GEOMETRICAL AND TOPOLOGICAL PROPERTIES OF POLYNOMIAL FIBRES

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Abstract. We establish a Gauss-Bonnet type formula for a smooth fibre of a non-proper real polynomial of $\mathbb{R}^n$. For this we need to study topological properties of a generic hyperplane section of this fibre.

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1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d and let us assume that 0 is a regular value of f. This implies that $V = f^{-1}(0)$ is a smooth orientable manifold of dimension $n-1$ (possibly empty). We choose the orientation given by ∇f, the gradient vector of f. Let k be the curvature of V. It is well known from Morse theory that for a generic line $L \in G_n^1$, G_n^1 being the Grassman manifold of lines in $\mathbb{R}^n$, the orthogonal projection $P_L : V \mapsto L$ has only non-degenerate, hence isolated, critical points. For any subset $U \subset V$, we write $C(U, L)$ for the number of those points lying inside U. Since we are in an algebraic setting, $C(U, L)$ is finite. Let $(K_r)_{r>0}$ be an exhaustive sequence of compact sets of V, we mean that $\cup_{r>0} K_r = V$ and $K_r \subsetneq K_{r'}$ if $r < r'$. Using an exchange formula, we have

$$\int_{K_r} |k| dv = \int_{G_n^1} C(K_r, L) dL,$$

dv being the volume form of V and dL the one of G_n^1.

As r tends to infinity, $C(K_r, L)$ tends to $C(V, L)$ and, since by Bezout's theorem $C(K_r, L) \leq d(d-1)^{n-1}$, we can use Lebesgue's theorem to get that $\lim_{r \to +\infty} \int_{K_r} |k| dv$ does exist, is equal to $\int_{G_1^n} C(V, L) dL$ and therefore is independent of the choice of the exhaustive sequence of compact sets. Hence it makes sense to define $\int_V |k| dv := \lim_{r \to +\infty} \int_{K_r} |k| dv$ where $(K_r)_{r>0}$ is an exhaustive sequence of compact sets. We have proved that

$$\int_V |k| dv = \int_{G_n^1} C(V,L) dL \leq \mathrm{Vol}(S^{n-1}) d(d-1)^{n-1}.$$

A more precise study of $\int_V |k| dv$ has been started by Risler [Ri]. In the case $n = 2$, he gives a finer bound for $\int_V |k| dv$ for a generic polynomial f and studies its sharpness.

It seems natural after that to search similar results for the curvature k instead of the absolute curvature $|k|$. One has to give first a suitable definition for the integral of k over V and then, having in mind the Gauss-Bonnet formula, to relate this integral to topological invariants associated with V. This is the aim of this paper. We will prove that $\lim_{r \to +\infty} \int_{K_r} k dv$, with $(K_r)_{r>0}$ an exhaustive sequence of compact sets, does exist and does not depend on the choice of the sequence. Furthermore, putting $\int_V k dv$ for this limit, we show that (**Theorem 4.5**)

$$\text{if } n \text{ is odd, } \int_V k dv = \frac{1}{2} \mathrm{Vol}(S^{n-1}) \chi(V) - \frac{1}{2} \int_{S^{n-1}} \chi(V \cap \{P_u = 0\}) du,$$

$$\text{if } n \text{ is even, } \int_V k dv = -\frac{1}{2} \mathrm{Vol}(S^{n-1}) \big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) \big]$$

$$+ \frac{1}{2} \int_{S^{n-1}} \big[\chi(\{f \geq 0\} \cap \{P_u = 0\}) - \chi(\{f \leq 0\} \cap \{P_u = 0\}) \big] du,$$

where du is the volume form of S^{n-1} and $P_u(x) = \langle x, u \rangle$ for all $u \in S^{n-1}$. Then we explain how these formulas generalize in the algebraic case the well known Gauss-Bonnet formula, due to Hopf [Ho], for even-dimensional hypersurfaces and the less known Gauss-Bonnet formula, due to Haefliger [Hae] and Samelson [Sa], for odd-dimensional hypersurfaces.

In order to establish our formulas, we use an exchange formula. It relates the integral of the curvature over a compact subset with non-empty interior K of V, to the critical points of orthogonal projections onto generic lines. It states that

$$\int_K k dv = \frac{1}{2} \int_{S^{n-1}} I(K,u) du,$$

where $I(K,u)$ is the sum of the Gauss indices of the critical points of the function $P_u : V \to \mathbb{R}$ lying inside K. By the Gauss index at a point in V, we mean the local degree of the Gauss map. Since our goal is to compute the limit of $\int_K k dv$ as K "tends" to V, we see that we have to evaluate $I(V,u)$ for a generic $u \in S^{n-1}$. But for such an u, the Gauss index at a critical point of P_u is strongly related to the Morse index of $P_{u|V}$ (see Lemma 4.3). That is why we are leaded to study topological properties of generic hyperplane sections of V. This is actually the first part of our work.

We show that there exists a semi-algebraic set Σ of S^{n-1} of dimension less than $n-2$ such that for each $u \notin \Sigma$ the projection P_u has a nice behaviour. More precisely, if $\{P_i\}$ is the set of critical points of $P_{u|V}$ and $\{\lambda_i\}$ the set of their respective Morse indices, then for every regular value c of $P_u : V \to \mathbb{R}$ we have (**Theorem 3.8**), if n is odd:

$$\chi(V) - \chi(V \cap \{P_u = c\}) = \sum_i (-1)^{\lambda_i},$$

$$\chi(\{f \geq 0\} \cap \{P_u \geq c\}) - \chi(\{f \geq 0\} \cap \{P_u \leq c\}) - \chi(\{f \leq 0\} \cap \{P_u \geq c\}) + \chi(\{f \leq 0\} \cap \{P_u \leq c\}) = \sum_i \operatorname{sign}\langle u, \nabla f(p_i)\rangle \cdot (-1)^{\lambda_i},$$

if n is even:

$$\chi(V \cap \{P_u \geq c\}) - \chi(V \cap \{P_u \leq c\}) = \sum_i (-1)^{\lambda_i},$$

$$\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big] - \big[\chi(\{f \geq 0\} \cap \{P_u = c\}) - \chi(\{f \leq 0\} \cap \{P_u = c\})\big] = \sum_i \operatorname{sign}\langle u, \nabla f(p_i)\rangle \cdot (-1)^{\lambda_i}.$$

