RATIONAL EXTENDED MACKEY FUNCTORS FOR THE CIRCLE GROUP.

J.P.C. GREENLEES AND J.-PH. HOFFMANN

Abstract. We give a simple description of tom Dieck's extended Mackey functors for the circle group when their values are rational vector spaces.

1. Introduction.

T. tom Dieck [1, IV.8] has introduced a self dual notion of Mackey functors for compact Lie groups. It coincides with the usual notion for finite groups, and the most obvious difference in general is that tom Dieck's functors have non-trivial induction maps even when a subgroup has infinite index in its normalizer. The purpose of this note is to record the structure of these extended Mackey functors when the ambient group is the circle group and the values are rational. We give proofs in some detail because we found many opportunities for confusion.

1.A. **Notation for subgroups.** We let T denote the circle group, and for each $a \geq 1$ we let $T[a]$ denote the subgroup of order a. We will therefore index the closed subgroups by $1, 2, 3, \cdots, T$. The degree zero T-maps between homogeneous spaces are (up to homotopy, and omitting identities) $\pi_a : T/T[a] \longrightarrow T/T$. The Burnside ring has idempotents $e_a : T/T[a]_+ \longrightarrow T/T[a]_+$, and we let $\sigma^0_a = e_a T/T[a]_+$ denote the corresponding summand, which we call a *basic cell*. The maps between the cells are the rational multiples of the identity maps and the maps $p_a : \sigma^0_a \longrightarrow \sigma^0_T$.

1.B. **Notation for Mackey functors.** A Mackey functor is an additive contravariant functors $M : s\mathcal{O}_T \longrightarrow \mathbb{Q}$-mod on the stable orbit category $s\mathcal{O}_T$. We let $M_H = M(T/H_+)$ denote the value on the homogeneous space T/H_+. When $K \subseteq H \subseteq T$ the projection map $T/K \longrightarrow T/H$ induces restriction $M_H \longrightarrow M_K$ and induction $M_K \longrightarrow M_H$, and the induction maps are zero when $H = T$. As in [2] it is more economical to record the values $M^e_H = e_H M_H = M(\sigma^0_H)$ on the basic cells σ^0_H. Thus a Mackey

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functor M is given by a contravariant functor M^e on the category $s\mathcal{O}^b_T$ with objects the basic cells σ^0_H. In very concrete terms, M is given by rational vector spaces $V_1, V_2, \ldots, V_T$ (with $V_a = M^e_a$) and morphisms the restriction maps

$$r_a : V_T \longrightarrow V_a.$$

In this idempotent form M^e there are no induction maps. We may recover the original Mackey functor using the formulae

$$M_T = V_T \text{ and } M_a = \bigoplus_{b|a} V_b,$$

with induction and restriction maps being given by inclusions and projections for the relevant factors. In this note we will work exclusively with the economical idempotent form of Mackey functors. We continue the alphabetical association, so that L, M and N are associated to U, V and W.

We let $\mathfrak{M}_T$ denote the category of rational Mackey functors.

1.C. **Notation for extended Mackey functors.** We turn now to tom Dieck's extended Mackey functors. Again there are idempotents acting on the values for finite subgroups $T[a]$. Thus an extended Mackey functor $\tilde{M}$ is again given by specifying rational vector spaces $V_1, V_2, \ldots, V_T$ (with $V_a = \tilde{M}^e_a$) together with restriction maps

$$r_a : V_T \longrightarrow V_a,$$

but now we must also specify induction maps

$$i_a : V_a \longrightarrow V_T.$$

These are required to satisfy the relations

$$r_a i_b = 0$$

for all a and b, since the double coset formula implies that $r_a i_b$ is given by

$$\chi(T[a]\backslash T/T[b]) e_a \mathrm{ind}_a^{a\cap b} \mathrm{res}^b_{a\cap b}$$

and $\chi(T[a]\backslash T/T[b]) = \chi(T) = 0$. Here res and ind are the usual restriction and induction maps and χ the Euler-Poincaré characteristic.

Now consider the kernels

$$K_a = \ker(V_T \longrightarrow V_a) \text{ and } KV = \bigcap_{a\geq 1} K_a.$$

We refer to KV as the *T-core* of $\tilde{M}$. Using this, we see that the induction maps must map into KV and that this guarantees the required relations.

Summary 1.1. An extended Mackey functor is given by restriction maps $r_a : V_T \longrightarrow V_a$, and induction maps $i_a : V_a \longrightarrow KV$, where $KV = \bigcap_{a\geq 1} \ker(r_a)$.

