ .

It is always possible to find real equations for germs of plane curves up to topological equivalence. In fact the proof is as follows: the topological type of a germ of plane curve  $(C, 0)$  is determined by the characteristic pairs of the Puiseux expansions of the irreducible branches and by the intersection multiplicities between this branches. Then we can choose the coefficients of the Puiseux expansions in  $\mathbb{R}$  (even in  $\mathbb{Z}$ ). Now it is possible (see [EN], appendix to chapter 1) to find a polynomial  $f \in \mathbb{R}[x, y]$  (even in  $\mathbb{Z}[x, y]$ ) such that the germ  $(f = 0, 0)$  is equivalent to the germ  $(C, 0)$ .

This property has been widely used by N. A'Campo and others (see [AC] for example) in the theory of divisors. Lee Rudolph asks the question whether it is true for polynomials. We give a negative answer:

---

*Date:* July 18, 2002.

**Theorem A.** *Up to topological equivalence it is not always possible to find real equations for complex polynomials.*

We now deal with another problem. In [Bo2] we proved that a family of polynomials with some constant numerical data are all topologically equivalent. More precisely for a polynomial let  $\mathbf{m} = (\mu, \#\mathcal{B}_{\text{aff}}, \lambda, \#\mathcal{B}_{\infty}, \#\mathcal{B})$  be the multi-integer respectively composed of the affine Milnor number, the number of affine critical values, the Milnor number at infinity, the number of critical values at infinity, the number of critical values (with  $\mathcal{B} = \mathcal{B}_{\text{aff}} \cup \mathcal{B}_{\infty}$ ). Then we have a global version of Lê-Ramanujam  $\mu$ -constant theorem:

**Theorem** ([Bo2]). *Let  $(f_t)_{t \in [0,1]}$  be a family of complex polynomials in two variables whose coefficients are polynomials in  $t$ . Suppose that the multi-integer  $\mathbf{m}(t)$  and the degree  $\deg f_t$  do not depend on  $t \in [0, 1]$ . Then the polynomials  $f_0$  and  $f_1$  are topologically equivalent.*

It is true that two topologically equivalent polynomials have the same multi-integers  $\mathbf{m}$ . A natural question is: can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials ?

**Theorem B.** *There exist two topologically equivalent polynomials  $f_0, f_1$  that can not be connected by a family of equivalent polynomials. That is to say for all continuous family  $(f_t)_{t \in [0,1]}$  there is a  $\tau \in ]0, 1[$  such that  $f_\tau$  is not topologically equivalent to  $f_0$ .*

It can be noticed that the answer is positive for algebraic equivalence. Two algebraically equivalent polynomials can be connected by algebraically equivalent polynomials. Let  $f_0, f_1$  be algebraically equivalent polynomials, then  $f_1 = f_0 \circ \Phi$  with  $\Phi \in \text{Aut } \mathbb{C}^2$ . But Jung's theorem asserts that  $\text{Aut } \mathbb{C}^2$  is generated by linear maps  $(x, y) \mapsto (ax + by + e, cx + dy + f)$  with  $ad - bc \neq 0$ , and triangular automorphisms  $(x, y) \mapsto (x, y + p(x))$ ,  $p \in \mathbb{C}[x]$ . Each of these maps can be connected to id in  $\text{Aut } \mathbb{C}^2$ . So  $\text{Aut } \mathbb{C}^2$  is connected and we have a path  $(\Phi_t)_{t \in [0,1]}$  in  $\text{Aut } \mathbb{C}^2$  such that  $\Phi_0 = \text{id}$ ,  $\Phi_1 = \Phi$ . Now  $f_t = f_0 \circ \Phi_t$  defines a continuous family between  $f_0$  and  $f_1$ .

Such kinds of problems have been studied by V. Kharlamov and V. Kulikov in [KK] for algebraic curve in  $\mathbb{P}^2$ . They give two complex conjugate projective curves that are not isotopic. The example with lowest degree has degree 825. In [ACA], E. Artal, J. Carmon and L. Cogolludo give examples of projective curves  $C, C'$  of degree 6 that have conjugate equations in  $\mathbb{Q}(\sqrt{2})$  but the pairs  $(\mathbb{P}^2, C)$  and  $(\mathbb{P}^2, C')$  are not homeomorphic by an orientation-preserving homeomorphism.

The method used in this paper is based on the relationship between topological and algebraic equivalence: we set a family  $(f_s)_{s \in \mathbb{C}}$  of polynomials such that  $(f_s = 0)$  is a line arrangement in  $\mathbb{C}^2$ . One of the line depends on a parameter  $s \in \mathbb{C}$ . There are enough lines in order that each polynomial is algebraically essentially unique. Moreover every polynomial topologically equivalent to  $f_s$  is algebraic equivalent to a  $f_{s'}$ , where  $s'$  maybe different from  $s$ .