We think that this theorem, key-point in the proof of the Gauss-Bonnet formula, has interest by itself. It means that for a generic u the projection $P_{u|V}$ does not have singularities at infinity and that the above linear combinations of Euler characteristics do not depend on the choice of the regular value. It can be viewed as a real version of previous results on hyperplane sections of a complex affine variety, given by Némethi [Ne1,Ne2] and Cassou-Nogues and Dimca [C-N.D] in the smooth case and by Tibar [Ti] in the case of isolated singularities. We prove it studying some polar set at infinity. The polar set considered is the set of points in V where ∇f, $\nabla \omega$ and u are not linearly independent, ω being the square of the euclidian distance function. Using a Curve Selection Lemma at infinity [NZ], we show that there exists no sequence of points (x^k) in this set such that $\|x^k\| \to +\infty$ and $|P_u(x^k)|$ tends to a finite value. This implies that given a regular value c of $P_{u|V}$, there exists $R_c > 0$ such that for all $R \geq R_c$, the functions $P_{u|V \cap B_R^n}$ and $P_{u|\{f?0\} \cap B_R^n}$, $? \in \{\leq, \geq\}$, (resp. $-P_{u|V \cap B_R^n}$ and $-P_{u|\{f?0\} \cap B_R^n}$) are not inward at the critical points of $P_{u|V \cap S_R^{n-1}}$ lying in $\{P_u > c\}$ (resp. $\{P_u < c\}$). Here $S_R^{n-1} = \partial B_R^n$ is the $(n-1)$-dimensional sphere of radius R. It remains to apply Morse theory for manifolds with corners to get our results, knowing that for any non-compact semi-algebraic set $W \subset \mathbb{R}^n$ the intersection $W \cap B_R^n$ is a retraction of W for R sufficiently big.

The paper is organized as follows: in Section 2, we recall some facts about Morse theory for manifolds with corners. In Section 3, we study hyperplane sections and in Section 4 we prove our Gauss-Bonnet formula. Section 5 is devoted to the special case of plane curves. Finally in Section 6 we explain briefly a complex version of our results. The author is grateful to Marcel Nicolau for his remarks and comments.

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2. Morse theory for manifolds with corners

We generalize the notion of correct critical points and Morse correct functions, defined for manifolds with boundary in [HL], to the case of manifolds with corners. Then we relate the Euler characteristic of a manifold with corners to the indices of correct critical points.

Let us start with some basic facts on manifolds with corners. Our reference is [Ce]. A manifold with corners is defined by an atlas of charts modelled on open subsets of $\mathbb{R}^n_+$. We write ∂M for its boundary. We will make the additional assumption that the boundary is partitioned into pieces $\partial_i M$, themselves manifolds with corners, such that in each chart, the intersections with the coordinate hyperplanes $x_j = 0$ correspond to distinct pieces $\partial_i M$ of the boundary. For any set I of suffices, we write $\partial_I M = \cap_{i \in I} \partial_i M$ and we make the convention that $\partial_\emptyset M = M \setminus \partial M$.

Any n-manifold M with corners can be embedded in a n-manifold M^+ without boundary so that the pieces $\partial_i M$ extend to submanifolds $\partial_i M^+$ of codimension 1 in M^+. We will assume that M^+ is provided with a Riemannian metric.

Let M be a manifold with corners and $\omega : M^+ \to \mathbb{R}$ a smooth map. We consider the points P which are critical points of $\omega_{|\partial_I M^+}$.

Definition 2.1. *A critical point P is correct (resp. Morse correct) if, taking $I(P) := \{i \ |P \in \partial_i M\}$, P is a critical point (resp. Morse critical point) of $\omega_{|\partial_{I(P)} M^+}$, and is not a critical point of $\omega_{|\partial_J M^+}$ for any proper subset J of $I(P)$.*

Definition 2.2. *The maps ω with all critical points Morse correct are called Morse correct.*

Proposition 2.3. *The set of Morse correct functions is dense and open in the space of all maps $M^+ \to \mathbb{R}$.*

Proof. It is clear from classical Morse theory, because there is a finite number of pieces $\partial_I M^+$. □

The index $\lambda(P)$ of ω at a Morse correct point P is defined to be that of $\omega_{|\partial_{I(P)}M^+}$. If P is a correct critical point of ω, $i \in I(P)$, and J is formed from $I(P)$ by deleting i, then in a chart at P with $\partial_J M$ mapping to $\mathbb{R}^p_+$ and $\partial_{I(P)}M$ to the subset $x_1 = 0$, the function ω is non-critical, but its restriction to $x_1 = 0$ is. Hence $\partial\omega/\partial x_1 \neq 0$.

Definition 2.4. *We say that ω is inward at P if, for each $i \in I(P)$, we have $\partial\omega/\partial x_1 > 0$.*

Remark 2.5. *By our convention, if $I(P) = \emptyset$, then ω is inward at P.*

Theorem 2.6. *If M is compact and ω is Morse correct,*

$$\chi(M) = \sum \left\{ (-1)^{\lambda(P)} \mid P \textit{ inward Morse critical point} \right\}.$$

Proof. This is a consequence of stratified Morse theory ([G.M-P], [Ham]). A good summary of the results we use can be found in [BK], Section 2.