It is convenient for later use to introduce the dual notations

$$I_a = \mathrm{im}(V_a \longrightarrow V_T) \text{ and } IV = \sum_{a\geq 1} I_a.$$

We refer to IV as the *induction image*. Any map of extended Mackey functors $\tilde{M} \longrightarrow \tilde{N}$ induce maps $KV \longrightarrow KW$ and $IV \longrightarrow IW$. Again, we may recover the original extended Mackey functor using the formulae

$$\tilde{M}_T = V_T \text{ and } \tilde{M}_a = \bigoplus_{b|a} V_b,$$

with induction and restriction maps being given by combining those for the relevant factors. In this note we will work exclusively with the economical idempotent form of extended Mackey functors.

We let $\tilde{\mathfrak{M}}_T$ denote the category of rational extended Mackey functors.

1.D. **Relation between Mackey functors and extended Mackey functors.** Our description makes clear that any Mackey functor can be regarded as an extended Mackey functor with trivial induction maps. We thus get a full embedding

$$i : \mathfrak{M}_T \longrightarrow \tilde{\mathfrak{M}}_T.$$

Any map $\tilde{M} \longrightarrow iN$ from an extended Mackey functor necessarily kills the T-core KV, and we quickly check that we may define a new extended Mackey functor $Q\tilde{M}$ by giving it the same values as $\tilde{M}$ on finite subgroups and $Q\tilde{M}_T = \tilde{M}_T/KV$, with zero induction maps.

Lemma 1.2. *The functor Q is left adjoint to i:*

$$\mathfrak{M}_T(Q\tilde{M}, N) = \tilde{\mathfrak{M}}_T(\tilde{M}, iN). \quad \square$$

2. Some simple examples of extended Mackey functors.

We introduce notation for some extended Mackey functors obtained from vector spaces in a simple way. Indeed, we shall see that these extended constructions give various adjoints to evaluation.

2.A. **Left adjoints.** First we construct a left adjoint to evaluation at T.

Construction 2.1. Given a vector space W, we may form the extended Mackey functor $p(W)$ by giving it value $W \oplus \bigoplus_{a \geq 1} W$ at T and value W at each $a \geq 1$. The restriction map

$$p(W)_T = W \oplus \bigoplus_{a \geq 1} W \longrightarrow W = p(W)_a$$

is projection onto the first factor, so the T-core is $\bigoplus_{a \geq 1} W$. The induction map

$$p(W)_a = W \longrightarrow W \oplus \bigoplus_{a \geq 1} W = p(W)_T$$

is inclusion of the ath term in the sum.

Lemma 2.2. *The functor p is left adjoint to evaluation at T in the sense that*

$$\tilde{\mathfrak{M}}_T(p(W), \tilde{M}) = \mathbb{Q}\text{-}mod(W, V_T). \quad \square$$

There is also a left adjoint to evaluation at finite subgroups.

Construction 2.3. Given vector spaces W_a for $a \geq 1$, we may form the extended Mackey functor $q(\mathbf{W})$ by giving it value W_a at a and $\bigoplus_{a \geq 1} W_a$ at T. The restriction map

$$q(\mathbf{W})_T = \bigoplus_{a \geq 1} W_a \longrightarrow W_a = q(\mathbf{W})_a$$

is zero (so the T-core is all of $q(\mathbf{W})_T$). The induction map

$$q(\mathbf{W})_a = W_a \longrightarrow \bigoplus_{a \geq 1} W_a = q(\mathbf{W})_T$$

is inclusion of the ath term in the sum.

Lemma 2.4. *The functor q is left adjoint to evaluation at finite subgroups in the sense that*

$$\tilde{\mathfrak{M}}_T(q(\mathbf{W}), \tilde{M}) = \prod_{a \geq 1} \mathbb{Q}\text{-}mod(W_a, V_a). \quad \square$$

Finally, there is a left adjoint to the T-core functor.

Construction 2.5. Given a vector space W we may form an extended Mackey functor $r(W)$ with T-core W by giving it value W at T and zero elsewhere.

Lemma 2.6. *The functor r is left adjoint to the T-core functor in the sense that*

$$\tilde{\mathfrak{M}}_T(r(W), \tilde{M}) = \mathbb{Q}\text{-}mod(W, KV). \quad \square$$

Remark 2.7. These constructions are related by a short exact sequence

$$0 \longrightarrow q(cW) \longrightarrow p(W) \longrightarrow r(W) \longrightarrow 0$$

where cW denotes the sequence constant at W. The sequence is not split.

2.B. **Right adjoints.** In fact r is also a right adjoint.

Lemma 2.8. *The functor r is right adjoint to the functor taking $\tilde{M}$ to V_T/IV:*

$$\tilde{\mathfrak{M}}_T(\tilde{M}, r(W)) = \mathbb{Q}\text{-}mod(V_T/IV, W). \quad \square$$

Next we construct a right adjoint to evaluation at T.

