Poincaré duality quivers

Sophie Dourlens

Abstract. The purpose of this paper is to establish a class of associative algebras which satisfy a kind of Poincaré duality on their Hochschild homology and cohomology. In order to do so, we build a class of quivers, whose path algebra, quotiented by a certain ideal, has a duality property. Van den Bergh in [**VdB**] has already established a kind of Poincaré duality on certain algebras, but without direct links with the topological Poincaré duality. By contrast, the classical Poincaré duality in simplicial homology and cohomology over $\mathbb{F}_2$ of finite dimensional compact manifolds is a direct corollary of the main theorem of this paper.

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Introduction

If X is a finite simplicial complex, and k a field, let us consider the *incidence algebra kX of X over k* : this algebra is generated by all the incidence relations $\sigma \subset \tau$, where σ and τ are simplexes of X. The product of two generators is given by:

$$(\sigma \subset \sigma').(\tau \subset \tau') = \begin{cases} \sigma \subset \tau' & \text{if } \sigma' = \tau, \\ 0 & \text{else.} \end{cases}$$

Gerstenhaber and Schack proved in [**GS2**] that the simplicial cohomology of X with coefficients in k is isomorphic to the Hochschild cohomology of kX with coefficients in itself:

$$H^*(X,k) \cong HH^*(kX,kX).$$

In the same way, there is an isomorphism in homology:

$$H_*(X,k) \cong HH_*(kX,kX'),$$

where kX' denotes the dual vector space of kX. Hence, when X is a triangulation of a compact orientable n-manifold, the Poincaré duality on X holds on kX:

$$HH^*(kX,kX) \cong HH_{n-*}(kX,kX').$$

In order to get such a duality property on other algebras, remark that the incidence algebra of X over k can be obtained by a different way. Consider the *quiver* $\mathcal{Q}$ *associated with* X: its vertices are the simplexes of X, and there is an arrow between the vertices σ and τ if $\sigma \subset \tau$ and $dim(\tau) = dim(\sigma) + 1$. Now consider the *path algebra* $k\mathcal{Q}$ *of* $\mathcal{Q}$ *over* k; this algebra is generated by the paths of $\mathcal{Q}$, and the product of two elements is induced by the paths composition if possible. There is a surjective morphism $k\mathcal{Q} \twoheadrightarrow kX$, whose kernel is generated by the differences $\gamma - \gamma'$, where γ and γ' are paths with same source and target. So kX can be viewed as the quotient $\overline{k\mathcal{Q}}$ of the path algebra of $\mathcal{Q}$ by this ideal (see [**C1**]).

Hence, when k is $\mathbb{F}_2$, and $\mathcal{Q}$ is the quiver of a triangulation of a compact n-manifold, the following duality property holds:

$$HH^*(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}) \cong HH_{n-*}(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}').$$

This remark suggests to search a class of quivers, whose path algebra over $\mathbb{F}_2$, quotiented by the above ideal, would have the same duality property; we show that it consists in the quivers satisfying hypothesis 1 to 7, given below. This class contains the quivers associated with triangulations of finite dimensional compact manifolds. So our main result (theorem 1) generalizes the Poincaré duality over $\mathbb{F}_2$ on these manifolds.

1 The class of Poincaré duality quivers

Let $\mathcal{Q}$ be a quiver, that is, a finite oriented graph. A *path of length* α in $\mathcal{Q}$ is a sequence of α consecutive arrows of $\mathcal{Q}$; a path of length 0 is just a vertex of $\mathcal{Q}$. If γ is a path, we note $s(\gamma)$ and $t(\gamma)$ its source and its target, respectively. Assume that $\mathcal{Q}$ is connected and has no oriented cycles, *i.e.* path γ of positive length such that $s(\gamma) = t(\gamma)$. Recall that such a quiver has a *level structure* (see [**D**]).

Suppose that $\mathcal{Q}$ satisfies the following seven hypothesis:

Hypothesis 1 *There is a level structure on* $\mathcal{Q}$ *such that there is an arrow between the vertex* e *and the vertex* e' only if *level* $(e') =$ *level* $(e) + 1$. *Then, the number of levels of* $\mathcal{Q}$ *is well defined; index the levels by* $0, \ldots, n$.