The manifold with corners M is a compact Whitney stratified set of M^+, with stratum the $\partial_I M$. The function $\omega : M \to \mathbb{R}$ is easily seen to be a Morse function in the sense of [G.M-P] and so

$$\chi(M) = \sum \{\alpha(\omega, P) \mid P \text{ correct critical point}\},$$

where

$$\alpha(\omega, P) = 1 - \chi\big(\omega^{-1}(\omega(P) - \delta) \cap B(P, \varepsilon)\big),$$

with $0 < \delta \ll \varepsilon \ll 1$. Here $B(P, \varepsilon)$ is the ball centered at P of radius ε in the Riemannian manifold M^+. If P belongs to $\partial_\emptyset M$ then $\alpha(\omega, P)$ is exactly $(-1)^{\lambda(P)}$. If P belongs to $\partial_I M$, $I \neq \emptyset$, then $\alpha(\omega, P) = (-1)^{\lambda(P)}.\alpha_{nor}(\omega, P)$, where $\alpha_{nor}(\omega, P)$ is the normal index of ω at P. It is defined as follows. Choose a normal slice N at P, that is a closed submanifold of M^+ of dimension $n - \dim \partial_I M$, which intersects $\partial_I M$ in P orthogonally, then

$$\alpha_{nor}(\omega, P) = 1 - \chi\big(\omega^{-1}(\omega(P) - \delta) \cap B(P, \varepsilon) \cap N\big).$$

Let us compute this normal index. We can assume that $\omega(P) = 0$. We can choose a local chart $(x_1, \ldots, x_n)$ centered at P such that $\partial_I M$ is given by $\{x_1 = \ldots = x_k = 0\}$, N is given by $\{x_{k+1} = \ldots = x_n = 0\}$, $k < n$. Locally M is the set $\{x_1 \geq 0, \ldots, x_k \geq 0\}$. Furthermore, since P is a correct point, $\partial\omega/\partial x_j(P) \neq 0$ for each $j \in \{1, \ldots, k\}$ and, by an appropriate change of coordinates, the restriction of ω to N is just the linear form

$$\sum_{j=1}^{k} \frac{\partial\omega}{\partial x_j}(P) x_j.$$

It is then straightforward to see that $\alpha_{nor}(\omega, P) = 1$ if $\partial\omega/\partial x_j(P) > 0$ for all $j \in \{1, \ldots, k\}$ and $\alpha_{nor}(\omega, P) = 0$ otherwise. This proves the theorem. $\square$

3. Hyperplane sections

Recall that V is the zero set of a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree d having 0 as a regular value.

Let $\Sigma_1 \subset S^{n-1}$ be defined as follows : $u \in \Sigma_1$ if and only if there exists a sequence $(x^k) \subset V$ such that $\|x^k\| \to +\infty$ and $\frac{\nabla f}{\|\nabla f\|}(x^k)$ tends to u or $-u$.

Proposition 3.1. *The set Σ_1 is a semi-algebraic set of dimension strictly lower than $n-1$.*

Proof. Let us take $x = (x_1, \ldots, x_n)$ as a coordinate system for $\mathbb{R}^n$ and (x_0, x) for $\mathbb{R}^{n+1}$. Let φ be the embedding of $\mathbb{R}^n$ in S^n given by

$$\varphi(x) = \left(\frac{1}{\sqrt{1+\|x\|^2}}, \frac{x_1}{\sqrt{1+\|x\|^2}}, \ldots, \frac{x_n}{\sqrt{1+\|x\|^2}} \right).$$

The homogenized polynomial F of f is $F(x_0, x) = x_0^d f\left(\frac{x}{x_0}\right) = x_0^d f(\varphi^{-1}(x_0, x))$. The set $F^{-1}(0) \cap S^n \cap \{x_0 > 0\}$ is homeomorphic to V, we denote it by W. It is an analytic manifold of dimension $n-1$, for it is a straight consequence of Euler identity that ∇F and (x_0, x) are colinear if and only if ∇F vanishes. By our assumptions on f, this latter never occurs. Let

$$M = \big\{(x_0, x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \mid (x_0, x) \in W \text{ and } \forall (i,j) \in \{1, \ldots, n\}^2 \ m_{ij}(x_0, x, y) = 0\big\},$$

where $m_{ij}(x_0, x, y) = \begin{vmatrix} \frac{\partial F}{\partial x_i}(x_0, x) & \frac{\partial F}{\partial x_j}(x_0, x) \\ y_i & y_j \end{vmatrix}$. This set is an analytic manifold of dimension n. To see this, take a point $p = (x_0, x, y) \in M$. Since ∇f does not vanish at $\varphi^{-1}(x_0, x)$, we can assume that $\frac{\partial F}{\partial x_1}(x_0, x) \neq 0$ ($\frac{\partial F}{\partial x_1}(x_0, x) = x_0^{d-1} \frac{\partial f}{\partial x_1}(x_0, x)$). This implies that around p, M is defined by the vanishing of $F, m_{12}, \ldots, m_{1n}$ and $x_0^2 + x_1^2 + \cdots + x_n^2 - 1$. A simple computation of determinant show that the gradient vectors of these functions are linearly independent. Then $\overline{M} \setminus M$ is a semi-algebraic of dimension lower than n (see [BCR], Proposition 2.8.12). If we call $\Pi : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R}^n$ the projection on the n last components, $\Pi(\overline{M} \setminus M)$ is semi-algebraic of dimension strictly less than n. We conclude with the fact that $\Sigma_1 = S^{n-1} \cap \Pi(\overline{M} \setminus M)$. $\square$

Lemma 3.2. *If $u \notin \Sigma_1$ then the set of critical points of $P_u : V \mapsto \mathbb{R}$ is compact.*

Proof. If it is not then there is a sequence of points $(x^k) \subset V$, $\|x^k\| \to +\infty$, such that $\nabla f(x^k)$ and u are colinear. □

Lemma 3.3. *There exists a semi-algebraic set $\Sigma_2 \subset S^{n-1}$ of dimension strictly less than $n-1$ such that if $u \notin \Sigma_2$ then $V \cap \{x \in \mathbb{R}^n \mid \mathrm{rank}(x,u) < 2\}$ is finite.*

Proof. Let

$$N = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in V \text{ and } \forall (i,j) \in \{1,\dots,n\}^2 \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = 0\right\}.$$

Using the same arguments as in Proposition 3.1, we see that $N \setminus \{\{0\} \times \mathbb{R}^n\}$ is a Nash manifold of dimension n. Let

$$\begin{array}{cccc} \Pi_y & : & N & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y. \end{array}$$