For generic parameters the polynomials are topologically equivalent and the function  $f_s$  is a Morse function outside on  $\mathbb{C}^2 \setminus f_s^{-1}(0)$ . We choose our counter-examples with non-generic parameter, for such an example  $f_k$  is not a Morse function on  $\mathbb{C}^2 \setminus f_k^{-1}(0)$ . The fact that non-generic parameter are finite enables us to prove the requested properties.

## 2. NON REALITY

Let

$$f_s = xy(x - y)(y - 1)(x - sy).$$

Let  $k, \bar{k}$  the roots of  $z^2 - z + 1$ .

**Theorem A.** *There does not exist a polynomial  $g$  with real coefficients such that  $g \approx f_k$ .*

Let  $\mathcal{C} = \{0, 1, k, \bar{k}\}$ . Then for  $s \notin \mathcal{C}$ ,  $f_s$  verifies  $\mu = 14$ ,  $\#\mathcal{B}_{\text{aff}} = 3$  and  $\mathcal{B}_\infty = \emptyset$ . By the connectivity of  $\mathbb{C} \setminus \mathcal{C}$  and the global version of Lê-Ramanujam theorem two polynomials  $f_s$  and  $f_{s'}$ , with  $s, s' \notin \mathcal{C}$ , are topologically equivalent.

The polynomials  $f_k$  and  $f_{\bar{k}}$  verify  $\mu = 14$ , but  $\#\mathcal{B}_{\text{aff}} = 2$ . Then such a polynomial is not topologically equivalent to a generic one  $f_s$ ,  $s \notin \mathcal{C}$ . In fact for  $s \notin \mathcal{C}$  there are two non-zero critical fibers with one double points for each one. For  $s = k$  or  $s = \bar{k}$ , there is only one non-zero critical fiber with an ordinary cusp.

**Lemma 1.** *Let  $s, s' \in \mathbb{C}$ . The polynomials  $f_s$  and  $f_{s'}$  are algebraically equivalent if and only if  $s = s'$  or  $s = 1 - s'$ .*

In particular the polynomials  $f_k$  and  $f_{\bar{k}}$  are algebraically equivalent.

*Proof.* Let us suppose that  $f_s$  and  $f_{s'}$  are algebraically equivalent. Then we can suppose that there exists  $\Phi \in \text{Aut } \mathbb{C}^2$  such that  $f_{s'} = f_s \circ \Phi$ . Such a  $\Phi$  must send the lines  $(x = 0), (y = 0)$  to two lines, then  $\Phi$  is linear:  $\Phi(x, y) = (ax + by, cx + dy)$ . A calculus proves that  $\Phi(x, y) = (x, y)$  or  $\Phi(x, y) = (y - x, y)$  that is to say  $s = s'$  or  $s = 1 - s'$ .  $\square$

**Lemma 2.** *Fix  $s \in \mathbb{C}$  and let  $f$  a polynomial such that  $f \approx f_s$ . Then it exists  $s'$  such that  $f \sim f_{s'}$ .*

From the former lemma it implies that there are only two choices for  $s'$  but it can be different from  $s$ .

*Proof.* The curve  $f_s^{-1}(0)$  contains the simply connected curve  $xy(x-y)(x-sy)$ , then the curve  $f^{-1}(0)$  contains also a simply connected curve (with 4 components), by the generalization of Zaidenberg-Lin theorem (see [Bo1]) this simply connected curve is algebraically equivalent to  $xy(x-y)(x-s'y)$ . Then the polynomial  $f_s$  is algebraically equivalent to  $xy(x-y)(x-s'y)P(x,y)$ , and a calculus gives  $P(x,y) = y - 1$ . That is to say  $f$  is algebraically equivalent to  $f_{s'}$ .  $\square$

Let  $g \in \mathbb{C}[x,y]$ , if  $g(x,y) = \sum a_{i,j}x^i y^j$  then we denote by  $\bar{g}$  the polynomial defined by  $\bar{g}(x,y) = \sum \bar{a}_{i,j}x^i y^j$ . Then  $g = \bar{g}$  if and only if all the coefficients of  $g$  are real.