Hypothesis 2 *Between two vertices (of consecutive levels) there is at most one arrow.*

Hypothesis 3 *Each vertex of level* $l < n$ *is the source of at least one arrow.*

Hypothesis 4 *Each vertex of level $l > 0$ is the target of at least one arrow.*

Hypothesis 5 *Between two vertices of non consecutive levels, there is an even number of path.*

Hypothesis 6 *Every vertex of level 1 is the target of an even number of arrows.*

Hypothesis 7 *Every vertex of level $n-1$ is the source of an even number of arrows.*

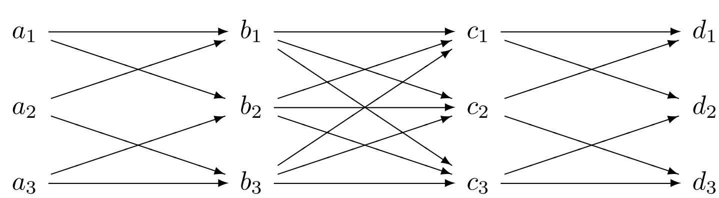
Remark 1 First, instead of hypothesis 5, it suffices to suppose that it holds between two vertices of respective levels l and $l+2$ $(0 \leq l \leq n-2)$. Second, the hypothesis 3 (*resp. hypothesis 4*) and hypothesis 5 imply that in fact each vertex of level $l < n$ (*resp.* $l > 0$) is the source (*resp. the target*) of at least two arrows.

The main example of such quivers is the one of quivers associated with a triangulation of a compact n-manifold:

Example 1 Let X be a finite simplicial complex of dimension n. Then the quiver $\mathcal{Q}$ associated with X has natural levels given by the dimension of simplexes, and hypothesis 1 is satisfied. Hypothesis 4 and 6 are obviously satisfied: each l-simplex contains exactly $l+1$ $(l-1)$-simplexes. Now if σ is any l-simplex and τ any $(l+k)$-simplex, then there is $k!$ path between σ and τ if σ is contained in τ, and 0 path if not. If k equals 1, this implies hypothesis 2, and, if k is greater or equal than 2, then $k!$ is an even number, and hypothesis 5 is satisfied. Now suppose that X is the triangulation of a compact n-manifold. Then an l-simplex with $l < n$ is necessarily contained in at least two $(l+1)$-simplexes, so hypothesis 3 is satisfied; furthermore, if $l = n-1$, each $(n-1)$-simplex is contained in exactly two n-simplexes, that is, hypothesis 7 is satisfied.

But the following example shows that quivers associated with triangulations of compact manifolds are not the only ones which satisfy the seven hypothesis:

Example 2



One can easily verify that this quiver satisfies the hypothesis above. Combinatorial arguments show that it does not come from a simplicial complex.

Let k be a field; the *path algebra of $\mathcal{Q}$ over k* is the k-algebra $k\mathcal{Q}$ generated by all the paths of $\mathcal{Q}$, with the relations:

$$\gamma \,.\, \gamma' = \begin{cases} \gamma \wedge \gamma' & \text{if } s(\gamma') = t(\gamma), \\ 0 & \text{else,} \end{cases}$$

where $\gamma \wedge \gamma'$ is the composite path of γ and γ'. Consider now the quotient $\overline{k\mathcal{Q}}$ of the path algebra of $\mathcal{Q}$ over k by the following equivalence relation on the paths of $\mathcal{Q}$:

$$\gamma \sim \gamma' \;\text{ if and only if }\; s(\gamma) = s(\gamma') \text{ and } t(\gamma) = t(\gamma') \tag{1}$$

(two such paths are said *parallel*, see [**C1**]).

Theorem 1 *Let $\mathcal{Q}$ be a connected quiver without oriented cycles satisfying hypothesis 1 to 7, with $n+1$ levels. Then the following duality property holds:*

$$HH^*(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}) \cong HH_{n-*}(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}\,'),$$

where $\overline{\mathbb{F}_2\mathcal{Q}}\,'$ is the dual vector space of $\overline{\mathbb{F}_2\mathcal{Q}}$.