Bertini-Sard's Theorem for Nash manifolds ([BCR], 9.5.2) implies that the set Σ_y of critical values of Π_y is semi-algebraic of dimension strictly less than n. We take $\Sigma_2 = \Sigma_y \cap S^{n-1}$. For all $u \notin \Sigma_2$, the set $V \cap \{x \in \mathbb{R}^n \mid \mathrm{rank}(x,u) < 2\}$ is $\Pi_y^{-1}(u)$ if $0 \notin V$ and $\Pi_y^{-1}(u) \cup \{0\}$ if $0 \in V$. □

Recall that $\omega(x) = x_1^2 + \cdots + x_n^2$. For each $u \in S^{n-1}$, let Γ_u be the following polar set:

$$\Gamma_u = \{x \in V \mid \mathrm{rank}(\nabla f, \nabla \omega, u) < 3\}.$$

For each $R > 0$, $\Gamma_u \cap S_R^{n-1}$ is the set of critical points of $P_{u|V \cap S_R^{n-1}}$. Lemma 3.2 and Lemma 3.3 imply that for R sufficiently big and for $u \notin \Sigma_1 \cup \Sigma_2$, these critical points are correct for $P_{u|V \cap B_R^n}$ and $P_{u|\{f?0\} \cap B_R^n}$, where $? \in \{\leq, \geq\}$. We are going to study the behaviour at infinity of Γ_u. We need a version at infinity of the Curve Selection Lemma.

Lemma 3.4. *Let $f_1,\dots,f_q, g_1,\dots,g_s, h_1,\dots,h_r \in \mathbb{R}[x_1,\dots,x_n]$ be polynomial functions. Let $U = \{x \in \mathbb{R}^n \mid f_i(x) = 0, i = 1,\dots,q\}$ and $W = \{x \in \mathbb{R}^n \mid g_i(x) > 0, i = 1,\dots,s\}$. Suppose that there exists a sequence $(x^k) \subset U \cap W$ such that $\lim_{k\to+\infty} \|x^k\| = +\infty$ and for all $j \in \{1,\dots,r\}$ $\lim_{k\to+\infty} h_j(x^k) = 0$. Then there exists a real analytic curve $p : (0,\varepsilon) \to U \cap W$ with $\lim_{t\to 0} \|p(t)\| = \infty$, $\lim_{t\to 0} h_j(p(t)) = 0$ for $1 \leq j \leq r$ and of the form $p(t) = a_0 t^\alpha + a_1 t^{\alpha+1} + \cdots$ with $a_0 \in \mathbb{R}^n \setminus \{0\}$ and $\alpha < 0$.*

Proof. See [NZ], Lemma 2. □

Lemma 3.5. *If there exists a sequence $(x^k) \subset \Gamma_u$ such that $\|x^k\| \to +\infty$ and $\lim_{k\to+\infty} |P_u(x^k)| < +\infty$ then $u \in \Sigma_1$.*

Proof. Without loss of generality we can assume that $u = e_1 := (1,0,\dots,0)$. In that case, $P_u = x_1$. By the previous lemma, there exists an analytic curve $p = (p_1,\dots,p_n) : (0,\varepsilon) \to \Gamma_u$ such that $\lim_{t\to 0} \|p(t)\| = +\infty$ and

$\lim_{t\to 0} p_1(t) < +\infty$. Let us consider the expansions as Laurent series of the p_i's:

$$p_i(t) = a_i t^{\alpha_i} + \cdots, \quad i = 1, \ldots, n.$$

Let $\alpha = \min\{\alpha_i \mid i = 1, \ldots, n\}$. Necessarily, $\alpha < 0$ and $\alpha_1 \geq 0$. The function $\omega \circ \gamma$ has then the following expansion:

$$(\omega \circ p)(t) = bt^{2\alpha} + \cdots, b > 0,$$

which gives

$$(\omega \circ p)'(t) = 2\alpha b t^{2\alpha - 1} + \cdots.$$

Since we are in an algebraic setting, the vectors $\nabla f(p(t))$ and $\nabla \omega(p(t))$ are linearly independent for t small enough. Hence we can decompose e_1 the following way:

$$e_1 = \lambda(t) \cdot \nabla f(p(t)) + \mu(t) \cdot \nabla \omega(p(t)). \tag{$*$}$$

Normalizing, we have

$$e_1 = \lambda(t) \cdot \|\nabla f(p(t))\| \cdot \frac{\nabla f(p(t))}{\|\nabla f(p(t))\|} + \mu(t) \cdot \|\nabla \omega(p(t))\| \cdot \frac{\nabla \omega(p(t))}{\|\nabla \omega(p(t))\|}.$$

We are going to prove that $\lim_{t\to 0} \mu(t) \cdot \|\nabla \omega(p(t))\| = 0$ which enables us to conclude. It is straight to see that $\|\nabla \omega(p(t))\|$ has the following expansion:

$$\|\nabla \omega(p(t))\| = ct^{\alpha} + \cdots, c > 0.$$

Together with relation $(*)$, the scalar product $\langle p'(t), e_1 \rangle$ gives:

$$p_1'(t) = \lambda(t) \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p(t)) \cdot p_i'(t) + 2\mu(t) \sum_{i=1}^{n} p_i(t) \cdot p_i'(t).$$

Finally we have $p_1'(t) = \mu(t) \cdot (\omega \circ p)'(t)$ because $0 = (f \circ p)'(t) = \sum_i \frac{\partial f}{\partial x_i}(p(t)) \cdot p_i'(t)$. We deduce that ord $\mu =$ ord $p_1' -$ ord $(\omega \circ p)' \geq \alpha_1 - 2\alpha$ and then ord $\mu \cdot \|\nabla \omega\| \geq \alpha_1 - \alpha > 0$. $\square$

Lemma 3.6. *Let $u \notin \Sigma_1 \cup \Sigma_2$ and let c be a regular value of $P_{u|V}$. There exists $R_c > 0$ such that for $R \geq R_c$ the functions $P_{u|V \cap B_R^n}$ and $P_{u|\{f?0\} \cap B_R^n}$, $? \in \{\leq, \geq\}$, (resp. $-P_{u|V \cap B_R^n}$ and $-P_{u|\{f?0\} \cap B_R^n}$) are not inward at the critical points of $P_{u|V \cap S_R^{n-1}}$ lying in $\{P_u > c\}$ (resp. $\{P_u < c\}$).*