Construction 2.9. Given vector spaces W we may form an extended Mackey functor $e(W)$ with T-core W by giving it value $W \oplus \prod_{a\geq 1} W$ at T and value W at a for all $a \geq 1$. The restriction map

$$e(W)_T = W \oplus \prod_{a\geq 1} W \longrightarrow W = e(W)_a$$

is the projection, onto the ath factor of the product. The T-core is the first factor W, and for each a, the induction map

$$e(W)_a = W \longrightarrow W \oplus \prod_a W = e(W)_T$$

is the inclusion of W as the first factor.

Lemma 2.10. *The functor e is right adjoint to evaluation at T in the sense that*

$$\tilde{\mathfrak{M}}_T(\tilde{M}, e(W)) = \mathbb{Q}\text{-}mod(V_T, W). \quad \square$$

Finally, we construct a right adjoint to evaluation on finite subgroups.

Construction 2.11. Given vector spaces W_a for $a \geq 1$ we may form the extended Mackey functor $f(\mathbf{W})$ with T-core 0 by giving it value $\prod_{a\geq 1} W_a$ at T and value W_a at a. The restriction maps

$$f(\mathbf{W})_T = \prod_{a\geq 1} W_a \longrightarrow W_a = f(\mathbf{W})_a$$

are projections, and the T-core is zero (so that the induction maps are necessarily zero).

Lemma 2.12. *The functor f is right adjoint to evaluation at finite subgroups in the sense that*

$$\tilde{\mathfrak{M}}_T(\tilde{M}, f(\mathbf{W})) = \prod_{a\geq 1} \mathbb{Q}\text{-}mod(V_a, W_a). \quad \square$$

In particular by taking sequences $\mathbf{W}$ with only one non-zero entry we obtain right adjoints to evaluation at a for each $a \geq 1$.

3. Some homological algebra.

3.A. **Generalities.** Basic homological algebra of functor categories leads to the following.

Lemma 3.1. *A map $g : \tilde{M} \longrightarrow \tilde{N}$ of extended Mackey functors is a monomorphism if and only if it gives a monomorphism of vector spaces $V_a \longrightarrow W_a$ for $a \geq 1$ and $V_T \longrightarrow W_T$. It is an epimorphism if and only if it gives an epimorphism of vector spaces $V_a \longrightarrow W_a$ for $a \geq 1$ and $V_T \longrightarrow W_T$.*

We deduce some exactness statements.

Corollary 3.2. *(i) The T-core functor $\tilde{M} \longmapsto KV$ is left exact, but not exact.*
(ii) The induction image functor $\tilde{M} \longmapsto IV$ preserves monomorphisms and epimorphisms but is not half-exact.

Proof: The left exactness of the T-core follows from the Snake Lemma, and the fact that the induction image functor preserves monomorphisms and epimorphisms is elementary.

To see that the T-core functor is not right exact, take a sequence $\mathbf{W}$ of vector spaces and consider the map $q(\mathbf{W}) \longrightarrow f(\mathbf{W})$ which is the identify at each finite subgroup. This gives the exact sequence

$$0 \longrightarrow q(\mathbf{W}) \longrightarrow f(\mathbf{W}) \longrightarrow r(\prod_a W_a / \bigoplus_a W_a) \longrightarrow 0.$$

Now $K(r(V)) = V$ for any vector space V, and $K(f(\mathbf{W})) = 0$, so if infinitely many of the W_a are non-zero, the map on T-cores is not an epimorphism.

To see that the induction image functor is not half exact, we consider a map $r(\mathbb{Q}) \longrightarrow q(\mathbf{W})$ corresponding to a non-zero element x of $Kq(\mathbf{W}) = \bigoplus_a W_a$; this gives a short exact sequence

$$0 \longrightarrow r(\mathbb{Q}) \longrightarrow q(\mathbf{W}) \longrightarrow \tilde{M} \longrightarrow 0.$$

However $I(q(\mathbf{W})) = \bigoplus_a W_a$ and the element x maps to zero but does not lift to an element of $I(r(\mathbb{Q})) = 0$. $\square$

Remark 3.3. The counterpart of 2.7 is the statement that for any vector space W, there is an exact sequence

$$0 \longrightarrow r(W) \longrightarrow e(W) \longrightarrow f(cW) \longrightarrow 0$$

where cW is the constant sequence at W. This sequence does not split.

3.B. **Injective resolutions.** ¿From 2.10 and 2.12 we acquire a supply of injectives.

Corollary 3.4. *The extended Mackey functors $e(W)$ and $f(\mathbf{W})$ are injective.* □

It is easy to check that the category $\mathfrak{M}_T$ of conventional Mackey functors has injective dimension 1 since all the morphisms go in one direction (see [2, Appendix A]). Perhaps it is more surprising that the category of extended Mackey functors is also of finite injective dimension.