Remark 2 As we hoped, the classical Poincaré duality over $\mathbb{F}_2$ for a compact n-manifold is a consequence of this theorem: just take for $\mathcal{Q}$ the quiver associated with any triangulation X of the manifold, and apply the result of Gerstenhaber and Schack [**GS2**] (detailed in section 2.1 of the present paper):

$$H^*(X, \mathbb{F}_2) \cong HH^*(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}) \cong HH_{n-*}(\overline{\mathbb{F}_2\mathcal{Q}}, \overline{\mathbb{F}_2\mathcal{Q}}\,') \cong H_{n-*}(X, \mathbb{F}_2).$$

2 Proof of the theorem

In this section k denotes the field $\mathbb{F}_2$.

2.1 Notation

Let $\mathcal{Q}$ be a quiver. An *l-trajectory of $\mathcal{Q}$* is a sequence of l consecutive paths $(\gamma_1, \ldots, \gamma_l)$ in $\mathcal{Q}$. Let $k\mathcal{Q}_0$ be the separable subalgebra of $k\mathcal{Q}$ generated by the vertices of $\mathcal{Q}$. The tensor product $(k\mathcal{Q})^{\otimes_{(k\mathcal{Q}_0)} l}$ is isomorphic to the

k-vector space whose basis is the set of all l-trajectories of $\mathcal{Q}$ (see [**C2**]). Here, in order to get an analogous decomposition of $(\overline{k\mathcal{Q}})^{\otimes_{(k\mathcal{Q}_0)}l}$, we need another definition of trajectory. So, in this paper, an *l-trajectory* will be a sequence of l consecutive *classes* of paths $([\gamma_1], \dots, [\gamma_l])$, under the equivalence relation 1; such a trajectory can be written $(e_0 \to e_1 \to \cdots \to e_l)$, where e_k is the source of the path γ_{k+1}, which is the target of the path γ_k. We obtain that $(\overline{k\mathcal{Q}})^{\otimes_{(k\mathcal{Q}_0)}l}$ is isomorphic to the k-vector space whose basis is the set of all l-trajectories (in the new sense) of $\mathcal{Q}$.

Let $\overline{C}^{\,*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$ be the $(k\mathcal{Q}_0)$-relative normalized Hochschild cochain complex of $\overline{k\mathcal{Q}}$ with coefficients in itself (see [**GS1**]); an element of $\overline{C}^{\,*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$ is a $(k\mathcal{Q}_0)^e$-linear morphism $\varphi : (\overline{k\mathcal{Q}})^{\otimes_{(k\mathcal{Q}_0)}l} \to \overline{k\mathcal{Q}}$ such that $\varphi(e_0 \to \cdots \to e_l) = 0$ if one of the e_i appears twice, $(k\mathcal{Q}_0)^e$ being the enveloping algebra of $k\mathcal{Q}_0$. Such a morphism is equivalent to a k-linear morphism $\varphi : (\overline{k\mathcal{Q}})^{\otimes_{(k\mathcal{Q}_0)}l} \to k$, still with $\varphi(e_0 \to \cdots \to e_l) = 0$ if one of the e_i appears twice. If $(e_0 \to \cdots \to e_l)$ is an l-trajectory without repetition of vertices, we note $(e_0 \to \cdots \to e_l)'$ the corresponding element of $\overline{C}^{\,l}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$:

$$(e_0 \to \cdots \to e_l)' : \quad \begin{array}{ccc} (\overline{k\mathcal{Q}})^{\otimes_{(k\mathcal{Q}_0)}l} & \longrightarrow & k \\ \left\{ \begin{array}{c} (e_0 \to \cdots \to e_l) \\ \text{others } l\text{-trajectories} \end{array} \right. & \begin{array}{c} \longmapsto \\ \longmapsto \end{array} & \begin{array}{c} 1 \\ 0. \end{array} \end{array}$$

Such elements form a basis of $\overline{C}^{\,l}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$.