Proof. We prove the case of $P_{u|V \cap B_R^n}$ and $\{P_u > c\}$. First remark that for R sufficiently big, $\{P_u = c\} \cap V$ and S_R^{n-1} intersect transversally so that the critical points mentioned above lie in $\{P_u \neq c\}$. They are correct as already explained before. Put $u = e_1$. Assume that there exist a sequence $(x^k) \subset \Gamma_{e_1}$ lying in $\{x_1 > c\}$ such that $\|x^k\| \to +\infty$ and $x_{1|V}$ is inward at x^k. This means that in the above decomposition $(*)$, $\mu < 0$ at x^k. By the Curve Selection Lemma, there exists an analytic curve $p(t) \subset \Gamma_{e_1}$ with $\lim_{t\to 0} \|p(t)\| = +\infty$ and $\mu(p(t)) < 0$. Since $p_1'(t) = \mu(t) \cdot (\omega \circ p)'(t)$, we see that p_1 is decreasing and bounded as t tends to 0 and then $\lim_{t\to 0}|p_1(t)| < +\infty$, a contradiction. □

Proposition 3.7. *There exists a semi-algebraic set $\Sigma_3 \subset S^{n-1}$ of dimension strictly less than $n-1$ such that if $u \notin \Sigma_3$ then $P_{u|V}$ has only Morse critical points.*

Proof. Let

$$X = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in V \text{ and } \forall (i,j) \in \{1,\dots,n\}^2 \; n_{ij}(x,y) = 0\},$$

where $n_{ij}(x,y) = \begin{vmatrix} \frac{\partial f}{\partial x_i}(x) & \frac{\partial f}{\partial x_j}(x) \\ y_i & y_j \end{vmatrix}$. Using the same arguments as in Proposition 3.1, we see that X is a smooth analytic variety of dimension n. Let

$$\begin{array}{cccc} \Pi_y & : & X & \to & \mathbb{R}^n \\ & & (x,y) & \mapsto & y. \end{array}$$

Bertini-Sard's Theorem for Nash manifolds implies that the set Σ_y of critical values of Π_y is semi-algebraic of dimension strictly less than n. We take $\Sigma_3 = \Sigma_y \cap S^{n-1}$. Then for all $u \notin \Sigma_3$, the set of critical points of $P_{u|V}$, which is exactly $\Pi_y^{-1}(u)$, consists of non-degenerated isolated points. This means that $P_u : V \to \mathbb{R}$ is Morse.

□

Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and let $u \notin \Sigma$. Let $\{p_i\}$ be the set of critical points of $P_{u|V}$ with respective Morse indices $\{\lambda_i\}$. We have:

Theorem 3.8. *Let c be a regular value of $P_{u|V}$, $u \notin \Sigma$. If n is odd:*

$$\chi(V) - \chi(V \cap \{P_u = c\}) = \sum_i (-1)^{\lambda_i},$$

$$\chi(\{f \geq 0\} \cap \{P_u \geq c\}) - \chi(\{f \geq 0\} \cap \{P_u \leq c\}) - \chi(\{f \leq 0\} \cap \{P_u \geq c\}) + \chi(\{f \leq 0\} \cap \{P_u \leq c\}) = \sum_i \operatorname{sign}\langle u, \nabla f(p_i)\rangle \cdot (-1)^{\lambda_i}.$$

If n is even:

$$\chi(V \cap \{P_u \geq c\}) - \chi(V \cap \{P_u \leq c\}) = \sum_i (-1)^{\lambda_i},$$

$$\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big] - \big[\chi(\{f \geq 0\} \cap \{P_u = c\}) - \chi(\{f \leq 0\} \cap \{P_u = c\})\big] = \sum_i \operatorname{sign}\langle u, \nabla f(p_i)\rangle \cdot (-1)^{\lambda_i}.$$

Proof. We assume $u = e_1$ and we prove the case n odd. There exists $R > 0$ sufficiently big such that $\chi(V \cap B_R^n) = \chi(V)$ and $\chi(V \cap B_R^n \cap \{x_1 =$

$c\}) = \chi(V \cap \{x_1 = c\})$. Furthermore by Lemma 3.6, we can assume that the gradient of $x_{1|V\cap S_R^{n-1}}$ at the critical points in $\{x_1 \geq c\}$ (resp. $\{x_1 \leq c\}$) is outward pointing (resp. inward pointing). Applying Theorem 2.6, we get

$$\chi(V \cap B_R^n \cap \{x_1 \geq c\}, V \cap B_R^n \cap \{x_1 = c\}) = \sum_{x_1(p_i)>c} (-1)^{\lambda_i}, \quad (1)$$

$$\chi(V \cap B_R^n \cap \{x_1 \leq c\}, V \cap B_R^n \cap \{x_1 = c\}) = \sum_{x_1(p_i)<c} (-1)^{\lambda_i}. \quad (2)$$

The combination (1)+(2) together with Mayer-Vietoris sequence gives $\chi(V \cap B_R^n) - \chi(V \cap B_R^n \cap \{x_1 = c\}) = \sum_i (-1)^{\lambda_i}$.