*Proof of theorem A.* Let suppose that there exists a polynomial  $g$  such that  $g = \bar{g}$  and that  $g \approx f_k$ . There exists  $s \in \mathbb{C}$  such that  $g \sim f_s$ . But  $f_k$  has only two critical values, so  $g$  and  $f_s$  have two critical values. Then  $s = k$  or  $s = \bar{k}$  ( $s = 0$  or  $s = 1$  gives a polynomial with non-isolated singularities). As  $f_k \sim f_{\bar{k}}$  we can choose  $s = k$ . As a consequence we have  $\Phi \in \text{Aut } \mathbb{C}^2$  such that:

$$g = f_k \circ \Phi.$$

Let  $\Phi$  be  $\Phi = (p, q)$ . Then  $g = pq(p-q)(q-1)(p-kq)$ . As  $g = \bar{g}$  we have :

$$\{p, q, p-q, q-1, p-kq\} = \{\bar{p}, \bar{q}, \bar{p}-\bar{q}, \bar{q}-1, \bar{p}-\bar{k}\bar{q}\}.$$

Moreover by the configuration of the lines we have that  $q-1 = \bar{q}-1$ . So  $q = \bar{q}$ . Hence  $q \in \mathbb{R}[x,y]$ . So

$$\{p, p-q, p-kq\} = \{\bar{p}, \bar{p}-\bar{q}, \bar{p}-\bar{k}\bar{q}\}.$$

Let suppose that  $p \neq \bar{p}$ . Then  $p = \bar{p} - q$  or  $p = \bar{p} - \bar{k}q$ . So  $p - \bar{p}$  equals  $-q$  or  $-\bar{k}q$ . But  $p - \bar{p}$  has coefficients in  $i\mathbb{R}$ , which is not the case of  $q \in \mathbb{R}[x,y]$  nor of  $\bar{k}q$ . Then  $p = \bar{p}$ . We have proved that  $\Phi = (p, q)$  has real coefficients. From  $g = f_k \circ \Phi$  we get  $\bar{g} = \bar{f}_k \circ \bar{\Phi}$ . So  $g = f_{\bar{k}} \circ \bar{\Phi}$ . Then from one hand  $f_{\bar{k}} = g \circ \Phi^{-1}$  and from the other hand  $f_{\bar{k}} = g \circ \bar{\Phi}^{-1}$ . So  $f_k = f_{\bar{k}}$ , then  $k = \bar{k}$  which is false. It ends the proof.

We could have end in the following way:  $\Phi = (p, q)$  is in  $\text{Aut } \mathbb{C}^2$  with real coefficients, then  $\bar{\Phi}$ , considered as a real map, is in  $\text{Aut } \mathbb{R}^2$  (see [BCW, Theorem 2.1] for example). Then  $f_k = g \circ \Phi^{-1}$  with  $g, \Phi^{-1}$  with real coefficients, then  $f_k$  has real coefficients which provides the contradiction.  $\square$

3. NON CONNECTIVITY

Let

$$f_s = xy(y-1)(x+y-1)(x-sy).$$

Let  $\mathcal{C}$  be the roots of

$$s(s-1)(s+1)(256s^4 + 736s^3 + 825s^2 + 736s + 256) \\ (256s^4 + 448s^3 + 789s^2 + 448s + 256).$$

Then for  $s \notin \mathcal{C}$ ,  $f_s$  verifies  $\mu = 14$ ,  $\#\mathcal{B}_{\text{aff}} = 4$  and  $\mathcal{B}_\infty = \emptyset$ . For  $s, s' \notin \mathcal{C}$   $f_s$  and  $f_{s'}$  are topologically equivalent. The roots of  $256s^4 + 448s^3 + 789s^2 + 448s + 256$  are of the form  $\{k, \bar{k}, 1/k, 1/\bar{k}\}$ . The polynomials  $f_k$  and  $f_{\bar{k}}$  verify  $\mu = 14$ , but  $\#\mathcal{B}_{\text{aff}} = 3$ . Then such a polynomial is not topologically equivalent to a generic one  $f_s$ ,  $s \notin \mathcal{C}$

**Theorem B.** *The polynomials  $f_k$  and  $f_{\bar{k}}$  are topologically equivalent and it is not possible to find a continuous family  $(g_t)_{t \in [0,1]}$  such that  $g_0 = f_k$ ,  $g_1 = f_{\bar{k}}$  and  $g_t \approx f_k$  for all  $t \in [0, 1]$ .*

The polynomials  $f_k$  and  $f_{\bar{k}}$  are topologically equivalent (by complex conjugation). We have the formula

$$f_{\bar{k}}(\bar{x}, \bar{y}) = \overline{f_k(x, y)}.$$

Let  $\Phi(x, y) = (\bar{x}, \bar{y})$  and  $\Psi(z) = \bar{z}$ , these are homeomorphisms. The polynomials  $f_k$  and  $f_{\bar{k}}$  are topologically equivalent because we have  $f_{\bar{k}} \circ \Phi = \Psi \circ f_k$ .