The differential d of $\overline{C}^{\,*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$ is defined on a basis element $(e_0 \to \cdots \to e_l)'$ by:

$$\begin{aligned} d((e_0 \to \ldots \to e_l)') = & \sum_{\epsilon \to e_0} (\epsilon \to e_0 \to \cdots \to e_l)' \\ & + \sum_{i=0}^{l-1} \sum_{e_i \to \epsilon \to e_{i+1}} (e_0 \to \cdots \to e_i \to \epsilon \to e_{i+1} \to \cdots \to e_l)' \\ & + \sum_{e_l \to \epsilon} (e_0 \to \cdots \to e_l \to \epsilon)'. \end{aligned}$$

Because $k\mathcal{Q}_0$ is a separable algebra, the cohomology of $\overline{C}^{\,*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$ is $HH^*(\overline{k\mathcal{Q}}, \overline{k\mathcal{Q}})$ (see [**GS1**]). If $\mathcal{Q}$ is the quiver of a simplicial complex X, then there is a 1-1 correspondence between the l-flags of simplexes of X and the l-trajectories of $\mathcal{Q}$, compatible with the differentials. Thus this correspondence gives an isomorphism of cochain complexes [**GS2**]:

$$\overline{C}^{\,*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}) \cong \overline{C}^{\,*}(sd\, X, k),$$

where $\overline{C}^*(sd\,X, k)$ is the normalized cochain complex over k of the barycentric subdivision of X. This implies the isomorphism in cohomology:

$$HH^*(\overline{k\mathcal{Q}}, \overline{k\mathcal{Q}}) \cong H^*(X, k).$$

In the same way we note $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$ the $(k\mathcal{Q}_0)$-relative normalized Hochschild chain complex of $\overline{k\mathcal{Q}}$ with coefficients in its dual. If $(e \to f)$ is a class of paths of $\mathcal{Q}$, we note $(e \to f)'$ the corresponding element in $\overline{k\mathcal{Q}}'$. A basis element of $\overline{C}_l(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$ is a tensor product over $(k\mathcal{Q}_0)^e$ of the type $(e_0 \to \cdots \to e_l) \otimes (e_0 \to e_l)'$, with no repetition of vertices in the trajectory. As for cochains, such an element may be replaced by the trajectory $(e_0 \to \cdots \to e_l)$, with no repetition of vertices. The differential is given by:

$$d(e_0 \to \cdots \to e_l) \quad = \quad \sum_{i=0}^{l} (e_0 \to \cdots \to \widehat{e_i} \to \cdots \to e_l).$$

The homology of $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$ is $HH_*(\overline{k\mathcal{Q}}; \overline{k\mathcal{Q}}')$. If $\mathcal{Q}$ comes from a simplicial complex X, then we have the isomorphism of chain complexes:

$$\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \cong \overline{C}_*(sd\,X, k),$$

and thus the isomorphism in homology:

$$HH_*(\overline{k\mathcal{Q}}; \overline{k\mathcal{Q}}') \cong H_*(X, k)$$

(see [**C1**]).

In order to prove the theorem, we will need also the following complexes $\mathcal{C}^*$ and $\mathcal{D}_*$. Let $\mathcal{C}_l$ be the k-vector space with basis the set of all vertices of level l in $\mathcal{Q}$. Let $\mathcal{C}^l$ be the dual of $\mathcal{C}_l$. If e is a vertex of degree l, we note e' the element associated with e in $\mathcal{C}^l$; such elements form a basis of $\mathcal{C}^l$. Consider the following morphism:

$$\begin{array}{rccc} d: & \mathcal{C}^l & \longrightarrow & \mathcal{C}^{l+1} \\ & e' & \longmapsto & \sum_{e \to f} f'. \end{array}$$

Because k is $\mathbb{F}_2$, hypothesis 5 implies $d \circ d = 0$, so in this case $\mathcal{C}^*$ becomes a cochain complex. If k was not $\mathbb{F}_2$, then one would have to define some signs for d, that is not clearly feasible. Let now $\mathcal{D}_{n-l}$ be the k-vector space whose basis is the set of elements

$$\left\{ \sum_{\substack{(e_l \to \cdots \to e_n) \\ (n-l)\text{-traj.} \\ \text{starting with } e_l}} (e_l \to \cdots \to e_n) \quad \middle| \quad e_l \in \mathcal{C}_l \right\};$$

note that this set is never empty because of hypothesis 3: each vertex of level $0 \leq l \leq n$ is even the source of at least two $(n-l)$-trajectories (remark 1). We define the following differential on $\mathcal{D}_*$:

$$\begin{array}{rccc} d: & \mathcal{D}_{n-l} & \longrightarrow & \mathcal{D}_{n-l-1} \\ & \sum\limits_{e_l \text{ fixed}} (e_l \to \cdots \to e_n) & \longmapsto & \sum\limits_{e_l \to e_{l+1}} \sum\limits_{e_{l+1} \text{ fixed}} (e_{l+1} \to \cdots \to e_n). \end{array}$$

Once again, because k is $\mathbb{F}_2$, hypothesis 5 implies $d \circ d = 0$, and $\mathcal{D}_*$ is a chain complex.