Similarly there exists $R > 0$ big enough such that $\chi(\{f?0\} \cap B_R^n) = \chi(\{f?0\})$ and $\chi(\{f?0\} \cap B_R^n \cap \{x_1 = c\}) = \chi(\{f?0\} \cap \{x_1 = c\})$ with $? \in \{\leq, \geq\}$. By Theorem 2.6 and Lemma 3.6,

$$\chi(\{f \geq 0\} \cap B_R^n \cap \{x_1 \geq c\}, \{f \geq 0\} \cap B_R^n \cap \{x_1 = c\}) = \sum_{\substack{x_1(p_i)>c \\ \frac{\partial f}{\partial x_1}(p_i)>0}} (-1)^{\lambda_i}, \quad (3)$$

$$\chi(\{f \geq 0\} \cap B_R^n \cap \{x_1 \leq c\}, \{f \geq 0\} \cap B_R^n \cap \{x_1 = c\}) = \sum_{\substack{x_1(p_i)<c \\ \frac{\partial f}{\partial x_1}(p_i)<0}} (-1)^{\lambda_i}, \quad (4)$$

$$\chi(\{f \leq 0\} \cap B_R^n \cap \{x_1 \geq c\}, \{f \leq 0\} \cap B_R^n \cap \{x_1 = c\}) = \sum_{\substack{x_1(p_i)>c \\ \frac{\partial f}{\partial x_1}(p_i)<0}} (-1)^{\lambda_i}, \quad (5)$$

$$\chi(\{f \leq 0\} \cap B_R^n \cap \{x_1 \leq c\}, \{f \leq 0\} \cap B_R^n \cap \{x_1 = c\}) = \sum_{\substack{x_1(p_i)<c \\ \frac{\partial f}{\partial x_1}(p_i)>0}} (-1)^{\lambda_i}. \quad (6)$$

The combination $(3) - (4) - (5) + (6)$ together with Mayer-Vietoris sequence gives the result. The case n even is proved in the same way, taking into account that if λ is the Morse index of $x_{1|V}$ at p, $n - 1 - \lambda$ is the index of $-x_{1|V}$. $\square$

Remark 3.9. *This result is still true with the weaker assumption that* $u \notin \Sigma_1 \cup \Sigma_2$*, replacing the critical points of* $P_{u|V}$*, which may be degenerated, by the ones of* $P_{\tilde{u}|V}$ *with* $\tilde{u} \notin \Sigma$ *and close to* u.

Corollary 3.10. *For any* $u \notin \Sigma_1 \cup \Sigma_2$*, the expressions*

$$\chi(V \cap \{P_u = c\})$$

and

$$\chi(\{f \geq 0\} \cap \{P_u \geq c\}) - \chi(\{f \geq 0\} \cap \{P_u \leq c\}) - \chi(\{f \leq 0\} \cap \{P_u \geq c\}) + \chi(\{f \leq 0\} \cap \{P_u \leq c\})$$

do not depend on the choice of the regular value c for n odd. The expressions

$$\chi(V \cap \{P_u \geq c\}) - \chi(V \cap \{P_u \leq c\})$$

and

$$\chi(\{f \geq 0\} \cap \{P_u = c\}) - \chi(\{f \leq 0\} \cap \{P_u = c\})$$

do not depend on the choice of the regular value c for n even.

4. The Gauss-Bonnet formula

We prove in this section our Gauss-Bonnet type formula. We need some lemmas.

Lemma 4.1. *There exists a semi-algebraic set $\Sigma_4 \subset S^{n-1}$ of dimension strictly less than $n-1$ such that for all $u \notin \Sigma_4$, 0 is a regular value of $P_{u|V}$.*

Proof. Let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $g(x,y) = \langle x, y\rangle$. Let

$$Y = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid f(x) = 0 \text{ and } g(x,y) = 0\}.$$

Since ∇f and ∇g are linearly independent for $x \neq 0$, $Y \setminus \{\{0\} \times \mathbb{R}^n\}$ is a Nash manifold of dimension $2n-2$. Using the projection Π_y, as in Proposition 3.7, we can conclude that there exists a semi-algebraic set $\Sigma' \subset S^{n-1}$ of dimension lower than $n-2$ such that for all $u \notin \Sigma'$, 0 is a regular value of $P_{u|V\setminus\{0\}}$. It is enough to take $\Sigma_4 = \Sigma'$, if $0 \notin V$ and $\Sigma_4 = \Sigma' \cup \{\pm\frac{\nabla f(0)}{\|\nabla f(0)\|}\}$, if $0 \in V$. □

Let γ be the Gauss mapping defined by

$$\begin{array}{rccc} \gamma & : & V & \to & S^{n-1} \\ & & x & \mapsto & \frac{\nabla f(x)}{\|\nabla f(x)\|}. \end{array}$$

Lemma 4.2. *A point $p \in V$ is a critical point of $P_{u|V}$ if and only if $\gamma(p) = \pm u$.*

Proof. It is clear. □

For the following lemma, we fix $u = (1, 0, \ldots, 0)$. Let p be such that $\gamma(p) = \pm u$ and let $\deg(\gamma, p)$ be the local degree of γ at p.

Lemma 4.3. *The critical point p of $x_{1|V}$ is non-degenerated if and only if p is a regular point of γ. In that case,*

$$\deg(\gamma, p) = (-1)^{n-1} \text{sign} \left(\frac{\partial f}{\partial x_1}\right)^{n-1} (p) \cdot (-1)^{\lambda},$$

where λ is the Morse index of $x_{1|V}$ at p.

Proof. See [Du1], Lemma 2.3. □

Let $(K_r)_{r>0}$ be an exhaustive sequence of compact sets of V. For each $u \in S^{n-1} \setminus (\Sigma \cup \Sigma_4)$ and each $r > 0$, we define $I(r,u) = \sum_i \deg(\gamma, p_i)$ où $\{p_i\} = \gamma^{-1}\{\pm u\} \cap K_r$. Our first aim is to evaluate $\lim_{r\to+\infty} I(r,u)$. We have

Lemma 4.4. *For each $u \notin \Sigma \cup \Sigma_4$, if n is odd*

$$\lim_{r \to +\infty} I(r,u) = \chi(V) - \chi(V \cap \{P_u = 0\}).$$

If n is even

$$\lim_{r \to +\infty} I(r,u) = -\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big] + \big[\chi(\{f \geq 0\} \cap \{P_u = 0\}) - \chi(\{f \leq 0\} \cap \{P_u = 0\})\big].$$

Proof. This is a combination of Theorem 3.8 and the three previous lemmas. □

By Bezout theorem, $|I(r,u)| \leq d(d-1)^{n-1}$ for all $u \notin \Sigma \cup \Sigma_4$. Since $\Sigma \cup \Sigma_4$ is of measure zero, we can apply Lebesgue dominated convergence theorem to get $\lim_{r \to +\infty} \int_{S^{n-1}} I(r,u)du = \int_{S^{n-1}} \lim_{r \to +\infty} I(r,u)du$. Now by an exchange formula ([Du1], Theorem 5.2), we have $\int_{K_r} kdv = \frac{1}{2} \int_{S^{n-1}} I(r,u)du$. Taking the limit leads to

$$\lim_{r \to +\infty} \int_{K_r} kdv = \frac{1}{2} \int_{S^{n-1}} \lim_{r \to +\infty} I(r,u)du.$$

By Lemma 4.4 this shows that $\lim_{r \to +\infty} \int_{K_r} kdv$ does exist and does not depend on the exhaustive sequence chosen. We denote this limit $\int_V kdv$. We have proved:

Theorem 4.5.