The two following lemmas are similar to lemmas 1 and 2.

**Lemma 3.** *The polynomials  $f_s$  and  $f_{s'}$  are algebraically equivalent if and only if  $s = s'$  or  $s = 1/s'$ .*

**Lemma 4.** *Fix  $s$  and let  $f$  a polynomial such that  $f \approx f_s$ . Then it exists  $s'$  such that  $f \sim f_{s'}$ .*

*Proof of theorem B.* Let us suppose that such a family  $(g_t)$  does exist. Then by lemma 4 for each  $t \in [0, 1]$  there exists a unique  $s(t) \in \mathbb{C}$  such that  $g_t$  is algebraically equivalent to  $f_{s(t)}$ . We can suppose that there exists  $\Phi_t \in \text{Aut } \mathbb{C}^2$  such that  $f_{s(t)} = g_t \circ \Phi_t$ .

We now prove that the map  $t \mapsto \Phi_t$  can be chosen continuous, that is to say the coefficient of the defining polynomials are continuous functions of  $t$ . We write  $g_t = A_t B_t G_t$  such that  $A_0(x, y) = x$ ,  $B_0(x, y) = y$  and the maps  $t \mapsto A_t$ ,  $t \mapsto B_t$  are continuous. So the automorphism  $\Phi_t^{-1}$  is defined by

$$\Phi_t^{-1}(x, y) = (A_t(x, y), B_t(x, y)).$$

By the inverse local theorem with parameter  $t$ , we have that  $t \mapsto \Phi_t$  is a continuous function. Then the map  $t \mapsto f_{s(t)}$  is a continuous function, as

the composition of two continuous functions. As  $s(t)$  is a coefficient of the polynomial  $f_{s(t)}$ , the map  $t \mapsto s(t)$  is a continuous function.

As a conclusion we have a map  $t \mapsto s(t)$  which is continuous and such that  $s(0) = k$  and  $s(1) = \bar{k}$ . It implies that there exists  $\tau \in ]0, 1[$  such that  $s(\tau) \notin \mathcal{C}$ . Then  $g_\tau$  is algebraically, hence topologically, equivalent to  $f_{s(\tau)}$ ; on the other hand  $g_\tau$  is topologically equivalent to  $f_k$  (by hypothesis). As  $s(\tau) \notin \mathcal{C}$ ,  $f_{s(\tau)}$  and  $f_k$  are not topologically equivalent (because  $\#\mathcal{B}_{\text{aff}}$  are different), it provides a contradiction.  $\square$

The calculus have been done with the help of SINGULAR, [GPS], and especially with author's library `critic` described in [Bo3].

This research has been supported by a Marie Curie Individual Fellowship of the European Community (HPMF-CT-2001-01246).

## REFERENCES

- [AC] N. A'CAMPO, Real deformations and complex topology of plane curve singularities, *Ann. Fac. Sci. Toulouse Math.*, 8, 5-23, 1999.
- [ACA] E. ARTAL, J. CARMONA and J. COGOLLUDO, Effective invariants of braid monodromy and topology of plane curves, *preprint*.
- [BCW] H. BASS, E. CONNELL and D. WRIGHT, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc.*, 7, 287-330, 1982.
- [Bo1] A. BODIN, Classification of polynomials from  $\mathbb{C}^2$  to  $\mathbb{C}$  with one critical value, *Math. Z.*, 242, 303-322, 2002.
- [Bo2] A. BODIN, Invariance of Milnor numbers and topology of complex polynomials, to appear in *Commetarii Mathematici Helvetici*.
- [Bo3] A. BODIN, Computation of Milnor numbers and critical values in affine space and at infinity, *preprint*.
- [EN] D. EISENBUD and W. NEUMANN, Three-dimensional link theory and invariants of plane curve singularities, *Annals of Mathematics Studies*, 110, Princeton University Press, 1985.
- [GPS] G.-M. GREUEL, G. PFISTER, and H. SCHÖNEMANN, SINGULAR 2.0: a computer algebra system for polynomial computations. Centre for computer algebra, university of Kaiserslautern, 2001. <http://www.singular.uni-kl.de>.
- [KK] V. KHARLAMOV and VIK. KULIKOV, Diffeomorphisms, isotopies, and braid monodromy factorizations of plane cuspidal curves. *C. R. Acad. Sci.*, 333, 855-859, 2001.

Arnaud Bodin

Centre de Recerca Matemàtica, Apartat 50, 08193 Bellaterra, Spain

abodin@crm.es