2.2 The quasi-isomorphism ϕ

If $\mathcal{Q}$ is a quiver satisfying hypothesis 1 to 7, we define a morphism:

$$\phi: \overline{C}^*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}) \longrightarrow \overline{C}_{n-*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$$

and show that it induces an isomorphism in homology. We let in degree l:

$\phi: \overline{C}^l(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}) \longrightarrow \overline{C}_{n-l}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$

$$(e_0 \to \cdots \to e_l)' \longmapsto \begin{cases} \sum\limits_{\substack{(e_l \to \cdots \to e_n) \\ (n-l)\text{-trajectories}}} (e_l \to e_{l+1} \to \cdots \to e_n) & \text{if level } (e_l) = l \\ 0 & \text{if not.} \end{cases}$$

2.2.1 ϕ is a complex morphism

We have to prove that ϕ commutes to the differentials of $\overline{C}^*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$ and $\overline{C}_{n-*}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$; it suffices to prove it on the element $(e_0 \to \cdots \to e_l)'$.

On the first hand, recall the definition of the differential d of $\overline{C}^*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}})$:

$$\begin{aligned} d((e_0 \to \ldots \to e_l)') = & \sum_{\epsilon \to e_0} (\epsilon \to e_0 \to \cdots \to e_l)' \\ & + \sum_{i=0}^{l-1} \sum_{e_i \to \epsilon \to e_{i+1}} (e_0 \to \cdots \to e_i \to \epsilon \to e_{i+1} \to \cdots \to e_l)' \\ & + \sum_{e_l \to \epsilon} (e_0 \to \cdots \to e_l \to \epsilon)'. \end{aligned}$$

If the level of the vertex e_l is l, then the two first terms are zero, and applying ϕ gives:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = \sum_{e_l \to \epsilon} \sum_{\substack{(\epsilon \to \cdots \to e_{n-1}) \\ (n-l-1)\text{-traj.}}} (\epsilon \to e_{l+1} \to \cdots \to e_{n-1}).$$

If the level of e_l is $l+1$, and the level of e_0 is 1, then the second term is zero, and when applying ϕ, the third term vanishes. Hence we get:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = \sum_{\epsilon \to e_0} \sum_{\substack{(e_l \to \cdots \to e_{n-1}) \\ (n-l-1)\text{-traj.}}} (e_l \to \cdots \to e_{n-1}).$$

But, by hypothesis 6, there is an even number of such ϵ with an arrow $\epsilon \to e_0$; hence in this case we get:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = 0.$$

If the level of e_l is $l+1$, and the level of e_0 is 0, then the first term is zero, and when applying ϕ, the third term vanishes. Then we get:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = \sum_{i=0}^{l-1} \sum_{e_i \to \epsilon \to e_{i+1}} \sum_{\substack{(e_l \to \cdots \to e_{n-1}) \\ (n-l-1)\text{-traj.}}} (e_l \to \cdots \to e_{n-1}).$$

As before, by hypothesis 5, there is an even number of vertices ϵ included in a path $e_i \to \epsilon \to e_{i+1}$, and then in this case:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = 0.$$

In all other cases, we have $\phi \circ d((e_0 \to \cdots \to e_l)') = 0$.