If n is odd, $$\int_V kdv = \frac{1}{2}\mathrm{Vol}(S^{n-1})\chi(V) - \frac{1}{2} \int_{S^{n-1}} \chi(V \cap \{P_u = 0\})du.$$

if n is even, $$\int_V kdv = -\frac{1}{2}\mathrm{Vol}(S^{n-1})\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big] + \frac{1}{2} \int_{S^{n-1}} \big[\chi(\{f \geq 0\} \cap \{P_u = 0\}) - \chi(\{f \leq 0\} \cap \{P_u = 0\})\big]du.$$ □

Let us explain why this generalizes the Gauss-Bonnet formula for compact manifolds, which we recall here. If M is a smooth compact $(n-1)$-dimensional manifold in $\mathbb{R}^n$, oriented by its outward-pointing normal vector, and if we denote by N the compact part of $\mathbb{R}^n$ it bounds then:

$$\int_M kdv = \mathrm{Vol}(S^{n-1}) \times \chi(N).$$

If M is even-dimensional, $\chi(N) = \frac{1}{2}\chi(M)$ and then

$$\int_M kdv = \frac{1}{2}\mathrm{Vol}(S^{n-1}) \times \chi(M).$$

The first equality is due to Haefliger [Hae] and Samelson [Sa] and the second one to Hopf [Ho]. It should be noticed that, in [Hae], [Sa] and [Ho], the results deal with the degree of the Gauss map. However, we can deduce easily the above formulas from them for the manifold M is compact.

Let us assume that f is proper and that $f > 0$ at infinity. If n is odd, $\chi(V \cap \{P_u = 0\}) = 0$ for almost all $u \in S^{n-1}$ so we recover easily Hopf's result. If n is even then $\{f = 0\}$ bounds $\{f \leq 0\}$. By Mayer-Vietoris sequence,

$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = 1 - 2\chi(\{f \leq 0\}),$$

and, for almost all $u \in S^{n-1}$,

$$\chi(\{f \geq 0\} \cap \{P_u = 0\}) - \chi(\{f \leq 0\} \cap \{P_u = 0\}) = \\ 1 - 2\chi(\{f \leq 0\} \cap \{P_u = 0\}) + \chi(\{f = 0\} \cap \{P_u = 0\}).$$

Since $\{f = 0\} \cap \{P_u = 0\}$ is even-dimensional,

$$\chi(\{f \geq 0\} \cap \{P_u = 0\}) - \chi(\{f \leq 0\} \cap \{P_u = 0\}) = 1.$$

Using Theorem 4.5, we recover Haefliger and Samelson's result.

Here are examples. Let $V_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\}$ be the 1-sheeted hyperboloid. We have $\chi(V_1) = 0$ and it remains to compute $\int_{S^2} \chi(V_1 \cap \{P_v = 0\}) dv$. Let us write $v = (a, b, c)$ with $a^2 + b^2 + c^2 = 1$. For v such that $a^2 - c^2 \neq 0$ and $c \neq 0$, a point (x_1, x_2, x_3) belongs to $V_1 \cap \{P_v = 0\}$ if and only if:

$$\left(x_1 - \frac{ab}{c^2 - a^2} x_2\right)^2 + \frac{c^2(c^2 - a^2 - b^2)}{(c^2 - a^2)^2} x_2^2 = \frac{c^2}{c^2 - a^2} \quad \text{and}$$

$$ax_1 + bx_2 + cx_3 = 0.$$

If $c^2 - a^2 - b^2 > 0$ then $\chi(V_1 \cap \{P_v = 0\}) = 0$ since $V_1 \cap \{P_v = 0\}$ is either empty or a circle. If $c^2 - a^2 - b^2 < 0$ then $V_1 \cap \{P_v = 0\}$ is a hyperbola and $\chi(V_1 \cap \{P_v = 0\}) = 2$. Now $c^2 - a^2 - b^2 < 0$ if and only if $c^2 < \frac{1}{2}$ and the area of $S^2 \cap \{c^2 < \frac{1}{2}\}$ is $2\pi\sqrt{2}$. Using Theorem 4.5, we find $\int_{V_1} k dv = -2\pi\sqrt{2}$.

Let $V_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 1\}$ be the 2-sheeted hyperboloid. We have $\chi(V_2) = 2$ and it remains to compute $\int_{S^2} \chi(V_2 \cap \{P_v = 0\}) dv$. For v such that $a^2 - c^2 \neq 0$ and $c \neq 0$, a point (x_1, x_2, x_3) belongs to $V_2 \cap \{P_v = 0\}$ if and only if:

$$\left(x_1 - \frac{ab}{c^2 - a^2} x_2\right)^2 - \frac{c^2(c^2 - a^2 + b^2)}{(c^2 - a^2)^2} x_2^2 = \frac{c^2}{c^2 - a^2} \quad \text{and}$$

$$ax_1 + bx_2 + cx_3 = 0.$$

As above, if $a^2 < \frac{1}{2}$ then $\chi(V_2 \cap \{P_v = 0\}) = 2$ and if $a^2 > \frac{1}{2}$ then $\chi(V_2 \cap \{P_v = 0\}) = 0$. Using Theorem 4.5, we find $\int_{V_2} k dv = 2\pi(2 - \sqrt{2})$.