Proposition 3.5. *The category of extended Mackey functors has injective dimension 2.*

Proof: It is easy to see there are enough injectives since $\tilde{M}$ embeds in $e(V_T) \times f(\mathbf{V})$.

Now consider the map $\tilde{M} \longrightarrow f(\mathbf{V})$, and note that its kernel and cokernel are supported at T. We therefore obtain exact sequence

$$0 \longrightarrow r(U') \longrightarrow \tilde{M} \longrightarrow \tilde{M}' \longrightarrow 0 \qquad \text{and}$$
$$0 \longrightarrow \tilde{M}' \longrightarrow f(\mathbf{V}) \longrightarrow r(U'') \longrightarrow 0.$$

Since $f(\mathbf{V})$ is injective the result follows from the fact that $r(U)$ is of injective dimension ≤ 1 by 3.3.

It remains to give an example where the dimension is attained. For example if $\mathbf{V}$ is any sequence $V_1, V_2, \cdots$ and $\tilde{M}$ is defined to be zero at T and V_a at the finite subgroup of order a we have the injective resolution

$$0 \longrightarrow \tilde{M} \longrightarrow f(\mathbf{V}) \longrightarrow e(W) \longrightarrow f(cW) \longrightarrow 0,$$

where $W = \prod_{a \geq 1} V_a$ and cW is constant at W. Applying $\tilde{\mathfrak{M}}_T(\tilde{L}, \cdot)$ to the resolution we obtain the complex

$$\prod_a \mathbb{Q}\text{-mod}(U_a, V_a) \longrightarrow \mathbb{Q}\text{-mod}(U_T, \prod_b V_b) \longrightarrow \prod_a \mathbb{Q}\text{-mod}(U_a, \prod_b V_b)$$

for calculating the derived functors of $\tilde{\mathfrak{M}}_T(\cdot, \tilde{M})$ evaluated at $\tilde{L}$. Evidently, this has cohomology in codegree 2 for any non-zero $\tilde{L}$ with $U_T = 0$ (for example if $\tilde{L} = \tilde{M}$). □

3.C. **Projective resolutions.** ¿From 2.2 and 2.4 we acquire a supply of projectives.

Corollary 3.6. *The extended Mackey functors* $p(W)$ *and* $q(\mathbf{W})$ *are projective.* □

It is easy to check that the category $\mathfrak{M}_T$ of conventional Mackey functors has projective dimension 1 since all the morphisms go in one direction (see [2, Appendix A]). Perhaps it is more surprising that the category of extended Mackey functors has the same finite projective dimension.

Proposition 3.7. *The category of extended Mackey functors has projective dimension 2.*

Proof: It is easy to see there are enough projectives since there is an epimorphism $p(V_T) \oplus q(\mathbf{V}) \longrightarrow \tilde{M}$.

Now consider the map $q(\mathbf{V}) \longrightarrow \tilde{M}$, and note that its kernel and cokernel are supported at T. We therefore obtain exact sequence

$$0 \longrightarrow r(U') \longrightarrow q(\mathbf{V}) \longrightarrow \tilde{M}' \longrightarrow 0 \text{ and } 0 \longrightarrow \tilde{M}' \longrightarrow \tilde{M} \longrightarrow r(U'') \longrightarrow 0.$$

Since $q(\mathbf{V})$ is projective the fact that the projective dimension is ≤ 2 follows from the fact that $r(U)$ is of projective dimension ≤ 1 by 2.7.

In particular the object $\tilde{M}$ with $V_T = 0$ has projective resolution

$$0 \longrightarrow q(cU) \longrightarrow p(U) \longrightarrow q(\mathbf{V}) \longrightarrow \tilde{M} \longrightarrow 0,$$

where $U = \bigoplus_{a \geq 1} V_a$. Applying $\tilde{\mathfrak{M}}_T(\cdot, \tilde{N})$ we obtain the complex

$$\prod_a \mathbb{Q}\text{-mod}(\bigoplus_{b \geq 1} V_b, W_a) \longleftarrow \mathbb{Q}\text{-mod}(\bigoplus_{b \geq 1} V_b, W_T) \longleftarrow \prod_{a \geq 1} \mathbb{Q}\text{-mod}(V_a, W_a)$$

for calculating the right derived functors of $\tilde{\mathfrak{M}}_T(\tilde{M}, \cdot)$ evaluated at $\tilde{N}$. Choosing a non-zero $\tilde{N}$ with $W_T = 0$ (for instance $\tilde{N} = \tilde{M}$) we see that the second derived functor is non-zero. □

References

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School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: j.greenlees@sheffield.ac.uk

Mathematisches Institut, Universität Göttingen, D-37073 Göttingen
E-mail address: phoffman@uni-math.gwdg.de