In conclusion, we have:

$$\phi \circ d((e_0 \to \cdots \to e_l)') = \begin{cases} \sum_{e_l \to \epsilon} \sum_{\substack{(\epsilon \to \cdots \to e_{n-1}) \\ (n-l-1)\text{-traj.}}} (\epsilon \to e_{l+1} \to & \cdots \to e_{n-1} \\ & \text{if level } (e_l) = l \\ 0 & \text{else.} \end{cases}$$

On the other hand, by definition of ϕ, we have $d \circ \phi((e_0 \to \cdots \to e_l)') = 0$

if the level of the vertex e_l is not l, and else:

$$d \circ \phi((e_0 \to \cdots \to e_l)') = d\Big(\sum_{\substack{(e_l \to \cdots \to e_n) \\ (n-l)\text{-traj.}}} (e_l \to e_{l+1} \to \cdots \to e_n) \Big)$$

$$= \sum_{e_l \to e_{l+1}} \sum_{\substack{(e_{l+1} \to \cdots \to e_n) \\ (n-l-1)\text{-traj.}}} (e_{l+1} \to \cdots \to e_n)$$

$$+ \sum_{\substack{(e_l \to \cdots \to e_n) \\ (n-l)\text{-traj.}}} \sum_{i=1}^{n-2} (e_l \to \cdots \to e_i \to e_{i+2} \to \cdots \to e_n)$$

$$+ \sum_{\substack{(e_l \to \cdots \to e_{n-1}) \\ (n-l-1)\text{-traj.}}} \sum_{e_{n-1} \to e_n} (e_l \to \cdots \to e_{n-1}).$$

As before, the second term vanishes by hypothesis 5, and the third term also vanishes by hypothesis 7.

So, we have found the same result for $\phi \circ d$ and $d \circ \phi$ on the basis element $(e_0 \to \cdots \to e_l)'$. Hence ϕ is a complex morphism.

2.2.2 ϕ induces an isomorphism in homology

Consider the following decomposition of ϕ:

$$\phi : \overline{C}^{\,*}(\overline{kQ}, kQ_0; \overline{kQ}) \xrightarrow{\phi_1} \mathcal{C}^* \xrightarrow{\phi_2} \mathcal{D}_{n-*} \xrightarrow{\phi_3} \overline{C}_{\,n-*}(\overline{kQ}, kQ_0; \overline{kQ}\,'),$$

where the three morphisms ϕ_1, ϕ_2 and ϕ_3 are defined in the following way:

$$\begin{array}{rccc} \phi_1 : & \overline{C}^{\,l}(\overline{kQ}, kQ_0; \overline{kQ}) & \longrightarrow & \mathcal{C}^l \\ & (e_0 \to \cdots \to e_l)' & \longmapsto & \begin{cases} e_l' & \text{if level } (e_l) = l \\ 0 & \text{else.} \end{cases} \end{array}$$

Basic verifications prove that ϕ_1 is a complex morphism. In order to prove that ϕ_1 induces an isomorphism in cohomology, we will consider the chain morphism whose dual is ϕ_1. We will prove that this last morphism is a quasi-isomorphism by building, using an acyclic carrier, an homotopy inverse.

Now, ϕ_2 is the morphism:

$$\begin{array}{rccl} \phi_2 : & \mathcal{C}^l & \longrightarrow & \mathcal{D}_{n-l} \\ & e'_l & \longmapsto & \displaystyle\sum_{\substack{(e_l \to \cdots \to e_n) \\ l\text{-traj.}}} (e_l \to \cdots \to e_n). \end{array}$$

It is easy to see that ϕ_2 is a complex morphism; moreover, because of hypothesis 3, ϕ_2 is an isomorphism. Last, ϕ_3 is the inclusion:

$$\begin{array}{rccl} \phi_3 : & \mathcal{D}_{n-l} & \hookrightarrow & \overline{C}_{n-l}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \\ & \displaystyle\sum_{e_l \text{ fixed}} (e_l \to \cdots \to e_n) & \mapsto & \displaystyle\sum_{e_l \text{ fixed}} (e_l \to \cdots \to e_n). \end{array}$$

The map ϕ_3 commutes with the differentials because ϕ_1, ϕ_2 and $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ do. Once again, we will construct for ϕ_3 a homotopy inverse, by using an acyclic carrier.

For the definition of the homotopy inverses, we will have to make some choices on the vertices of $\mathcal{Q}$. In order to do that we suppose that there is a lexicographical order on the vertices of $\mathcal{Q}$.