Let $V_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3 = 0\}$ be the elliptic paraboloid. We have $\chi(V_3) = 1$. For v such that $c \neq 0$, a point (x_1, x_2, x_3) belongs to $V_3 \cap \{P_v = 0\}$ if and only if:

$$\left(x_1 + \frac{a}{2c}\right)^2 + \left(x_2 + \frac{b}{2c}\right)^2 = \frac{a^2 + b^2}{c^2} \qquad \text{and}$$

$$ax_1 + bx_2 + cx_3 = 0.$$

Hence $V_3 \cap \{P_v = 0\}$ is a circle and $\chi(V_3 \cap \{P_v = 0\}) = 0$. Using Theorem 4.5, we find $\int_{V_3} kdv = 2\pi$.

Let $V_4 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3 = 0\}$ be the hyperbolic paraboloid. We have $\chi(V_4) = 1$. For v such that $c \neq 0$, a point (x_1, x_2, x_3) belongs to $V_4 \cap \{P_v = 0\}$ if and only if:

$$\left(x_1 + \frac{a}{2c}\right)^2 - \left(x_2 - \frac{b}{2c}\right)^2 = \frac{a^2 - b^2}{c^2} \qquad \text{and}$$

$$ax_1 + bx_2 + cx_3 = 0.$$

Hence $V_4 \cap \{P_v = 2\}$ is a hyperbola and $\chi(V_4 \cap \{P_v = 0\}) = 2$. Using Theorem 4.5, we find $\int_{V_4} kdv = -2\pi$.

5. Case of plane curves

Let $f : \mathbb{R}^2 \to \mathbb{R}$ of degree d be such that 0 is a regular value of f. Let f_d be the homogeneous component of degree d of f. Theorem 4.5 becomes:

Theorem 5.1. *If d is odd,*

$$\int_v kdv = -\pi\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big].$$

If d is even,

$$\int_v kdv = -\pi\big[\chi(\{f \geq 0\}) - \chi(\{f \leq 0\})\big] + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sign} f_d(-\tan\theta, 1) d\theta.$$

Proof. We have to compute $\chi(\{g \geq 0\}) - \chi(\{g \leq 0\})$ for a polynomial $g : \mathbb{R} \to \mathbb{R}$ of degree δ with simple roots. This is exactly $I_g^+ - I_g^-$, where I_g^+ (resp. I_g^-) is the number of intervals where g is positive (resp. negative). Since the number of roots of g is congruent to δ modulo 2, we see that $I_g^+ - I_g^- = 0$ if δ is odd and $I_g^+ - I_g^- = \operatorname{sign} f(\infty) = \operatorname{sign} g_\delta(1)$ if δ is even. We conclude as we did in [Du1] using the usual parametrization of S^1 and the fact that for almost all $u \in S^1$, $f_{|\{P_u=0\}}$ has simple roots (Lemma 4.1) □

Here is an example. The following polynomial $f(x, y) = x^2y^2 + 2xy + (y^2 - 1)^2$ appears in [TZ]. They study the topology of the fibres $f^{-1}(t)$ for $|t|$ small. If $t < 0$ then $f^{-1}(t)$ has two circle components. For $t \geq 0$, $f^{-1}(t)$

has two lines components. Let us apply Theorem 5.1 to the fibres $f^{-1}(0)$ and $f^{-1}(10^{-3})$. Using results from [Du2] and a computer program due to Lecki and Szafraniec, we find:

$$\chi(\{f \geq 0\}) - \chi(\{f \leq 0\}) = -1,$$

$$\chi(\{f \geq 10^{-3}\}) - \chi(\{f \leq 10^{-3}\}) = -3.$$

Hence $\int_{f^{-1}(0)} kdv = 2\pi$ and $\int_{f^{-1}(10^{-3})} kdv = 4\pi$. These difference between the two fibres is explained by the results of [TZ].

6. The complex case

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial such that 0 is a regular value of f. Let $V = f^{-1}(0)$ and let K be the Lipschitz-Killing curvature of V viewed as a $(2n-2)$-dimensional real manifold embedded in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Let $(K_r)_{r>0}$ be an exhaustive sequence of compact sets that cover V. We have:

Theorem 6.1. *The limit* $\lim_{r\to+\infty} \int_{K_r} Kdv$ *does exist and does not depend on the sequence chosen. Furthermore, putting* $\int_V Kdv$ *for this limit,*

$$\int_V Kdv = \frac{\text{Vol}(S^{2n-2})}{2} \left[\chi(V) - \chi(V \cap H)\right],$$

where H *is a generic linear hyperplane.*

Proof. For any hyperplane H, defined by $H = \{x \in \mathbb{C}^n \mid h(x) = 0\}$ where h is a linear form, let Γ_H be the polar variety given by:

$$\Gamma_H = \{x \in \mathbb{C}^n \mid \text{rank}(df(x), dh(x)) \leq 2\}.$$

It is known (see [Ti]) that generically the intersection $\Gamma_H \cap V$ is finite, consists of non-degenerate points and $\sharp\Gamma_H \cap V$ does not depend on H. Let us put $\gamma = \sharp\Gamma_H \cap V$. Then for a generic H (see [Ti]):

$$\chi(V) - \chi(V \cap H) = (-1)^{n-1}\gamma.$$

Now if $\gamma_{H,r} = \sharp\Gamma_H \cap V \cap K_r$, we have $\lim_{r\to+\infty} \gamma_{H,r} = (-1)^{n-1}\left[\chi(V) -\chi(V \cap H)\right]$, hence, by Bezout's theorem and Lebesgue's theorem:

$$\lim_{r\to+\infty} \int_{G_{n-1}} \gamma_{H,r} dH = \frac{(-1)^{n-1}\pi^{n-1}}{(n-1)!} \left[\chi(V) - \chi(V \cap H)\right],$$

where G_{n-1} is the Grassmanian of complex hyperplanes in $\mathbb{C}^n$. By an exchange formula due to Langevin [La],

$$\int_{V\cap K_r} a_{n-1}^{-1}(-1)^{n-1} Kdv = \int_{G_{n-1}} \gamma_{H,r} dH,$$

where a_{n-1} is defined in [La]. Combining these two formulas gives the result. □

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