First, recall the notion of acyclic carrier. Let C_* and C'_* be two augmented chain complexes; suppose that C_* is free, and fix a basis in each C_l. An *acyclic carrier from* C_* *into* C'_* is a map Λ which associates with any basis element s of C_l an acyclic subcomplex $\Lambda(s)$ of C'_*, such that if the basis element t appears in the decomposition of ds, then $\Lambda(t)$ is a subcomplex of $\Lambda(s)$. Let $\varphi : C_* \longrightarrow C'_*$ be a chain morphism. We said that φ *is carried by* Λ if, for every basis element s of C_*, its image $\varphi(s)$ belongs to $\Lambda(s)$. Two such morphisms, carried by the same acyclic carrier, are homotopic (see [**M**]).

Now consider the morphism

$$\begin{array}{rccl} \psi_1 : & \mathcal{C}_l & \longrightarrow & \overline{C}_l(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \\ & e_l & \longmapsto & \displaystyle\sum_{\substack{(e_0 \to \cdots \to e_l) \\ l\text{-traj.}}} (e_0 \to \cdots \to e_{l-1} \to e_l)\,; \end{array}$$

the dual of this morphism is ϕ_1. Note that the image by ψ_1 of a vertex e_l of $\mathcal{C}_l$ is never 0 because of hypothesis 4: there is at least one l-trajectory ending with e_l. We define

$$\begin{array}{rcl} \mu_1 : \overline{C}_l(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') & \longrightarrow & \mathcal{C}_l \\ (e_0 \to \cdots \to e_l) & \longmapsto & \begin{cases} e_l & \text{if level } (e_l) = l \text{ and} \\ & (e_0 \to \ldots \to e_l) = (\epsilon_0 \to \ldots \to \epsilon_{l-1} \to e_l) \\ 0 & \text{else,} \end{cases} \end{array}$$

where $(\epsilon_0 \to \cdots \to \epsilon_{l-1} \to e_l)$ is the first l-trajectory ending with e_l in the lexicographical order. Precisely, ϵ_{l-1} is the first vertex of level $l-1$ with an arrow $\epsilon_{l-1} \to e_l$, and, for all i from $l-1$ down to 1, ϵ_{i-1} is the first vertex in the lexicographical order of level $i-1$ with an arrow $\epsilon_{i-1} \to \epsilon_i$. One can easily verify that this choice is compatible with the differentials. The composed morphism $\mu_1 \circ \psi_1$ is equal to the identity morphism of $\mathcal{C}^l$, but the other composed morphism is not: $\psi_1 \circ \mu_1(e_0 \to \cdots \to e_l)$ is equal to

$$\sum_{\substack{(f_0 \to \cdots \to e_l) \\ l\text{-traj.}}} (f_0 \to \cdots \to f_{l-1} \to e_l)$$

if $e_l \in \mathcal{C}_l$ and $(e_0 \to \cdots \to e_l) = (\epsilon_0 \to \cdots \to \epsilon_{l-1} \to e_l)$, and 0 else. In order to prove that ψ_1 is a quasi-isomorphism, we have just to define an acyclic carrier

$$\Lambda_1 : \overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \longrightarrow \overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$$

which carries both $\psi_1 \circ \mu_1$ and the identity morphism of $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$. First, remark that $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$ can be augmented by the following map:

$$\begin{array}{rcl} \overline{C}_0(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') & \longrightarrow & k \\ e & \longmapsto & 1. \end{array}$$

Let $\Lambda_1(e_0 \to \cdots \to e_l)$ be the chain complex whose degree l is the k-vector space generated by the set of all the l-trajectories ending with e_l, and all their faces in degree less than l. If $(f_0 \to \cdots \to f_{l-1})$ is an $(l-1)$-trajectory which appears in the decomposition of $d(e_0 \to \cdots \to e_l)$, that is with $\{f_0, \ldots, f_{l-1}\} \subset \{e_0, \ldots, e_l\}$, then one can easily see that $\Lambda_1(f_0 \to \cdots \to f_{l-1})$ is a subcomplex of $\Lambda_1(e_0 \to \cdots \to e_l)$. Moreover, Λ_1 carries both $\psi_1 \circ \mu_1$ and the identity of $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$. So, in order to prove that these two morphisms are homotopic, we have just to prove that Λ_1 is acyclic. In fact, it suffices to see that the following map is an homotopy for

$\Lambda_1(e_0 \to \cdots \to e_l)$:

$$h_1 : \quad \big(\Lambda_1(e_0 \to \ldots \to e_l)\big)_j \longrightarrow \big(\Lambda_1(e_0 \to \ldots \to e_l)\big)_{j+1}$$

$$(f_0 \to \ldots \to f_j) \longmapsto \begin{cases} (f_0 \to \ldots \to f_j \to e_l) & \text{if } f_j \neq e_l \\ 0 & \text{else.} \end{cases}$$

A simple computation shows that h_1 is an homotopy. So, we proved that ψ_1, hence ϕ_1, is a quasi-isomorphism.

Now, for ϕ_3, the computations are symmetric than the ones for ψ_1; we define

$$\mu_3 : \overline{C}_{n-l}(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \longrightarrow \mathcal{D}_{n-l}$$

by

$$\mu_3(e_l \to \cdots \to e_n) = \sum_{\substack{(e_l \to \cdots \to f_n) \\ (n-l)\text{-traj.}}} (e_l \to f_{l+1} \to \cdots \to f_n)$$

if level $(e_l) = l$ and $(e_l \to \cdots \to e_n) = (e_l \to \epsilon_{l+1} \to \cdots \to \epsilon_n)$ is the first $(n-l)$-trajectory starting with e_l in the lexicographical order: ϵ_{l+1} is the first vertex of level $l+1$ with an arrow $e_l \to \epsilon_{l+1}$, and for all i from $l+1$ to $n-1$, ϵ_{i+1} is the first vertex of level $i+1$ with an arrow $\epsilon_i \to \epsilon_{i+1}$. Else, we let $\mu_3(e_l \to \cdots \to e_n) = 0$. So the composite $\mu_3 \circ \phi_3$ equals the identity morphism of $\mathcal{D}_*$, and $\phi_3 \circ \mu_3$ is defined by:

$$\phi_3 \circ \mu_3(e_l \to \epsilon_{l+1} \to \ldots \to \epsilon_n) = \sum_{\substack{(e_l \to \cdots \to f_n) \\ (n-l)\text{-traj.}}} (e_l \to f_{l+1} \to \ldots \to f_n);$$

$\phi_3 \circ \mu_3$ is zero on the other trajectories.

We build now an acyclic carrier

$$\Lambda_3 : \overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}') \longrightarrow \overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$$

in such a way: let $\Lambda_3(e_l \to \ldots \to e_n)$ be in degree $n-l$ the k-vector space generated by all the $(n-l)$-trajectories starting with e_l, and all their faces in degree less than $n-l$. So Λ_3 carries both $\phi_3 \circ \mu_3$ and the identity of $\overline{C}_*(\overline{k\mathcal{Q}}, k\mathcal{Q}_0; \overline{k\mathcal{Q}}')$, and, as before, in order to finish the proof, it is sufficient to observe that the following map is an homotopy for $\Lambda_3(e_l \to \ldots \to e_n)$:

$$h_3 : \quad \big(\Lambda_3(e_l \to \ldots \to e_n)\big)_j \quad \longrightarrow \quad \big(\Lambda_3(e_l \to \ldots \to e_n)\big)_{j+1}$$

$$(f_0 \to \ldots \to f_j) \quad \longmapsto \quad \begin{cases} (e_l \to f_0 \to \ldots \to f_j) & \text{if } f_0 \neq e_l \\ 0 & \text{else.} \end{cases}$$

Remark 3 The acyclic carrier Λ_1 generalizes the acyclic carrier used to prove that a simplicial complex has the same homology than its barycentric subdivision (see [**ES**]). In the same way, Λ_3 can be used to prove that a simplicial complex has the same homology than its dual block complex. The homotopies h_1 and h_3 are copies of the one used in the simplicial framework to prove that a cone is acyclic.

Remark 4 The proof of the theorem did not use hypothesis 2. Actually this hypothesis is not necessary, but permits to simplify the other ones; for example, hypothesis 6 would be: *Every vertex of level 1 is the target of an even number of* classes *of arrows*, two arrows with same source and target being identified by the equivalence relation 1. So, adding this hypothesis restricts the class of quivers, but, because of this identification in $\overline{k\mathcal{Q}}$, does not restrict the class of algebras that verify the duality property.

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Laboratoire Dieudonné, Parc Valrose, Université de Nice-Sophia Antipolis, 06108 Nice cedex 2, France.
E-mail address: dourlens@math.unice.fr